From Corecursive Algebras to Corecursive Monads

Jiří Adámek, Mahdieh Haddadi, and Stefan Milius

Institut für Theoretische Informatik, Technische Universität Braunschweig, Germany
adamek@iti.cs.tu-bs.de,mail@stefan-milius.eu,
haddadi_1360@yahoo.com

To the memory of Stephen Bloom

Abstract. An algebra is called corecursive if from every coalgebra a unique coalgebra-to-algebra homomorphism exists into it. We prove that free corecursive algebras are obtained as a coproduct of the final coalgebra (considered as an algebra) and free algebras. The monad of free corecursive algebras is proved to be the free corecursive monad, where the concept of corecursive monad is a generalization of Elgot's iterative monads, analogous to corecursive algebras generalizing completely iterative algebras. We also characterize the Eilenberg-Moore algebras for the free corecursive monad and call them Bloom algebras.

1 Introduction

The study of structured recursive definitions is fundamental in many areas of computer science. It can use algebraic methods extended by suitable recursion concepts. One such example are completely iterative algebras: these are algebras in which recursive equations with parameters have unique solutions, see [16]. In the present paper we study corecursive algebras, which are algebras for a given endofunctor H in which recursive equations without parameters have unique solutions or, equivalently, which for every coalgebra have a unique coalgebra-to-algebra morphism. V. Capretta, T. Uustalu and V. Vene [12] present applications of corecursive algebras for the semantics of structural corecursion in languages of total functional programming [20], such as R. Cockett's charity [13], and other settings, where unrestricted general recursion is unavailable. The dual concept, recursive coalgebra, was introduced by G. Osius in [17] to categorically capture well-founded induction. For endofunctors weakly preserving pullbacks P. Taylor proved that corecursive algebras are equivalent to parametrically recursive ones, see [18]. Recursive coalgebras were also studied by V. Capretta et al. [11]. In the dual situation, since weak preservation of pushouts is rare, the concepts of corecursive algebra and completely iterative one usually do not coincide. The former was studied by V. Capretta et al. [12], and various counter-examples demonstrating e.g. the difference of the two concepts for algebras can be found there.

In the present paper we contribute to the development of the mathematical theory of corecursive algebras. The goal is to eventually arrive at a useful body of results and constructions for these algebras. A major ingredient of any theory of algebraic structures is the study of how to freely endow an object with the structure of interest. So the main focus of the present paper are corecursive H-algebras freely generated by an object Y. Let FY denote the free H-algebra on Y and T the final H-coalgebra (which, due to Lambek's Lemma, can be regarded as an algebra). We prove that the coproduct of these two algebras

$$MY = T \oplus FY$$

is the free corecursive algebra on Y. Here $\oplus$ is the coproduct in the category of H-algebras. For example for the endofunctor $HX = X \times X$ the algebra MY consists of all (finite and infinite) binary trees with finitely many leaves labelled in Y.

We also introduce the concept of a corecursive monad. This is a weakening of completely iterative monads of C. Elgot, S. Bloom and R. Tindell [15] analogous to corecursive algebras as a weakening of completely iterative ones. The monad $Y \mapsto MY$ of free corecursive algebras is proved to be corecursive, actually, this is the free corecursive monad generated by H. For endofunctors of **Set** we also prove the converse: whenever H generates a free corecursive monad, then it has free corecursive algebras (and the free monad is then given by the corresponding adjunction).

Finally, we study the equational properties of the solution operation in corecursive algebras. In category-theoretic terms, we characterize the Eilenberg-Moore algebras for the free corecursive monad: they are H-algebras equipped with an operation † that assigns to every recursive equation without parameters a solution, where † is subject to one axiom stating that the assignment of solutions is functorial (or uniform). We call these algebras Bloom algebras; they are analogous to the complete Elgot algebras of [5] where the corresponding monad was the free completely iterative monad on H. The characterization of the Eilenberg-Moore algebras for the free corecursive monad can be understood as a kind of completeness result: all equational properties of † that hold in every corecursive algebra follow from the properties of † given in the definition of a Bloom algebra.

Acknowledgment. We are grateful to Paul Levy for interesting discussions and his formulation of Proposition 3.8.

2 Corecursive Algebras

The following definition is the dual of the concept introduced by G. Osius in [17] and studied by P. Taylor [18]. We assume throughout the paper that a category $\mathcal{A}$ and an endofunctor $H : \mathcal{A} \to \mathcal{A}$ are given. We denote by $\mathsf{Alg}\, H$ the category of algebras $a : HA \to A$ and homomorphisms, and by $\mathsf{Coalg}\, H$ the category of coalgebras $e : X \to HX$ and homomorphisms. A coalgebra-to-algebra morphism from the latter to the former is a morphism $f : X \to A$ such that $f = a.Hf.e$.

Definition 2.1. *An algebra $a : HA \to A$ is called* corecursive *if for every coalgebra $e : X \to HX$ there exists a unique coalgebra-to-algebra homomorphism $e^\dagger : X \to A$. That is, the square*

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle a} \\ HX & \xrightarrow[He^\dagger]{} & HA \end{array}$$

commutes. We call e *an* equation morphism *and* $e^\dagger$ *its* solution.

Example 2.2. (1) V. Capretta et al. [12] studied this concept of corecursive algebras and compared it with a number of related concepts. A concrete example of corecursive algebra from [12], for the endofunctor $HX = E \times X \times X$ of **Set**, is the set E^ω of all streams. The operation $a : E \times E^\omega \times E^\omega \to E^\omega$ is given by $a(e, u, v)$ having head e and continuing by the merge of u and v.

(2) If H has a final coalgebra $\tau : T \to HT$, then by Lambek's Lemma τ is invertible and the resulting algebra $\tau^{-1} : HT \to T$ is corecursive. In fact, it is the initial corecursive algebra, that is, for every corecursive algebra (A, a) a unique algebra homomorphism from (T, τ^{-1}) exists, see (the dual of) Proposition 2 in [11]. There also the converse is proved (dual of Proposition 7), that is, if the initial corecursive algebra exists, then it is a final coalgebra (via the inverse of the algebra structure).

(3) The trivial final algebra $H1 \to 1$, where 1 is the final object in $\mathcal{A}$, is clearly corecursive.

(4) If $a : HA \to A$ is a corecursive algebra, then so is $Ha : HHA \to HA$, see the dual of Proposition 6 in [11].

(5) Combining (3) and (4) we conclude that the final ω^{op}-chain

$$1 \xleftarrow{a} H1 \xleftarrow{Ha} HH1 \xleftarrow{HHa} \dots$$

consists of corecursive algebras. Indeed, the continuation to $H^i 1$ for all ordinals (with $H^i 1 = \lim_{k \leq i} H^k 1$ for all limit ordinals) also yields corecursive algebras. This follows from Proposition 2.6 below.

Remark 2.3. For an endofunctor of **Set**, we can view $e : X \to HX$ as a system of recursive equations using variables from the set X and $e^\dagger : X \to A$ as the solution of the system. We illustrate this on classical algebras for a signature Σ. Equivalently, these are the algebras for the polynomial set functor

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^n$$

where n is the arity of σ. For every set X (of recursion variables) and every system of mutually recursive equations

$$x = \sigma(x_1, \dots, x_n),$$

one for every $x \in X$, where $\sigma \in \Sigma$ has arity n and $x_i \in X$, we get the corresponding coalgebra $e : X \to H_\Sigma X$with $x \mapsto (x_1, \dots, x_n)$ in the σ-summand X^n. The square in Definition 2.1 tells us that the substitution of $e^\dagger(x)$ for $x \in X$ makes the formal equations $x = \sigma(x_1, \dots, x_n)$ identities in A:

$$e^\dagger(x) = \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_n)).$$

Example 2.4. Binary algebras: For $HX = X \times X$, every algebra (given by the binary operation "$*$" on a set A) which is corecursive has a unique *idempotent* $i = i * i$. This is the solution of the recursive equation

$$x = x * x$$

expressed by the isomorphism $e : 1 \xrightarrow{\sim} 1 \times 1$. Moreover the idempotent is *completely factorizable*, where the set of all completely factorizable elements is the largest subset of A such that every element a in it can be factorized as $a = b * c$, with b, c completely factorizable. The corecursiveness of A implies that no other element but i is completely factorizable: consider the system of recursive equations

$$x_\epsilon = x_0 * x_1, \qquad x_0 = x_{00} * x_{01}, \qquad \cdots \qquad x_w = x_{w0} * x_{w1}, \qquad \cdots$$

for all binary words w. Every completely factorizable element a provides a solution $e^\dagger$ with $e^\dagger(x_\epsilon) = a$. Since solutions are unique, $a = i$.

