

On Iteratable Endofunctors

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Abstract

Completely iterative monads of Elgot et al. are the monads such that every guarded iterative equation has a unique solution. Free completely iterative monads are known to exist on every iteratable endofunctor H , i. e., one with final coalgebras of all functors $H(-) + X$. We show that conversely, if H generates a free completely iterative monad, then it is iteratable.

Key words: monad, completely iterative, iterable

1 Introduction

There have been various attempts to algebraically capture the concept of computations on data through a program, taking into account that such computations are potentially infinite. During the 1970's the ADJ-group studied continuous algebras, i. e., algebras endowed with a CPO structure. There, an infinite computation is a join of the directed set of its finite approximations, see e. g. [ADJ]. Later, algebras over complete metric spaces were considered, where an infinite computation is a limit of a Cauchy sequence of finite approximations, see e. g. [ARu].

Another approach to infinite computations are iterative algebraic theories, introduced by Calvin C. Elgot in [E]. This notion has been extended to the notion of completely iterative theories by Elgot, Bloom and Tindell, see [EBT]. The latter are algebraic theories (in the sense of Lawvere and Linton [Lin]) that allow for unique solutions of fixed point equations. An important example of a completely iterative theory is the theory of finite and infinite trees over a given signature Σ . In [EBT] it is shown that this is the free completely iterative theory over Σ .

It has recently been discovered by Peter Aczel, Jiří Adámek, Jiří Velebil, and the present author [AAMV], that the above fact is a special case of a much more general categorical result using a coalgebraic approach to infinite

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computation. This coalgebraic approach has also independently been studied by Larry Moss in [M]. Here one considers a category \mathcal{A} with binary coproducts, and an *iteratable* endofunctor H on \mathcal{A} , i.e., such that for every object X a final coalgebra

$$TX$$

of $H(-) + X$ exists. In [AAMV] the notion of a completely iterative monad is introduced. Informally, this is a monad that allows for unique solutions of systems of equations of a certain liberal type. It has been shown that the mapping $X \mapsto TX$ is the object assignment of a completely iterative monad. Moreover, it was proved that this monad T is a free completely iterative monad on H .

In the present paper we investigate the exact relationship between the notion of iteratability and the existence of free completely iterative monads for an endofunctor. The main result of [AAMV] shows that iteratable endofunctors admit free completely iterative monads. Here we prove that no other functors do so. More precisely, if S is a free completely iterative monad over an endofunctor H on \mathcal{A} , then H is iteratable, and for all objects X of \mathcal{A} , SX is a final coalgebra of $H(-) + X$.

Before we prove our main result in Section 3 we shall recall the results of [AAMV] and give some motivation for the notion of completely iterative monad in Section 2.

2 Iteratable Endofunctors and Completely Iterative Monads

2.1 A Motivating Example

We take from [AMV] a motivating example for the coalgebraic approach of [AAMV]. Consider the algebra of finite and infinite trees over a given signature Σ . This algebra allows for the unique solution of systems of so-called guarded equations. Let us give the details of this. Denote by

$$T_{\Sigma}X$$

the algebra of all finite and infinite Σ -labelled trees with variables from X . That is, trees labelled so that a node with $n > 0$ children is labelled by an n -ary operation symbol (an element of Σ_n) and a leaf is labelled by a variable or a constant (an element of $X + \Sigma_0$). The operations on $T_{\Sigma}X$ are given by tree-tupling. Furthermore, consider a system of equations

$$(1) \quad \begin{array}{l} x_0 \approx t_0(x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots) \\ x_1 \approx t_1(x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots) \\ \vdots \\ x_n \approx t_n(x_0, x_1, x_2, \dots, y_0, y_1, y_2, \dots) \\ \vdots \end{array}$$

where t_i are trees with variables from $X = \{x_0, x_1, x_2, \dots\}$ and parameters from $Y = \{y_0, y_1, y_2, \dots\}$, i.e.,

$$t_i \in T_\Sigma(X + Y) \quad \text{for } i = 0, 1, 2, \dots$$

Notice that in a system we denote by \approx formal equations and $=$ is the identity of the two sides. A system is called *guarded* provided that none of the trees t_i is just a variable from X . This condition is enough to force the existence of a unique *solution* of (1), i.e., a unique tuple $x_i^\dagger(y_0, y_1, y_2, \dots)$ of trees in $T_\Sigma Y$ such that the identities

$$\begin{aligned} x_0^\dagger &= t_0(x_0^\dagger, x_1^\dagger, x_2^\dagger, \dots, y_0, y_1, y_2, \dots) \\ x_1^\dagger &= t_1(x_0^\dagger, x_1^\dagger, x_2^\dagger, \dots, y_0, y_1, y_2, \dots) \\ &\vdots \\ x_n^\dagger &= t_n(x_0^\dagger, x_1^\dagger, x_2^\dagger, \dots, y_0, y_1, y_2, \dots) \\ &\vdots \end{aligned}$$

hold.

