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Fundamental Study

Infinite trees and completely iterative theories: a coalgebraic view

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Abstract

Infinite trees form a free completely iterative theory over any given signature—this fact, proved by Elgot, Bloom and Tindell, turns out to be a special case of a much more general categorical result exhibited in the present paper. We prove that whenever an endofunctor H of a category has final coalgebras for all functors $H(\_) + X$, then those coalgebras, TX, form a monad. This monad is completely iterative, i.e., every guarded system of recursive equations has a unique solution. And it is a free completely iterative monad on H. The special case of polynomial endofunctors of the category Set is the above mentioned theory, or monad, of infinite trees.

This procedure can be generalized to monoidal categories satisfying a mild side condition: if, for an object H, the endofunctor $H \otimes \_ + I$ has a final coalgebra, T, then T is a monoid. This specializes to the above case for the monoidal category of all endofunctors.

Keywords: Completely iterative theory; Monad; Coalgebra; Solution Theorem; Monoidal category

1. Introduction

Our paper presents an application of corecursion, i.e., of the construction method using final coalgebras, to the theory of iterative equation systems. Recall that equations

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such as

$$\begin{aligned} x_1 &\approx x_2 \diamond a \\ x_2 &\approx x_1 \diamond b \end{aligned} \tag{1.1}$$

have unique solutions in the realm of infinite expressions. In our case, the solution is ${x_1}^\dagger = (((\ldots \diamond b) \diamond a) \diamond b) \diamond a$ and ${x_2}^\dagger = (((\ldots \diamond a) \diamond b) \diamond a) \diamond b$. Such infinite expressions, or infinite trees, have been studied in the 1970s in connection with (potentially infinite) computations, where various additional structures were introduced with the aim of formalizing an infinite computation as a join of finite approximations in a CPO, see e.g. [18], or as a limit of a Cauchy sequence of approximations in a complete metric space, see e.g. [10]. A different approach, not using additional structures such as ordering or metric, has been taken by Elgot and his co-authors, see, e.g. [15,16]. The above system (1.1) is an example of a system of *iterative equations* using a set $X = \{x_1, x_2\}$ of variables and a set $Y = \{a, b\}$ of parameters. Given a signature Σ (here consisting of a single binary symbol $\diamond$) a system of iterative equations consists of equations

$$x \approx e(x) \quad \text{(one for every variable } x \text{ in } X)$$

whose right-hand sides are finite or infinite Σ-labelled trees $e(x)$ over the set $X + Y$. That is, trees with leaves labelled by variables, parameters or nullary symbols, and internal nodes with n children labelled by n-ary symbols. The symbol $\approx$ indicates a formal equation, whereas $=$ means the identity of the two sides. A *solution* of the system of equations is a collection

$$e^\dagger(x) \quad (x \in X)$$

of Σ-labelled trees over Y, i.e., trees without variables, such that the substitution of $e^\dagger(x)$ for x, for all variables x, turns the formal equations into identities. That is, for every $x_0 \in X$ we have

$$e^\dagger(x_0) = e(x_0)[e^\dagger(x)/x].$$

The given system is called *guarded* provided that none of the right-hand sides is a single variable. Every guarded system has a unique solution.

In the present paper we show that a coalgebraic approach makes it possible to study solutions of iterative equations without any additional (always a bit arbitrary) structure—that is, we can simply work in Set, the category of sets. We use the simple and well-known fact that for polynomial endofunctors H of Set the algebra of all (finite and infinite) properly labelled trees is a final H-coalgebra. Well, this is not enough: what we need is working with "trees with variables", i.e., given a set X of variables, we work with trees whose internal nodes are labelled by operations, and leaves are labelled by variables and constants. This is a final coalgebra again: not for the original functor, but for the functor

$$H(\_) + X : \mathsf{Set} \to \mathsf{Set}$$

We are going to show that for every polynomial functor $H : \mathsf{Set} \to \mathsf{Set}$

(a) final coalgebras TX of the functors $H(\_)+X$ form a monad, called the *completely iterative monad generated by* H,

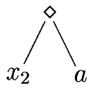
(b) there is also a canonical structure of an H-algebra on each TX, and all these canonical H-algebras form the Kleisli category of the completely iterative monad,

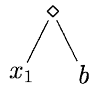
and

(c) the H-algebra TX has unique solutions of all guarded systems of iterative equations.

A surprising feature of the result we prove is its generality: this has nothing to do with polynomiality of H, nor with the base category Set. In fact, given an endofunctor H of any category $\mathscr{A}$ with binary coproducts, and assuming that each $H(\_)+X$ has a final coalgebra (such functors are called *iteratable*) then (a)–(c) hold.