Conversely, every binary algebra A with an idempotent i which is the only completely factorizable element is corecursive. Indeed, given a morphism $e : X \to X \times X$, the constant map $e^\dagger : X \to A$ with value i is a coalgebra-to-algebra morphism. Conversely, if $e^\dagger$ is a coalgebra-to-algebra morphism, then for every $x \in X$, the element $e^\dagger(x)$ is clearly completely factorizable. Therefore, $e^\dagger(x) = i$.

Remark 2.5. Recall the concept of *completely iterative algebra* (*cia* for short) from [16]: it is an algebra $a : HA \to A$ such that for every "flat equation" morphism $e : X \to HX + A$ there exists a unique solution, i.e. a unique morphism $e^\dagger$ such that

$$e^\dagger = (X \xrightarrow{e} HX + A \xrightarrow{He^\dagger + A} HA + A \xrightarrow{[a,A]} A).$$

This is obviously stronger than corecursiveness, because every coalgebra $e : X \to HX$ yields a flat equation morphism $\mathsf{inl}.e : X \to HX + A$, where $\mathsf{inl} : X \to X + A$ denotes the left-hand coproduct injection.[1] Then solutions of e are in bijective correspondence with coalgebra-to-algebra homomorphisms from e to a. Since the former exists uniquely so does the latter. Thus, for example, in the category of complete metric spaces with distance less than one and nonexpanding functions, all algebras for contracting endofunctors (in the sense of P. America and J. Rutten [9]) are corecursive, because, as proved in [16], they are cia's. Here is a concrete example: $HX = X \times X$ equipped with the metric taking $1/2$ of the maximum of the two distances is contracting. Thus every binary algebra whose operation is contracting is corecursive.

Proposition 2.6. *Let $\mathcal{A}$ be a complete category. Then corecursive algebras are closed under limits in $\mathsf{Alg}\,H$. Thus, limits of corecursive algebras are formed on the level of $\mathcal{A}$.*

Lemma 2.7. *Every homomorphism $h : (A, a) \to (B, b)$ in $\mathsf{Alg}\,H$ with (A, a) and (B, b) corecursive, preserves solutions. That is, given a coalgebra $e : X \to HX$ with a solution $e^\dagger : X \to A$ in the domain algebra, then $h.e^\dagger : X \to B$ is the solution in the codomain one.*

[1] Similarly, $\mathsf{inr} : A \to X + A$ denotes the right-hand coproduct injection.

We thus consider corecursive algebras as a full subcategory $\mathsf{Alg}_C\, H$ of $\mathsf{Alg}\, H$. We obtain a forgetful functor $\mathsf{Alg}_C\, H \to \mathcal{A}$ with $(A, a) \mapsto A$. In Section 4 we prove that this forgetful functor has a left adjoint, that is, free corecursive algebras exist, if and only if a terminal coalgebra T exists and every object Y generates a free algebra FY (i.e., the forgetful functor $\mathsf{Alg}\, H \to \mathcal{A}$ has a left adjoint). This equivalence holds for all set functors, and for them the formula for the free corecursive algebra is $T \oplus FY$, where $\oplus$ is the coproduct in $\mathsf{Alg}\, H$.

3 Bloom Algebras

For iterative algebras, it was proved in [6] that every finitary functor H of $\mathcal{A}$ has a free iterative algebra, and the resulting monad $\mathbb{R}$ of $\mathcal{A}$ is a free iterative monad. The next step was a characterization of the Eilenberg-Moore algebras for $\mathbb{R}$ that were called Elgot algebras, see [5]. An Elgot algebra has for every finitary flat equation e a solution $e^\dagger$, but not necessarily unique. Instead, Elgot algebras are equipped with a solution operation $e \mapsto e^\dagger$ satisfying two "natural" axioms.

In the present section we take the corresponding step for corecursive algebras. We introduce Bloom algebras as algebras equipped with an operation assigning to every coalgebra e a solution $e^\dagger$, and the operation $\dagger$ forms a functor. Later we prove that Bloom algebras are (analogously to Elgot algebras) precisely the Eilenberg-Moore algebras for the free corecursive monad, see Theorem 4.13.

Definition 3.1. *A* Bloom algebra *is a triple $(A, a, \dagger)$ where $a : HA \to A$ is an H-algebra and $\dagger$ is an operation assigning to every coalgebra $e : X \to HX$ a coalgebra-to-algebra homomorphism $e^\dagger : X \to A$ so that $\dagger$ is* functorial. *This means that we can define a functor $\dagger : \mathsf{Coalg}\, H \to \mathcal{A}/A$. More explicitly, for every H-coalgebra homomorphism $h : (X, e) \to (Y, f)$ we have $f^\dagger.h = e^\dagger : X \to A$.*

Example 3.2. (1) Every corecursive algebra can be considered as a Bloom algebra. Indeed, functoriality easily follows from the uniqueness of solutions due to the diagram

$$\begin{array}{ccccc} X & \xrightarrow{h} & X' & \xrightarrow{f^\dagger} & A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle Hf} & & \uparrow{\scriptstyle a} \\ HX & \xrightarrow[Hh]{} & HX' & \xrightarrow[Hf^\dagger]{} & HA \end{array}$$

(2) Let $\mathcal{A}$ have finite products. An algebra $a : A \times A \to A$ for $HX = X \times X$ can be equipped with a Bloom algebra structure if and only if it has an idempotent global element, that is $i : 1 \to A$ satisfying $a.(i \times i) = i$ (recall that $1 \times 1 = 1$). More precisely:

(a) Given an idempotent i, we have a Bloom algebra $(A, a, \dagger)$, where $\dagger$ is given by

$$e^\dagger = (X \xrightarrow{!} 1 \xrightarrow{i} A).$$

(b) Given a Bloom algebra $(A, a, \dagger)$, there exists an idempotent i such that $\dagger$ is the constant function with value $e^\dagger = i.!$.

(3) Every group, considered as a binary algebra in **Set**, is thus a Bloom algebra in a unique sense.

(4) Every continuous algebra is a Bloom algebra if we define $e^\dagger$ to be the least solution of e. More detailed, let H be a locally continuous endofunctor of the category CPO of complete ordered sets (i.e., partially ordered sets with a least element and with joins of ω-chains) and continuous functions. For every continuous algebra $a : HA \to A$ and every equation morphism $e : X \to HA$, we can define in $\mathsf{CPO}(X, A)$ a function $e^\dagger : X \to A$ as the join of the sequence $e^\dagger_n$ defined by $e^\dagger_0 = \mathsf{const}_\perp$ and $e^\dagger_{n+1} = a.He^\dagger_n.e$. Thus, the least solution is $e^\dagger = \bigvee e^\dagger_n$ and $(A, a, \dagger)$ is a Bloom algebra.

(5) Every limit of Bloom algebras is a Bloom algebra. Indeed, this is proved precisely as Proposition 2.6.

(6) Every complete Elgot algebra in the sense of [5] is a Bloom algebra.

Definition 3.3. *By a* homomorphism *of Bloom algebras from $(A, a, \dagger)$ to $(B, b, \ddagger)$ is meant an algebra homomorphism $h : (A, a) \to (B, b)$ preserving solutions, that is, for every coalgebra $e : X \to HX$ we have*

$$e^\ddagger = (X \xrightarrow{e^\dagger} A \xrightarrow{h} B).$$

We denote by $\mathsf{Alg}_B\, H$ the corresponding category of Bloom algebras.

Proposition 3.4. *An initial Bloom algebra is precisely a final coalgebra.*

More precisely, the statement in Example 2.2(2) generalizes from corecursive algebras to Bloom algebras. In fact, the proof in [11] can be used again.

Lemma 3.5. *If $(A, a, \dagger)$ is a Bloom algebra and $h : (A, a) \to (B, b)$ is a homomorphism of algebras, then there is a unique structure of a Bloom algebra on (B, b) such that h is a solution preserving morphism. We call it, the Bloom algebra* induced *by h.*

Remark 3.6. We are going to characterize the left adjoint of the forgetful functor

$$U : \mathsf{Alg}_B\, H \to \mathcal{A} \text{ with } (A, a, \dagger) \mapsto A.$$

In other words, free Bloom algebras exists. Moreover, these are coproducts $T \oplus FY$ of free algebras and the final coalgebra. For that we first attend to the existence of those ingredients.