Theorem 2.1 *Every guarded system of equations has a unique solution.*

In fact, this is a special case of a much more general Solution Theorem we mention in Subsection 2.2 below.

Example 2.2 Let Σ consist of binary operations $+$ and $*$ and a constant \perp . The following system of equations

$$x_0 \approx \begin{array}{c} \dagger \\ \swarrow \quad \searrow \\ x_1 \quad * \\ \swarrow \quad \searrow \\ y \quad \perp \end{array} \quad x_1 \approx \begin{array}{c} * \\ \swarrow \quad \searrow \\ x_0 \quad \perp \end{array}$$

is guarded. The solution is given by the following trees in $T_\Sigma Y$:

$$x_0^\dagger = \begin{array}{c} \dagger \\ \swarrow \quad \searrow \\ * \quad * \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \dagger \quad \perp \quad y \quad \perp \\ \swarrow \quad \searrow \\ \perp \quad \perp \end{array} \quad x_1^\dagger = \begin{array}{c} * \\ \swarrow \quad \searrow \\ \triangle x_0^\dagger \quad \perp \end{array}$$

2.2 Substitutions and Solutions Coalgebraically

The coalgebraic approach of [AAMV] and [M] relies on the following observation. To any signature Σ there is an associated polynomial endofunctor $H_\Sigma : \text{Set} \rightarrow \text{Set}$ defined by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \dots$$

Recall that H_Σ -algebras are just the classical universal Σ -algebras. A final H_Σ -coalgebra is well-known to be the coalgebra $T_\Sigma\emptyset$ of all finite and infinite Σ -labelled trees without variables, see [AK]. Now $H_\Sigma(-) + X$ is also polynomial (for the signature obtained from Σ by adding a constant symbol for every element of X), thus, $T_\Sigma X$ is a final coalgebra of $H_\Sigma(-) + X$.

Taking the existence of such a parametrized family of final coalgebras as the primitive notion, one can abstract away from signatures (=polynomial endofunctors) and from the category **Set**.

Assumption 2.3 *For the rest of this section we assume that \mathcal{A} denotes a category with binary coproducts whose injections are monomorphic, and H is an endofunctor on \mathcal{A} .*

Definition 2.4 An endofunctor H of \mathcal{A} is called *iteratable* if for every object X of \mathcal{A} there exists a final coalgebra of $H(-) + X$.

The following examples of iteratable endofunctors have been taken from [AAMV].

Example 2.5

- (i) Accessible (=bounded) endofunctors on **Set**. An endofunctor is called accessible if it preserves λ -filtered colimits for some infinite cardinal λ . In [AP], it was shown that those are precisely the so-called bounded endofunctors. This example subsumes all the following ones.
- (ii) (Generalized) polynomial endofunctors on **Set**, i. e., H is defined by

$$HZ = \coprod_{i < \lambda} A_i \times Z^i$$

for some cardinal λ ; for $\lambda = \omega$ one has a polynomial endofunctor associated to a finitary signature as in 2.1.

- (iii) The bounded power set functors defined on objects by

$$\mathcal{P}_\lambda X = \{Y \subseteq X \mid |Y| < \lambda\}$$

for some cardinal λ . Notice that the (unbounded) power set functor $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ does not allow for a final coalgebra, and hence, it is not iteratable.

Note that the notions of accessibility and iteratability are not equivalent. In fact, there are examples of non-accessible endofunctors that are iteratable (see [AAMV]).

Remark 2.6 If H is an iteratable endofunctor on \mathcal{A} we denote by

$$TX$$

the final coalgebra of $H(-) + X$. By the Lambek Lemma (see [L]), the structure map of that final coalgebra is an isomorphism, and consequently, TX is a

coproduct of HTX and X with injections

$$\begin{aligned} \eta_X : X &\longrightarrow TX && \text{“injection of variables”} \\ \tau_X : HTX &\longrightarrow TX && \text{“}TX \text{ is an } H\text{-algebra”} \end{aligned}$$

The final coalgebras TX have a rich structure. Firstly, the way how substitution works on trees in $T_\Sigma X$ generalizes smoothly to the categorical setting. Recall here that given an interpretation of variables $x \in X$ as trees $s(x)$ over Y , i.e., a function $s : X \longrightarrow T_\Sigma Y$, then the corresponding substitution of trees from $T_\Sigma Y$ into (leaves of) trees of $T_\Sigma X$ is a homomorphism

$$\widehat{s} : T_\Sigma X \longrightarrow T_\Sigma Y$$

of Σ -algebras. Moreover, \widehat{s} is the unique extension of s . This can be generalized to all iterable endofunctors:

Substitution Theorem 2.7 *For any arrow $s : X \longrightarrow TY$ there exists a unique homomorphism $\widehat{s} : TX \longrightarrow TY$ of H -algebras extending s , i. e., such that $\widehat{s} \cdot \eta_X = s$.*

The proof can be found in [M] or [AAV] (slightly improved in [AAMV]).