The above system (1.1) corresponds to the polynomial functor expressing one binary operation, $\diamond$, i.e., to the functor $HZ = Z \times Z$. A final coalgebra TX of $Z \mapsto Z \times Z + X$ can be described as the coalgebra of all finite and infinite binary trees with leaves labelled in X. System (1.1) describes a function from $X = \{x_1, x_2\}$ to the set $T(X+Y)$ of trees over variables from X and parameters from $Y = \{a, b\}$. Here we have

$e : X \longrightarrow T(X+Y),\quad x_1 \mapsto$ 

$,\quad x_2 \mapsto$ 

The above concept of solution is categorically expressed by a morphism

$$e^\dagger : X \to TY$$

characterized by the property that $e^\dagger$ is equal to the composite of $e : X \to T(X+Y)$ and the substitution morphism $T(X+Y) \to TY$ leaving parameters intact and substituting $e^\dagger(x)$ for $x \in X$. This substitution is given by the function $s = [e^\dagger, \eta_Y] : X + Y \to TY$ (taking a variable x to the tree $e^\dagger(x)$ and a parameter y to the trivial tree $\eta_Y(y)$). This extends to the unique homomorphism

$$\hat{s} : T(X+Y) \to TY$$

of H-algebras taking a tree over $X+Y$ and substituting the leaves according to s. The property defining a solution, $e^\dagger$, is thus that the following triangle

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & TY \\ T(X+Y) \arrow[ur, "\widehat{[e^\dagger, \eta_Y]}"'] & \end{tikzcd} \tag{1.2}$$

commutes. As mentioned above, T is a part of a monad, so that the substitution corresponding to $s : Z \to TY$ is given by $TZ \xrightarrow{Ts} TTY \xrightarrow{\mu_Y} TY$, where $\mu : TT \to T$ is the monad multiplication. Thus, (1.2) is the following square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & TY \\ T(X+Y) \arrow[r, "{T[e^\dagger, \eta_Y]}"'] & TTY \arrow[u, "\mu_Y"'] \end{tikzcd} \tag{1.3}$$

We are going to prove that "almost" all equations expressed by $e : X \to T(X+Y)$ have a unique solution $e^\dagger : X \to TY$. Exceptions are equations such as

$$x \approx x$$

What we want to avoid is that the right-hand side of an equation is a variable from X. This can be expressed categorically as follows: the final coalgebra TY is a fixed point of $H(\_) + Y$ (by Lambek's lemma [20]), therefore, TY is a coproduct of HTY and Y. Let us denote the coproduct injections by

$$\begin{tikzcd} HTY \arrow[dr, "\tau_Y"'] & & Y \arrow[dl, "\eta_Y"] \\ & TY & \end{tikzcd}$$

where the right-hand injection is the unit of the monad T, and the left-hand one is the structure of an H-algebra mentioned in (b) above. The object $T(X+Y)$ is, thus, a coproduct of $HT(X+Y)+Y$ and X:

$$\begin{tikzcd} HT(X+Y)+Y \arrow[dr, "{[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]}"'] & & X \arrow[dl, "\eta_{X+Y} \cdot \mathsf{inl}"] \\ & T(X+Y) & \end{tikzcd}$$

We can think of $HT(X+Y)+Y$ as the "rest" of $T(X+Y)$ when single variables from X have been removed. The equations we would like to solve are then the guarded ones:

Definition. By a *guarded equation morphism* is meant a morphism

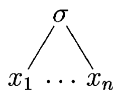
$$e : X \to T(X+Y)$$

(for an arbitrary object X "of variables" and an arbitrary object Y "of parameters") which factors through $HT(X+Y)+Y$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & T(X+Y) \\ & HT(X+Y)+Y \arrow[u, "{[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]}"'] \end{tikzcd}$$

Although guarded equation morphisms are allowed to have, on the right-hand sides, trees of arbitrary depth over X and Y, it is actually sufficient to solve *flat equations* where the right-hand sides are allowed to be only

(a) flat trees



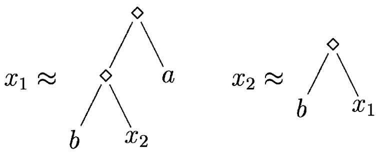
for an n-ary operation symbol σ and n variables $x_1,\dots,x_n \in X$ (including $n=0$ where we have just σ)

or

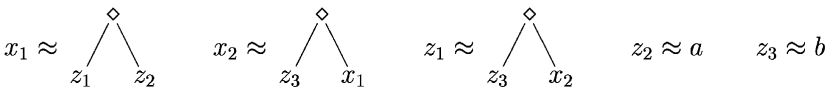
(b) single parameters from Y.

In fact, every guarded system can be "flattened" by adding auxilliary variables.

Example. To solve the following system



where $\diamond$ is a binary operation we flatten it by introducing new variables z_1, z_2, z_3 as follows:



Now for general functors H, *flat equation morphisms* have the form

$$e : X \to HX + Y.$$

But these are simply coalgebras of $H(\_) + Y$! And indeed, to solve e means precisely to use corecursion: a morphism $X \to TY$ is a solution of e iff it is the unique homomorphism from the coalgebra e into TY (the final coalgebra). This is our Solution Lemma, see Lemma 3.4.