Lemma 3.7. *Let $\mathcal{A}$ be a complete category. If H has a free Bloom algebra on an object Y with $\mathcal{A}(Y, HY) \neq \emptyset$, then H has a final coalgebra.*

Proof. The free Bloom algebra $(A, a, \dagger)$ on Y is weakly initial in $\mathsf{Alg}_B\, H$. To see this, choose a morphism $e : Y \to HY$. For every Bloom algebra $(B, b, \ddagger)$ the solution $e^\ddagger : Y \to B$ extends to a homomorphism $h : (A, a, \dagger) \to (B, b, \ddagger)$ of Bloom algebras.

Since $\mathsf{Alg}_B\, H$ is complete by Example 3.2(5), we can use Freyd's Adjoint Functor Theorem. The existence of a weakly initial object implies that $\mathsf{Alg}_B\, H$ has an initial object. Now apply Proposition 3.4. □

Proposition 3.8. *Let (T, τ) be a final coalgebra for H. The category of Bloom algebras for H is isomorphic to the slice category $(T, \tau^{-1})/\mathsf{Alg}\, H$.*

Construction 3.9. Free-algebra chain. Recall from [3] that if $\mathcal{A}$ is cocomplete, we can define a chain constructing the free algebra on Y as follows:

$$Y \xrightarrow{\mathsf{inr}} HY + Y \xrightarrow{H\mathsf{inr}+Y} H(HY+Y)+Y \longrightarrow \cdots$$

We mean the essentially unique chain $V : \mathsf{Ord} \to \mathcal{A}$ with

$$V_0 = Y, \quad V_{i+1} = HV_i + Y, \quad \text{and} \quad V_j = \mathrm{colim}_{k<j} V_k \text{ for limit ordinals } j,$$

whose connecting morphisms $v_{ij} : V_i \to V_j$ are defined by

$$v_{0,1} \equiv \mathsf{inr} : Y \to HY + Y \qquad \text{and} \qquad v_{i+1,j+1} \equiv Hv_{i,j} + Y,$$

and for limit ordinals j, $(v_{k,j})_{k<j}$ is the colimit cocone.

This chain is called the *free-algebra chain*. If it *converges* at some ordinal λ, that means that if $v_{\lambda,\lambda+1}$ is an isomorphism, then V_λ is a free algebra on Y. More detailed: this isomorphism turns V_λ into a coproduct $V_\lambda = HV_\lambda + Y$ and thus V_λ is an H-algebra via the right-hand injection, and the left-hand one $Y \to V_\lambda$ is the universal arrow.

Definition 3.10. *(See [19]) We say that monomorphisms are* constructive *provided that*

(a) *if $m_i : A_i \to B_i$ are monomorphisms for $i = 1, 2$ then $m_1 + m_2 : A_1 + A_2 \to B_1 + B_2$ is a monomorphism,*
(b) *coproduct injections are monomorphisms, and*
(c) *if $a_i : A_i \to A$, ($i < \alpha$), is a colimit of an α-chain and $m : A \to B$ has all composites $m.a_i$ monic, then m is monic.*

Example 3.11. Sets, posets, graphs and abelian groups have constructive monomorphisms. If $\mathcal{A}$ has constructive monomorphisms, then all functor categories $\mathcal{A}^{\mathcal{C}}$ do. In all locally finitely presentable categories (c) holds (see [8]) but (a) and (b) can fail. For example, in the category of rings (b) fails because the initial ring $\mathbb{Z}$ has morphisms $f : \mathbb{Z} \to A$ that fail to be monomorphisms. And f is a coproduct injection of $A = A + \mathbb{Z}$.

Proposition 3.12. *Let $\mathcal{A}$ be a cocomplete, wellpowered category with constructive monomorphisms. If H has a free Bloom algebra on Y and preserves monomorphisms, then it also has a free algebra on Y.*

Sketch of proof. Given a free Bloom algebra MY the coproduct $HMY + Y$ also carries the structure of a Bloom algebra. From that we derive $MY \simeq HMY + Y$ and obtain a cone of the free-algebra chain formed by monomorphisms $m_i : V_i \to MY$. Since $\mathcal{A}$ is wellpowered, the chain converges. ☐

Notation 3.13. The coproduct in $\mathsf{Alg}\, H$ is denoted by $(A, a) \oplus (B, b)$.

Theorem 3.14. *Suppose that H has a terminal coalgebra T, a free algebra FY on Y, and their coproduct $T \oplus FY$. Then the last algebra is the free Bloom algebra induced by* $\mathsf{inl} : T \to T \oplus FY$ *(cf. Lemma 3.5) with the universal arrow* $\mathsf{inr}.\eta : Y \to T \oplus FY$.

Proof. Given a Bloom algebra $(B, b, \ddagger)$ and morphism $g : Y \to B$, we obtain a unique homomorphism $\overline{g} : (FY, \varphi_Y) \to (B, b)$ with $g = \overline{g}.\eta$. We also have a unique solution preserving homomorphism $f : (T, \tau^{-1}, \dagger) \to (B, b, \ddagger)$, see Proposition 3.4. This yields a homomorphism $[f, \overline{g}] : T \oplus FY \to B$ which is clearly solution preserving: recall from Lemma 3.5 that solutions in $T \oplus FY$ have the form $\mathsf{inl}.e^\dagger$. Thus, $[f, \overline{g}].(\mathsf{inl}.e^\dagger) = f.e^\dagger = e^\ddagger$. And this is the desired morphism since $[f, g].\mathsf{inr}.\eta = \overline{g}.\eta = g$.

Conversely, given a solution preserving homomorphism $h : T \oplus FY \to B$ with $h.\mathsf{inr}.\eta = g$, then $h = [f, \overline{g}]$, because $h.\mathsf{inl} : T \to B$ is clearly solution preserving, hence $h.\mathsf{inl} = f$. Also $h.\mathsf{inr}$ is a homomorphism from FY with $h.\mathsf{inr}.\eta = g$, thus $h.\mathsf{inr} = \overline{g}$. □

Recall from [8] that given an infinite cardinal number λ, a functor is called λ-*accessible* if it preserves λ-filtered colimits. An object X whose hom-functor $\mathcal{A}(X, -)$ is λ-accessible is called λ-*presentable*. A category $\mathcal{A}$ is *locally* λ-*presentable* if it has (a) colimits, and (b) a set of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathcal{A}$. For a λ-accessible endofunctor H, the category $\mathsf{Alg}\, H$ is also locally λ-presentable (see [8]).

Corollary 3.15. *Every accessible endofunctor of a locally presentable category has free Bloom algebras. They have the form $T \oplus FY$.*

4 Free Corecursive Algebras

For accessible functors H we now prove that free corecursive algebras MY exist and, if H preserves monomorphisms, they coincide with the free Bloom algebras $MY = T \oplus FY$. Moreover an iterative construction of these free algebras (closely related to the free algebra chain in Construction 3.9) is presented.

We first prove that the category of corecursive algebras is strongly epireflective in the category of Bloom algebras. That is, the full embedding is a right adjoint, and the components of the unit of the adjunction are strong epimorphisms.

Proposition 4.1. *For every accessible endofunctor of a locally presentable category, corecursive algebras form a strongly epireflective subcategory of the category of Bloom algebras. In particular, every Bloom subalgebra of a corecursive algebra is corecursive.*

Corollary 4.2. *Every accessible endofunctor of a locally presentable category has free corecursive algebras.*

Indeed, since the functors $\mathsf{Alg}_C\, H \hookrightarrow \mathsf{Alg}_B\, H$ and $\mathsf{Alg}_B\, H \to \mathcal{A}$ have left adjoints by Corollary 3.15 and Proposition 4.1, this composite has a left adjoint, too.

Remark 4.3. We conjecture that in the generality of the above corollary, the free corecursive algebras are $T \oplus FY$ (as in Corollary 3.15). But we can only prove this in case

H preserves monomorphisms and monomorphisms are constructive. We are going to apply the following transfinite construction of free corecursive algebras closely related to the free-algebra construction of 3.9

Construction 4.4. Free Corecursive Chain. Let $\mathcal{A}$ be cocomplete and H have a final coalgebra (T, τ). We define an essentially unique chain $U : \mathsf{Ord} \to \mathcal{A}$ by

$$U_0 = T, \quad U_{i+1} = HU_i + Y, \quad \text{and} \quad U_j = \mathrm{colim}_{k<j} U_k \text{ for limit ordinals } j.$$

The connecting morphisms $u_{i,j} : U_i \to U_j$ are defined by

$$u_{0,1} = (T \xrightarrow{\tau} HT \xrightarrow{\mathsf{inl}} HT + Y), \quad u_{i+1,j+1} = Hu_{i,j} + id_Y$$

and for limit ordinals j, $(u_{k,j})_{k<j}$ is the colimit cocone.
We say that the chain *converges* at λ if the connecting morphism $u_{\lambda,\lambda+1}$ is an isomorphism, thus $U_\lambda = HU_\lambda + Y$ so that U_λ is an H-algebra (via inl) connected to Y via $\mathsf{inr} : Y \to U_\lambda$.