Next, one can generalize in a straightforward way the notion of a system of equations. An *equation arrow* is a morphism

$$e : X \longrightarrow T(X + Y)$$

in \mathcal{A} . It is called *guarded* if it factors as follows

$$\begin{array}{ccc} X & \xrightarrow{e} & T(X + Y) \\ & \searrow & \uparrow [\tau_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \\ & & HT(X + Y) + Y. \end{array}$$

Notice that for a polynomial endofunctor $H = H_\Sigma$ on **Set** this is precisely the notion of a guarded system as presented above, since $T(X+Y) = HT(X+Y) + X + Y$. Finally, a *solution* for an equation arrow e is an arrow $e^\dagger : X \longrightarrow TY$ such that the following triangle

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & TY \\ \downarrow e & \nearrow [\widehat{e^\dagger}, \eta_Y] & \\ T(X + Y) & & \end{array}$$

commutes. Again, this corresponds precisely to the notion of solution for systems of equations in case of polynomial endofunctors on **Set**.

The following result is called Parametric Corecursion in [M] and Solution Theorem in [AAV]; see also an improved version of the proof in [AAMV]:

Solution Theorem 2.8 *Given an iterable endofunctor H , every guarded equation morphism has a unique solution.*

Remark 2.9 It is an easy consequence of the Substitution Theorem that $(T, \eta, \widehat{_})$ forms a Kleisli triple, i. e., the following three conditions are satisfied

- (i) $\widehat{\eta}_X = id_{TX}$ for all objects X ,
- (ii) $\widehat{s} \cdot \eta_X = s$ for all arrows $s : X \rightarrow TY$,
- (iii) $\widehat{r} \cdot \widehat{s} = \widehat{r \cdot s}$ for any morphisms $s : X \rightarrow TY$ and $r : Y \rightarrow TZ$.

Thus setting $\mu_X = \widehat{id_{TX}} : TTX \rightarrow TX$ we obtain a monad (T, η, μ) , and we call it the *completely iterative monad* generated by H .

2.3 The free Completely Iterative Monad

Based on the consideration in the previous section 2.2 it is quite natural to call for any monad (S, η, μ) on \mathcal{A} a morphism

$$e : X \rightarrow S(X + Y)$$

an *equation arrow*. Recall that for any monad there is an associated Kleisli triple, where for $s : X \rightarrow SY$ we have $\widehat{s} = \mu_Y \cdot Ss$. Hence, a morphism $e^\dagger : X \rightarrow SY$ with

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ e \downarrow & \nearrow [\widehat{e^\dagger, \eta_Y}] & \\ S(X + Y) & & \end{array}$$

will be called a *solution*. However, it is in general not obvious how the property of e being guarded is to be expressed for an arbitrary monad.

Elgot, Bloom and Tindell [EBT] use, in their setting of algebraic theories, the notion of an ideal theory introduced by Elgot in [E]. For finitary monads on **Set** this notion is equivalent to the following notion of ideal monad (see [AAMV] for a simple proof of this fact):

Definition 2.10

- (i) Let (S, η, μ) be a monad. A (right) ideal of S is a subfunctor $\sigma : S' \rightarrow S$ such that there exists a (necessarily unique) restriction $\mu' : S'S \rightarrow S'$ of μ , i. e., the following square

$$\begin{array}{ccc} S'S & \xrightarrow{\mu'} & S' \\ \sigma S \downarrow & & \downarrow \sigma \\ SS & \xrightarrow{\mu} & S \end{array}$$

commutes.

- (ii) A monad together with an ideal of it is called an *idealized monad*. If furthermore we have $S = S' + Id$, i. e., $[\sigma, \eta] : S' + Id \rightarrow S$ is an

isomorphism, then S is called an *ideal monad*.

- (iii) An *idealized-monad morphism* between idealized monads S_1 and S_2 with chosen ideals $\sigma_i : S'_i \twoheadrightarrow S_i$, $i = 1, 2$, is a monad morphism $h : S_1 \rightarrow S_2$ that preserves the chosen ideals, i. e., there exists a (necessarily unique) natural transformation $h' : S'_1 \rightarrow S'_2$ such that the following square

$$\begin{array}{ccc} S'_1 & \xrightarrow{h'} & S'_2 \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ S_1 & \xrightarrow{h} & S_2 \end{array}$$

commutes.

Example 2.11

- (i) Recall that the monad T is a coproduct of HT and Id . Hence the ideal $\tau : HT \twoheadrightarrow T$, where μ' is given by $H\mu$ makes T into an ideal monad.
- (ii) Any monad S has ideals, e. g., the largest one (S itself). If \mathcal{A} has a strict initial object, then the smallest ideal is given by the constant functor on the initial object.