The above flattening can also be performed quite generally, thus, the Solution Lemma implies the following

Solution Theorem. *Given an iteratable endofunctor, every guarded equation morphism has a unique solution.*

Now in [16] a theory (or monad) $\mathbb{T}$ on Set is called *completely iterative* provided that every guarded system of equations, $e : X \to T(X+Y)$, has a unique solution $e^\dagger : X \to TY$. Thus, our monad T is completely iterative. For example, if we start with a polynomial functor $H : \mathsf{Set} \to \mathsf{Set}$, then T is the monad of infinite properly labelled trees. This is a free completely iterative monad on H, as proved in [16]. The proof there is very involved. We present here a considerably shorter and conceptually clearer proof. And moreover, the same proof works for all iteratable endofunctors of Set (not just the polynomial ones), in fact, all iteratable endofunctors of any category $\mathscr{A}$ with finite coproducts.

We can also view the completely iterative monad $T:\mathscr{A}\to\mathscr{A}$ as an object of the endofunctor category $[\mathscr{A},\mathscr{A}]$. We prove that T is a final coalgebra of the following endofunctor $\hat{H}$ of $[\mathscr{A},\mathscr{A}]$:

$$\hat{H}(B) = H\cdot B + Id \quad \text{for all } B:\mathscr{A}\to\mathscr{A}.$$

Now $[\mathscr{A},\mathscr{A}]$ is a monoidal category whose tensor product $\otimes$ is composition and unit I is the identity functor Id. And the completely iterative monad generated by H is a monoid in $[\mathscr{A},\mathscr{A}]$. We thus turn to the more general problem: given a monoidal category $\mathscr{B}$, we call an object H *iteratable* provided that the endofunctor $\hat{H}:\mathscr{B}\to\mathscr{B}$ given by $\hat{H}(B)=H\otimes B+I$ has a final coalgebra T. Assuming that binary coproducts of $\mathscr{B}$ distribute on the left with the tensor product, we deduce that T has a structure of a monoid, called the *completely iterative monoid generated by the object H*.

Throughout the paper we use the concept of category as "category in some universe". Thus, we can form, e.g., the category $[\mathscr{A},\mathscr{A}]$ of all endofunctors for any category $\mathscr{A}$. As usual, a universe of "small sets" is supposed to be chosen, and the corresponding category is called Set. On two occasions we mention non-well-founded set theory briefly; there we denote by Class the category of classes and class functions.

Related work. The present paper is an expanded and improved version of the extended abstract [2].

In the very inspiring papers [24] and [25] of Moss, which we have discovered after completing [2], the Solution Theorem and Substitution Theorem we prove below have already been formulated and proved. In the setting of those papers, one works with final coalgebras of $H(\_+X)$, but Moss already discussed in [24] the fact that these two approaches are equivalent; we state that explicitly below for the sake of completeness. Thus, the fact that the monad $\mathbb{T}$ we construct is completely iterative is due to Moss, whereas the result that $\mathbb{T}$ is free on H is new. And our proof of the complete iterativeness, presented here, is a happy combination of the proofs presented in [24] and [2].

The question of infinite trees forming a monad has been asked by Ghani and de Marchi, see also [17]. We acknowledge interesting discussion on that topic with them.

2. Iteratable functors

Assumption 2.1. Throughout this section, H denotes an endofunctor of a category $\mathscr{A}$ with finite coproducts. Whenever possible we denote by

$$\mathsf{inl}: X\to X+Y \quad\text{and}\quad \mathsf{inr}: Y\to X+Y$$

the first and the second coproduct injection respectively. Recall that, since coproducts are determined up to isomorphism only, equations such as $Z = X + Y$ are always meant as an isomorphism.

Remark 2.2. For the functor

$$H(\_) + X : \mathscr{A} \to \mathscr{A}$$

(i.e., for the coproduct of H with the constant functor of value X) it is well-known that

$$\text{initial } (H(\_) + X)\text{-algebra} \equiv \text{free } H\text{-algebra on } X.$$

See e.g. [9]. More precisely, suppose that FX together with

$$\alpha_X : HFX + X \to FX$$

is an initial algebra of $H(\_) + X$. The components of α_X then form

$$\text{an } H\text{-algebra } \varphi_X : HFX \to FX$$

and

$$\text{a universal arrow } \eta_X^F : X \to FX.$$

That is, for every H-algebra

$$HA \to A$$

and for every morphism $f : X \to A$ there exists a unique homomorphism $f^\sharp : FX \to A$ of H-algebras with

$$f = f^\sharp \cdot \eta_X^F.$$

Example 2.3. *Polynomial endofunctors of* Set.

These are the endofunctors of the form

$$H_\Sigma Z = A_0 + A_1 \times Z + A_2 \times Z \times Z + \cdots = \coprod_{n<\omega} A_n \times Z^n,$$

where

$$\Sigma = (A_0, A_1, A_2, \ldots)$$

is a sequence of pairwise disjoint sets called the *signature*. An initial H-algebra can be described as the algebra of all finite Σ-labelled trees. Here a Σ-*labelled tree* t is represented by a partial function

$$t : \omega^* \to \bigcup_{n<\omega} A_n$$

whose domain of definition D_t is a nonempty and prefix-closed subset of ω^* (the set of all finite sequences of natural numbers), such that for any $i_1 i_2 \dots i_r \in D_t$ with $t(i_1 \dots i_r) \in A_n$ we have

$$i_1 i_2 \dots i_r i \in D_t \quad \text{iff} \quad i < n \quad (\text{for all } i < \omega).$$

The tree t is called finite if D_t is a finite set.