Proposition 4.5. *Let $\mathcal{A}$ be a cocomplete and wellpowered category with constructive monomorphisms, and let H preserve monomorphisms and have a final coalgebra. If the corecursive chain for Y converges in λ steps, then $U_\lambda = T \oplus FY$.*

Sketch of proof. The free algebra FY exists because we have a natural transformation $m_i : V_i \to U_{i+1}$ from the free-algebra chain to the corecursive chain: put $m_0 = inr : Y \to HT + Y$ and $m_{i+1} = Hm_i + id_Y$. Since the m_i's are monomorphisms and U_i converges, so does V_i. Thus $FY = V_\rho$ for some $\rho \geq \lambda$. One readily proves that U_λ is a coproduct (in $\mathsf{Alg}\, H$) of V_ρ and T w.r.t. the injections $m_\rho : FY \to MY$ and $u_{0,\lambda} : T \to MY$. □

Theorem 4.6. *Let $\mathcal{A}$ be a locally presentable category with constructive monomorphisms. Every accessible endofunctor preserving monomorphisms has free corecursive algebras $MY = T \oplus FY$.*

Sketch of proof. Let X be an accessible endofunctor of $\mathcal{A}$. From Theorem 3.14 we know that $T \oplus FY$ is a free Bloom algebra, thus, it is sufficient to prove that this algebra is corecursive. For that, we use Proposition 4.1 and find a corecursive algebra such that $T \oplus FY$ is its subalgebra; this will finish the proof.

The endofunctor $H(-) + Y$ is also accessible. Thus, it also has a final coalgebra. We denote it by TY. The components of the inverse of its algebra structure $TY \xrightarrow{\sim} H(TY) + Y$ are denoted by $\tau_Y : H(TY) \to TY$ and $\eta_Y : Y \to TY$. As proved in [16], the algebra TY is a cia for H, cf. Remark 2.5. Next one proves that $T \oplus FY$ is a subalgebra of this H-algebra TY.

Then we use the final opchain of $H(-) + Y$ which converges and yields TY. This yield a canonical monomorphism from T to TY in $\mathsf{Alg}\, H$, and this is used to start a cocone of monomorphisms in $\mathsf{Alg}\, H$ on the corecursive chain U_i for H with vertex TY. Since $\mathcal{A}$ is wellpowered, this implies that U_i converges. □

Example 4.7. Free corecursive algebras MY obtained as U_ω.
(1) For $H = Id$ we have

$$MY = U_\omega = 1 + Y + Y + Y + \cdots$$

Indeed, the final H-coalgebra is $T = 1$, and

$$U_1 = T + Y, \quad U_2 = T + Y + Y, \quad \cdots$$

with colimit $U_\omega = 1 + Y + Y + Y + \cdots$.
(2) More generally, let $H : \mathcal{A} \to \mathcal{A}$ preserve countable coproducts and have a final coalgebra T. Then $MY = U_\omega = T + Y + HY + H^2Y + \cdots$.
(3) For the endofunctor $HX = X \times X$ of **Set** (of binary algebras) we have that the free corecursive algebra MY consists of

$$MY = \text{all binary trees with finitely many leaves which are labelled in } Y.$$

(4) More generally, let $\Sigma = (\Sigma_k)_{k<\omega}$ be a signature. Then Σ-algebras are precisely the algebras for the polynomial endofunctor H_Σ as explained in Remark 2.3.

Recall that the final H_Σ-coalgebra is the coalgebra of all Σ-*trees*, that is, (possibly infinite, rooted and ordered) trees labelled in Σ so that every node with a label of arity n has precisely n children. And FY is the algebra of all finite $(\Sigma + Y)$-trees, where members of Y are considered to have arity 0. Then U_n is the set of all $(\Sigma + Y)$-trees with no leaf of depth greater than n having a label from Y. (That means that all leaves with level n or more are labelled by a nullary symbol in Σ_0.) Consequently, the free corecursive algebra is

$$MY = T \oplus FY = \text{ all } (\Sigma + Y)\text{-trees with only finitely many } Y\text{-labelled leaves.}$$

Remark 4.8. (1) A *pre-fixed point* of a functor H is an object A such that HA is a subobject of A.
(2) A fixed point, i.e. an object $A \simeq HA$, can be considered as an algebra or a coalgebra for H. When we speak about *corecursive fixed points*, we mean fixed points $HA \xrightarrow{\sim} A$ that are corecursive algebras.

Theorem 4.9. *For every set endofunctor, the following statements are equivalent:*

(i) *all free corecursive algebras exist,*
(ii) *all free algebras and a final coalgebra exist, and*
(iii) *arbitrarily large pre-fixed points and a corecursive fixed point exist.*

They imply that the free corecursive algebra on Y *is* $T \oplus FY$.

Sketch of proof. Without loss of generality, H preserves monomorphisms. Then (i)⇒(ii) by Propositions 3.8 and 3.12 (which are true for corecursive algebras). The proof of (ii)⇒(i) is analogous to that of Theorem 4.6, just in lieu of the final coalgebra TY for $H(-) + Y$ we use members of the final opchain of that functor. The equivalence of (ii) and (iii) follows from the fact that (a) arbitrarily large pre-fixed points are necessary and sufficient for the existence of free algebras, see Theorem II.4 in [19] and (b) every corecursive fixed point is an initial corecursive algebra, thus, a final coalgebra, see Proposition 7 in [11]. □

Notation 4.10. Let H have free corecursive algebras MY. Denote by $\delta_Y : HMY \to MY$ the algebra structure and by $\eta_Y : Y \to MY$ the universal map. Then we obtain a unique homomorphism $\mu_Y : (MMY, \delta_{MY}) \to (MY, \delta_Y)$ with $\mu_Y.\eta_{MY} = id$:

$$\begin{array}{ccccc} HMMY & \xrightarrow{\delta_{MY}} & MMY & \xleftarrow{\eta_{MY}} & MY \\ {\scriptstyle H\mu_Y}\downarrow & & \downarrow{\scriptstyle \mu_Y} & \swarrow{\scriptstyle id_{MY}} & \\ HMY & \xrightarrow[\delta_Y]{} & MY & & \end{array} \tag{4.1}$$

The triple $\mathbb{M} = (M, \mu, \eta)$ is the monad generated by the adjoint situation $\mathsf{Alg}_C\, H \leftrightarrows \mathcal{A}$.

Example 4.11. We have $MY = \mathbb{N} \times Y + 1$ for the identity functor on **Set**, see Example 4.7(1). And for $HX = X \times X$ we have

$$MY = \text{binary trees with finitely many leaves labelled in } Y;$$

see Example 4.7(3). The functor $HX = \coprod_{n<\omega} X^n$ generates the monad

$$MY = \text{finitely branching trees with finitely many leaves labelled in } Y,$$

cf. Example 4.7(4).

Remark 4.12. Since μ_Y is, by definition, a homomorphism we have $\mu_Y.\delta_{MY} = \delta_Y.H\mu_Y$, and the unit law $\mu_Y.\eta_{MY} = id$ yields $\delta_Y = \mu_Y.\delta_{MY}.H\eta_{MY}$. It easy to prove that the δ_Y's are the components of a natural transformation $\delta : HM \to M$, and so are η and μ being part of the monad $\mathbb{M}$.

Theorem 4.13. *Let $\mathcal{A}$ be a locally presentable category with constructive monomorphisms. Let H be an accessible endofunctor preserving monomorphisms. Then Bloom algebras are precisely the Eilenberg-Moore algebras for $\mathbb{M}$, i.e., the category $\mathsf{Alg}_B\, H$ is isomorphic to $\mathcal{A}^{\mathbb{M}}$.*

Sketch of proof. From Theorem 3.14 and Proposition 4.5 we know that the monad generated by free Bloom algebras is $\mathbb{M}$. By Beck's Theorem we only need to verify that the forgetful functor U of $\mathsf{Alg}_B\, H$ creates coequalizers of U-split pairs. □

5 Corecursive monads

The iterative theories (or iterative monads) of C. Elgot [14] were introduced as a formalization of iteration in an algebraic setting, and in [15] completely iterative theories are studied. We first recall the concept of a completely iterative monad, and then introduce the weaker concept of a corecursive monad. The relationship between these two concepts is analogous to the relationship between cia's (see Remark 2.5) and corecursive algebras. The following definition is, for the base category **Set**, equivalent to completely iterative theories, as shown in [1].