Remark 2.12

- (i) Notice that the notion of an ideal of a monad corresponds precisely to the notion of a right ideal for a monoid. Indeed, recall that a right ideal of a monoid M is a subset I of M such that $I \cdot M \subseteq I$. Now a monad is just a monoid in the monoidal category $[\mathcal{A}, \mathcal{A}]$ of endofunctors on \mathcal{A} with tensor product being given by composition of functors.
- (ii) It is not difficult to show that the category of ideal monads and ideal monad homomorphisms is a coreflective subcategory of the category of idealized monads with the same morphisms. In fact, if (S, η, μ) is a monad with ideal $\sigma : S' \twoheadrightarrow S$ the coreflection arrow is given by

$$S' + Id \xrightarrow{[\sigma, \eta]} S.$$

Since this is not needed here, the proof is omitted.

Remark 2.13 Observe that the completely iterative monad T generated by H comes with a natural “embedding of H ”

$$\tau^* \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T$$

into it. More generally, we call for any endofunctor H and idealized monad S a natural transformation $\sigma^* : H \rightarrow S$ *ideal* if it factors through the ideal

$\sigma : S' \twoheadrightarrow S$ as follows

$$\begin{array}{ccc} H & \xrightarrow{\sigma^*} & S \\ & \searrow^{(\sigma^*)'} & \uparrow \sigma \\ & & S' \end{array}$$

For an idealized monad S we define the notion of a *guarded* equation arrow as a morphism e that factors

$$\begin{array}{ccc} X & \xrightarrow{e} & S(X + Y) \\ & \searrow & \uparrow [\sigma_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \\ & & S'(X + Y) + Y. \end{array}$$

Definition 2.14 An idealized monad S is called *completely iterative* if every guarded equation arrow has a unique solution.

Remark 2.15 In [AAMV] a completely iterative monad is required to be ideal. Observe however, that this is an unnecessary restriction. In fact, all of the proofs of [AAMV] use only properties of idealized monads.

The following is the main result of [AAMV].

Theorem 2.16 *For any iterable endofunctor H , the monad T is the free completely iterative monad on H . More precisely, for all completely iterative monads S and ideal transformations $\lambda : H \rightarrow S$ there exists a unique idealized-monad morphism $\bar{\lambda} : T \rightarrow S$ such that $\bar{\lambda} \cdot \tau^* = \lambda$:*

$$\begin{array}{ccc} H & \xrightarrow{\tau^*} & T \\ & \searrow^{\forall \lambda} & \downarrow \exists! \bar{\lambda} \\ & & S. \end{array}$$

Remark 2.17

- (i) Since the inclusion of the ideal $\sigma : S' \twoheadrightarrow S$ is a monomorphism, the last condition is equivalent to stating that

$$\begin{array}{ccc} H & \xrightarrow{H\eta} & HT \\ & \searrow^{\lambda'} & \downarrow \bar{\lambda}' \\ & & S' \end{array}$$

commutes, where $\bar{\lambda}' : HT \rightarrow S'$ is the restriction of $\bar{\lambda}$ to the ideal of T .

- (ii) Categorically, the statement of the theorem says that every iterable functor H in $[\mathcal{A}, \mathcal{A}]$ has a universal arrow w. r. t. the forgetful functor

$$U : \mathbf{CIM}(\mathcal{A}) \rightarrow [\mathcal{A}, \mathcal{A}]$$

of the category $\mathbf{CIM}(\mathcal{A})$ of all completely iterative monads and idealized-monad morphisms. Beware! The functor U assigns to every completely

iterative monad S its ideal S' , not the underlying functor S . This choice of U corresponds to the requirement that $\lambda : H \rightarrow S$ be an ideal transformation.

The above result states that any iterable endofunctor admits a free completely iterative monad. However, the obvious question whether these are the only endofunctors with this property remains unanswered in [AAMV]. We will present this answer in the next section.

3 Iterability is necessary

We have seen above that any iterable endofunctor admits a free completely iterative monad. We shall prove in this section that endofunctors that admit a free completely iterative monad are precisely the iterable ones.

Throughout this section we shall denote by \mathcal{A} a category with binary co-products such that injections are monomorphic.

Theorem 3.1 *Every endofunctor generating a free completely iterative monad is iterable.*

Remark 3.2 More detailed, suppose that H is an endofunctor on \mathcal{A} and

$$\sigma^* : H \rightarrow S$$

is a free completely iterative monad on H (where σ^* is an ideal transformation), then H is iterable and for all objects X of \mathcal{A} , SX is a final coalgebra of $H(-) + X$.

Before we proceed with the proof of this theorem, let us prove two auxiliary results. First we establish that for any natural transformation $H \rightarrow S$, where H is any endofunctor and S any monad on \mathcal{A} , one can easily obtain an ideal monad \tilde{S} and an ideal transformation $H \rightarrow \tilde{S}$ as follows.

