FROM ITERATIVE ALGEBRAS TO ITERATIVE THEORIES

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Abstract. Iterative theories introduced by Calvin Elgot formalize potentially infinite computations as unique solutions of recursive equations. One of the main results of Elgot and his coauthors is a description of a free iterative theory as the theory of all rational trees. Their algebraic proof of this fact is extremely complicated. In our paper we show that by starting with "iterative algebras", i. e., algebras admitting a unique solution of all systems of flat recursive equations, a free iterative theory is obtained as the theory of free iterative algebras. The (coalgebraic) proof we present is dramatically simpler than the original algebraic one. And our result is, nevertheless, much more general: we describe a free iterative theory on any finitary endofunctor of every locally presentable category $\mathcal{A}$. This allows us, e. g., to consider iterative algebras over every equationally specified class $\mathcal{A}$ of finitary algebras.

Reportedly, a blow from the welterweight boxer Norman Selby, also known as *Kid McCoy*, left one victim proclaiming, "*It's the real McCoy!*"

[TPT]

1. Introduction

Iterative theories have been introduced by Calvin C. Elgot [E] as a model of computation given by a sequence of instantaneous descriptions of an abstract machine. He and his co-authors then proved that for every signature Σ a free iterative theory on Σ exists [BE] and that it consists of all rational Σ-trees [EBT]. Recall that a Σ-tree (i. e., a tree, possibly infinite, labelled by operation symbols in Σ so that every node with n children is labelled by an n-ary symbol) is *rational* if it has up to isomorphism only finitely many subtrees, see [G].

In the present paper we introduce *iterative algebras* rather than iterative theories, and we show that the theory formed by all free iterative algebras is Elgot's free iterative theory. In the classical case of Σ-algebras, iterativity has been introduced by Evelyn Nelson [N] as follows: given a Σ-algebra A, let us consider an arbitrary system of recursive equations

$$x_i \approx t_i, \qquad i = 1, \dots, n\,, \tag{1.1}$$

where $X = \{x_1, x_2, \dots, x_n\}$ is a finite set of variables and t_1, t_2, …, t_n are terms over $X + A$, none of which is a single variable x_i. The algebra A is called *iterative* provided that for every such system of equations there exists a unique *solution*. That is, there exists a unique n-tuple ${x_1}^\dagger$, ${x_2}^\dagger$, …, ${x_n}^\dagger$ of elements of A such that each of the formal equations in (1.1) becomes an equality after the substitution ${x_i}^\dagger/x_i$:

$${x_i}^\dagger = t_i({x_1}^\dagger/x_1, {x_2}^\dagger/x_2, \dots, {x_n}^\dagger/x_n)\,, \qquad i = 1, \dots, n.$$

Example: let Σ consist of a single binary operation symbol, $*$, then the algebra A of all (finite and infinite) binary trees is iterative. For example, the system

$$\begin{array}{lll} x_1 & \approx & x_2 * t \\ x_2 & \approx & (x_1 * s) * t \end{array} \tag{1.2}$$

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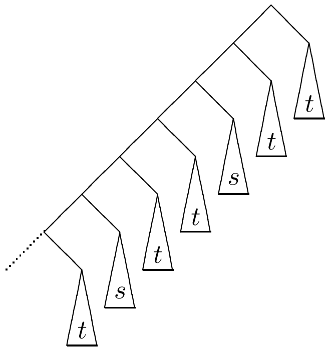
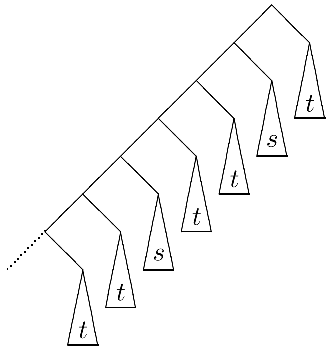
where s and t are trees in A has the unique solution

$x_1{}^\dagger =$  $x_2{}^\dagger =$  (1.3)

Every system (1.1) above can be modified to a *flat system*, i.e., one where each right-hand side is either a *flat term*

$$t_i = \sigma(y_1, \dots, y_k), \quad \text{for } \sigma \in \Sigma_k \text{ and } y_1, \dots, y_k \in X,$$

or an element of A

$$t_i \in A.$$

For example, the above system (1.2) has the following modification to a flat system:

$$\begin{array}{llllllll} x_1 & \approx & x_2 * x_3 & x_3 & \approx & t & x_5 \approx s \\ x_2 & \approx & x_4 * x_3 & x_4 & \approx & x_1 * x_5 & \end{array}$$

Therefore, an algebra is iterative iff every flat equation system has a unique solution.

Now Σ-algebras are a special case of algebras for an endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$ (which are pairs consisting of an object A of $\mathcal{A}$ and a morphism $\alpha : HA \longrightarrow A$): here $\mathcal{A}$ is the category of sets and $H = H_\Sigma$ is the *polynomial functor* given on objects by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

For a Σ-algebra (A, α) observe that a flat equation system has its right-hand sides in $H_\Sigma X + A$, thus, it can be represented by a morphism

$$e : X \longrightarrow H_\Sigma X + A, \quad e(x_i) = t_i.$$

A *solution* of e is then a morphism

$$e^\dagger : X \longrightarrow A, \quad e^\dagger(x_i) = x_i{}^\dagger,$$

with the property that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ H_\Sigma X + A & \xrightarrow[H_\Sigma e^\dagger + A]{} & H_\Sigma A + A \end{array} \tag{1.4}$$

commutes. This leads to the following definition concerning H-algebras for any endofunctor H of Set:

Definition 1.1. An H-algebra (A, α) is called *iterative* provided that for every flat equation morphism $e : X \longrightarrow HX + A$, where X is a finite set, there exists a unique solution, i.e., a unique morphism $e^\dagger : X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array}$$

commutes.

"Classical" algebras are seldom iterative. But there are enough interesting iterative algebras. For example, the Σ-algebra

$$T_\Sigma$$

of all (finite and infinite) Σ-trees is iterative. And so is its subalgebra

$$R_\Sigma$$

of all rational Σ-trees. In fact, the full subcategory $\mathsf{Alg}_{it}\ \Sigma$ of $\mathsf{Alg}\ \Sigma$ formed by all iterative Σ-algebras is rich enough: a limit or a filtered colimit of iterative algebras is always iterative, thus $\mathsf{Alg}_{it}\ \Sigma$ is reflective in $\mathsf{Alg}\ \Sigma$. From this it follows that every set generates a free iterative algebra, i. e., the forgetful functor $\mathsf{Alg}_{it}\ \Sigma \longrightarrow \mathsf{Set}$ is a right-adjoint. This defines a monad $\mathbb{R}_\Sigma$ on Set. We prove that

(i) $\mathbb{R}_\Sigma$ is a free iterative monad on H_Σ,

and

(ii) $\mathbb{R}_\Sigma$ assigns to every set X the algebra $R_\Sigma X$ of all rational Σ-trees on X, i. e., rational trees where leaves are labelled by constant symbols from Σ or elements of X.

In this way a new proof of the result of Elgot et al. describing a free iterative monad (or theory) is achieved.

In our proof we work with an arbitrary endofunctor H of the category of sets which is *finitary*, i. e., preserves filtered colimits. The main technical result is coalgebraic: in order to describe a free iterative algebra on a set Y, we form the diagram Eq_Y of all coalgebras $e : X \longrightarrow HX + Y$ of the endofunctor $H(-) + Y$ on finite sets X. We prove that a colimit of that diagram

$$RY = \operatorname{colim} \mathrm{Eq}_Y$$

carries naturally the structure of an algebra, and that RY is a free iterative H-algebra on Y. From that we derive that the monad $R(-)$ is a free iterative monad on H. In our proof the fact that H is a finitary endofunctor of Set plays no rôle: the same result holds for finitary endofunctors of all locally finitely presentable categories. Thus, if we start e. g. with an equational class $\mathcal{A}$ of finitary algebras then, again, for every finitary endofunctor H the free iterative algebras RY are constructed as colimits of coalgebras of $H(-) + Y$ on finitely presentable objects of $\mathcal{A}$, and they form a free iterative theory on H.

Related Work. In the classical setting, i. e., for polynomial endofunctors of Set, iterative algebras were introduced by Evelyn Nelson [N] to obtain a short proof of Elgot's free iterative theories. Our paper can be seen as a categorical generalization of that paper with distinctive coalgebraic "flavour". Also Jerzy Tiuryn introduced a concept of iterative algebra in [T] with the same aim as ours: to relate iterative theories of Elgot to properties of algebras. But the approach of [T] is different from ours; e.g., the trivial, one-element, algebra is not iterative in the sense of Tiuryn, thus, his iterative algebras are not closed under limits.

The description of the rational monad as a colimit is also presented in [GLM].

The present paper is a dramatic improvement of our previous description of the rational monad in [AMV_1], [AMV_2] where we assumed that the endofunctor preserves monomorphisms and the underlying category satisfies three rather technical conditions, and the proof was much more involved. The current approach includes all equationally defined algebraic categories as base categories (whereas in [AMV_2] we still needed strong side conditions which only hold in very few algebraic categories). We believe that with this paper we have the "real McCoy". Simultaneously to the present paper the paper [Mi] devoted to completely iterative algebras evolved.

2. Iterative Algebras

Notation 2.1. Throughout the paper all categories are assumed to have finite coproducts. We denote by inl and inr the coproduct injections of $A + B$. For an endofunctor H, let $\mathsf{can} : HA + HB \longrightarrow H(A + B)$ denote the canonical morphism $\mathsf{can} = [H\mathsf{inl}, H\mathsf{inr}]$.

In order to define the concept of a flat equation morphism as in the introduction (a morphism $e : X \longrightarrow HX + A$ in Set where X is finite) in a general category, we need the appropriate generalization of finiteness. Recall that a functor is called *finitary* provided that it preserves filtered colimits. A set is finite if and only if its hom-functor is finitary. This has inspired Gabriel and Ulmer [GU] to the following

Definition 2.2. An object of A a category $\mathcal{A}$ is *finitely presentable* if its hom-functor $\mathcal{A}(A, -) : \mathcal{A} \longrightarrow \mathsf{Set}$ is finitary.

A category $\mathcal{A}$ is called *locally finitely presentable* provided that it has colimits and a (small) set of finitely presentable objects whose closure under filtered colimits is all of $\mathcal{A}$.

Examples 2.3.

(1) A poset is finitely presentable in Pos, the category of posets and order-preserving functions, if and only if it is finite. Pos is a locally finitely presentable category.
(2) The category CPO of complete partial orders and continuous functions is not locally finitely presentable: it has no nontrivial finitely presentable objects.

(3) Every variety of finitary algebras is locally finitely presentable. The categorical concept of finitely presentable object coincides with the algebraic one (of having finitely many generators and finitely many presenting equations), see [AR].
(4) Let H be a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$. Then the category $\mathsf{Alg}\, H$ of H-algebras and homomorphisms is also locally finitely presentable, see [AR].

Definition 2.4. Given an endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$, by a *finitary flat equation morphism* (later just: *equation morphism*) in an object A we mean a morphism $e : X \longrightarrow HX + A$ of $\mathcal{A}$, where X is a finitely presentable object of $\mathcal{A}$.

Suppose that A is an underlying object of an H-algebra $\alpha : HA \longrightarrow A$. Then by a *solution* of e in the algebra A is meant a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array} \tag{2.1}$$

commutes.

An H-algebra is called *iterative* provided that every finitary flat equation morphism has a unique solution.

Example 2.5. The algebra $T_\Sigma Y$ of all Σ-*trees on* Y (i. e., trees with leaves labelled by constant symbols in Σ_0 or by elements of Y, and inner nodes with n children labelled in Σ_n) is iterative. And so is the subalgebra $R_\Sigma Y$ of all rational trees on Y.

Example 2.6. Groups, lattices etc. considered as Σ-algebras are seldom iterative. For example, if a group is iterative, then its unique element is the unit element 1, since the recursive equations $x \approx x \cdot y,\ y \approx 1$ have a unique solution. If a lattice is iterative, then it has a unique element: consider $x \approx x \vee x$.

Example 2.7. The algebra of addition on the set

$$\widetilde{\mathbb{N}} = \{\, 1, 2, 3, \dots \} \cup \{\, \infty \,\}$$

is iterative (and "almost classical"). (Observe that 0 is not included. This is forced by the uniqueness of solutions of $x \approx x + x$.)

To prove the iterativity of $\widetilde{\mathbb{N}}$ denote by $h : T_\Sigma \widetilde{\mathbb{N}} \longrightarrow \widetilde{\mathbb{N}}$ the homomorphism which to every finite tree assigns the result of computing the corresponding term in $\widetilde{\mathbb{N}}$ and to every infinite tree assigns ∞. Observe that the canonical embedding $\eta : \widetilde{\mathbb{N}} \longrightarrow T_\Sigma \widetilde{\mathbb{N}}$ satisfies $h \cdot \eta = id$. Let

$$e : X \longrightarrow X \times X + \widetilde{\mathbb{N}}$$

be an equation morphism. The derived equation morphism

$$\overline{e} \equiv X \xrightarrow{\ e\ } X \times X + \widetilde{\mathbb{N}} \xrightarrow{X \times X + \eta} X \times X + T_\Sigma \widetilde{\mathbb{N}}$$

has a unique solution $\overline{e}^\dagger : X \longrightarrow T_\Sigma \widetilde{\mathbb{N}}$ in the tree algebra. This yields a solution $e^\dagger$ in $\widetilde{\mathbb{N}}$ as follows:

$$e^\dagger \equiv X \xrightarrow{\ \overline{e}^\dagger\ } T_\Sigma \widetilde{\mathbb{N}} \xrightarrow{\ h\ } \widetilde{\mathbb{N}}\,.$$

To prove that solutions in $\widetilde{\mathbb{N}}$ are unique, let $e^\ddagger : X \longrightarrow \widetilde{\mathbb{N}}$ be a solution of e. For every $x \in X$ with $\overline{e}^\dagger(x)$ finite we have $e^\ddagger(x)$ as the computation of $\overline{e}^\dagger(x)$, i. e., $e^\ddagger(x) = e^\dagger(x)$ (easy proof by induction on the cardinality of the set of nodes of $\overline{e}^\dagger(x)$). And for every x with $\overline{e}^\dagger(x)$ infinite we prove $e^\ddagger(x) = \infty\ (= e^\dagger(x))$. This follows from the next Lemma since $\overline{e}^\dagger(x)$ has either infinitely many leaves or a complete binary subtree.

Lemma.

(1) Suppose that the tree $\overline{e}^\dagger(x)$ has (at least) k leaves labelled by $r_1, \dots, r_k \in \widetilde{\mathbb{N}}$, then

$$e^\dagger(x) \geq r_1 + \cdots + r_k\,.$$

(2) Suppose that the tree $\overline{e}^\dagger(x)$ has a node whose subtree is a complete binary tree (no leaves), then $e^\dagger(x) = \infty$.