Now the functor

$$H_\Sigma(\_) + X$$

is also polynomial of signature

$$\Sigma_X = (X + A_0, A_1, A_2, \dots).$$

Therefore,

$$FX$$

can be described as the algebra of all finite Σ_X-labelled trees, i.e., trees with leaves labelled by variables or nullary operation symbols, and nodes with $n > 0$ successors labelled by n-ary operation symbols.

Remark 2.4.

(1) Dualizing the concept of a free H-algebra, we can study cofree H-coalgebras. A cofree H-coalgebra on an object X of $\mathscr{A}$ is just a free H^{op}-algebra on X in $\mathscr{A}^{op}$, where $H^{op} : \mathscr{A}^{op} \to \mathscr{A}^{op}$ is the obvious endofunctor. If $\mathscr{A}$ has finite products, then, by dualizing 2.2, we see that

final $(H(\_)\times X)$-coalgebra $\equiv$ cofree H-coalgebra on X.

Example: let H_Σ be a polynomial functor on Set. Then

$$H_\Sigma(\_) \times X$$

is also a polynomial functor, since

$$H_\Sigma Z \times X = \coprod_{n<\omega} X \times A_n \times Z^n.$$

This is the polynomial functor of signature

$$\Sigma^X = (X \times A_0, X \times A_1, X \times A_2, \dots).$$

A cofree H_Σ-coalgebra can be described as the coalgebra $\tilde{T}X$ of all (finite and infinite) Σ^X-labelled trees. Every node with n successors is labelled by (i) an n-ary operation symbol and (ii) a variable from X.

(2) Besides a free H-algebra on X and a cofree H-coalgebra on X, we have a third structure associated with X: a final coalgebra of $H(\_) + X$. We will show that it has an important universal property.

Definition 2.5. An endofunctor H of $\mathscr{A}$ is called *iteratable* provided that for every object X of $\mathscr{A}$ the endofunctor

$$H(\_) + X$$

has a final coalgebra.

Notation 2.6. Let

$$TX$$

denote a final coalgebra of $H(\_) + X$. The coalgebra map

$$\alpha_X : TX \to H(TX) + X$$

is, by Lambek's lemma [20], an isomorphism. Thus, TX is a coproduct of HTX and X; we denote the coproduct injections by

$$\tau_X : H(TX) \to TX \quad \text{and} \quad \eta_X : X \to TX.$$

Thus $[\tau_X, \eta_X] = \alpha_X^{-1} : H(TX) + X \to TX$.

In particular, TX is an H-algebra via τ_X.

Example 2.7. *Polynomial endofunctors of* Set *are iteratable.*

A final coalgebra

$$TX$$

of the (polynomial!) functor $H_\Sigma(\_) + X$ of signature Σ_X is the algebra of all finite and infinite Σ_X-labelled trees. That is, unlike the coalgebra

$$\tilde{T}X$$

of all Σ^X-labelled trees, see Remark 2.4, where every node carries a label from X and one from A_n (for the case of n children), the trees in TX have leaves labelled by variables or nullary operation symbols, and nodes with $n > 0$ successors labelled by n-ary operation symbols.

As a concrete example, consider a unary signature:

$$HZ = A \times Z.$$

We have defined three algebras for a set X of variables: the free algebra

$$FX = A^* \times X$$

of all finite Σ-labelled trees for $\Sigma = (\emptyset, A, \emptyset, \emptyset, \ldots)$, the cofree coalgebra

$$\tilde{T}X = (A \times X)^\infty$$

(where $(\_)^\infty$ denotes the set of all finite and infinite words in the given alphabet), and the coalgebra

$$TX = A^* \times X + A^\omega$$

(where $(\_)^\omega$ denotes the set of all infinite words in the given alphabet).

Example 2.8. *Generalized polynomial functors are iteratable.*

We want to include functors such as $HZ = Z^B$, where B is a (not necessarily finite) set; the description of TX is quite analogous to the preceding case. Here we introduce a *generalized signature* as a collection

$$\Sigma = (A_i)_{i \in Card}$$

of pairwise disjoint sets indexed by all cardinals such that for some cardinal λ we have

$$i \geqslant \lambda \quad \text{implies} \quad A_i = \emptyset.$$

(We say that Σ is a λ*-ary* generalized signature; the case $\lambda = \omega$ being the above one.) The generalized polynomial functor of generalized signature Σ is defined on objects by

$$H_\Sigma Z = \coprod_{j<\lambda} A_j \times Z^j$$

and analogously on morphisms.