Definition 3.3 Let (S, η, μ) be a monad on \mathcal{A} and let

$$\sigma^* : H \rightarrow S$$

be a natural transformation from an endofunctor H on \mathcal{A} . Define $(\tilde{S}, \tilde{\eta}, \tilde{\mu})$ as follows:

- (i) $\tilde{S} = HS + Id$
- (ii) $\tilde{\eta} \equiv \text{inr} : Id \rightarrow HS + Id$

$$\begin{aligned}
 \text{(iii)} \quad \tilde{\mu} &\equiv \tilde{S}^2 = (HS + Id)^2 = HS(HS + Id) + HS + Id \\
 &\quad \downarrow HS(\sigma^* S + Id) + HS + Id \\
 &HS(S^2 + Id) + HS + Id \\
 &\quad \downarrow HS[\mu, \eta] + HS + Id \\
 &HS^2 + HS + Id \\
 &\quad \downarrow [H\mu, \text{inl}] + Id \\
 &HS + Id = \tilde{S}
 \end{aligned}$$

Lemma 3.4 *The triple $(\tilde{S}, \tilde{\eta}, \tilde{\mu})$ is an ideal monad.*

Proof. Once we have established that \tilde{S} is a monad, it is obvious that it is ideal: Note that for $\tilde{S}' = HS$ we have

$$\tilde{\mu}' \equiv \tilde{S}'\tilde{S} = HS(HS + Id) \xrightarrow{HS(\sigma^* S + Id)} HS(S^2 + Id) \xrightarrow{HS[\mu, \eta]} HS^2 \xrightarrow{H\mu} HS = \tilde{S}'.$$

Hence, it is sufficient to show that $\tilde{\eta}$ and $\tilde{\mu}$ satisfy the three axioms of a monad.

(i) $\tilde{\mu} \cdot \tilde{\eta}_{\tilde{S}} = 1_{\tilde{S}}$: This is obvious since

$$HS + Id \xrightarrow{\text{inr}} HS(HS + Id) + HS + Id \xrightarrow{\tilde{\mu}} HS + Id \equiv 1_{HS + Id}.$$

(ii) $\tilde{\mu} \cdot \tilde{S}\tilde{\eta} = 1_{\tilde{S}}$: Observe that

$$\tilde{S}\tilde{\eta} \equiv HS + Id \xrightarrow{HS\text{inr} + \text{inr}} HS(HS + Id) + HS + Id.$$

We compose this with $\tilde{\mu}$ and consider the components of the coproduct $HS + Id$ separately. On the right-hand component we obviously obtain $\text{inr} : Id \rightarrow HS + Id$. For the left-hand one we drop H and consider the resulting commutative diagram

$$\begin{array}{ccc}
 S & \xrightarrow{S\text{inr}} & S(HS + Id) \\
 & \searrow^{S\text{inr}} & \downarrow S(\sigma^* S + Id) \\
 & & S(S^2 + Id) \\
 & \searrow^{S\eta} & \downarrow S[\mu, \eta] \\
 & & S^2 \\
 & & \downarrow \mu \\
 & & S.
 \end{array}$$

(iii) $\tilde{\mu} \cdot \tilde{S}\tilde{\mu} = \tilde{\mu} \cdot \tilde{\mu}\tilde{S}$: This is a straightforward and not particularly enlightening chase through rather huge diagrams. Since it only involves naturality and

the equation $\mu \cdot S\mu = \mu \cdot \mu S$, we leave this as an easy exercise for the Reader. \square

Lemma 3.5 *If S in Definition 3.3 above is a completely iterative monad and $\sigma^* : H \rightarrow S$ is an ideal transformation, then \tilde{S} is completely iterative, too.*

Proof. We have to show that for each guarded equation morphism $e : X \rightarrow \tilde{S}(X + Y)$ with a factorization

$$\begin{array}{ccc} X & \xrightarrow{e} & HS(X + Y) + X + Y \\ & \searrow f & \uparrow [\text{inl}, \text{inr}] \\ & & HS(X + Y) + Y \end{array}$$

we have a unique solution $e^\dagger : X \rightarrow \tilde{S}Y$. We define a guarded equation arrow $\bar{e} : X \rightarrow S(X + Y)$ as follows

$$\bar{e} \equiv X \xrightarrow{f} HS(X + Y) + Y \xrightarrow{\sigma^* S + \eta} S^2(X + Y) + SY \xrightarrow{[\mu, S\text{inr}]} S(X + Y).$$

In order to see that \bar{e} is indeed guarded, use that S is an idealized monad and that σ^* is an ideal transformation.