Proof. (a) is proved by induction on the maximum depth d of the k leaves: The case $d = 0$, i.e., where $\overline{e}^\dagger(x)$ is a single root labelled by r_1, is clear: $e^\dagger(x) = r_1$. In the induction step let $d > 0$. Then certainly $e(x) \in X \times X$ say, $e(x) = (y_1, y_2)$, and each of the k leaves is a leaf of $\overline{e}^\dagger(y_i)$, $i = 1$ or 2. Since the maximum depth in $\overline{e}^\dagger(y_i)$ is one less than that in $\overline{e}^\dagger(x)$, we can use the induction hypothesis to conclude

$$e^\dagger(y_1) + e^\dagger(y_2) \geq r_1 + \cdots + r_k\,.$$

And from $e(x) = (y_1, y_2)$ we obtain, due to $e^\dagger = [\alpha, id] \cdot (H_\Sigma e^\dagger + id) \cdot e$,

$$e^\dagger(x) = e^\dagger(y_1) + e^\dagger(y_2) \geq r_1 + \cdots + r_k\,.$$

(b) is proved by induction on the depth of the given node j. The case $d = 0$ means that $\overline{e}^\dagger(x)$ is a complete binary tree, thus $e^\dagger(x)$ is an idempotent of $\widetilde{\mathbb{N}}$ — the unique idempotent is ∞. In the induction step we have $e(x) = (y, z)$ and the node j lies in $\overline{e}^\dagger(y)$ or $\overline{e}^\dagger(z)$ where it has smaller depth than in $\overline{e}^\dagger(x)$, thus $e^\dagger(y) = \infty$ or $e^\dagger(z) = \infty$. Consequently,

$$e^\dagger(x) = e^\dagger(y) + e^\dagger(z) = \infty\,.$$

□

Example 2.8. The algebra of addition of extended real numbers of the interval

$$I = (0, \infty]$$

is iterative.

The proof that equation morphisms have solutions is completely analogous to (2) above. The uniqueness is proved as follows: we first establish the above Lemma. Next we use (unlike in (2)!) the finiteness of the set X: since X is finite, the tree $\overline{e}^\dagger(x)$ is rational. If it has a subtree that is a complete binary tree, then $e^\dagger(x) = \infty$. Otherwise, every subtree of $\overline{e}^\dagger(x)$ contains a leaf, and the rationality of $\overline{e}^\dagger(x)$ then implies that infinitely many leaves of $\overline{e}^\dagger(x)$ carry the same label, say, $r \in I$. The Lemma, applied to k of these leaves, implies $e^\dagger(x) \geq k \cdot r$, for any $k = 1, 2, 3, \dots$ — thus, $e^\dagger(x) = \infty$.

Remark 2.9. Uniqueness of solutions is sometimes subtle. In Example 2.7 above we need not assume that X is a finite set, but Example 2.8 would be false without this assumption: consider the system

$$\begin{array}{rcl} x_0 & \approx & x_1 + \frac{1}{2} \\ x_1 & \approx & x_2 + \frac{1}{4} \\ x_2 & \approx & x_3 + \frac{1}{8} \\ & \vdots & \end{array}$$

One solution is $x_n^\dagger = \infty$ $(n \in \mathbb{N})$, another is $x_n^\dagger = 2^{-n}$ $(n \in \mathbb{N})$.

Example 2.10. *Unary algebras in* Set.

Let us consider the endofunctor

$$HA = \Sigma \times A$$

corresponding to unary Σ-algebras: every algebra $\alpha : \Sigma \times A \longrightarrow A$ is given by unary operations

$$s^A = \alpha(s, -) : A \longrightarrow A \quad \text{for } s \in \Sigma.$$

Such an algebra is iterative if and only if the operation

$$s_1^A \cdot s_2^A \cdot \cdots \cdot s_n^A : A \longrightarrow A$$

has a unique fixed point for every nonempty word $s_1 s_2 \cdots s_n$ over Σ.

In fact, the above condition is necessary because the solution of the following system

$$e : \{x_0, \dots, x_{n-1}\} \longrightarrow \Sigma \times \{x_0, \dots, x_{n-1}\} + A$$

where

$$e(x_i) = (s_i, x_{i+1}) \qquad \text{for } i < n-1, \text{ and } e(x_{n-1}) = (s_n, x_0)$$

is precisely a fixed point, a, of $s_1^A \cdot \cdots \cdot s_n^A$. More precisely, the corresponding map $e^\dagger : \{x_0, \dots, x_{n-1}\} \longrightarrow A$ with

$$e^\dagger(x_i) = s_{i+1}^A \cdot \cdots \cdot s_n^A(a) \quad (i = 0, \dots, n-1)$$

solves e.

To prove that the above condition is sufficient, consider a finitary equation morphism

$$e : X \longrightarrow \Sigma \times X .$$

Let us call a variable $x_0 \in X$ *cyclic* if the values of e always stay in the first summand, i.e., we have

$$e(x_i) = (s_{i+1}, x_{i+1}) \qquad i = 0, 1, 2, \dots$$

for an infinite sequence $(s_n, x_n) \in \Sigma \times X$. Since X is finite, there exists $p < q$ with

$$x_p = x_q .$$

Every solution

$$e^\dagger : X \longrightarrow A$$

assigns to x_i elements $a_i = e^\dagger(x_i)$ such that

$$a_i = \alpha(s_{i+1}, a_{i+1})$$

in other words

$$a_i = s^A_{i+1}(a_{i+1})$$

Therefore $a_p = a_q$ implies that a_p is a fixed point of $s^A_{p+1} \cdot \dots \cdot s^A_q$, and this fixed point determines the value

$$a_0 = s^A_1 \cdot \dots \cdot s^A_p(a_p) .$$

Consequently, if the fixed point is unique, $e^\dagger(x_0)$ is uniquely determined.

The non-cyclic variables x_0 present no problem: here we have, for some $k \geq 0$,

$$\begin{aligned} e(x_i) &= (s_{i+1}, x_{i+1}) \qquad i = 0, \dots, k-1 \\ e(x_k) &= a \in A \end{aligned}$$

which implies

$$e^\dagger(x_0) = s^A_1 \cdot \dots \cdot s^A_k(a) .$$

Remark 2.11. In particular, for $Id : \mathsf{Set} \longrightarrow \mathsf{Set}$, an algebra $\alpha : A \longrightarrow A$ is iterative if and only if α has a unique fixed point and none of α^n, $n \geq 2$, has a different fixed point.

Example 2.12. *Ordered unary algebras.*

Here we consider, for a set Σ with discrete ordering, the endofunctor

$$HA = \Sigma \times A$$

on the category Pos of partially ordered sets and order-preserving functions. An ordered unary Σ-algebra is iterative if and only if the operation $s^A_1 \cdot \dots \cdot s^A_n$ has a unique fixed point for every nonempty word $s_1 \cdots s_n$ over Σ.

The argument is as before, we just have to verify that the function

$$e^\dagger(x_0) = \begin{cases} s^A_1 \cdot \dots \cdot s^A_p(a_p), & x_0 \text{ cyclic} \\ \\ s^A_1 \cdot \dots \cdot s^A_p(a), & \text{else} \end{cases}$$

is order-preserving (whenever $e : X \longrightarrow \Sigma \times X + A$ is), which is easy.

Example 2.13. *Unary algebras in* Un.

Here the base category Un is that of unary algebras on one operation $\sigma_A : A \longrightarrow A$ and homomorphisms. We consider H-algebras for the identity endofunctor Id_{Un}. That is, algebras

$$\alpha : (A, \sigma_A) \longrightarrow (A, \sigma_A) ,$$

where α is another unary operation on A, and since α is a homomorphism, it commutes with σ_A:

$$\alpha \cdot \sigma_A = \sigma_A \cdot \alpha .$$

Finitely presentable objects of Un are precisely the unary algebras given by finitely many generators and finitely many equations; for example, free algebras on n generators for $n \in \mathbb{N}$. We prove that an algebra is iterative if and only if

$$\sigma^k_A \alpha^n : A \longrightarrow A \text{ has a unique fixed point for all } n \geq 1 \text{ and } k \geq 0. \qquad (*)$$

The necessity of $(*)$ follows from solutions of the equation morphisms

$$e : X \longrightarrow X + A$$

where X is a free unary algebra on n generators, $x_1, \ldots, x_n$, and e is determined by

$$e(x_i) = x_{i+1} \quad \text{for } i < n, \quad e(x_n) = \sigma_X^k(x_1)$$

In fact, a solution $e^\dagger : X \longrightarrow A$ is determined by elements $a_i = e^\dagger(x_i)$, $i = 1, \ldots, n$ satisfying

$$a_i = \alpha(a_{i+1}) \quad \text{for } i < n,\ a_n = \sigma_A^k(a_1)$$

Thus, a_1 is a fixed point of $\sigma_A^k \alpha^n$, and conversely, every fixed point corresponds to a solution of e.

The sufficiency of $(*)$: given an equation morphism

$$e : X \longrightarrow X + A \quad \text{with } X \text{ generated by } y_1, \ldots, y_r$$

we can describe a solution analogously to in Example 2.10 above. Given a "non-cyclic" variable $x_0 \in X$, i.e., one with

$$\begin{aligned} e(x_i) &= x_{i+1} \qquad i = 0, \ldots, k-1 \\ e(x_k) &= a \in A \end{aligned}$$

we necessarily have $e^\dagger(x_k) = a$, $e^\dagger(x_{k-1}) = \alpha(a)$ etc., thus here

$$e^\dagger(x_0) = \alpha^k(a)$$

For a "cyclic" variable $x_0 \in X$ we have an infinite sequence $x_0, x_1, x_2, \ldots$ in X with $e(x_i) = x_{i+1}$. A solution $e^\dagger$ assigns to x_i an element $a_i \in A$ with

$$a_i = \alpha(a_{i+1}) = \alpha^2(a_{i+2}) = \ldots$$

On the other hand, we can express each x_i via the generators $y_1, \ldots, y_r$ in the form

$$x_i = \sigma_X^{c(i)}(y_{d(i)}) \quad c(i) \geq 0,\ d(i) \in \{1, \ldots, r\}.$$

This implies $a_i = \sigma_A^{c(i)}(b_{d(i)})$ where $b_1, \ldots, b_r$ are the elements $e^\dagger(y_1), \ldots, e^\dagger(y_r)$. We can certainly choose $p < q$ such that

$$d(p) = d(q) \quad \text{and} \quad c(p) \leq c(q).$$

Then the equality $a_p = \alpha^{q-p}(a_q)$ yields

$$\sigma_A^{c(p)}(b_{d(p)}) = \alpha^{q-p}\sigma_A^{c(q)}(b_{d(p)})$$

Put $n = q - p$ and $k = c(q) - c(p)$ to conclude that

$$a_p = \sigma_A^{c(p)}(b_{d(p)}) \quad \text{is a fixed point of } \alpha^n \sigma_A^k.$$

Consequently, if a^* denotes the unique fixed point of $\alpha^n \sigma_A^k$, we conclude $a_1 = \alpha^p(a_p) = \alpha^p(a^*)$. Thus, we have to define

$$e^\dagger(x_0) = \alpha^p(a^*)$$

In summary, the unique solution of e is defined as follows:

$$e^\dagger(x_0) = \begin{cases} \alpha^k(a), & \text{if } x_0 \text{ is not cyclic} \\ \alpha^p(a^*), & \text{if } x_0 \text{ is cyclic.} \end{cases}$$

Remark 2.14. We denote by

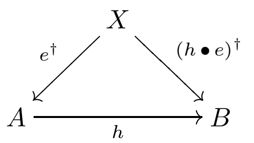
$$\mathsf{Alg}_{it}\ H$$

the category of all iterative algebras and all homomorphisms. The following lemma shows that this choice of morphisms is "right".

Lemma 2.15. (Homomorphisms = solutions-preserving morphisms.) *Let $h : A \longrightarrow B$ be an H-algebra homomorphism between iterative algebras. For every equation morphism $e : X \longrightarrow HX + A$ the solution of e in A yields a solution of the equation morphism*

$$h \bullet e \equiv X \xrightarrow{\ e\ } HX + A \xrightarrow{\ HX+h\ } HX + B$$

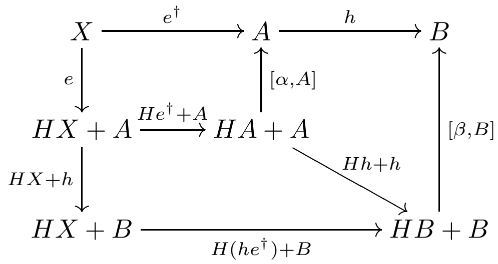
in B via the commutative triangle



(2.2)

Conversely, any morphism h so that this triangle commutes for every equation morphism is an algebra homomorphism.

Proof. The following commutative diagram shows that $h \cdot e^\dagger$ solves $h \bullet e$:



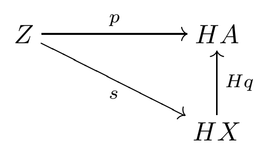
The upper left-hand part commutes since $e^\dagger$ is a solution of e, the right-hand part commutes since h is an H-algebra homomorphism, and the lower part is obvious. Thus, by the uniqueness of solutions we know that the triangle (2.2) commutes.

For the converse, let $\mathcal{A}_{fp}$ be a set of representative finitely presentable objects of $\mathcal{A}$, and let $\mathcal{A}_{fp}/A$ be the comma-category of all arrows $q : X \longrightarrow A$ with X in $\mathcal{A}_{fp}$. Since $\mathcal{A}$ is locally finitely presentable, A is a filtered colimit of the canonical diagram $D_A : \mathcal{A}_{fp}/A \longrightarrow \mathcal{A}$ given by $(q : X \longrightarrow A) \longmapsto X$.