An initial algebra of $H_\Sigma(\_) + X$, i.e., a free Σ-algebra, FX, on a set X of variables, can be described as the algebra of all *well-founded* Σ_X-labelled trees (i.e., Σ_X-labelled trees in which every branch is finite). For a λ-ary signature, a Σ_X-labelled tree can be formalized as follows: Let λ^* be the set of all words (= finite sequences) of ordinals smaller than λ. A Σ_X-labelled tree is a partial function

$$t : \lambda^* \to X + \coprod_{j<\lambda} A_j$$

defined on a nonempty, prefixed-closed subset D_t of λ^* such that for all $i_1 \dots i_r \in D_t$ we have: if $t(i_1 \dots i_r) \in X$, then $i_1 \dots i_r i \notin D_t$ for any i, and if $t(i_1 \dots i_r) \in A_j$, then

$$i_1 \dots i_r i \in D_t \quad \text{iff} \quad i < j \quad (\text{for all } i < \lambda).$$

The tree t is well-founded if D_t does not contain any infinite sequence of the form $i_1, i_1 i_2, i_1 i_2 i_3, \dots$, see, e.g., [9, II.3.6].

A *final coalgebra*, TX, of $H(\_) + X$ is, analogously to the finitary case, the coalgebra of *all* Σ_X-labelled trees, as proved, e.g., in [5].

Example 2.9. *Accessible* (= *bounded*) *endofunctors are iteratable.*

Recall that an endofunctor of Set is called accessible if it preserves λ-filtered colimits for some infinite cardinal λ. These are precisely the so-called bounded endofunctors, see [6]. This generalizes Examples 2.7 and 2.8 above.

Every accessible endofunctor has a final coalgebra: see a simple, explicit proof in [11, Proposition 1.3]. That proof applies, in fact, to accessible endofunctors of all locally presentable categories.

Since for H accessible also the functors $H(\_)+X$ are accessible, we conclude that

accessible $\Rightarrow$ iteratable.

Example 2.10. *Power-set functor and subfunctors.*

The power-set functor $\mathscr{P}:\mathsf{Set}\to\mathsf{Set}$ is not iteratable, in fact, it does not have a final coalgebra $T\emptyset$ (because there are no sets X isomorphic to $\mathscr{P}X$).

For every cardinal number κ the subfunctor $\mathscr{P}_\kappa$ of $\mathscr{P}$ defined on objects by

$$\mathscr{P}_\kappa Z = \{A \mid A \subseteq Z \text{ and } \operatorname{card} A < \kappa\}$$

is iteratable because it is accessible: for every cardinal λ with cofinality bigger than κ it is clear that $\mathscr{P}_\kappa$ preserves λ-filtered colimits.

For $\kappa = \aleph_0$ we use the notation $\mathscr{P}_f$. A final coalgebra of $\mathscr{P}_f$ has been described by Barr [11] as the coalgebra of all finitely-branching extensional trees (i.e., non-ordered trees such that any two distinct siblings yield non-isomorphic subtrees) modulo the following equivalence $\equiv$:

$t \equiv s$ iff for every $n\in\omega$ the cuttings $t|_n$ and $s|_n$ at level n have isomorphic extensional quotients.

This can be generalized to the following description of TX for $\mathscr{P}_f$: TX is the coalgebra of all finitely-branching extensional trees with leaves labelled in $X+\{\emptyset\}$ modulo the above congruence $\equiv$ (where the cutting $t|_n$ is understood to have all new leaves labelled by $\emptyset$).

Example 2.11. *A non-accessible iteratable functor* $H:\mathsf{Set}\to\mathsf{Set}$ (*see Example 4.2 in* [6]).

We assume the Generalized Continuum Hypothesis (GCH) here. Let M be a class of cardinal numbers containing 1. Define

$$\mathscr{P}_M : \mathsf{Set} \to \mathsf{Set}$$

on sets A by

$$\mathscr{P}_M A = \{B \subseteq A \mid B=\emptyset \text{ or } \operatorname{card}(B)\in M\}$$

and on functions $f:A\to A'$ by

$$\mathscr{P}_M f : B \mapsto \begin{cases} f[B] & \text{if } f \text{ restricted to } B \text{ is one-to-one,}\\ \emptyset & \text{otherwise.}\end{cases}$$

Then $\mathscr{P}_M$ is accessible iff M is a set. In fact, if M is a set with supremum smaller than λ, then $\mathscr{P}_M$ preserves λ-filtered colimits; if M is a proper class then $\mathscr{P}_M$ does not preserve λ-filtered colimits for any λ.

Now let M be a proper class of cardinals such that there exist arbitrarily large regular cardinals α with the property

$$\alpha \notin M \quad \text{and} \quad 2^\alpha \notin M. \tag{2.1}$$

Then the following lemma shows that the functor $\mathscr{P}_M(\_)+X$ has, for every set X, "fixed points" α and 2^α, where $\alpha \geqslant \mathrm{card}(X)$ is any regular cardinal number with $\alpha \notin M$ and $2^\alpha \notin M$. It follows from [5] that, then, a final coalgebra of $\mathscr{P}_M(\_)+X$ exists, i.e., that $\mathscr{P}_M$ is iteratable (but not accessible). For the proof of the lemma we use the following result: if α is a regular, infinite cardinal number and $\beta < \alpha$, then $\alpha^\beta = \alpha$ (under GCH), see [19].