We solve \bar{e} to obtain a unique arrow $\bar{e}^\dagger : X \rightarrow SY$ such that the outer shape of the following diagram

$$(2) \quad \begin{array}{ccc} X & \xrightarrow{\bar{e}^\dagger} & SY \\ \downarrow f & \text{(II)} & \uparrow [\mu, \eta] \\ HS(X + Y) + Y & \xrightarrow{HS[\bar{e}^\dagger, \eta_Y] + Y} & HS^2 + Y \xrightarrow{\sigma^* S^2 + Y} S^3Y + Y \\ \downarrow \sigma^* S + \eta & \uparrow H\mu + Y & \uparrow S\mu + Y \\ S^2(X + Y) + SY & \xrightarrow{S^2[\bar{e}^\dagger, \eta_Y] + SY} & S^3Y + SY \\ \downarrow [\mu, S\text{inr}] & & \downarrow S^3Y + \eta \\ S(X + Y) & \xrightarrow{S[\bar{e}^\dagger, \eta_Y]} & S^2Y \\ & & \uparrow \mu_Y \end{array}$$

commutes. To see that square (I) commutes consider the components of the coproduct $HS(X + Y) + Y$ separately. The left-hand component commutes by naturality of σ^* , whereas the right-hand one obviously does. Hence, region (II) commutes since all other parts of the above Diagram (2) clearly do.

We define

$$e^\dagger \equiv X \xrightarrow{f} HS(X + Y) + Y \xrightarrow{HS[\bar{e}^\dagger, \eta_Y] + Y} HS^2Y + Y \xrightarrow{H\mu + Y} HSY + Y = \tilde{S}Y,$$

and check that this yields a solution for e . Indeed, consider the following

commutes.

Below we will show that

$$(4) \quad X \xrightarrow{h} HSY + Y \xrightarrow{\sigma^* S + Y} S^2Y + Y \xrightarrow{[\mu, \eta]} SY$$

solves \bar{e} . But then it is not difficult to show that $e^\dagger = h$. In fact, we start with the definition of the solution e^\dagger

$$e^\dagger = (H\mu_Y + Y) \cdot (HS[\bar{e}^\dagger, \eta_Y] + Y) \cdot f,$$

then substitute (4) for \bar{e}^\dagger to obtain

$$(5) \quad (H\mu_Y + Y) \cdot (HS[[\mu_Y, \eta_Y] \cdot (\sigma_{SY}^* + Y) \cdot h, \eta_Y] + Y) \cdot f,$$

and finally, we use the equation

$$(6) \quad \eta_Y = [\mu_Y, \eta_Y] \cdot (\sigma_{SY}^* + Y) \cdot \text{inr}.$$

in order to see that (5) is the same as

$$(H\mu_Y + Y) \cdot (HS[\mu_Y, \eta_Y] + Y) \cdot (HS(\sigma_{SY}^* + Y) + Y) \cdot (HS[h, \text{inr}] + Y) \cdot f,$$

which according to the upper left-hand part of Diagram (3) is just h .

Let us complete our proof by showing that the arrow (4) solves \bar{e} . In fact, the following diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{h} & HSY + Y & \xrightarrow{\sigma^* S + Y} & S^2Y + Y & \xrightarrow{[\mu, \eta]} & SY \\
 & & \uparrow H\mu + Y & & \uparrow S\mu_Y + Y & & \uparrow \mu_Y \\
 & & HS^2Y + Y & \xrightarrow{\sigma^* S^2 + Y} & S^3Y + Y & & \\
 & & \uparrow HS[\mu, \eta] + Y & & \downarrow S^3Y + \eta_Y & & \\
 & & HS(S^2Y + Y) + Y & & & & \\
 & & \uparrow HS(\sigma^* S + Y) + Y & & & & \\
 HS(X + Y) + Y & \xrightarrow{HS[h, \text{inr}] + Y} & HS(HSY + Y) + Y & & & & \\
 \downarrow \sigma^* S + \eta_Y & & & & & & \\
 S^2(X + Y) + SY & \xrightarrow{S^2[[\mu_Y, \eta_Y] \cdot (\sigma_{SY}^* + Y) \cdot h, \eta_Y] + SY} & S^3Y + SY & & & & \\
 \downarrow [\mu, \text{Sinr}] & & \searrow [\mu_{SY}, S\eta_Y] & & & & \\
 S(X + Y) & \xrightarrow{S[[\mu_Y, \eta_Y] \cdot (\sigma_{SY}^* + Y) \cdot h, \eta_Y]} & S^2Y & & & &
 \end{array}$$

commutes. The upper left-hand square is just the upper left-hand square of Diagram (3). For the inner part (I), consider the components of the coproduct $HS(X + Y) + Y$ separately. The right-hand components obviously commute, for the left-hand ones use naturality of σ^* and Equation (6). All other parts clearly commute. \square

Proof of Theorem 3.1. Suppose that (S, η, μ) is a free completely iterative

monad on H , i. e., there exists a universal ideal transformation

$$\sigma^* : H \longrightarrow S.$$

By Lemma 3.4, $\tilde{S} = HS + Id$ is an ideal monad, and by Lemma 3.5 it is completely iterative. Then by the universal property we have a unique idealized-monad morphism $\alpha : S \longrightarrow HS + Id$ such that the following diagram

$$(7) \quad \begin{array}{ccc} H & \xrightarrow{\sigma^*} & S \\ H\eta \searrow & & \downarrow \alpha \\ & HS & \\ & \text{inl} \searrow & \\ & & HS + Id \end{array}$$

commutes.