Now $\mathcal{A}_{fp}$ is a generator of $\mathcal{A}$, thus, in order to prove the lemma it is sufficient to prove that for every morphism $p : Z \longrightarrow HA$ with Z in $\mathcal{A}_{fp}$ we have

$$h \cdot \alpha \cdot p = \beta \cdot Hh \cdot p. \tag{2.3}$$

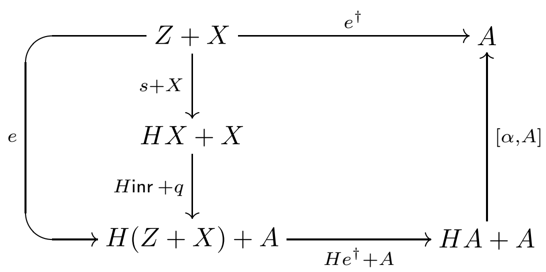
Since H is finitary, it preserves the above colimit D_A. This implies, since $\mathcal{A}(Z, -)$ preserves filtered colimits, that p has a factorization



for some $q : X \longrightarrow A$ in $\mathcal{A}_{fp}/A$ and some s. For the following equation morphism

$$e \equiv Z + X \xrightarrow{s+X} HX + X \xrightarrow{H\mathsf{inr}\,+q} H(Z + X) + A$$

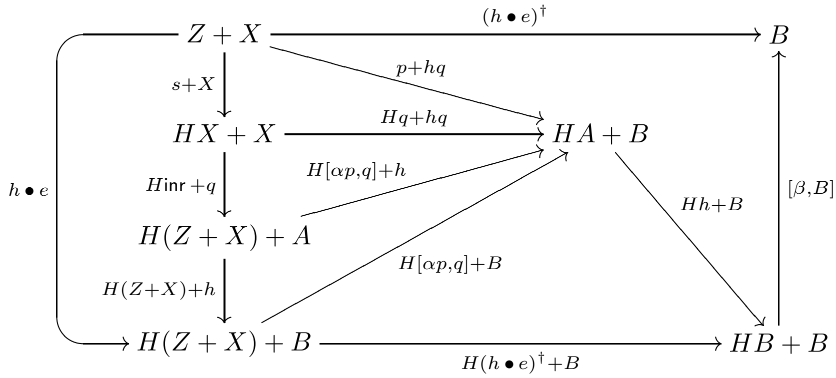
we have a commutative square



Consequently, $e^\dagger \cdot \mathsf{inr} = q$, and this implies $e^\dagger \cdot \mathsf{inl} = \alpha \cdot H(e^\dagger \cdot \mathsf{inr}) \cdot s = \alpha \cdot p$. By (2.2), we have $h \cdot e^\dagger = (h \bullet e)^\dagger$ and therefore

$$(h \bullet e)^\dagger = [h \cdot \alpha \cdot p, h \cdot q]. \tag{2.4}$$

On the other hand, consider the following diagram



It commutes: the outward square commutes since $(h \bullet e)^\dagger$ is a solution, for the lower triangle use equation (2.4), and the remaining triangles are trivial. Thus, the upper right-hand part commutes:

$$(h \bullet e)^\dagger = [\beta \cdot Hh \cdot p, h \cdot q]. \tag{2.5}$$

Now the left-hand components of (2.4) and (2.5) establish the desired equality (2.3). □

Proposition 2.16. *Iterative algebras are closed under limits and filtered colimits in* $\mathsf{Alg}\ H$*.*

Proof. (1) Let (A, α) be a limit, in $\mathsf{Alg}\ H$, of iterative algebras with a limit cone $h_i : (A, \alpha) \longrightarrow (A_i, \alpha_i)$, $i \in I$. It then easily follows that $A = \lim A_i$ in $\mathcal{A}$ with the limit cone $(h_i)_{i \in I}$. For every equation morphism $e : X \longrightarrow HX + A$ the uniqueness of its solution in A follows from Lemma 2.15: given $e^\dagger : X \longrightarrow A$, then each $h_i e^\dagger$ is the unique solution of $e_i = (HX + h_i) \cdot e$ in A_i, thus, $h_i e^\dagger$ is unique, and since $(h_i)_{i \in I}$ is a limit cone in $\mathcal{A}$, we conclude that $e^\dagger$ is unique. To prove the existence, let ${e_i}^\dagger : X \longrightarrow A_i$ denote the solution of e_i in A_i. This is a cone of the given diagram, i.e., for every connecting homomorphism

$$f : (A_i, \alpha_i) \longrightarrow (A_j, \alpha_j)$$

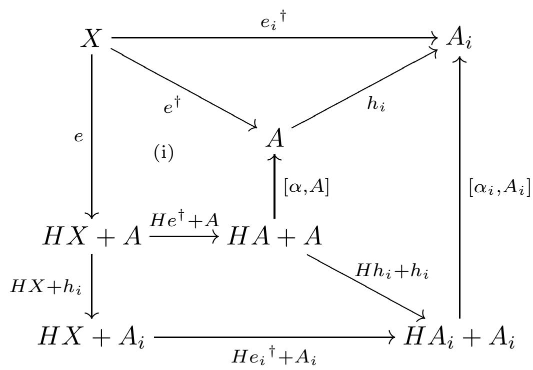
we have

$$f {e_i}^\dagger = {e_j}^\dagger.$$

This, again, follows from Lemma 2.15 and $f h_i = h_j$ (which implies $(HX + f) \cdot e_i = e_j$). Thus, there exists a unique morphism $e^\dagger : X \longrightarrow A$ with

$${e_i}^\dagger = h_i e^\dagger \quad (i \in I).$$

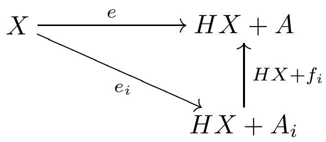
To prove that $e^\dagger$ solves e, it is sufficient to verify that $h_i e^\dagger = h_i \cdot [\alpha, A] \cdot (He^\dagger + A) \cdot e$ for all $i \in I$. In fact, the outer square of the following diagram



commutes, and so do the upper triangle, the right-hand and lower parts. Thus, part (i) commutes when extended by h_i as desired.

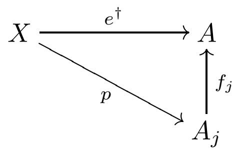
(2) Let (A, α) be a filtered colimit, in $\mathsf{Alg}\ H$, of iterative algebras with a colimit cocone $f_i : (A_i, \alpha_i) \longrightarrow (A, \alpha)$, $i \in I$. Since H is finitary, filtered colimits of H-algebras are formed on the level of $\mathcal{A}$. Given an equation

morphism $e : X \longrightarrow HX + A = \operatorname{colim}(HX + A_i)$, since X is finitely presentable, e factors through one of the colimit morphisms $HX + f_i$:

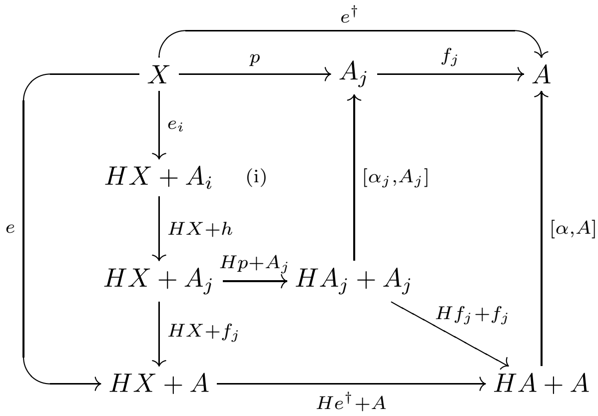


If ${e_i}^\dagger : X \longrightarrow A_i$ is the solution of e_i in A_i, then $f_i{e_i}^\dagger : X \longrightarrow A$ is a solution of e in A by Lemma 2.15.

Conversely, for every solution $e^\dagger : X \longrightarrow A$ of e in A we prove $e^\dagger = f_i{e_i}^\dagger$. Factorize $e^\dagger$ through one of the colimit morphisms:



Since the given diagram is filtered, we can suppose that the choice of $j \in I$ is such that a connecting homomorphism $h : (A_i, \alpha_i) \longrightarrow (A_j, \alpha_j)$ of our diagram exists. Then the morphism $e_j = (HX + h)e_i : X \longrightarrow HX + A_j$ has the solution ${e_j}^\dagger = p$. In fact, all parts of the following diagram



except (i) commute. Therefore (i) commutes when extended by f_j. By filteredness we can therefore suppose that (i) commutes (otherwise choose a connecting morphism $g : (A_j, \alpha_j) \longrightarrow (A_k, \alpha_k)$ equating the sides of (i) and work with k in lieu of j). But it follows from Lemma 2.15 that ${e_j}^\dagger = h{e_i}^\dagger$, therefore $p = h{e_i}^\dagger$. This proves

$$e^\dagger = f_j p = f_j h {e_i}^\dagger = f_i {e_i}^\dagger,$$

as desired. □

Corollary 2.17. *The category* $\mathsf{Alg}_{it}\, H$ *is a reflective subcategory of* $\mathsf{Alg}\, H$*.*

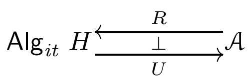
Proof. In fact, $\mathsf{Alg}\, H$ is locally finitely presentable, see Example 2.3(4). Thus, by the Reflection Theorem of [AR], every full subcategory closed under limits and filtered colimits is reflective. □

Corollary 2.18. *Every object of* $\mathcal{A}$ *generates a free iterative* H*-algebra.*

In other words, the natural forgetful functor $U : \mathsf{Alg}_{it}\, H \longrightarrow \mathcal{A}$ has a left adjoint.

Definition 2.19. The finitary monad on $\mathcal{A}$ formed by free iterative H-algebras is called the *rational monad* of H and is denoted by $\mathbb{R} = (R, \eta, \mu)$.

Thus, $\mathbb{R}$ is the monad of the above adjunction



More detailed, for every object Z of $\mathcal{A}$ we denote by RZ a free iterative H-algebra on Z with the universal arrow

$$\eta_Z : Z \longrightarrow RZ\,,$$

and the algebra structure

$$\rho_Z : HRZ \longrightarrow RZ\,.$$

Then $\mu_Z : RRZ \longrightarrow RZ$ is the unique homomorphism of H-algebras with $\mu_Z \cdot \eta_{RZ} = id$.

Before turning to concrete examples of free iterative algebras, we will show that it is sufficient to describe the initial one:

Proposition 2.20. *For any object Y of $\mathcal{A}$ the following are equivalent:*

(1) *RY is an initial iterative algebra of $H(-) + Y$,*
(2) *RY is a free iterative H-algebra on Y.*

In fact, this was proved for completely iterative algebras in [Mi]; the proof for iterative algebras is the same.

Example 2.21. *The rational monad of Id.*

(a) For the identity functor on Set it follows from Example 2.10 that RZ is obtained from the free unary algebra $\mathbb{N} \times Z$ by adding a single element, say, a_0:

$$RZ = \mathbb{N} \times Z + 1$$

with

$$\eta_Z : Z \longrightarrow \mathbb{N} \times Z + 1 \quad z \longmapsto (0, z)$$

and

$$\rho_Z : \mathbb{N} \times Z + 1 \longrightarrow \mathbb{N} \times Z + 1 \quad (n, z) \longmapsto (n+1, z),\ a_0 \longmapsto a_0.$$

(b) Analogously for the rational monad of Id on Pos we have

$$R(Z, \leq) = \mathbb{N} \times (Z, \leq) + 1 \quad \text{with } \mathbb{N} \text{ discretely ordered.}$$

This follows from Example 2.12.

(c) The rational monad of $Id : \mathsf{Un} \longrightarrow \mathsf{Un}$, see Example 2.13, is obtained as follows: given an object (Z, σ_Z) of Un, we first freely "add" a unary operation α which commutes with σ_Z by forming the algebra $Z \times \mathbb{N}$ with the operations σ given by $(z, n) \longmapsto (\sigma_Z(z), n)$ and α given by $(z, n) \longmapsto (z, n+1)$. Then we add a single element, a_0, say, which is the joint fixed point of both operations. Thus,

$$R(Z, \sigma_Z) = (Z \times \mathbb{N} + 1, \sigma_{R(Z, \sigma_Z)})$$

where

$$\sigma_{R(Z,\sigma_Z)} : \begin{cases} (z, n) \longmapsto (\sigma_Z(z), n) \\ \\ a_0 \longmapsto a_0 \quad \text{where } 1 = \{a_0\}, \end{cases}$$

and with $\eta_{(Z,\sigma_Z)} : z \longmapsto (z, 0)$ and $\rho_{(Z,\sigma_Z)} : (z, n) \longmapsto (z, n+1)$, $a_0 \longmapsto a_0$.

Example 2.22. *The rational monad of $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$.*

Recall from Example 2.5 that for every set Z the algebra $R_\Sigma Z$ of all rational Σ-trees over Z, i.e., Σ-trees over Z which have only finitely many subtrees (up to isomorphism), is iterative. As proved in [N], $R_\Sigma Z$ is a free iterative Σ-algebra on Z.

Corollary 2.23. *The rational monad $\mathbb{R}_\Sigma$ of the polynomial endofunctor H_Σ of Set is given by the formation of the Σ-algebras $R_\Sigma(Z)$ of all rational Σ-trees over Z.*

More precisely, the rational trees over Z (see Introduction) form an endofunctor $Z \longmapsto R_\Sigma(Z)$ of Set which is the underlying endofunctor of the monad $\mathbb{R}_\Sigma$. This follows from Proposition 2.20 and Example 2.22.

Example 2.24. The rational monad of $\mathcal{P}_{\text{fin}} : \mathsf{Set} \longrightarrow \mathsf{Set}$, the finite power-set functor was described in [A_2]: it assigns to a set X the algebra of all rational strongly extensional finitely-branching trees (where "strongly extensional" means that every pair of distinct siblings define subtrees which are not bisimilar).

Remark 2.25. A special case of a recursive equation morphism is that where no parameters appear, i. e., simply coalgebras $e : X \longrightarrow HX$ with X finitely presentable. They appear in various contexts, e. g., in non-wellfounded set theory [BM] or, dually, in the theory of transitive sets [O]. Let us explain here why solutions of these special equation morphisms are not sufficient for our purposes. Let us (just in the present remark) call an algebra *weakly iterative* if every equation morphism $e : X \longrightarrow HX$, X finitely presentable, has a unique solution $e^\dagger : X \longrightarrow A$ (i. e., $e^\dagger = \alpha \cdot He^\dagger \cdot e$). For example in case $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ represents a binary operation, $H_\Sigma X = X \times X$, the free iterative algebra $R_\Sigma\{\, a \,\}$ on one generator has the property that every equation $e : X \longrightarrow X \times X$ has the solution $e^\dagger : x \longmapsto t_0$, the constant function to the complete binary tree t_0. Consequently, every subalgebra of $R_\Sigma\{\, a \,\}$ containing t_0 and all finite trees is weakly iterative, although $R_\Sigma\{\, a \,\}$ has no proper iterative subalgebra containing finite trees.

3. A Coalgebraic Construction

The aim of this section is to describe an initial iterative H-algebra as a colimit of a simple diagram Eq in the given base category $\mathcal{A}$. We assume throughout this section that

(a) $\mathcal{A}$ is a locally finitely presentable category, see Definition 2.2,

and

(b) H is a finitary endofunctor of $\mathcal{A}$.

We choose a set $\mathcal{A}_{fp}$ of representatives of finitely presentable objects of $\mathcal{A}$ w.r.t. isomorphism.