Lemma. *Let X be a set and $\alpha \notin M$ an infinite regular cardinal number with* $\mathrm{card}(X) \leqslant \alpha$. *Then every set A of cardinality α is a "fixed point" of $\mathscr{P}_M(\_)+X$, i.e.,*

$$A \cong \mathscr{P}_M(A)+X.$$

Proof. Since $1 \in M$, we have $\mathrm{card}(\mathscr{P}_M(A)) \geqslant \mathrm{card}(A)$, thus, it is sufficient to prove

$$\mathrm{card}(A) \geqslant \mathrm{card}(\mathscr{P}_M(A)+X).$$

Since $\mathrm{card}(A)=\alpha \notin M$, we have

$$\mathscr{P}_M(A) \subseteq \bigcup_{\beta<\alpha} \{B \subseteq A \mid \mathrm{card}(B)=\beta\}$$

therefore

$$\begin{aligned}\mathrm{card}(\mathscr{P}_M(A)+X) &\leqslant \left(\sum_{\beta<\alpha} \alpha^\beta\right) + \mathrm{card}(X)\\ &\leqslant \left(\sum_{\beta<\alpha} \alpha\right) + \alpha = \alpha \times \alpha + \alpha = \alpha. \qquad \square\end{aligned}$$

Example 2.12. Iteratable endofunctors of Set do not have desired stability properties. For example, if F and G are iteratable, then neither $F \cdot G$ nor $F+G$ need to be iteratable. In fact, in the notation of Example 2.11, consider classes M and M' of cardinal numbers containing 1 and such that
(1) $M \cup M'$ is the class of all cardinal numbers
(2) there exist arbitrarily large cardinals α with $\alpha \notin M$ and $2^\alpha \notin M$
and
(3) there exist arbitrarily large cardinals β with $\beta \notin M'$ and $2^\beta \notin M'$
Then $\mathscr{P}_M$ and $\mathscr{P}_{M'}$ are both iteratable by Example 2.11. But $\mathscr{P}_M + \mathscr{P}_{M'}$ does not have any fixed point (for every set A either $\mathrm{card}(\mathscr{P}_M A) > \mathrm{card}(A)$, or $\mathrm{card}(\mathscr{P}_{M'} A) > \mathrm{card}(A)$), hence, $\mathscr{P}_M + \mathscr{P}_{M'}$ it is not iteratable, having no final coalgebra. Analogously with $\mathscr{P}_M \cdot \mathscr{P}_{M'}$.

Example 2.13. All set functors are "almost" iteratable. There are, of course, non-iteratable endofunctors of Set, e.g., the power-set functor $\mathscr{P}$. However, every functor $H:\mathsf{Set}\to\mathsf{Set}$ can be extended (uniquely up to natural isomorphism) to an endofunctor H^∞ of Class, the category of all large sets (= classes) and functions so that H^∞ preserves colimits, of transfinite chain, see [11].

Applying this to $H(\_)+X$ we see that a final coalgebra, TX, always exists, but it can be a proper class.

Example 2.14. *Power-set functor in non-well-founded set theory.*

The power-set functor $\mathscr{P}:\mathsf{Class}\to\mathsf{Class}$ (assigning to every class the class of its sub*sets*) is iteratable. Assuming the anti-foundation axiom (AFA), for every class X we can describe TX as the so called hyperuniverse of sets built up using the elements of X as atoms. In Chapter 1 of [1] the sets of this hyperuniverse were called the X-sets and they form the class V_X of [12]. The Substitution and Solution theorems have been exploited in the context of these hyperuniverses by applying them to Milner's CCS approach to concurrency, the Liar Paradox and Situation Theory. See also [13].

Example 2.15. *Continuous functors are iteratable.*

Recall that a functor is called *continuous* if it preserves limits of ω^{op}-sequences.

Here we assume that our base category $\mathscr{A}$ has

1. a terminal object 1
2. limits of ω^{op}-sequences

and

3. binary coproducts commuting with ω^{op}-limits.

(Set fulfills these requirements, of course.) Every continuous endofunctor F has a final coalgebra $\lim_{n<\omega}F^n 1$—this is dual to the famous construction of an initial algebra as $\mathrm{colim}_{n<\omega}F^n 0$ first formulated in [3].