Note that for all objects Y of \mathcal{A} the arrows

$$\alpha_Y : SY \longrightarrow HSY + Y$$

define a coalgebra structure for $H(-) + Y$ on SY . We shall establish below that α is an isomorphism with an inverse given by the natural transformation

$$\beta \equiv HS + Id \xrightarrow{\sigma^* S + Y} S^2 + Id \xrightarrow{[\mu, \eta]} S.$$

In order to establish that (SY, α_Y) is a final coalgebra suppose that $\gamma : A \longrightarrow HA + Y$ is any coalgebra of $H(-) + Y$. Then

$$\bar{\gamma} \equiv A \xrightarrow{\gamma} HA + Y \xrightarrow{\sigma^* + \eta} SA + SY \xrightarrow{[S\text{inl}, S\text{inr}]} S(A + Y)$$

is a guarded equation arrow (since σ^* is ideal, i. e., it factors through $\sigma : S' \longrightarrow S$) whose solution $\bar{\gamma}^\dagger : A \longrightarrow SY$ yields the desired unique homomorphism of coalgebras. Indeed, consider the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\gamma} & HA + Y & \xrightarrow{\sigma^* + \eta} & SA + SY & \xrightarrow{[S\text{inl}, S\text{inr}]} & S(A + Y) \\ \downarrow x & & \downarrow Hx + Y & & \downarrow Sx + SY & & \downarrow S[x, \eta_Y] \\ & & HSY + Y & & & & \\ & & \downarrow \sigma_{SY}^* + \eta & & & & \\ & & S^2Y + SY & & & & \\ & \swarrow \beta_Y = \alpha_Y^{-1} & & \searrow [\mu, SY] & & & \\ SY & & & & SY & & S^2Y \end{array}$$

Suppose we put $\bar{\gamma}^\dagger$ in place of x in the diagram. Then the outer square commutes, and we conclude that the upper left-hand part commutes, since all other parts obviously do. This shows that $\bar{\gamma}^\dagger$ is a coalgebra homomorphism.

Conversely, put any coalgebra homomorphism $h : (A, \gamma) \longrightarrow (SY, \alpha_Y)$ in place of x . Then the upper left-hand part commutes, and therefore the whole diagram does. But then $h = \overline{\gamma}^\dagger$, by the uniqueness of solutions. This concludes the proof.

Finally, we show that β is the inverse of α .

- (i) $\beta \cdot \alpha = 1_S$: We will first show that $\beta : HS + Id \longrightarrow S$ is an idealized-monad morphism. In fact, once we know it is a monad morphism, it is easily established that it is ideal. To see this, consider the following commutative diagram:

$$\begin{array}{ccc}
 HS & \xrightarrow{\text{inl}} & HS + Id \\
 (\sigma^*)'S \downarrow & \searrow \sigma^*S & \downarrow \sigma^*S + Id \\
 S'S & \xrightarrow{\sigma S} & S^2 \xrightarrow{\text{inl}} S^2 + id \\
 \mu' \downarrow & & \searrow \mu \downarrow [\mu, \eta] \\
 S' & \xrightarrow{\sigma} & S
 \end{array}$$

Let us show that β is a monad homomorphism. We clearly have

$$\begin{aligned}
 \beta \cdot \tilde{\eta} &= [\mu, \eta] \cdot (\sigma S + Id) \cdot \text{inr} \\
 &= [\mu, \eta] \cdot \text{inr} \\
 &= \eta.
 \end{aligned}$$

Hence, it suffices to prove that the following square

$$\begin{array}{ccc}
 \tilde{S}^2 & \xrightarrow{\tilde{\mu}} & \tilde{S} \\
 \tilde{S}\beta \downarrow & & \downarrow \beta \\
 \tilde{S}S & & \\
 \beta S \downarrow & & \\
 S^2 & \xrightarrow{\mu} & S
 \end{array}$$

is commutative. We apply the definition of $(\tilde{S}, \tilde{\eta}, \tilde{\mu})$ and consider the components of the coproduct

$$\tilde{S}^2 = (HS + Id)^2 = HS(HS + Id) + HS + Id$$

separately. For the right-hand component $HS + Id$ we obtain

$$\begin{aligned}
 \beta \cdot \tilde{\mu} \cdot \text{inr} &= \beta & (\text{inr} = \tilde{\eta}\tilde{S}) \\
 &= \mu \cdot \eta S \cdot \beta & (\mu \cdot \eta S = 1_S) \\
 &= \mu \cdot \beta S \cdot \tilde{\eta}S \cdot \beta & (\text{since } \beta \cdot \tilde{\eta} = \eta) \\
 &= \mu \cdot \beta S \cdot \tilde{S}\beta \cdot \text{inr} & (\text{naturality of } \tilde{\eta}).
 \end{aligned}$$