Recall that (a) and (b) allow a simple description of the *initial H-algebra* as a colimit of the ω-chain

$$0 \xrightarrow{t} H0 \xrightarrow{Ht} HH0 \xrightarrow{HHt} \cdots$$

where t is the unique morphism from 0, an initial object of $\mathcal{A}$. More precisely, if $I = \operatorname{colim}_{n<\omega} H^n 0$ is this colimit, then the chain above defines a canonical morphism $i : I \longrightarrow HI$ — and one proves that i is invertible, yielding an initial H-algebra structure on I, see [A$_1$].

Analogously, the initial iterative algebra will be proved to be a colimit of the diagram

$$\mathrm{Eq} : \mathsf{EQ} \longrightarrow \mathcal{A}$$

whose objects are all H-coalgebras carried by finitely presentable objects of $\mathcal{A}$:

$$e : X \longrightarrow HX \quad \text{with } X \text{ in } \mathcal{A}_{fp},$$

with the usual coalgebra homomorphisms as morphisms, and with Eq the obvious forgetful functor $e \longmapsto X$.

A colimit

$$R_0 = \operatorname{colim} \mathrm{Eq}$$

of this diagram (with colimit morphisms $e^\sharp : X \longrightarrow R_0$ for all $e : X \longrightarrow HX$ in EQ) yields, again, a canonical morphism

$$i : R_0 \longrightarrow HR_0$$

Namely, i is the unique morphism such that every $e^\sharp$ becomes a coalgebra homomorphism, i.e., the squares

$$\begin{array}{ccc} X & \xrightarrow{e} & HX \\ {\scriptstyle e^\sharp}\downarrow & & \downarrow{\scriptstyle He^\sharp} \\ R_0 & \xrightarrow[i]{} & HR_0 \end{array} \tag{3.1}$$

commute. (In fact, the forgetful functor $\mathsf{Coalg}\, H \longrightarrow \mathcal{A}$ creates colimits.) The aim of the present section is to prove the following

Theorem 3.1. *R_0 is the initial iterative H-algebra. More precisely, the morphism i is an isomorphism and $i^{-1} : HR_0 \longrightarrow R_0$ is an initial iterative H-algebra.*

We establish some auxilliary facts first.

Remark 3.2. The diagram Eq is filtered. In fact, the category of all coalgebras is cocomplete, with colimits formed at the level of $\mathcal{A}$. Since $\mathcal{A}_{fp}$ is well-known to be closed under finite colimits, it follows that the category EQ is closed under finite colimits in the category of all H-coalgebras — thus, EQ is finitely cocomplete, hence, filtered.

Consequently, H preserves the colimit of Eq:

$$HR_0 = \operatorname{colim} H \cdot \mathrm{Eq}$$

with the colimit cocone $He^\sharp$.

Lemma 3.3. *$i : R_0 \longrightarrow HR_0$ is an isomorphism.*

Proof. (a) We define a morphism

$$j : HR_0 \longrightarrow R_0$$

We use the fact that in a locally finitely presentable category the given object HR_0 is a colimit of the diagram of all arrows $p : P \longrightarrow HR_0$ where P is in $\mathcal{A}_{fp}$. More precisely, let $\mathcal{A}_{fp}/HR_0$ denote the comma-category (of all these arrows p), then the forgetful functor $D_{HR_0} : \mathcal{A}_{fp}/HR_0 \longrightarrow \mathcal{A}$ has, in $\mathcal{A}$, the colimit cocone formed by all $p : P \longrightarrow HR_0$. Thus, in order to define j we need to define morphisms $jp : P \longrightarrow R_0$ forming a cocone of the diagram D_{HR_0}. We know that HR_0 is a filtered colimit of $H \cdot \mathrm{Eq}$ and that $\mathcal{A}(P,-)$ preserves this colimit, since P is in $\mathcal{A}_{fp}$. Therefore, p factors through one of the colimit morphisms

$$\begin{array}{ccc} P & \xrightarrow{\ p\ } & HR_0 \\ & \searrow^{p'} & \uparrow{\scriptstyle Hg^\sharp} \\ & & HW \end{array} \tag{3.2}$$

for some $g : W \longrightarrow HW$ in EQ. We form a new object

$$e_{p'} \equiv P + W \xrightarrow{[p',g]} HW \xrightarrow{H\mathsf{inr}} H(P+W)$$

of EQ and define j to be the unique morphism such that the following square

$$\begin{array}{ccc} P & \xrightarrow{\ \mathsf{inl}\ } & P+W \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle e_{p'}{}^\sharp} \\ HR_0 & \xrightarrow[\ j\ ]{} & R_0 \end{array} \tag{3.3}$$

commutes for every p in $\mathcal{A}_{fp}/HR_0$. To prove that j is well-defined we need to show that

(i) $e_{p'}{}^\sharp \cdot \mathsf{inl}$ is independent of the choice of factorization (3.2),

and

(ii) the morphisms $e_{p'}{}^\sharp \cdot \mathsf{inl}$ form a cocone of $\mathcal{A}_{fp}/HR_0$.

For (i), consider another factorization

$$\begin{array}{ccc} P & \xrightarrow{\ p\ } & HR_0 \\ & \searrow^{q'} & \uparrow{\scriptstyle Hf^\sharp} \\ & & HV \end{array}$$

for $f : V \longrightarrow HV$ in EQ. Using the fact that the diagram $H\mathrm{Eq}$ is filtered, we conclude that, without loss of generality, this new factorization can be assumed to posses a coalgebra homomorphism $h : W \longrightarrow V$ from the first one with $q' = Hh \cdot p'$:

$$\begin{array}{ccccc} W & \xrightarrow{\ g\ } & HW & \xleftarrow{\ p'\ } & \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh} & & P \\ V & \xrightarrow[\ f\ ]{} & HV & \xleftarrow[\ q'\ ]{} & \end{array}$$

This yields a coalgebra homomorphism $P + h$ from $e_{p'}$ to $e_{q'}$:

$$\begin{array}{ccccc} P+W & \xrightarrow{[p',g]} & HW & \xrightarrow{H\mathsf{inr}} & H(P+W) \\ {\scriptstyle P+h}\downarrow & & \downarrow{\scriptstyle Hh} & & \downarrow{\scriptstyle H(P+h)} \\ P+V & \xrightarrow[{[q',f]}]{} & HV & \xrightarrow[H\mathsf{inr}]{} & H(P+V) \end{array}$$

which proves

$$e_{p'}{}^\sharp = e_{q'}{}^\sharp \cdot (P+h)\,.$$

Consequently,

$$e_{p'}{}^\sharp \cdot \mathsf{inl} = e_{q'}{}^\sharp \cdot (P+h) \cdot \mathsf{inl} = e_{q'}{}^\sharp \cdot \mathsf{inl}$$

as requested.

To prove (ii), consider a morphism r in $\mathcal{A}_{fp}/HR_0$:

$$\begin{array}{ccc} Q & \xrightarrow{\ r\ } & P \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & HR_0 & \end{array}$$

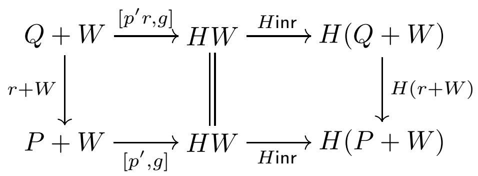
We have defined $jp = {e_{p'}}^\sharp \cdot \mathsf{inl}$ for the factorization (3.2) and, due to (i) above, we can use the factorization

$$q = Hg^\sharp \cdot (p' \cdot r)$$

for the definition of $jq = {e_{p'r}}^\sharp \cdot \mathsf{inl}$. It is our task to prove

$$ {e_{p'}}^\sharp \cdot \mathsf{inl} \cdot r = {e_{p'r}}^\sharp \cdot \mathsf{inl}\,. \tag{3.4}$$

Observe that $r + W$ is a coalgebra homomorphism from $e_{p'r}$ to $e_{p'}$:

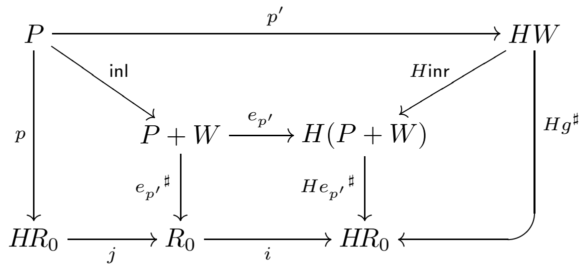


Thus ${e_{p'r}}^\sharp = {e_{p'}}^\sharp \cdot (r + W)$ which proves (3.4).

(b) The proof of $ij = id$. It is our task to prove that $ijp = p$ for every $p : P \longrightarrow HR_0$ in $\mathcal{A}_{fp}/HR_0$. Observe that $\mathsf{inr} : W \longrightarrow P + W$ is a coalgebra homomorphism from $g : W \longrightarrow HW$ to $e_{p'} : P + W \longrightarrow H(P + W)$, thus,

$$g^\sharp = {e_{p'}}^\sharp \cdot \mathsf{inr}\ .$$

The desired equality $ijp = p$ follows from (3.2) and the fact that the following diagram



commutes.

(c) The proof of $ji = id$. It is our task to prove that $jie^\sharp = e^\sharp$ for every $e : X \longrightarrow HX$ in EQ. In order to do so, apply (3.3) to $p = He^\sharp \cdot e : X \longrightarrow HR_0$ with $p' = e$ and $g = e$ to obtain

$$ j \cdot He^\sharp \cdot e = {e_{p'}}^\sharp \cdot \mathsf{inl} \tag{3.5}$$

for $e_{p'} \equiv X + X \xrightarrow{[e,e]} HX \xrightarrow{H\mathsf{inr}} H(X + X)$. It is easily checked that the codiagonal $\nabla : X + X \longrightarrow X$ is a coalgebra homomorphism from $e_{p'}$ to e, thus,

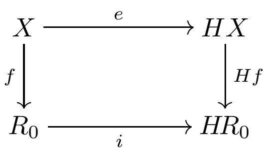
$$e^\sharp \cdot \nabla = {e_{p'}}^\sharp.$$

Now use $i \cdot e^\sharp = He^\sharp \cdot e$, see (3.1), and (3.5) to conclude

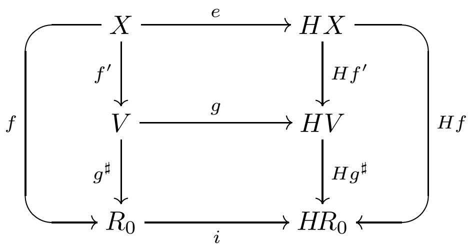
$$ j \cdot (i \cdot e^\sharp) = j \cdot He^\sharp \cdot e = {e_{p'}}^\sharp \cdot \mathsf{inl} = e^\sharp \cdot \nabla \cdot \mathsf{inl} = e^\sharp.$$

□

Remark 3.4. The coalgebra homomorphisms of (3.1) are unique: given an object $e : X \longrightarrow HX$ of EQ and a coalgebra homomorphism into R_0



then $f = e^\sharp$. In fact, since X is finitely presentable, the morphism $f : X \longrightarrow \operatorname{colim} \mathrm{Eq}$ factors through the colimit morphism $g^\sharp$ for some $g : V \longrightarrow HV$: $f = g^\sharp f'$. In the following diagram



the outward square commutes, and so do all inner parts except possibly for the upper square. This implies that $Hg^\sharp$ merges the two sides of that square. Now $Hg^\sharp$ is a colimit morphism of $HR_0 = H \operatorname{colim} \mathrm{Eq} = \operatorname{colim} H\mathrm{Eq}$ (recall that Eq is a filtered diagram; thus H preserves its colimit). Since X is finitely presentable, $\mathcal{A}(X, -)$ preserves the colimit of $H\mathrm{Eq}$ — thus, if $Hg^\sharp$ merges two morphisms, then one of the connecting maps Hp, where p is a morphism in EQ, i. e., the following square

$$\begin{array}{ccc} V & \xrightarrow{\quad g \quad} & HV \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle Hp} \\ W & \xrightarrow[\quad h \quad]{} & HW \end{array}$$

commutes, also merges those morphisms. That is, we have

$$Hp \cdot (Hf' \cdot e) = Hp \cdot (g \cdot f'),$$

from which we conclude that pf' is a morphism of EQ from e to h since

$$H(p \cdot f') \cdot e = Hp \cdot g \cdot f' = h \cdot (p \cdot f')\,.$$

Thus, $e^\sharp = h^\sharp \cdot (pf')$. Now p being a morphism of EQ implies $g^\sharp = h^\sharp \cdot p$, and consequently

$$f = g^\sharp f' = h^\sharp p f' = e^\sharp.$$

Lemma 3.5. *The H-algebra $i^{-1} : HR_0 \longrightarrow R_0$ is iterative.*

Proof. (1) *Existence of solutions.* For every equation morphism

$$e : X \longrightarrow HX + R_0 = \operatorname{colim}(HX + \mathrm{Eq})$$

there exists, since X is finitely presentable, a factorization through the colimit morphism $HX + f^\sharp$ (for some $f : V \longrightarrow HV$ in EQ):

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & HX + R_0 \\ & {\scriptstyle e_0}\searrow & \big\uparrow{\scriptstyle HX + f^\sharp} \\ & & HX + V \end{array} \tag{3.6}$$

Recall from 2.1 that $\mathsf{can} : HX + HV \longrightarrow H(X + V)$ denotes the canonical morphism. Define a new object, $\overline{e}$, of EQ as follows:

$$\overline{e} \equiv X + V \xrightarrow{[e_0, \mathsf{inr}]} HX + V \xrightarrow{HX + f} HX + HV \xrightarrow{\mathsf{can}} H(X + V)\,. \tag{3.7}$$

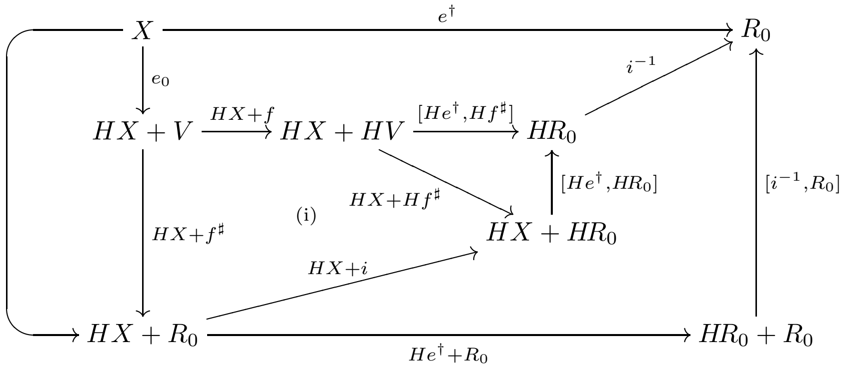
Observe that

$$f^\sharp = \overline{e}^\sharp \cdot \mathsf{inr} \tag{3.8}$$

because $\mathsf{inr} : V \longrightarrow X + V$ is a coalgebra morphism (in EQ) from f to $\overline{e}$. We define a solution of e by

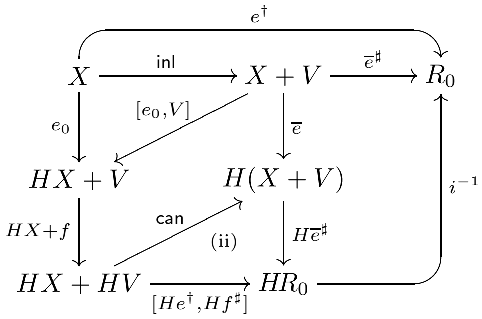
$$e^\dagger \equiv X \xrightarrow{\mathsf{inl}} X + V \xrightarrow{\overline{e}^\sharp} R_0\,. \tag{3.9}$$

In fact, in the following diagram



(3.10)

all inner parts commute: see (3.6) for the left-hand part, (3.1) for part (i), whereas the right-hand part commutes trivially (analyze the two components separately) and so does the middle triangle. It remains to verify the upper part: here we use (3.1) and (3.7) to conclude that the following diagram

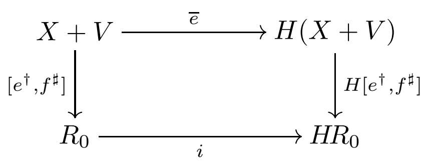


commutes. In fact, the left-hand component of (ii) commutes by definition of $e^\dagger$ and the right-hand one does by (3.8). Thus, (3.10) commutes, proving that $e^\dagger$ is a solution of e.