If H is continuous, then due to 3., all functors $H(\_)+X$ are continuous, thus, have a final coalgebra

$$TX=\lim_{n<\omega}(H(\_)+X)^n 1.$$

Remark 2.16. Denote by $U:H\text{-}\mathsf{Alg}\to\mathscr{A}$ the forgetful functor of the category of all H-algebras and homomorphisms. The universal property of free H-algebras $\varphi_X:HFX\to FX$ (provided they exist on all objects X of $\mathscr{A}$) makes U a right adjoint. The left adjoint is the functor

$$X\mapsto(FX,\varphi_X).$$

We now show a related universal property of the H-algebras $\tau_X:HTX\to TX$ of 2.6: given a morphism $s:X\to TY$ we prove that there is a unique homomorphism $\hat{s}:TX\to TY$ of H-algebras extending s. This is interesting even for the basic case of the polynomial endofunctors of **Set**: here a morphism $s: X \to TY$ can be viewed as a substitution rule, substituting a variable $x \in X$ by the Σ_Y-labelled tree $s(x)$. We obviously have a homomorphism $\hat{s}: TX \to TY$ extending s: take a tree $t \in TX$, substitute every variable $x \in X$ on any leaf of t by the tree $s(x)$ and obtain a tree

$$t' = Ts(t) \quad \text{in } TTY$$

over TY. Now forget that t' is a tree of trees and obtain a tree $\hat{s}(t)$ in TY. However, it is not obvious that such a homomorphism is unique. This is what we prove now:

Substitution Theorem 2.17. *For every iteratable endofunctor H of $\mathscr{A}$ and any morphism*

$$s : X \to TY \quad \textit{in } \mathscr{A}$$

there exists a unique extension into a homomorphism

$$\hat{s} : TX \to TY$$

of H-algebras. That is, a unique homomorphism $\hat{s}: (TX, \tau_X) \to (TY, \tau_Y)$ with $s = \hat{s} \cdot \eta_X$.

Proof. We turn $TX + TY$ into a coalgebra of type $H(\_) + Y$ as follows: the coalgebra map is

$$TX+TY = HTX+X+TY \xrightarrow{id+[s,id]} HTX+TY = HTX+HTY+Y \xrightarrow{[H\mathsf{inl},H\mathsf{inr}]+id} H(TX+TY)+Y$$

There exists a unique homomorphism

$$f : TX + TY \to TY$$

of $(H(\_) + Y)$-coalgebras. Equivalently, a unique morphism

$$f = [f_1, f_2] : TX + TY \to TY$$

in $\mathscr{A}$ for which the following two squares

$$\begin{tikzcd}
HTX + X \arrow[r, equal] \arrow[d, "{[\mathsf{inl}\cdot H\mathsf{inl},(H\mathsf{inr}+id)\cdot s]}"'] & TX \arrow[dd, "f_1"] & HTY + Y \arrow[r, equal] \arrow[dd, "Hf_2+id"'] & TY \arrow[dd, "f_2"] \\
H(TX + TY) + Y \arrow[d, "Hf+id"'] & & & \\
HTY + Y \arrow[r, equal] & TY & HTY + Y \arrow[r, equal] & TY
\end{tikzcd}$$

commute. The right-hand square shows that f_2 is an endomorphism of the final $(H(\_)+Y)$-coalgebra—thus,

$$f_2 = id.$$

The left-hand square is equivalent to the commutativity of the following two squares:

$$\begin{tikzcd} HTX \arrow[r, "\tau_X"] \arrow[d, "Hf_1"'] & TX \arrow[d, "f_1"] \\ HTY \arrow[r, "\tau_Y"'] & TY \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "s"'] & TX \arrow[ddd, "f_1"] \\ TY \arrow[d, "\alpha_Y"'] & \\ HTY+Y \arrow[d, "Hf_2+id"'] & \\ HTY+Y \arrow[r, equal] & TY \end{tikzcd}$$

The square on the left tells us that f_1 is a homomorphism of H-algebras. And since $f_2 = id$ (thus $Hf_2 + id = id$) and $\alpha_Y^{-1} = [\tau_Y, \eta_Y]$, the square on the right states $f_1 \cdot \eta_X = s$, i.e., f_1 extends s. This proves that there is a unique extension of s to a homomorphism: put $\hat{s} = f_1$. $\square$

Corollary 2.18. *The formation of TX and η_X (for all objects X) and of $\hat{s}$ (for all morphisms $s : X \to TY$) is a Kleisli triple on $\mathscr{A}$.*

In fact, the axioms of Kleisli triples (i.e., $\hat{s} \cdot \eta_X = s$, $\widehat{\eta_X} = id$, and $\hat{s} \cdot \hat{t} = \widehat{\hat{s} \cdot t}$) follow immediately from the uniqueness of $\hat{s}$ in the Substitution Theorem.

In other words, TX is the object part of a functor T, such that η_X are the components of a natural transformation $\eta : Id \to T$, and we have a natural transformation $\mu : TT \to T$ defined by

$$\mu_X = \widehat{id} : TTX \to TX$$

forming a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathscr{A}$. Observe that

μ_X is a homomorphism of H-algebras

since each $\hat{s}$ is. Also, for every morphism $f : A \to B$ in $\mathscr{A}$, $Tf : TA \to TB$ is a homomorphism of H-algebras (because $Tf = \widehat{\eta_B \cdot f}$). Thus,

$$\tau : HT \to T$$

is a natural transformation.