For the left-hand component we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 HS(HS + Id) & \xrightarrow{HS(\sigma^*S + Id)} & HS(S^2 + Id) & \xrightarrow{HS[\mu, \eta]} & HS^2 & \xrightarrow{H\mu} & HS \\
 \downarrow HS(\sigma^*S + Id) & & & \nearrow & & & \downarrow \sigma^*S \\
 HS(S^2 + Id) & & & & & & \\
 \downarrow HS[\mu, \eta] & & & & & & \\
 HS^2 & & & & & & \\
 \downarrow \sigma^*S^2 & & & & & & \\
 S^3 & \xrightarrow{S\mu} & & & & & S^2 \\
 \downarrow \mu S & & & & & & \downarrow \mu \\
 S^2 & \xrightarrow{\mu} & & & & & S.
 \end{array}$$

Now $\beta \cdot \alpha$ is an idealized-monad morphism such that $\beta \cdot \alpha \cdot \sigma^* = \sigma^*$. In fact, the following diagram

$$\begin{array}{ccccc}
 H & \xrightarrow{\sigma^*} & S & & \\
 \downarrow \sigma^* & \searrow H\eta & \downarrow \alpha & & \\
 S & \xrightarrow{S\eta} & HS & \xrightarrow{\text{inl}} & HS + Id \\
 & \searrow \sigma^*S & \downarrow \sigma^*S + Id & & \downarrow \beta \\
 & & S^2 & \xrightarrow{\text{inl}} & S^2 + Id \\
 & & \downarrow \mu & & \downarrow [\mu, \eta] \\
 & & S & & S
 \end{array}$$

commutes. Therefore, by the freeness of S on H , we have $\beta \cdot \alpha = 1_S$, as desired.

- (ii) $\alpha \cdot \beta = 1_{HS+Id}$: We check this on the components of $HS + Id$. For the right-hand component we obtain

$$\begin{aligned}
 \alpha \cdot \beta \cdot \text{inr} &= \alpha \cdot [\mu, \eta] \cdot (\sigma^*S + Id) \cdot \text{inr} && \text{(definition of } \beta) \\
 &= \alpha \cdot \eta \\
 &= \text{inr} && (\alpha \text{ is a monad morphism),}
 \end{aligned}$$

For the left-hand component we have

$$\begin{aligned}
 \alpha \cdot \beta \cdot \text{inl} &= \alpha \cdot [\mu, \eta] \cdot (\sigma^* S + \text{Id}) \cdot \text{inl} && \text{(definition of } \beta) \\
 &= \alpha \cdot \mu \cdot \sigma^* S \\
 &= \tilde{\mu} \cdot \tilde{S} \alpha \cdot \alpha S \cdot \sigma^* S && (\alpha \text{ is a monad morphism}) \\
 &= \tilde{\mu} \cdot \tilde{S} \alpha \cdot \text{inl} \cdot H\eta S && (\alpha \cdot \sigma^* = \text{inl} \cdot H\eta, \text{ see (7)}).
 \end{aligned}$$

We analyze the last expression further and obtain the following commutative diagram:

$$\begin{array}{ccc}
 HS & \xrightarrow{\quad\quad\quad} & HS \\
 H\eta S \downarrow & & \downarrow H\mu \\
 HS^2 & \xrightarrow{\quad\quad\quad} & HS^2 \\
 \text{inl} \downarrow & \searrow HS\alpha & \nearrow HS\beta \\
 HS^2 + S & \xrightarrow{HS(HS + \text{Id})} & HS(S^2 + \text{Id}) \\
 \tilde{S}\alpha = HS\alpha + \alpha \downarrow & \nearrow \text{inl} & \uparrow HS[\mu, \eta] \\
 \tilde{S}^2 & \xrightarrow{\quad\quad\quad} & \tilde{S} = HS + \text{Id} \\
 & \searrow \tilde{\mu} & \downarrow \text{inl}
 \end{array}$$

Note that the inner triangle commutes since $\beta \cdot \alpha = 1_S$, and the other parts obviously commute. Thus we have shown that

$$\alpha \cdot \beta = [\text{inl}, \text{inr}] = 1_{HS + \text{Id}}$$

as required. □

Acknowledgments

I would like to thank Jiří Adámek for many useful suggestions that helped me improving this current text.

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