(2) *Uniqueness.* Suppose that $e^\dagger : X \longrightarrow R_0$ is a solution of e. Then in (3.10) the outward square commutes. Since all the inner parts except the upper one commute, this proves that the upper part commutes, too. Consequently,

$$i \cdot e^\dagger = [He^\dagger, Hf^\sharp] \cdot (HX + f) \cdot e_0 = H[e^\dagger, f^\sharp] \cdot \overline{e} \cdot \mathsf{inl}\ .$$

This equality implies that in the following square

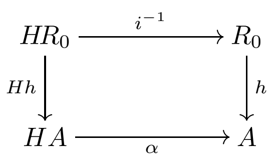


the left-hand components commute. Since $\overline{e} \cdot \mathsf{inr} = H\mathsf{inr} \cdot f$, the right-hand ones commute by (3.1). Therefore, the square commutes, which, by Remark 3.4, proves

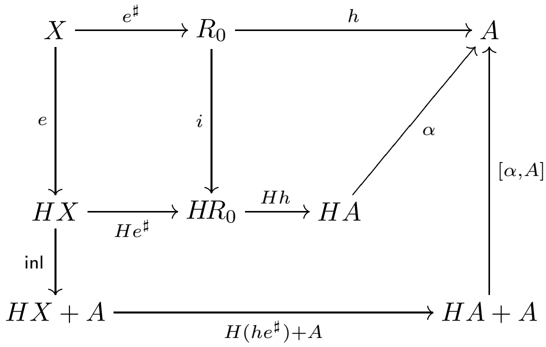
$$\overline{e}^\sharp = [e^\dagger, f^\sharp]\,.$$

Thus, the given solution is the previous one: $e^\dagger = \overline{e}^\sharp \cdot \mathsf{inl}$. ☐

Proof of Theorem 3.1. Let $\alpha : HA \longrightarrow A$ be an iterative H-algebra. We prove first that there is at most one H-algebra homomorphism from R_0. Let



be a homomorphism. For every object $e : X \longrightarrow HX$ of EQ the following diagram



(3.11)

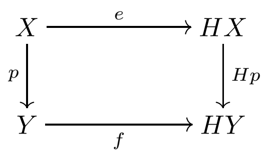
commutes, see (3.1), which proves that $he^\sharp$ is a solution of $\mathsf{inl}\, e$ in A.

This determines h uniquely, since the $e^\sharp$'s form a colimit cocone of $R_0 = \operatorname{colim} \mathrm{Eq}$.

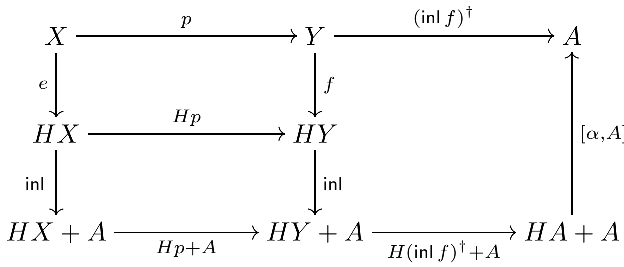
Conversely, let us define a morphism $h : R_0 \longrightarrow A$ by the above rule

$$he^\sharp = (\mathsf{inl}\, e)^\dagger \quad \text{for all } e : X \longrightarrow HX \text{ in } \mathsf{EQ}$$

where $(-)^\dagger$ is the unique solution in A. This is well-defined since the morphisms $(\mathsf{inl}\, e)^\dagger$ form a cocone of the diagram Eq: in fact, let



be a morphism of EQ. We prove that $(\mathsf{inl}\, f)^\dagger p$ is a solution of $\mathsf{inl}\, e$ by considering the corresponding diagram:



This proves

$$(\mathsf{inl}\, e)^\dagger = (\mathsf{inl}\, f)^\dagger p\,.$$

The morphism h above is a homomorphism of algebras because the diagram (3.11) commutes: the outward square commutes by definition of h, the upper left-hand square by (3.1), and the lower part is obvious. This shows that the upper right-hand part commutes when precomposed with $e^\sharp$, e in EQ. Since the $e^\sharp$'s form a colimit cocone, it follows that h is a homomorphism. □

Corollary 3.6. *A free iterative H-algebra RZ is a colimit*

$$RZ = \operatorname{colim} \mathrm{Eq}_Z$$

of the diagram

$$\mathrm{Eq}_Z : \mathsf{EQ}_Z \longrightarrow \mathcal{A}$$

where EQ_Z consists of all equation morphisms $e : X \longrightarrow HX+Z$, $X \in \mathcal{A}_{fp}$, and all coalgebra homomorphisms w.r.t. $H(-)+Z$, and Eq_Z sends e to X.

In fact, this is a consequence of Proposition 2.20 and Theorem 3.1.

Remark 3.7. We denote, again, the colimit morphisms of Eq_Z by

$$e^\sharp : X \longrightarrow RZ$$

for all $e : X \longrightarrow HX + Z$ in EQ_Z. The appropriate isomorphism is denoted by

$$i_Z : RZ \longrightarrow HRZ + Z$$

It is characterized by the fact that the two coproduct injections of $HRZ + Z$ are (in the notation of Definition 2.19)

$$\mathsf{inl} = i_Z \rho_Z \quad \text{and} \quad \mathsf{inr} = i_Z \eta_Z$$

In other words, $i_Z = [\rho_Z, \eta_Z]^{-1}$.

4. An Alternative Definition of Iterativity

In the Introduction we considered non-flat systems (1.1) of recursive equations for Σ-algebras. And we argued that, due to the possibility of flattening such a system, we will just have to consider the flat equation morphism $e : X \longrightarrow H_\Sigma X + A$. We are going to make that statement precise by showing that in iterative algebras (in general, not only in Set) much more general systems of recursive equations than the flat ones are uniquely solvable. This implies that, for polynomial endofunctors of Set, our definition of iterative algebras coincides with that presented by Evelyn Nelson [N]. And as we explain in the next section, this also implies that the rational monad is iterative in the sense of Calvin Elgot [E].

Let us first remark that the condition stated for (1.1) in the Introduction, that no right-hand side be a single variable, is substantial: the equation $x \approx x$ has a unique solution only in the trivial terminal algebras. Systems satisfying the above condition are called *guarded*.

We first consider guarded systems where the right-hand sides live in the free H-algebra (i. e., they are finite trees in case $H = H_\Sigma$). Such systems are called *finitary*.

Remark 4.1. Since H is finitary, free H-algebras exist, see [A$_1$]. We denote for every object X in $\mathcal{A}$ a free algebra by $\varphi^0_X : HFX \longrightarrow FX$ with universal arrow $\eta^0_X : X \longrightarrow FX$. This defines a monad $\mathbb{F} = (F, \eta^0, \mu^0)$ where the component μ^0_X is the unique homomorphism $\mu^0_X : FFX \longrightarrow FX$ with $\mu^0_X \cdot \eta^0_{FX} = id$. It is easy to see that analogously to Proposition 2.20, FX is an initial algebra of $H(-) + X$; thus, by Lambek's Lemma [L]

$$FX = HFX + X\,. \tag{4.1}$$

More precisely, the morphism

$$j_X = [\varphi^0_X, \eta^0_X] : HFX + X \longrightarrow FX$$

is an isomorphism. For every H-algebra $\alpha : HA \longrightarrow A$ we have the unique homomorphism

$$\widehat{\alpha} : FA \longrightarrow A \qquad \text{with} \quad \widehat{\alpha} \cdot \eta_A = id$$

(which, in case of H_Σ, is the computation of (finite) terms over A in the Σ-algebra A). This allows us to define solutions of finitary equations morphisms in A as follows:

Definition 4.2.

(1) By a *finitary equation morphism* in an object A is meant a morphism

$$e : X \longrightarrow F(X + A), \quad X \text{ finitely presentable.}$$

(2) We call e *guarded* provided that it factors through the summand $HF(X + A) + A$ of $F(X + A) = HF(X + A) + X + A$ (see (4.1) above):

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & F(X+A) \\ & \searrow & \uparrow{\scriptstyle [\varphi^0, \eta^0 \cdot \mathsf{inr}]} \\ & & HF(X+A) + A \end{array}$$

(3) Suppose that A is an underlying object of an H-algebra $\alpha : HA \longrightarrow A$. Then by a *solution* of e in the algebra A is meant a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widehat{\alpha}} \\ F(X+A) & \xrightarrow[F[e^\dagger, A]]{} & FA \end{array} \tag{4.2}$$

commutes.

Remark 4.3. The square (4.2) in Definition 4.2 means, for polynomial functors, that the assignment $e^\dagger$ of variables $x \in X$ to elements of A has the following property: form the “substitution” mapping $[e^\dagger, A] : X + A \longrightarrow A$ (which interprets the variables as $e^\dagger$ does, and leaves elements of A unchanged). Extend it to the unique homomorphism

$$\widehat{\alpha} \cdot F[e^\dagger, A] : F(X + A) \longrightarrow A$$

of the free algebra. Then the (formal) equations $x \approx e(x)$ become actual identities in A after the substitution $x \longmapsto e^\dagger(x)$ is performed for all $x \in X$, and the right-hand sides are computed in A. This is precisely the definition of solution of (1.1) in the Introduction.

Theorem 4.4. *An H-algebra A is iterative if and only if every guarded finitary equation morphism in A has a unique solution.*

The proof of Theorem 4.4 follows from the next result, generalizing “finitary” to “rational”.

Definition 4.5. By a *rational equation morphism* in an object A we mean a morphism

$$e : X \longrightarrow R(X + A), \qquad X \text{ finitely presentable},$$

and e is called *guarded* if it factors through the summand $HR(X + A) + A$ of $R(X+A) = HR(X+A)+X+A$ (see Remark 3.7):

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & R(X+A) \\ & \searrow & \uparrow {\scriptstyle [\rho,\eta\cdot\mathsf{inr}]} \\ & & HR(X+A)+A \end{array}$$

Suppose that A is an underlying object of an iterative H-algebra $\alpha : HA \longrightarrow A$. We denote (analogously to $\widehat{\alpha}$ above) by

$$\widetilde{\alpha} : RA \longrightarrow A$$

the unique homomorphism of H-algebras with $\widetilde{\alpha} \cdot \eta_A = id$. Then by a *solution* of e in the iterative algebra A is meant a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widetilde{\alpha}} \\ R(X+A) & \xrightarrow[R[e^\dagger, A]]{} & RA \end{array}$$

commutes.

Theorem 4.6. *If A is an iterative H-algebra, then every guarded rational equation morphism e in A has a unique solution.*

Proof. Let $\alpha : HA \longrightarrow A$ be an iterative algebra. Given a guarded rational equation morphism

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & R(X+A) \\ & \searrow^{e_0} & \uparrow {\scriptstyle [\rho_{X+A},\eta_{X+A}\cdot\mathsf{inr}]} \\ & & HR(X+A)+A \end{array}$$

we will prove that e has a unique solution $e^\dagger$.

(1) *Existence.* Recall from Corollary 3.6 that $R(X + A) = \operatorname{colim} \mathrm{Eq}_{X+A}$ with colimit cocone $g^\sharp : W \longrightarrow R(X + A)$ for all $g : W \longrightarrow HW + X + A$ in EQ_{X+A}. Since this colimit is filtered and H is finitary, we have a filtered colimit

$$HR(X + A) + A = \operatorname{colim} H\mathrm{Eq}_{X+A} + A$$

with the colimit cocone formed by all $Hg^\sharp + A$. Since X is a finitely presentable object, the morphism

$$e_0 : X \longrightarrow \operatorname{colim} H\mathrm{Eq}_{X+A} + A$$

factors through the colimit cocone:

$$\begin{array}{ccc} X & \xrightarrow{\;e_0\;} & HR(X+A)+A \\ & \searrow_{w} & \uparrow {\scriptstyle Hg^\sharp + A} \\ & & HW+A \end{array}$$

for some object $g : W \longrightarrow HW + X + A$ of EQ_{X+A} and some morphism w.

We define a finitary flat equation morphism as follows:

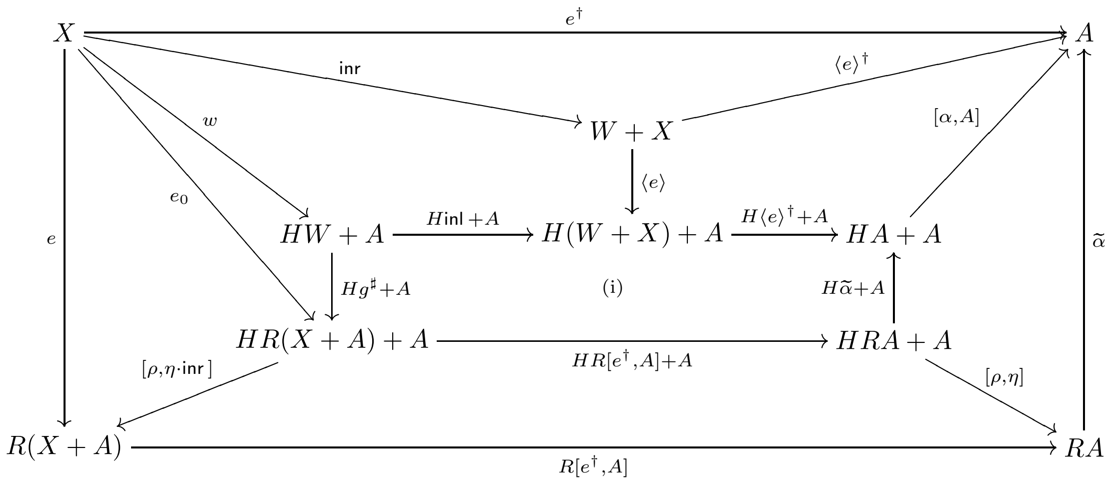
$$\langle e \rangle \equiv W + X \xrightarrow{[g, \mathsf{inm}]} HW + X + A \xrightarrow{[\mathsf{inl}, w, \mathsf{inr}]} HW + A \xrightarrow{H\mathsf{inl} + A} H(W + X) + A \tag{4.3}$$

where $\mathsf{inm} : X \longrightarrow HW + X + A$ is the middle coproduct injection. We obtain a unique solution $\langle e \rangle^\dagger : W + X \longrightarrow A$ and prove that the following morphism

$$e^\dagger \equiv X \xrightarrow{\mathsf{inr}} W + X \xrightarrow{\langle e \rangle^\dagger} A \tag{4.4}$$

is a solution of e.