Definition 5.1. *(See [1]) (1) An* ideal monad *is a six-tuple* $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ *consisting of a monad* (S, η, μ)*, a subfunctor* $\sigma : S' \to S$ *(called the* ideal of $\mathbb{S}$ *) such that* $S = S' + Id$ *with injections* σ *and* η*, and a natural transformation* $\mu' : S'S \to S'$ *restricting* μ*, i.e., with* $\sigma.\mu' = \mu.\sigma S$*.*

(2) An equation morphism with parameters *for* $\mathbb{S}$ *is a morphism* $e : X \to S(X+Y)$*, we call* X *the* variables *and* Y *the* parameters *of* e*. It is called* ideal *if it factorizes through* σ_{X+Y}*. A solution of* e *is a morphism* $e^\dagger : X \to SY$ *such that*

$$e^\dagger = (X \xrightarrow{e} S(X+Y) \xrightarrow{S[e^\dagger, \eta_Y]} SSY \xrightarrow{\mu_Y} SY) \tag{5.1}$$

(3) An ideal monad is called completely iterative *provided that every ideal equation morphism has a unique solution.*

Example 5.2. (See [1]) Let H be an endofunctor of $\mathcal{A}$ such that for every object Y a final coalgebra TY of $H(-)+Y$ exists. Then the assignment $Y \mapsto TY$ yields a monad (T, η, μ), which is the monad of free cia's for H. This is an ideal monad w.r.t. $T' = HT$ and $\mu' = H\mu$. Moreover, this monad is completely iterative, indeed, the free completely iterative monad on H.

For example the set functor $HX = X \times X$ generates the free completely iterative monad $\mathbb{T}$ where TY consists of all binary trees with leaves labelled in Y.

Definition 5.3. *Let* $\mathbb{S}$ *be an ideal monad. An* equation morphism *(without parameters) is a morphism* $e : X \to SX$*, and* e *is called* ideal *if it factorizes through* σ_X*, i.e., there exist* $e_0 : X \to S'X$ *such that* $e = \sigma_X \cdot e_0$*. The monad* $\mathbb{S}$ *is called* corecursive *if every ideal equation morphism* e *has a unique solution* $e^\dagger$*, i.e.,* $e^\dagger : X \to SY$ *such that* $e^\dagger = \mu_Y \cdot Se^\dagger \cdot e$*.*

Example 5.4. Examples of corecursive monads on **Set**.

(1) All the monads of Example 4.11 are corecursive, as we will see in Theorem 6.4 below.

(2) All completely iterative monads are corecursive, e.g, $\mathbb{S}$ where SY consists of all finitely branching trees with leaves labelled in Y. This is the free completely iterative monad on the functor $HX = \coprod_{n<\omega} X^n$.

(3) The monad

$$RY = \text{all rational, finitely branching trees with leaves labelled in } Y,$$

where *rational* means that the tree has up to isomorphism only finitely many subtrees. This is a corecursive monad that is neither free on any endofunctor, nor completely iterative.

(4) More generally, every submonad of S in item (2) containing the complete binary tree is corecursive.

Proposition 5.5. *The monad* $\mathbb{M} = (M, \eta, \mu)$ *of free corecursive algebras (of Notation 4.10) is ideal w.r.t. the ideal* $M' = HM$ *where* $\sigma = \delta : HM \to M$ *and* $\mu' = H\mu : HMM \to HM$*.*

Theorem 5.6. *The monad $\mathbb{M}$ of free corecursive algebras is corecursive.*

Sketch of proof. For every ideal equation morphism $e : X \to MX$ we form an equation morphism $\overline{e} : MX \to HMX$ by composing the isomorphism $MX \simeq HMX + X$ with $[HMX, e_0] : HMX + X \to MX$. Every algebra MY has a unique solution of $\overline{e}$, then one proves that $\overline{e}^{\dagger}.\eta_X$ is the unique solution of e. □

6 Free Corecursive Monad

In this section we prove that the corecursive monad $\mathbb{M}$ given by the free corecursive algebras is a free corecursive monad. For that we need the appropriate concept of morphism:

Definition 6.1. *(1) An* ideal monad morphism *from an ideal monad $(S, \eta^S, \mu^S, S', \sigma, \mu'^S)$ to an ideal monad $(U, \eta^U, \mu^U, U', \omega, \mu'^U)$ is a monad morphism $(S, \eta^S, \mu^S) \to (U, \eta^U, \mu^U)$ which has a domain-codomain restriction to the ideal, that is, there exists a natural transformation $\lambda' : S' \to U'$ with $\lambda.\sigma = \omega.\lambda'$.*

(2) Given a functor H, a natural transformation $\lambda : H \to S$ is called ideal *if it factors through $\sigma : S' \to S$.*

(3) By a free corecursive monad *on an endofunctor H is meant a corecursive monad $\mathbb{S} = (S, \mu, \eta, S', \sigma, \mu')$ together with an ideal natural transformation $\kappa : H \to S$ with the following universal property: For every ideal natural transformation $\lambda : H \to \overline{\mathbb{S}}$, where $\overline{\mathbb{S}}$ is a corecursive monad, there exists a unique ideal monad morphism $\hat{\lambda} : \mathbb{S} \to \overline{\mathbb{S}}$ such that $\lambda = \hat{\lambda} \cdot \kappa$.*

Remark 6.2. Let $\mathsf{CMon}(\mathcal{A})$ denote the category of corecursive monads and ideal monad morphisms. We have a forgetful functor to $\mathsf{Fun}(\mathcal{A})$, the category of all endofunctors of $\mathcal{A}$, assigning to every corecursive monad $\mathbb{S}$ its ideal S'. A free corecursive monad on $H \in \mathsf{Fun}(\mathcal{A})$ is precisely a universal arrow from H to the above forgetful functor.

Example 6.3. If H has free corecursive algebras, then we have the corecursive monad $\mathbb{M}$ of Proposition 5.5. And the natural transformation

$$\kappa = (H \xrightarrow{H\eta} HM \xrightarrow{\delta} M)$$

is obviously ideal. We prove that κ has the universal property:

Theorem 6.4. *If an endofunctor H has free corecursive algebras, then the corresponding monad $\mathbb{M}$ is the free corecursive monad on H.*

Remark 6.5. The proof is analogous to the corresponding theorem for free completely iterative monads, see [16], Theorem 4.3.

Are there any other free corecursive monads than the monads $\mathbb{M}$ of free corecursive algebras? Not for endofunctors of **Set**:

Proposition 6.6. *If a set functor generates a free corecursive monad, then it has free corecursive algebras.*

Proof. Let $H : \mathbf{Set} \to \mathbf{Set}$ generate a free corecursive monad $\mathbb{S} = (S, \mu^S, \eta^S, S', \sigma, \mu')$, and let $\kappa : H \to S$ be the universal arrow. Following Theorem 4.9 we need to prove the existence of (a) arbitrary large pre-fixed points and (b) a corecursive fixed point.

The main technical statement is the fact that the ideal S' is naturally isomorphic to HS. This proof is analogous to the same proof concerning free completely iterative monads, see Sections 5 and 6 in [16]. We therefore omit it.

Ad (a). Arbitrarily large pre-fixed points. Since $SY = S'Y + Y = HSY + Y$ for every set Y, we see that SY is a pre-fixed point of cardinality at least $\mathsf{card}\, Y$.

Ad (b). A corecursive fixed point. The isomorphism $\sigma_\emptyset : HS\emptyset \to S\emptyset$ defines a corecursive algebra for H. To prove this, consider an arbitrary equation morphism $e : X \to HX$ and form the equation morphism $\overline{e} = \kappa_X.e : X \to SX$. Then solutions of $\overline{e}$ w.r.t $\mathbb{S}$ are precisely the solutions of e (in $S\emptyset$). This is easy to prove, the details are as in the proof of Theorem 6.1 of [16]. □

7 Conclusions and Further Work

Whereas for coalgebras the concept of recursivity has several equivalent formulations (assuming the given endofunctor weakly preserves pullbacks), in the dual situation we need to study non-equivalent variations. The present paper is dedicated to corecursive algebras A, where corecursity means that every recursive system of equations represented by a coalgebra has a unique solution in A. This is strictly weaker than the concept of a completely iterative algebra, where every *parameterized* recursive system of equations has a unique solutions. For example, if we consider the endofunctor $X \mapsto X \times X$ of one binary operation in **Set**, the algebra of all binary trees with finitely many leaves is corecursive, but not completely iterative.

The main result of our paper is the description of free corecursive algebras on Y as coproducts $MY = T \oplus FY$ of the final coalgebra T and a free algebra FY (in the category of all algebras). The above example of binary trees is the free algebra $M1$ on one generator. Our description is true for all accessible (= bounded) endofunctors of **Set** and, moreover, for all endofunctors of **Set** having free corecursive algebras. For accessible functors, more general base categories (posets, groups, monoids etc.) also allow for the above description of the free corecursive algebras.