Indeed, consider the following diagram:



(4.5)

All of its parts, except the square (i), clearly commute. The right-hand component of (i) is obvious. To prove the commutativity of the left-hand component of (i), we remove H and show that the equation

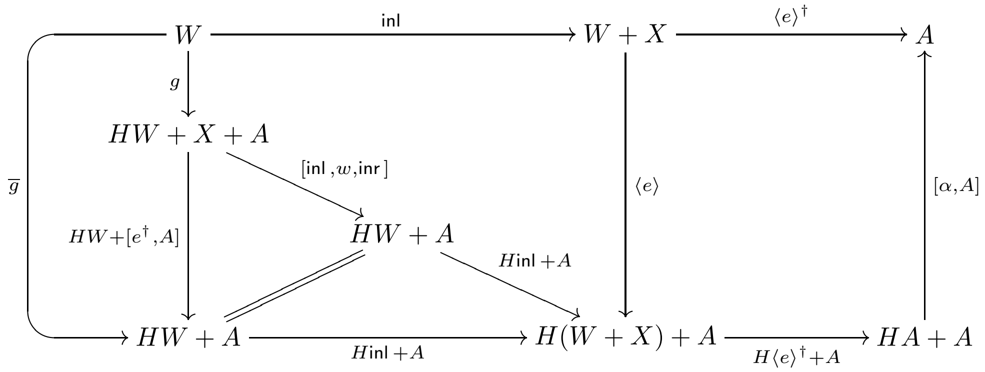
$$\langle e \rangle^\dagger \cdot \mathsf{inl} = \widetilde{\alpha} \cdot R[e^\dagger, A] \cdot g^\sharp \tag{4.6}$$

holds. To this end observe first that $\widetilde{\alpha} \cdot R[e^\dagger, A] : R(X + A) \longrightarrow A$ is an H-algebra homomorphism between iterative algebras extending $[e^\dagger, A]$. An inspection of the proof of Theorem 3.1 and Proposition 2.20 reveals that precomposing this homomorphism with the colimit injection $g^\sharp : W \longrightarrow R(X + A)$ yields the unique solution of the following equation morphism

$$\overline{g} \equiv W \xrightarrow{g} HW + X + A \xrightarrow{HW + [e^\dagger, A]} HW + A$$

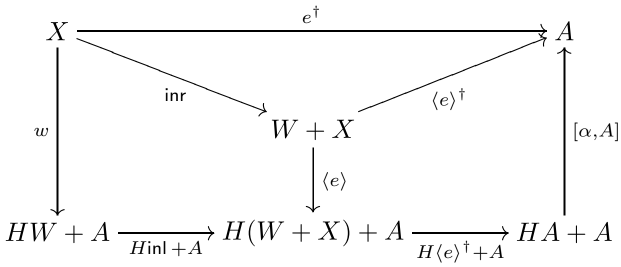
in the iterative algebra A.

Thus, to establish (4.6) it suffices to show that $\langle e \rangle^\dagger \cdot \mathsf{inl}$ is a solution of $\overline{g}$. In fact, the outward square of the following diagram



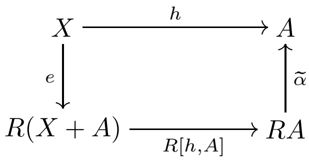
commutes. To prove this, observe that by (4.3) all parts except, perhaps, for the left-hand inner triangle, clearly commute. For that triangle consider the components of the coproduct separately. The left-hand and right-hand components are obviously commutative. We do not claim this for the middle component.

But this component commutes when extended to A in the upper right-hand corner. In fact, this yields the following square



which commutes: see the upper part of Diagram (4.5).

(2) *Uniqueness.* Let h be any solution of e, i.e., a morphism such that the following square



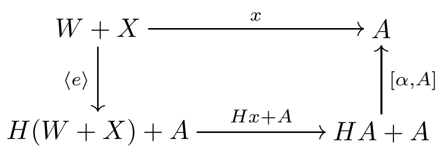
commutes. We shall show that

$$x \equiv W + X \xrightarrow{[\widetilde{\alpha}\cdot R[h,A]\cdot g^{\sharp},h]} A$$

is a solution of $\langle e\rangle$ in A, therefore

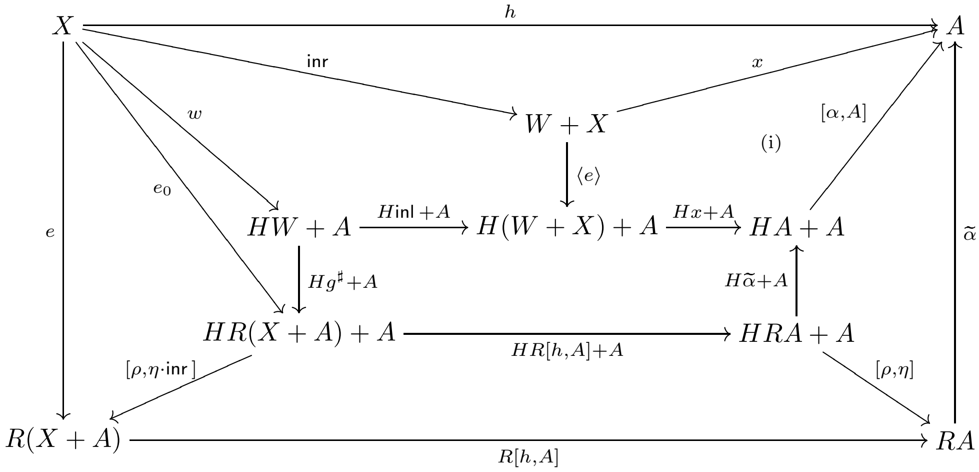
$$h = \langle e\rangle^{\dagger} \cdot \mathsf{inl} = e^{\dagger}$$

which completes the proof. Thus, it is our task to show that the following square



commutes.

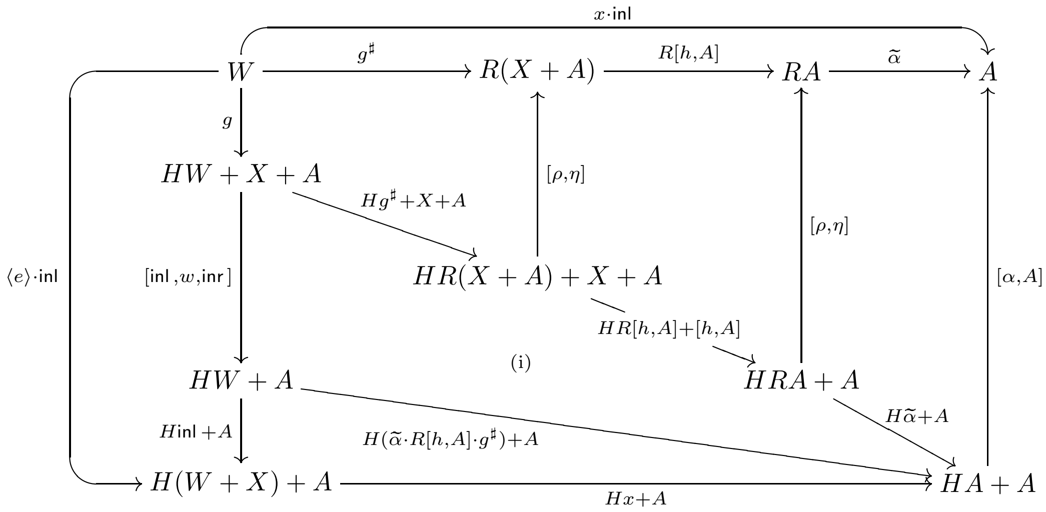
We consider the components of the coproduct $W+X$ separately. For the right-hand component we obtain the following commutative diagram



(4.7)

Since the outward square commutes, and all the inner parts but (i) clearly do, so must the right-hand component of part (i). (Notice that this diagram is precisely (4.5) with h for $e^{\dagger}$ and x for $\langle e\rangle^{\dagger}$.)

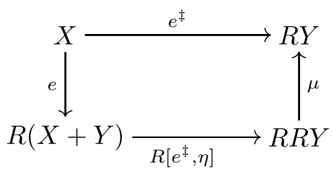
For the left-hand component consider the following diagram:



All of its parts commute, except possibly the middle component of (i), which commutes when extended by $[\alpha, A]$ to A in the upper right corner. In fact, this is easy to see by inspection of the upper three inner parts the of Diagram (4.7). ☐

The rational solution theorem we have proved in our previous work [AMV$_1$, AMV$_2$] is now an easy consequence of Theorem 4.6

Corollary 4.7. *Every rational guarded equation morphism* $e : X \longrightarrow R(X+Y)$ *has a unique solution in the algebra* RY*, i. e., there exists a unique* $e^\ddagger : X \longrightarrow RY$ *such that the square*

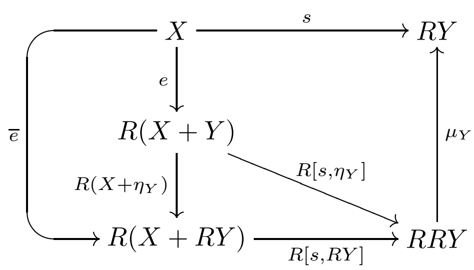


commutes.

Proof. Given a guarded rational equation morphism $e : X \longrightarrow R(X+Y)$ form the equation morphism

$$\overline{e} \equiv X \xrightarrow{\ e\ } R(X+Y) \xrightarrow{R(X+\eta_Y)} R(X+RY)\,.$$

This is a guarded equation morphism in the free iterative algebra RY. The result now follows from Theorem 4.6 applied to RY and to $\overline{e}$. In fact, there is a 1-1-correspondence between solutions of e and solutions of $\overline{e}$:

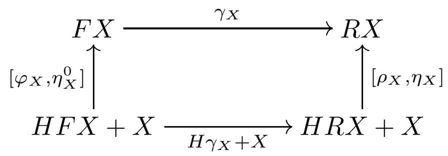


Observe first that $\widetilde{\rho_Y} = \mu_Y : RRY \longrightarrow RY$. Now since s is a solution of e, the upper inner part commutes, and equivalently, the outward square commutes, which is to say that s is a solution of $\overline{e}$. Since $\overline{e}$ has a unique solution, so does e. ☐

Proof of Theorem 4.4. (a) Sufficiency: let A be an iterative algebra. Denote by

$$\gamma : F \longrightarrow R$$

the natural transformation formed by the unique homomorphisms $\gamma_X : FX \longrightarrow RX$ of H-algebras with $\gamma_X \cdot \eta^0_X = \eta_X$. Observe that the following square



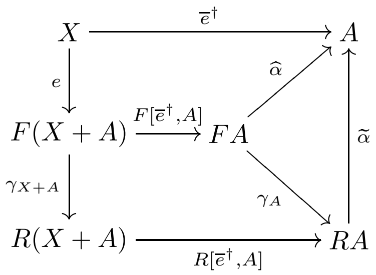
(4.8)

commutes.

Given a guarded finitary equation morphism $e : X \longrightarrow F(X + A)$, we prove that a unique solution $e^\dagger$ exists. To this end form the rational equation morphism

$$\overline{e} \equiv X \xrightarrow{e} F(X + A) \xrightarrow{\gamma_{X+A}} R(X + A)$$

and observe that it is guarded (use (4.8)). The unique solution $\overline{e}^\dagger$ solves e. In fact, in the following diagram

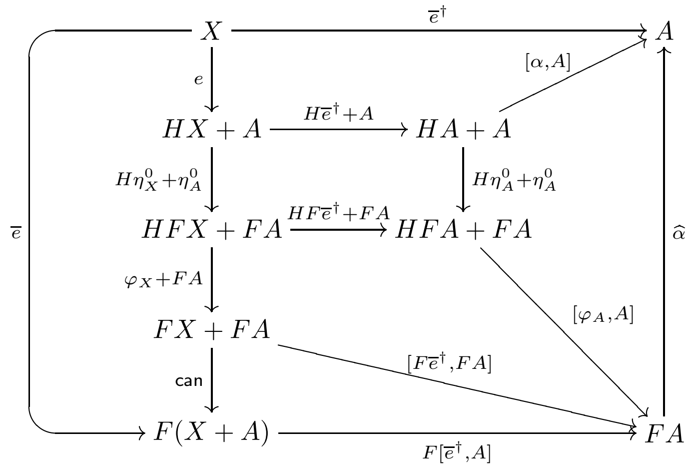


the outward square commutes (by definition of $\overline{e}^\dagger$), and the lower one does by the naturality of γ. The right-hand triangle commutes because both paths are homomorphisms extending id_A. Consequently, the upper square commutes, too. As for the uniqueness of solutions, suppose that in the above diagram $\overline{e}^\dagger : X \longrightarrow A$ denotes a solution of e. Then all inner parts of the diagram commute, thus, so does the outward square, whence $\overline{e}^\dagger$ is the unique solution of $\overline{e}$ (see Theorem 4.6).

(b) Necessity: if $\alpha : HA \longrightarrow A$ is an H-algebra such that every finitary equation morphism e has a unique solution, then A is iterative. In fact, given a flat equation morphism $e : X \longrightarrow HX + A$, denote by $\overline{e}$ the following finitary equation morphism

$$\overline{e} \equiv X \xrightarrow{e} HX + A \xrightarrow{H\eta^0_X + \eta^0_A} HFX + FA \xrightarrow{\varphi_X + FA} FX + FA \xrightarrow{\mathsf{can}} F(X + A)$$

It is easy to see that $\overline{e}$ is guarded. We obtain a unique solution $\overline{e}^\dagger : X \longrightarrow A$, and we prove that this solves e uniquely (in the sense of Definition 2.4). In other words, in the following diagram



the upper square commutes. In fact, the outward square commutes by definition of $\overline{e}^\dagger$, the right-hand one commutes because $\widehat{\alpha} \cdot \eta^0_A = id$ and (since $\widehat{\alpha}$ is a homomorphism)

$$\widehat{\alpha} \cdot \varphi_A \cdot H\eta^0_A = \alpha \cdot H\widehat{\alpha} \cdot H\eta^0_A = \alpha\,.$$

Since the remaining inner parts commute (by naturality of η and φ), the commutativity of the upper square follows.

To prove that e has a unique solution, suppose that in the above diagram $\overline{e}^\dagger : X \longrightarrow A$ denotes a solution of e. Then all inner parts of the diagram commute, thus, the outward square does. This shows that $\overline{e}^\dagger$ is the unique solution of $\overline{e}$. □

5. Free Iterative Monads

Assumptions 5.1. Throughout this section H denotes a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$.

We are going to prove that the rational monad $\mathbb{R}$, introduced in Section 2, is iterative in the sense of Calvin Elgot, and that it can be characterized as a free iterative monad on H.

5.2. Iterative Monads. This is a concept that Elgot has introduced in [E] for the base category $\mathcal{A} = \mathsf{Set}$. He used the language of algebraic theories rather than monads, but we have proved in [AAMV] that the following concepts are equivalent to those of Elgot. For a monad $\mathbb{S} = (S, \eta, \mu)$ over Set we can form the complements of $\eta_X[X]$ in SX, say,

$$\sigma_X : S'X \longrightarrow SX$$

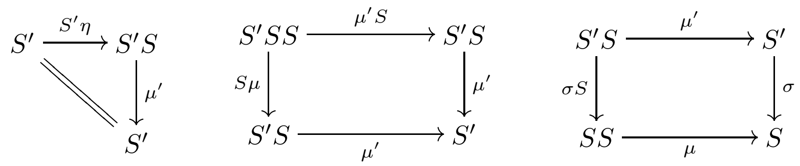
for all objects X. The monad $\mathbb{S}$ is called *ideal* provided $\sigma : S' \longrightarrow S$ is a subfunctor of S, and the monad multiplication has a domain-codomain restriction $\mu' : S'S \longrightarrow S'$. For general base categories in lieu of requiring a subfunctor S', we impose certain properties on μ' very similar to the monad laws for μ and η. The corresponding concept is as follows:

Definition 5.3. By an *ideal monad* is understood a sixtuple

$$\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$$

consisting of a monad (S, η, μ) and natural transformations $\sigma : S' \longrightarrow S$ and $\mu' : S'S \longrightarrow S'$ such that

1. $S = S' + Id$ with coproduct injections σ and η, and
2. The following three diagrams



(5.1)

   commute.

Remark 5.4. Notice that the left-hand and middle diagrams in (5.1) express that the pair (S', μ') is a right $\mathbb{S}$-module, and the right-hand diagram states that σ is morphism of $\mathbb{S}$-modules from (S', μ') to (S, μ). The notion of a module appears for a monoidal category and a monoid in that category under the name action in [M],VII.4. We chose the name module to remind of the classical example of abelian groups; in this category, a monoid is precisely a ring R and an R-module is precisely a module of the ring R. Here we work in the monoidal category of endofunctors on $\mathcal{A}$ with composition as the tensor product and the identity functor as the tensor unit.

Examples 5.5.

1. The rational monad is ideal. Recall from Remark 3.7 that $R = HR + Id$. Here we consider the natural transformation

$$\rho : HR \longrightarrow R$$

   expressing the H-algebra structure $\rho_Z : HRZ \longrightarrow RZ$ of each RZ, see Definition 2.19. The "restriction" of μ here is simply

$$\mu' = H\mu : HRR \longrightarrow HR\,.$$

   In fact, we know from Remark 3.7 that $RZ = HRZ + Z$ with the coproduct injections ρ_Z and η_Z. Next, $(HR, H\mu)$ is an $\mathbb{R}$-module: the first two diagram of (5.1) follow easily from the monad laws for μ and η; furthermore, the third square

$$\begin{array}{ccc} HRR & \xrightarrow{H\mu} & HR \\ {\scriptstyle H\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ RR & \xrightarrow[\mu]{} & R \end{array}$$

commutes because each μ_Z is a homomorphism of H-algebras, see Definition 2.19.

(2) The free-algebra monad $\mathbb{F}$ of Section 4 is ideal. Here analogously, we use $F = HF + Id$, see (4.1).

(3) Classical algebraic theories (groups, lattices, etc.) are usually not ideal.

Definition 5.6. Let $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ be an ideal monad on $\mathcal{A}$.

(1) By a *finitary equation morphism* is meant a morphism

$$e : X \longrightarrow S(X + Y)$$

in $\mathcal{A}$ where X is a finitely presentable object ("of variables") and Y is any object ("of parameters").

(2) By a *solution* of e is meant a morphism

$$e^\dagger : X \longrightarrow SY$$

for which the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ S(X+Y) & \xrightarrow[S[e^\dagger, \eta_Y]]{} & SSY \end{array}$$

commutes.

(3) The equation morphism e is called *guarded* if it factors through the summand $S'(X + Y) + Y$ of $S(X + Y) = S'(X + Y) + X + Y$:

$$\begin{array}{ccc} X & \xrightarrow{e} & S(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\sigma_{X+Y}, \eta_{X+Y} \mathsf{inr}]} \\ & & S'(X+Y) + Y \end{array}$$

(4) The ideal monad $\mathbb{S}$ is called *iterative* provided that every guarded finitary equation morphism has a unique solution.

Example 5.7. The rational monad of every finitary endofunctor is iterative, see Corollary 4.7.

Remark 5.8. Next we define morphisms of ideal monads. Whenever our base category $\mathcal{A}$ has the (very common) property that coproduct injections are monomorphic, then in an ideal monad $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ we automatically get a subfunctor $S' \hookrightarrow S$ and the module laws of (S', μ') follow automatically from the monad laws of S. This makes the definitions of morphisms easy and canonical: let $\mathbb{T} = (T, \eta^T, \mu^T, T', \tau, {\mu'}^T)$ be another ideal monad. An ideal monad morphism is a monad morphism

$$m : (S, \eta, \mu) \longrightarrow (T, \eta^T, \mu^T)$$

which has a restriction m' to the given subfunctors:

$$\begin{array}{ccc} S' & \xrightarrow{m'} & T' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow[m]{} & S' \end{array}$$

However, we do not want to impose any side conditions on $\mathcal{A}$. The prize is that ideal monad morphisms are defined as pairs (m, m'):

Definition 5.9.

(1) An *ideal monad morphism* from an ideal monad $(S, \eta, \mu, S', \sigma, \mu')$ to another one $(T, \eta^T, \mu^T, T', \tau, {\mu'}^T)$ is a pair (m, m') that consists of a monad morphism $m : (S, \eta, \mu) \longrightarrow (T, \eta^T, \mu^T)$ and a natural transformation $m' : S' \longrightarrow T'$ such that the diagrams

$$\begin{array}{ccc} S'S & \xrightarrow{m' * m} & T'T \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle {\mu'}^T} \\ S' & \xrightarrow[m']{} & T' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S' & \xrightarrow{m'} & T' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow[m]{} & T \end{array} \tag{5.2}$$

commute.

(2) Given a functor H, a natural transformation $\lambda : H \longrightarrow S$ is called *ideal* provided that it factors through $\sigma : S' \longrightarrow S$: $\lambda = \sigma \cdot \lambda'$ for some transformation $\lambda' : H \longrightarrow S'$.

Remark 5.10. The left-hand square in Diagram (5.2) expresses that $m' : S' \longrightarrow T'$ is a module morphism with change of base m. The right-hand one together with the preservation of the unit $m \cdot \eta = \eta^T$ expresses that $m = m' + Id$. In fact, every ideal monad morphism is determined by its second component m'.

Example 5.11. For the rational monad $\mathbb{R}$, the natural transformation

$$\kappa \equiv H \xrightarrow{H\eta} HR \xrightarrow{\rho} R$$

is ideal.

Remark 5.12.

(1) We are going to prove that, for every finitary endofunctor H, the rational monad $\mathbb{R}$ is a free iterative monad on H. Since ideal monad morphisms are pairs, the freeness is expressed by a pair of equations. Notice, however, that under the assumption that coproduct injections in the base category $\mathcal{A}$ are monomorphic, see Remark 5.8, the freeness of $\mathbb{R}$ means what one expects: for every iterative monad $\mathbb{S}$ and every ideal natural transformation $\lambda : H \longrightarrow S$ there exists a unique ideal monad morphism $\overline{\lambda} : \mathbb{R} \longrightarrow \mathbb{S}$ such that $\overline{\lambda} \cdot \kappa = \lambda$. The formulation below refrains from the assumption that coproduct injections are monomorphic.

(2) Parts of the following proof are identical to the corresponding parts of Theorem 5.14 of [Mi]; we already mentioned that that paper was written parallel to ours. We decided to present a complete proof, without referencing to [Mi], for the convenience of the reader.

Theorem 5.13. (Rational Monad as a Free Iterative Monad.) *For every iterative monad $\mathbb{S}$ and every ideal natural transformation $\lambda : H \longrightarrow S$ there exists a unique ideal monad morphism $(\overline{\lambda}, \overline{\lambda}')$:$\mathbb{R} \longrightarrow \mathbb{S}$ such that the diagrams*

$$\begin{array}{ccc} H & \xrightarrow{H\eta} & HR \\ & {\scriptstyle \lambda'}\searrow & \downarrow{\scriptstyle \overline{\lambda}'} \\ & & S' \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} H & \xrightarrow{\kappa} & R \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \overline{\lambda}} \\ & & S \end{array} \tag{5.3}$$

commute.

Remark. Let us form the category $\mathsf{Fin}(\mathcal{A}, \mathcal{A})$ of all finitary endofunctors and natural transformations. For the category

$$\mathsf{FIM}(\mathcal{A})$$

of all finitary iterative monads (i.e., iterative monads $(S, \eta, \mu, S', \sigma, \mu')$ with S and S' finitary) and ideal monad morphisms we have a forgetful functor

$$U : \mathsf{FIM}(\mathcal{A}) \longrightarrow \mathsf{Fin}(\mathcal{A}, \mathcal{A}), \qquad \mathbb{S} \longmapsto S'$$

The above theorem states that U has a left adjoint, viz, the functor $H \longmapsto \mathbb{R}$.

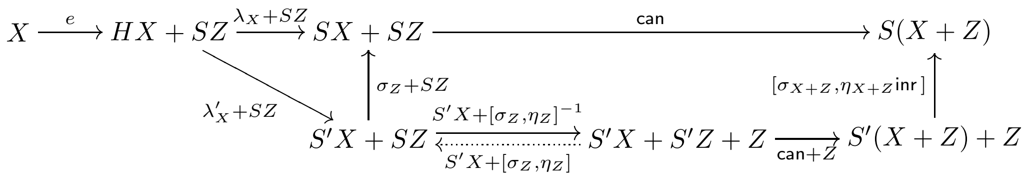
Proof. (1) For every object Z consider SZ as an H-algebra

$$HSZ \xrightarrow{\lambda_{SZ}} SSZ \xrightarrow{\mu_Z} SZ\,.$$

It is iterative. In fact, every equation morphism $e : X \longrightarrow HX + SZ$, X in $\mathcal{A}_{fp}$, yields the following equation morphism w.r.t. $\mathbb{S}$:

$$\overline{e} \equiv X \xrightarrow{\ e\ } HX + SZ \xrightarrow{\lambda_X + SZ} SX + SZ \xrightarrow{\ \mathsf{can}\ } S(X+Z)\,.$$

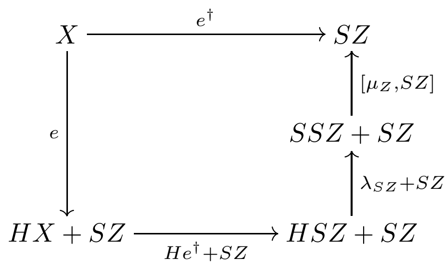
To verify that $\overline{e}$ is guarded, use the restriction $\lambda' : H \longrightarrow S'$ of λ:



To prove the commutativity of the square, consider the three components of $S'X + S'Z + Z$ separately, and use naturality of σ and η.

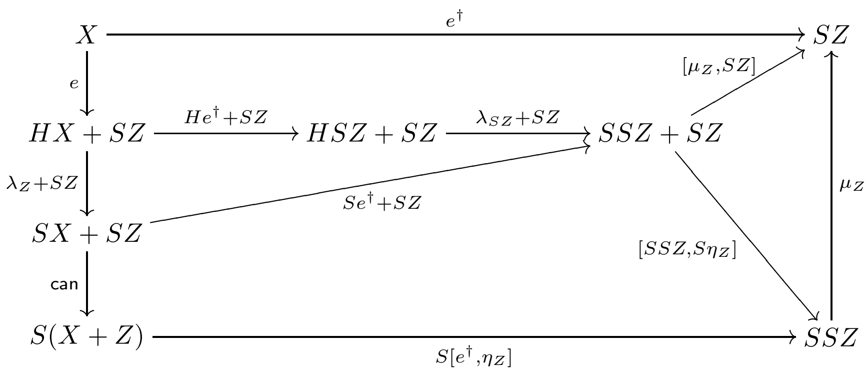
We prove that a morphism $e^\dagger : X \longrightarrow SZ$ is a solution of e in the H-algebra SZ if and only if it is a solution of $\overline{e}$ w.r.t. the iterative monad $\mathbb{S}$.

(1a) Let $e^\dagger$ be a solution of e in the algebra SZ, i.e., let



(5.4)

commute. We are to show that the following diagram



(5.5)

has the outward square commutative. The upper part is (5.4), the one directly below it is the naturality of λ. The lower part commutes obviously, and the right-hand one does due to $\mu_Z \cdot S\eta_Z = id$.

(1b) Let the outward square of (5.5) commute. Since the remaining three inner parts commute, so does the upper one, which is (5.4).