We introduced the concept of a corecursive monad, a weakening of completely iterative monad. We proved that the assignment $Y \mapsto MY = T \oplus FY$ is the free corecursive monad on the given accessible endofunctor. And we characterized the Eilenberg-Moore algebras for this monad. We called them Bloom algebras in honor of Stephen Bloom. They play the analogous role that Elgot algebras, studied in [5], play for iterative monads: solutions of recursive equations are not required to be unique, but have to satisfy some "basic" properties. In the case of Bloom algebras, the only property needed is functoriality.

One feature that has not been treated in our paper is that of finitariness of equations: If we consider systems of recursive equations as coalgebras $e : X \to HX$, then finite systems of recursive equation are represented by coalgebras in which X is a finite set (or more generally, a finitely presentable object). We can speak about finitely corecursive algebras as those in which these finite systems have unique solutions.

Another question is: what is the analogy of the notion of an iteration monad of S. Bloom and Z. Ésik [10] in the realm of corecursive algebras? We do not know the answer but at least we can formulate the question precisely. The idea of iteration monads is to collect all "equational" properties that the operation $e \mapsto e^\dagger$ of solving recursive systems e has in trees for a signature. This can be understood as forming the monad of free iterative theories (or monads) on the category $\mathsf{Fin}(\mathcal{A})$ of finitary endofunctors, and characterizing its Eilenberg-Moore algebras: these are, as recently proved in [4], precisely the iteration theories of S. Bloom and Z. Ésik that are functorial. So we state the following problem for future work: form the monad of free finitely corecursive theories on $\mathsf{Fin}(\mathcal{A})$, what are its Eilenberg-Moore algebras?

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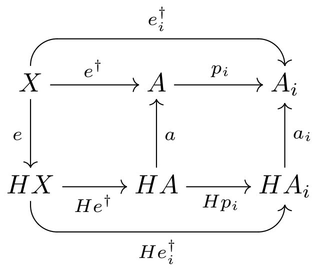
A Appendix

In this appendix some proofs are presented. Also three auxiliary results needed for the proofs are formulated.

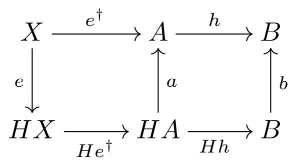
Proof of Proposition 2.6. It is easy to verify that limits in $\mathsf{Alg}\, H$ are formed on the level of $\mathcal{A}$. Let us prove that the a product of corecursive algebras is corecursive. The proof for general limits is analogous.

Let (A, a) be the product of corecursive algebras (A_i, a_i), with projections $p_i : A \to A_i$. For every coalgebra $e : X \to HX$ we have the unique coalgebra-to-algebra morphism $e_i^\dagger : X \to A_i$, for all $i \in I$, and the morphism $e^\dagger = \langle e_i^\dagger \rangle : X \to A = \prod_{i \in I} A_i$ is a coalgebra-to-algebra morphism. Indeed, for every $i \in I$, the diagram



commutes, except perhaps for the left hand inner square; but this suffices to establish the desired commutativity of the left hand square. Since all A_i are corecursive, the uniqueness of $e^\dagger$ follows from the observation that there is a one-one correspondence between solutions $s : X \to A$ of e in A and families of solutions $s_i : X \to A_i$ of e in A_i, for all $i \in I$. □

Proof of Lemma 2.7. This follows from the diagram

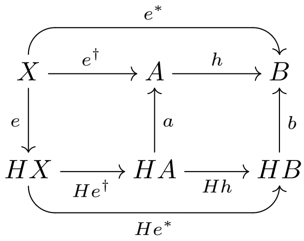


□

Proof of Lemma 3.5. For every coalgebra $e : X \to HX$ we define

$$e^* \equiv X \xrightarrow{e^\dagger} A \xrightarrow{h} B$$

and verify that e^* is a solution of e by the following commutative diagram



Functoriality is easily checked too: let $g : (X, e) \to (Y, f)$ be a coalgebra homomorphism. Then the following equations hold:

$$\begin{aligned} f^*.g &= h.f^\dagger.g && \text{by the definition of } (-)^* \\ &= h.e^\dagger && \text{by functoriality of } \dagger \\ &= e^* && \text{by the definition of } (-)^* \end{aligned}$$

Finally, h is clearly solution preserving.

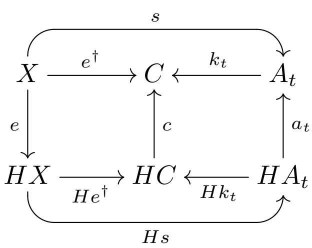
The unicity of the Bloom algebra structure given by $(-)^*$ is clear. □

Lemma A.1. *Let $\mathcal{A}$ be a locally presentable category. Then for every accessible endofunctor H, the category $\mathsf{Alg}_C\, H$ of corecursive algebras is locally presentable.*

Proof. Choose an uncountable cardinal number λ such that H preserves λ-filtered colimits and $\mathcal{A}$ is locally λ-presentable. Then λ-filtered colimits in $\mathsf{Alg}\, H$ are clearly formed on the level of $\mathcal{A}$. And as proved in [7], every coalgebra is a λ-filtered colimit of λ-presentable coalgebras and these are precisely the coalgebras carried by λ-presentable objects in $\mathcal{A}$.

(1) We prove that λ-filtered colimits of corecursive algebras in $\mathsf{Alg}\, H$ are corecursive. Indeed, let $(A_t, a_t)_{t \in T}$ be a λ-filtered diagram with colimit $k_t : (A_t, a_t) \to (C, c)$. For every coalgebra $e : X \to HX$, a solution $e^\dagger : X \to A_t$ exists in (A_t, a_t), and since k_t is a homomorphism, $k_t.e^\dagger$ is a solution in C, see Lemma 2.7.

To prove that solutions are unique, assume first that X is λ-presentable in $\mathcal{A}$. For every solution $e^\dagger : X \to C$ there exists a $t \in T$ such that $e^\dagger$ factorizes through k_t as follows



The morphism k_t merges s and $a_t.Hs.e$:

$$\begin{aligned} k_t.(a_t.Hs.e) &= c.Hk_t.Hs.e \\ &= c.He^\dagger.e \\ &= k_t.s \end{aligned}$$

Consequently, since k_t is a colimit morphism of a λ-filtered colimit, there exists an object $t' \in T$ and a connecting morphism $u : A_t \to A_{t'}$ which also merges s and $a_t.Hs.e$, that is,

$$u.s = u.a_t.Hs.e = a_{t'}.Hu.Hs.e$$

This last equation proves that $u.s$ is a solution of e in $A_{t'}$, thus $u.s$ is uniquely determined. Hence, $e^\dagger$ is uniquely determined from $e^\dagger = k_t.s = k_{t'}.u.s$.

Next let X be arbitrary. Express (X, e) in the category of coalgebras as a λ-filtered colimit of λ-presentable coalgebras (X_i, e_i). Let $x_i : X_i \to X$ be the corresponding colimit cocone. For every solution $e^\dagger : X \to C$ each $e^\dagger.x_i$ is a solution of e_i since we have the following commutative diagram

$$\begin{array}{ccccc} X_i & \xrightarrow{x_i} & X & \xrightarrow{e^\dagger} & C \\ {\scriptstyle e_i}\downarrow & & \downarrow{\scriptstyle e} & & \uparrow{\scriptstyle c} \\ HX_i & \xrightarrow[Hx_i]{} & HX & \xrightarrow[He^\dagger]{} & HC \end{array}$$

Thus, $e^\dagger.x_i$ is uniquely determined by the previous case. Since the cocone of all x_i's is collectively epic, this proves that $e^\dagger$ is uniquely determined. □

Lemma A.2. *Let $\mathcal{A}$ be a complete category. Then so is $\mathsf{Alg}_B\, H$, and limits of Bloom algebras are formed on the level of $\mathcal{A}$.*

Proof. This is completely analogous to the proof of Proposition 2.6. The verification that the function $e \mapsto \langle e_i^\dagger \rangle$ is functorial is trivial. □

Proof of Proposition 3.8. Let us, for a coalgebra (X, e), denote the unique coalgebra homomorphism from X to T by $e^\sharp : X \to T$. We shall define two functors between $\mathsf{Alg}_B\, H$ and $\mathcal{S} = (T, \tau^{-1})/\mathsf{Alg}\, H$ and show that they are mutually inverse.