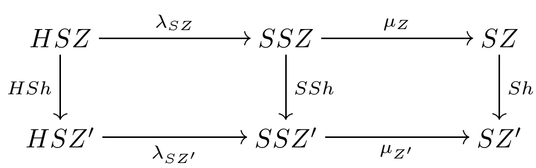
(2) *Existence of an ideal monad morphism $\overline{\lambda}$ such that (5.3) commute.* Denote by

$$\overline{\lambda}_Z : RZ \longrightarrow SZ$$

the unique homomorphism of H-algebras with

$$\overline{\lambda}_Z \cdot \eta_Z = \eta^S_Z\,.$$

We first observe that $\overline{\lambda}$ is a natural transformation. Given a morphism $h : Z \longrightarrow Z'$, then Sh is a homomorphism of H-algebras from SZ to SZ':

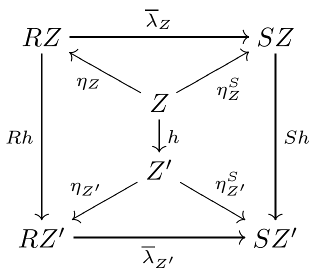


(5.6)

Thus, we have two parallel H-algebra homomorphisms from RZ to SZ':

$$Sh \cdot \overline{\lambda}_Z \qquad \text{and} \qquad \overline{\lambda}_{Z'} \cdot Rh\,.$$

They agree when precomposed with η_Z:

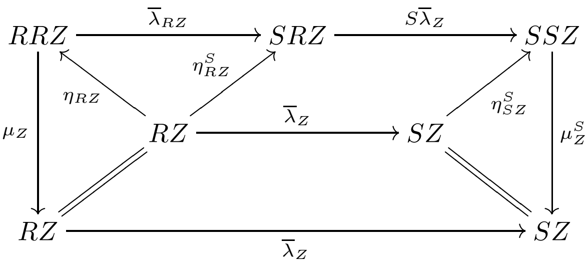


By the universal property of η_Z, and since SZ' is an iterative H-algebra, this proves that the above naturality square commutes.

Let us prove that $\overline{\lambda}$ is a monad morphism. Since $\overline{\lambda}\eta = \eta^S$ by definition, it remains to prove the commutativity of the following diagram

$$\begin{array}{ccccc} RRZ & \xrightarrow{\overline{\lambda}_{RZ}} & SRZ & \xrightarrow{S\overline{\lambda}_Z} & SSZ \\ {\scriptstyle \mu_Z}\big\downarrow & & & & \big\downarrow{\scriptstyle \mu^S_Z} \\ RZ & & \xrightarrow[\overline{\lambda}_Z]{} & & SZ \end{array} \tag{5.7}$$

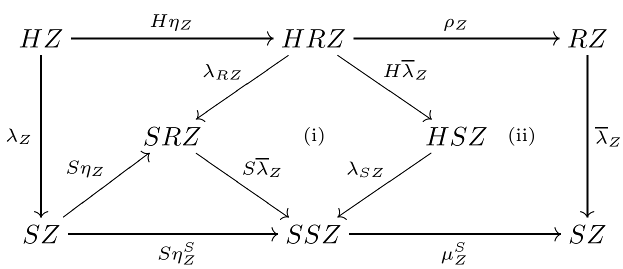
By (5.6), applied to $h = \overline{\lambda}_Z$, we see that $S\lambda_Z$ is a homomorphism of H-algebras. By the universal property of η_{RZ} it is sufficient to prove that (5.7) commutes when precomposed with η_{RZ}:



The equation

$$\lambda = \overline{\lambda} \cdot \kappa = \overline{\lambda} \cdot \rho \cdot H\eta$$

follows from the commutativity of the following diagram



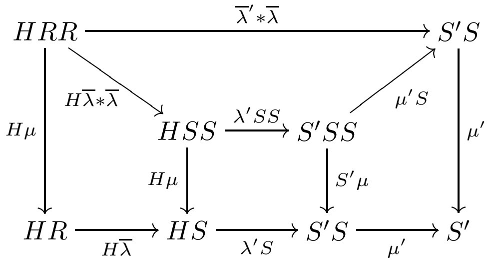
(5.8)

where (i) is naturality of λ and (ii) is clear since $\overline{\lambda}$ is a homomorphism. Now use that $\mu^S_Z \cdot S\eta^S_Z = id$.

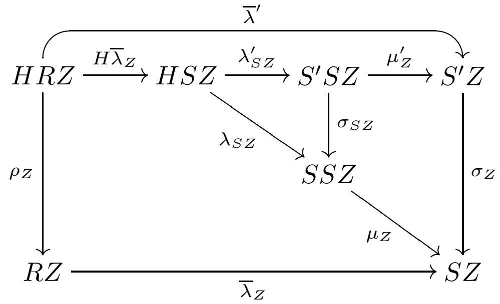
Thus, we have found a monad morphism $\overline{\lambda} : \mathbb{R} \longrightarrow \mathbb{S}$ with $\overline{\lambda} \cdot \kappa = \lambda$. It remains to verify that $\overline{\lambda}$ is part of an ideal monad morphism. Put

$$\overline{\lambda}' \equiv HR \xrightarrow{H\overline{\lambda}} HS \xrightarrow{\lambda' S} S'S \xrightarrow{\mu'} S'\,. \tag{5.9}$$

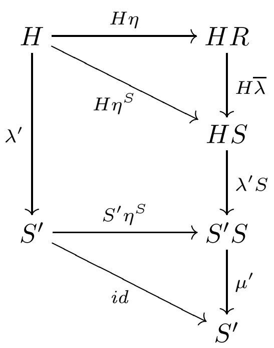
To see that the pair $(\overline{\lambda}, \overline{\lambda}')$ is an ideal monad morphism we have to verify the commutativity of the diagrams (5.2) for this pair. For the left-hand diagram of (5.2) consider the following diagram



The upper part clearly commutes by the definiton (5.9) of $\overline{\lambda}'$ and by the naturality of parallel composition. The other parts of the diagram are clear by invoking — from left to right — that $\overline{\lambda}$ is a monad morphism, the naturality of λ' and the module laws of S'. To verify the right-hand square of (5.2) consider the diagram



Finally, let us check the left-hand triangle of (5.3), i.e., we show that $\overline{\lambda}' \cdot H\eta = \lambda'$. To see this, consider the diagram



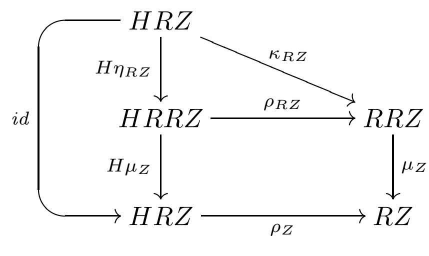
It commutes: for the upper triangle use that $\overline{\lambda}$ is a monad morphism, the middle part is naturality, and the lower triangle is the unit law of the $\mathbb{S}$-module S'.

(3) *Uniqueness of* $\overline{\lambda}$. Suppose that (m, m') is an ideal monad morphism from $\mathbb{R}$ to $\mathbb{S}$ such that the Diagrams (5.3) commute with (m, m') in lieu of $(\overline{\lambda}, \overline{\lambda}')$. We are going to show that for any object Z, m_Z is an H-algebra homomorphism extending η^S_Z, and then invoke the freeness of RZ as an iterative H-algebra, which implies that $m = \overline{\lambda}$, and then leads to the conclusion that $m' = \overline{\lambda}'$.

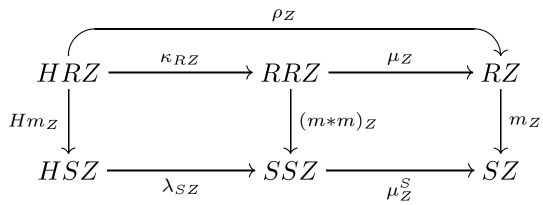
Firstly, notice that for any object Z we have

$$\rho_Z = \mu_Z \cdot \kappa_{RZ} \, . \tag{5.10}$$

Indeed, the diagram



commutes. Consequently, the following diagram

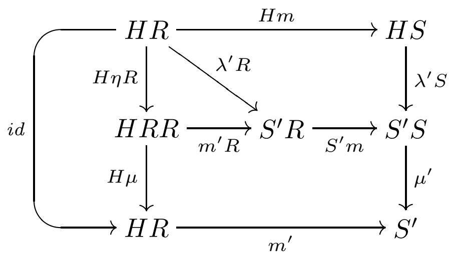


commutes: the upper part is (5.10), the right-hand square commutes since m is a monad morphism, and the left-hand one does since $m \cdot \kappa = \lambda$ and by naturality.

Thus $m_Z : RZ \longrightarrow SZ$ is an H-algebra homomorphism between iterative H-algebras such that $m_Z \cdot \eta_Z = \eta^S_Z$. This implies that $m = \overline{\lambda}$ and from this it follows that

$$m' = \mu' \cdot \lambda' S \cdot Hm = \mu' \cdot \lambda' S \cdot H\overline{\lambda}$$

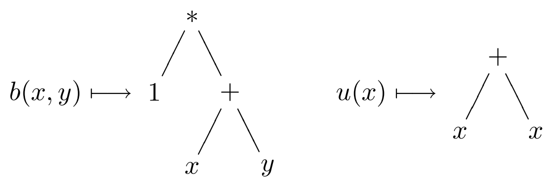
where the first equation holds due to the following diagram



The lower square commutes since m' is a module homomorphism with change of base m, the left-hand part by the unit law of the monad $\mathbb{R}$, the upper triangle by (5.3) and the upper right-hand part by naturality. This completes the proof. ☐

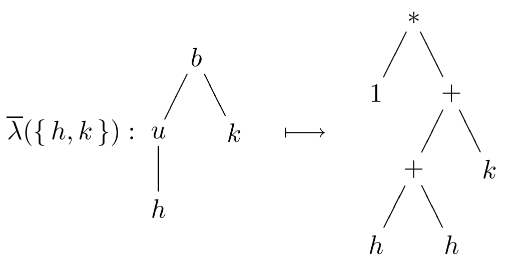
Remark 5.14. For polynomial endofunctors on Set, the freeness of $\mathbb{R}$ specializes to *second order substitution*, see [C], i. e., substitution of rational trees for operation symbols.

For example, consider a signature Σ with a binary operation symbol b, and a unary one u, and another signature Γ with two binary operation symbols $+$ and $*$ and a constant symbol 1. The assignment



(5.11)

of operation symbols in Σ to rational trees over Γ gives rise to a natural transformation $\lambda : H_\Sigma \longrightarrow R_\Gamma$. The induced monad morphism $\overline{\lambda} : \mathbb{R}_\Sigma \longrightarrow \mathbb{R}_\Gamma$ replaces, for any set of variables X, the operation symbols in trees of $R_\Sigma X$ according to λ. Example:



The requirement that λ be an ideal transformation means that no operation symbol of Σ is replaced by a single variable, i. e., that λ is a so-called *non-erasing* substitution.

Remark 5.15. We have defined a rational monad for every finitary endofunctor of a locally finitely presentable category. One may ask what happens if we "raise the index of presentability" to an uncountable regular cardinal λ. That is, what is the "λ-rational" monad of a λ-accessible endofunctor H (i.e., one, preserving λ-filtered colimits)?

It is easy to see that the main results above remain true if we systematically replace "finitely presentable" by "λ-presentable" and "finitary" by "λ-accessible". An H-algebra A might be called *λ-iterative* if every equation morphism $e : X \longrightarrow HX + A$ with X λ-presentable has a unique solution $e^\dagger : X \longrightarrow A$.

Then, for a λ-accessible endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$, one can prove the following

(1) The category of all λ-iterative H-algebras is reflective in $\mathsf{Alg}\, H$.

(2) The resulting λ-accessible monad $\mathbb{R}^\lambda$ on $\mathcal{A}$ is a free λ-iterative monad on H. Again, *λ-iterative* means unique solvability of equations $X \longrightarrow R^\lambda(X+Z)$ with X λ-presentable. Moreover, $R^\lambda Z = \operatorname{colim} \mathrm{Eq}_Z^\lambda$, where Eq_Z^λ is the obvious modification of the diagram Eq_Z from Corollary 3.6.

However, in case of uncountable λ such a monad $\mathbb{R}$ coincides with the *completely iterative monad* $\mathbb{T}$ of H which has been described in [AAMV, Mi]. This monad $\mathbb{T}$ is given object-wise by final coalgebras of the endofunctor $H(-) + Z : \mathcal{A} \longrightarrow \mathcal{A}$. To show that $\mathbb{T} \cong \mathbb{R}^\lambda$, it therefore suffices to prove the following:

Proposition. *For uncountable λ, the object $R^\lambda Z$ is a final coalgebra of $H(-) + Z$. More precisely, the isomorphism $i_Z : R^\lambda Z \longrightarrow HR^\lambda Z + Z$ is a final coalgebra of $H(-) + Z$.*

Proof. We use the fact proven in [AP] that the category EQ_Z^λ is a dense full subcategory of the locally λ-presentable category of all coalgebras of $H(-) + Z$. Here we use uncountability of λ. Thus, it suffices to prove that for every

$$e : X \longrightarrow HX + Z$$

in EQ_Z^λ there exists a unique homomorphism into $i_Z : R^\lambda Z \longrightarrow HR^\lambda Z + Z$. Since the colimit injection $e^\sharp : X \longrightarrow R^\lambda Z$ is such a homomorphism, it remains to verify uniqueness. This is done analogously to Remark 3.4. □

6. Conclusions and Future Work

We proved that all finitary endofunctors H generate a free iterative monad $\mathbb{R}$. All we needed in our proof was the assumptions that the base category is locally finitely presentable. This is the "real McCoy" that we tried to achieve in two preceeding papers [AMV_1] and [AMV_2]: there we obtained the same result only in the base category Set, and the proof was much more complicated. The reason was that when writing those papers we did not follow the footsteps of Evelyn Nelson and Jerzy Tiuryn who realized already more than twenty years ago that iterative algebras are more basic than Elgot's iterative theories.

The results of the present paper are analogous to results on completely iterative algebras and completely iterative theories. The latter were introduced in [EBT] in analogy to iterative theories by dropping the finiteness restriction on the objects of variables: one studies equation morphisms with arbitrary objects X of variables, and requests unique solutions of these more general equations. Stefan Milius [Mi] defines completely iterative algebras for a functor H on a category $\mathcal{A}$ with binary coproducts and he relates them to completely iterative monads: H has free completely iterative algebras TX if and only if H generates a free completely iterative monad T if and only if H has "enough final coalgebras", i. e., every functor $H(-) + X$ has a final coalgebra TX.

A natural question to ask, then, is whether there is a monad in between the free iterative monad R and the free completely iterative one T: how about considering, for an accessible functor and some uncountable cardinal λ, all equation morphisms with a λ-presentable object X of variables. We showed that the answer is negative: one gets the same monad, namely T, see Remark 5.15.

The main technical result of our paper is a description of an initial iterative algebra as a colimit of all H-coalgebras carried by finitely presentable objects. From this result we derived that the algebraic theory formed by all free iterative H-algebras is iterative in the sense of Calvin Elgot. In fact, that theory can be characterized as a free iterative theory on H. The freeness of the rational monad can be used to formulate clearly the "second-order substitution" described for rational Σ-trees by Bruno Courcelle [C], see Remark 5.14.

Our result can be applied to arbitrary base categories which are locally finitely presentable. For example, to the category of all finitary endofunctors of Set. In the future we intend to use this in an attempt to describe the monad of algebraic trees, see Courcelle [C], categorically.

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[TPT] Possible Origins of the Phrase *Real McCoy*, The Phrase Thesaurus, `http://phrases.shu.ac.uk`.

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