(a) From Bloom algebras to the slice category $\mathcal{S}$: given a Bloom algebra $(A, a, \dagger)$ we form the solution $\tau^\dagger : T \to A$ which clearly is an object $(A, \tau^\dagger)$ of $\mathcal{S}$. For a homomorphism $h : (A, a, \dagger) \to (B, b, \ddagger)$ of Bloom algebras, we clearly have a morphism $h : (A, \tau^\dagger) \to (B, \tau^\ddagger)$ of $\mathcal{S}$, since h is solution preserving. This defines a functor from $\mathsf{Alg}_B\, H$ to $\mathcal{S}$.

(b) From $\mathcal{S}$ to Bloom algebras: Suppose we are given an object (A, h) in $\mathcal{S}$, that is an algebra homomorphism:

$$\begin{array}{ccc} T & \xleftarrow{\tau^{-1}} & HT \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh} \\ A & \xleftarrow[a]{} & HA \end{array}$$

We define for every $e : X \to HX$ its dagger as $e^\dagger = h.e^\sharp$. This is functorial; indeed, for every coalgebra homomorphism $k : (X, e) \to (Y, f)$ we have $f^\sharp.k = e^\sharp$ by unicity of the universal property of the final coalgebra (T, τ), thus

$$\begin{aligned} f^\dagger.h &= h.f^\sharp.k \\ &= h.e^\sharp \\ &= e^\dagger \end{aligned}$$

In addition, every morphism $m : (A, h) \to (B, h')$ of $\mathcal{S}$ is a dagger preserving morphism $(A, a, \dagger) \to (B, b, \ddagger)$:

$$\begin{aligned} m.e^\dagger &= m.h.e^\sharp && \text{by definition of } \dagger \\ &= h'.e^\sharp && m \text{ is a morphism in } \mathcal{S} \\ &= e^\ddagger && \text{definition of } \ddagger \end{aligned}$$

That this gives a functor from $\mathsf{Alg}_B\, H$ to $\mathcal{S}$ is immediate.

(c) The two functors above are mutually inverse. Indeed, it suffices to show that we have a bijection on the level of objects, since both functors are clearly the identity on morphisms. So for (A, h) in $\mathcal{S}$ we form first $(A, a, \dagger)$ as in (b) and then $(A, \tau^\dagger)$ as in (a) and we have $\tau^\dagger = h.\tau^\sharp = h$ as $\tau^\sharp$ is the identity (being the unique coalgebra homomorphism from (T, τ) to itself). Finally, given a Bloom algebra $(A, a, \ddagger)$ we first form $(A, \tau^\ddagger)$ as in (a) and then $(A, a, \dagger)$ as in (b). Then we have

$$\begin{aligned} e^\dagger &= \tau^\ddagger.e^\sharp && \text{definition of } \dagger \text{ in (b)} \\ &= e^\ddagger && \text{functoriality of } \ddagger \end{aligned}$$

This completes the proof. □

Proof of Proposition 4.1. Let λ be an infinite cardinal such that $\mathcal{A}$ is a locally λ-presentable category and H preserves λ-filtered colimits. Since λ can be chosen arbitrarily large, we can assume that λ is uncountable. The category $\mathsf{Alg}_B H$ is locally λ-presentable. The proof is analogous to that of Lemma A.1 (the only difference is that in the proof of the uniqueness of $e^\dagger$ we simply observe that since k_t's are supposed to be solution preserving, we have $e^\dagger = k_t.s$, where s is the dagger of e in $A_t X$). Consequently, $\mathsf{Alg}_B\, H$ is a complete, well-powered, and cowellpowered category, and it has (strong epi-mono) factorization of morphisms, see [8]. The full subcategory of corecursive algebras is closed under products by Proposition 2.6. Thus, the proof will be completed when we prove that corecursive algebras are closed in $\mathsf{Alg}_B\, H$ under subalgebras, then it is reflective see [2], Theorem 16.8.

Let $m : (A, a, \dagger) \to (B, b, \ddagger)$ be a monomorphism with $(B, b, \ddagger)$ corecursive. From Lemma A.2 we have that m is a monomorphism in $\mathcal{A}$. It is our task to prove that for every coalgebra $e : X \to HX$ the morphism $e^\dagger$ is the unique solution in A. This follows from the fact that $m.e^\dagger = e^\ddagger$. Since $e^\ddagger$ is unique in B and m is a monomorphism in $\mathcal{A}$, the proof is concluded. □

Lemma A.3. *Let (A, a) be an algebra and $f : B \to A$ a morphism. Then (A, a) is a corecursive algebra if and only if the algebra*

$$\overline{a} \equiv H(HA + B) \xrightarrow{H[a,f]} HA \xrightarrow{\mathsf{inl}} HA + B$$

is corecursive.

Proof. Let (A, a) be a corecursive algebra and $e : X \to HX$ be an equation morphism. Then there is a unique solution:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle a} \\ HX & \xrightarrow[He^\dagger]{} & HA \end{array} \tag{A.1}$$

Now inspection of the following commutative diagram shows that $\mathsf{inl}.He^\dagger.e : X \to HA + B$ is a solution of e in $HA + B$.

$$\begin{array}{ccccccc} X & \xrightarrow{e} & HX & \xrightarrow{He^\dagger} & HA & \xrightarrow{\mathsf{inl}} & HA+B \\ {\scriptstyle e}\downarrow & // & \downarrow{\scriptstyle He} & & {\scriptstyle Ha}\uparrow & \nwarrow{\scriptstyle H[a,f]} & \uparrow{\scriptstyle \overline{a}} \\ HX & \xrightarrow[He]{} & HHX & \xrightarrow[HHe^\dagger]{} & HHA & \xrightarrow[H\mathsf{inl}]{} & H(HA+B) \end{array} \tag{A.2}$$

Indeed, commutativity of the middle rectangle follows from Diagram (A.1), the lower triangle is trivial and the upper triangle is the definition of $\overline{a}$. To show the uniqueness of solution, suppose that $s : X \to HA + B$ is a solution for e, so we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{s} & HA+B & \xrightarrow{[a,f]} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mathsf{inl}.H[a,f]} & & \uparrow{\scriptstyle a} \\ HX & \xrightarrow[Hs]{} & H(HA+B) & \xrightarrow[H[a,f]]{} & HA \end{array}$$

Since the solution $e^\dagger$ in A is unique, $e^\dagger = [a, f].s$ and hence

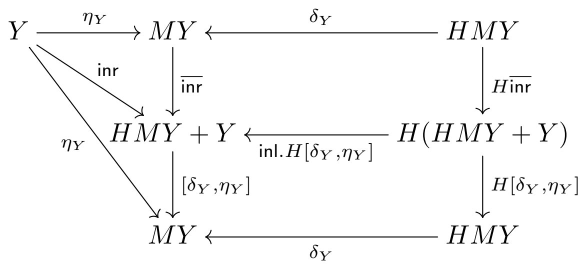
$$\mathsf{inl}.He^\dagger.e = \mathsf{inl}.H[a, f].Hs.e = s.$$

Conversely, let $(HA + B, \mathsf{inl}.H[a, f])$ be a corecursive algebra and $e : Y \to HY$ be an equation morphism. So there exists a unique solution $e^\dagger$ of e in $HA + B$, and hence we have the above commutative diagram with $e^\dagger$ in lieu of s. That is $[a, f].e^\dagger$ is a solution of e in the algebra (A, a). To show uniqueness suppose that $s : Y \to A$ is a solution of e, that is $s = a.Hs.e$. Then we have Diagrams (A.1) and (A.2) with the morphism s instead of $e^\dagger$. So, by uniqueness of solutions in the corecursive algebra $(HA + B, \mathsf{inl}.H[a, f])$, we have $\mathsf{inl}.Hs.e = e^\dagger$, and hence $s = a.Hs.e = [a, f].\mathsf{inl}.Hs.e = [a, f].e^\dagger$. □

Proof of Proposition 5.5. (1) We prove

$$MY = HMY + Y$$

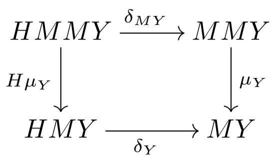
with injections δ_Y and η_Y. First apply Lemma A.3 to the corecursive algebra MY and $\eta_Y : Y \to MY$ and see that $HMY + Y$ is a corecursive algebra, too. Using the freeness of MY we see that for the right-hand injection morphism $\mathsf{inr} : Y \to HMY + Y$, there exists a unique homomorphism $\overline{\mathsf{inr}} : MY \to HMY + Y$ such that the upper part of Diagram (A.3) commutes. We show that $\overline{\mathsf{inr}}$ is an inverse for $[\delta_Y, \eta_Y]$. To prove this, consider the following diagram



(A.3)

Since the lower square is commutative, $[\delta_Y, \eta_Y]$ is homomorphism, and the outer square commutes as well as the right-hand triangle. So $[\delta_Y, \eta_Y].\overline{\mathsf{inr}} = id_{MY}$ follows from the universal property of η_Y. The upper square of (A.3) yields $\overline{\mathsf{inr}}.\delta_Y = \mathsf{inl}$, consequently $(\overline{\mathsf{inr}}.[\delta_Y, \eta_Y]).\mathsf{inl} = \overline{\mathsf{inr}}.\delta_Y = \mathsf{inl}$. Since also $(\overline{\mathsf{inr}}.[\delta_Y, \eta_Y]).\mathsf{inr} = \overline{\mathsf{inr}}.\eta_Y = \mathsf{inr}$, we conclude $\overline{\mathsf{inr}}.[\delta_Y, \eta_Y] = id$.

(2) It remains to prove $\sigma.\mu' = \mu.S\sigma$, that is $\delta.H\mu = \mu.HM\delta$. Recall that μ_Y is, by definition, a homomorphism of H-algebras:

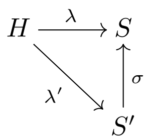


This concludes the proof. □

Proof of Theorem 6.4. For every corecursive monad

$$\mathbb{S} = (S, \mu^S, \eta^S, S', \sigma, \mu'^S)$$

and every ideal natural transformation

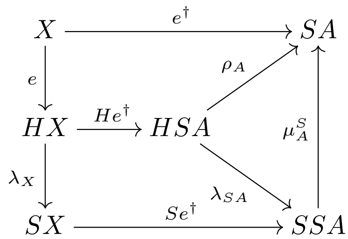


we are going to find an ideal monad morphism $\hat{\lambda} : M \to S$ with $\lambda = \hat{\lambda}.\kappa$, and prove that it is unique.

(a) Every object SA, $A \in \mathcal{A}$, is considered as an algebra for H via

$$\rho_A \equiv HSA \xrightarrow{\lambda_{SA}} SSA \xrightarrow{\mu^S_A} SA \tag{A.4}$$

We show that SA is corecursive. Every equation morphism $e : X \to HX$ yields the following equation morphism $\overline{e} : X \xrightarrow{e} HX \xrightarrow{\lambda_X} SX$ w.r.t. the monad $\mathbb{S}$ and, and since λ is an ideal natural transformation we see that e an ideal equation morphism. We prove that $e^\dagger$ is a solution of e in SA if and only if it is a solution of $\overline{e}$ with respect to the corecursive monad $\mathbb{S}$. Let $e^\dagger$ be a solution of e in SA, that is the upper part of the diagram below commutes.



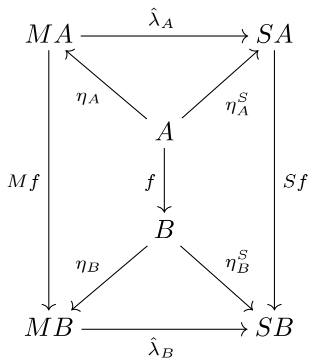
Since the lower part commutes because of naturality of λ we see that $e^\dagger$ is a solution of $\overline{e}$.

Conversely, if $e^\dagger$ is a solution of $\overline{e}$, then the outside of the above diagram commutes. The lower part is also commutative because of naturality of λ, and the right-hand triangle commutes by definition. So the upper part is commutative showing $e^\dagger$ to be a solution of e as desired. Thus SA is a corecursive algebra.

(b) There exists a unique homomorphism $\hat{\lambda}_A : MA \to SA$ of H-algebras such that $\hat{\lambda}_A.\eta_A = \eta^S_A$. Now we show that $\hat{\lambda}$ is a natural transformation. Consider $f : A \to B$, then $Sf : SA \to SB$ is a homomorphism:

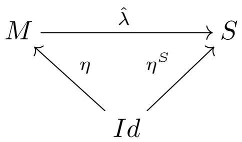
$$\begin{array}{ccccc} HSA & \xrightarrow{\lambda_{SA}} & SSA & \xrightarrow{\mu_A} & SA \\ {\scriptstyle HSf}\downarrow & & {\scriptstyle SSf}\downarrow & & \downarrow{\scriptstyle Sf} \\ HSB & \xrightarrow[\lambda_{SB}]{} & SSB & \xrightarrow[\mu_B]{} & SB \end{array} \tag{A.5}$$

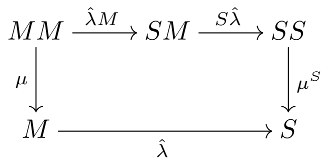
The outside of the following diagram



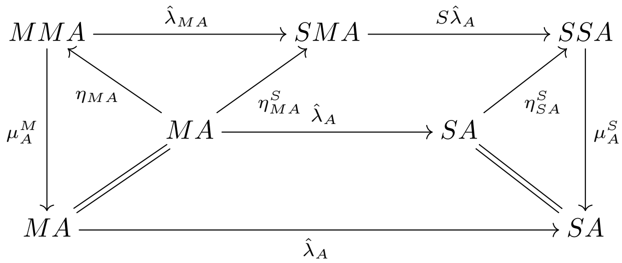
commutes, by the universal property of η_A.

(c) We prove next that $\hat{\lambda}$ is a monad morphism. That is, the following diagrams are commutative:



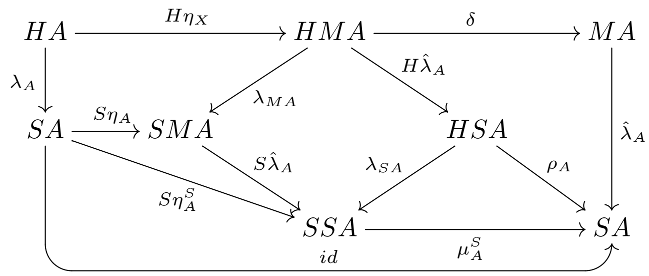


The left-hand triangle commutes because of the definition of $\hat{\lambda}$. For the right hand square we note that $S\hat{\lambda}_A$ is a homomorphism (cf. (A.5) with $f = \hat{\lambda}_A$), and we have the following diagram



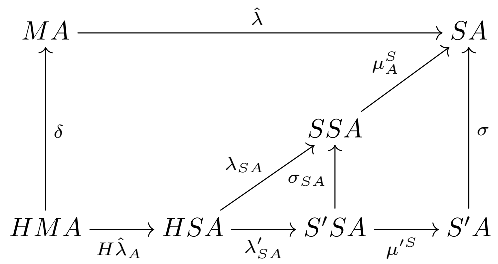
The right-hand square commutes by naturality of η^S, hence the outside is commutative.

(d) Now we have to show that $\lambda = \hat{\lambda}.\kappa = \hat{\lambda}.\delta.H\eta$ which follows from the commutativity of the diagram

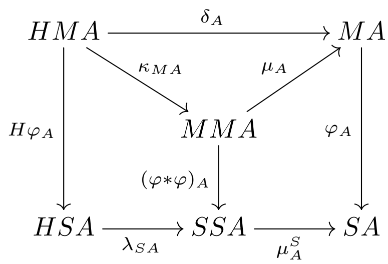


The left-hand upper part and the central one commute because λ is an ideal natural transformation. The right-hand upper part commutes by the definition of $\hat{\lambda}$. The lower right-hand triangle commutes by the definition of ρ, see (A.4) and the lowest part commutes by the monad laws of $\mathbb{S}$.

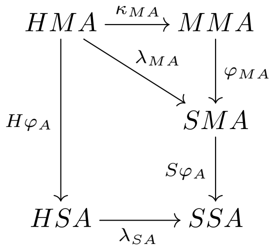
(e) Next we show that $\hat{\lambda}$ is an ideal monad morphism. This follows from



(f) It remains to prove that $\hat{\lambda}$ is unique. Let $\varphi : \mathbb{M} \to \mathbb{S}$ be an ideal monad morphism with $\varphi.\kappa = \lambda$. It is sufficient to prove that $\varphi_A : MA \to SA$ is a homomorphism of H-algebras w.r.t. the structure ρ_A above and $\varphi_A.\eta_A = \eta^S_A$, then $\varphi_A = \hat{\lambda}_A$ of (b) above. The equation $\varphi_A.\eta_A = \eta^S_A$ follows from φ preserving the units of the monad. And the fact that φ_A is a homomorphism follows from the following diagram:



For the upper triangle see Remark 4.12, the right-hand square is the preservation of the monad multiplication, and for the left-hand one we use $(\varphi.\kappa)M = \lambda M$ and the naturality of λ.



□