Remark 2.19. Our Substitution Theorem has been proved by Moss in [24] as Lemma 2.4, except that he works with final coalgebras of $H(\_ + X)$ rather than of $H(\_) + X$. However, in a remark preceding his 2.4 he shows the following:

Lemma. *An endofunctor H is iteratable iff for every object X the endofunctor $H(\_ + X)$ has a final coalgebra. In fact*

(i) *a final coalgebra of $H(\_ + X)$ is HTX with the structure map*

$$H\alpha_X : HTX \to H(HTX + X)$$

and, conversely,

(ii) *if* $\hat{\alpha}_X : \hat{T}X \to H(\hat{T}X + X)$ *is a final* $H(\_ + X)$*-coalgebra, then* $\hat{T}X + X$ *with the structure map*

$$\hat{\alpha}_X + X : \hat{T}X + X \to H(\hat{T}X + X) + X$$

is a final coalgebra for $H(\_) + X$.

Proof. Ad (i): given an $H(\_ + X)$-coalgebra

$$\rho : R \to H(R + X)$$

consider the $(H(\_) + X)$-coalgebra

$$\rho + id : R + X \to H(R + X) + X$$

The unique $(H(\_) + X)$-homomorphism $h : R + X \to TX = HTX + X$ has the form $h = h_1 + id_X$ where $h_1 : R \to HTX$ yields the desired $H(\_ + X)$-homomorphism.

Ad (ii): given an $(H(\_) + X)$-coalgebra

$$\rho : R \to HR + X$$

consider the $H(\_ + X)$-coalgebra

$$H\rho : HR \to H(HR + X).$$

The unique $H(\_ + X)$-homomorphism $h : HR \to \hat{T}X$ yields the desired unique $(H(\_) + X)$-homomorphism $g : R \to \hat{T}X + X$ as follows

$$g \equiv R \xrightarrow{\rho} HR + X \xrightarrow{h+id} \hat{T}X + X. \qquad \square$$

Remark 2.20. Note that the last result is an instance of a general fact about categories of fixed points of functors. Indeed, suppose that $F, G : \mathscr{A} \to \mathscr{A}$ are endofunctors. Then applying F and G respectively yields functors

$$FG\text{-}\mathsf{Coalg} \underset{F}{\overset{G}{\rightleftarrows}} GF\text{-}\mathsf{Coalg}$$

which preserve fixed points (i.e., coalgebras whose structure maps are isomorphisms). It is trivial to show that the restrictions of the latter to the full subcategories of fixed points of F-Coalg and G-Coalg respectively are equivalences of categories that are inverse to one another.

Definition 2.21. The above monad $\mathbb{T}$, associated with any iteratable endofunctor H, is called the *completely iterative monad generated by* H.

Examples 2.22.

(1) The completely iterative monad generated by the endofunctor

$$HZ = A \times Z$$

of Set is the monad

$$TX = A^* \times X + A^\omega.$$

This can be described as the free-algebra monad of the variety of algebras with
(a) unary operations f_a for $a \in A$,
(b) nullary operations indexed by A^ω (i.e., constants of the names $a_0 a_1 a_2 \ldots \in A^\omega$), and
(c) satisfying the equations

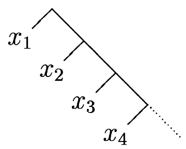
$$f_a(a_0 a_1 a_2 \ldots) = a a_0 a_1 a_2 \ldots \quad \text{for all } a, a_0, a_1, \ldots \in A$$

In this case, $\mathbb{T}$ is a finitary monad on Set.

(2) The completely iterative monad generated by the endofunctor

$$HZ = Z \times Z$$

of Set is the monad TX of all binary trees with leaves indexed in X. This is not finitary: consider the following element of TX:



in which all x_i are pairwise distinct.

(3) Let

$$\mathsf{CPO}$$

denote the category of CPO's (say, posets with a smallest element $\perp$ and joins of ω-chains) and strict continuous functions (i.e., those preserving $\perp$ and joins of ω-chains). For all *locally continuous* functors $H : \mathsf{CPO} \to \mathsf{CPO}$, i.e., such that the derived functions

$$\mathsf{CPO}(A, B) \to \mathsf{CPO}(HA, HB), \quad f \mapsto Hf$$

are all continuous, it is well-known that
initial H-algebra $\equiv$ final H-coalgebra,
see [26]. Since each $H(\_) + X$ is also locally continuous, we deduce that
locally continuous functors are iteratable,
and in this case

$$FX \equiv TX$$

that is, the completely iterative monad $\mathbb{T}$ is just the free algebra monad $\mathbb{F}$ on H.

(4) Analogously for the category

$$\mathsf{CMS}$$

of all complete metric spaces and contractions: every *contractive* endofunctor $H : \mathsf{CMS} \to \mathsf{CMS}$, i.e., such that the derived functions

$$\mathsf{CMS}(A,B) \to \mathsf{CMS}(HA,HB), \quad f \mapsto Hf$$

are all contractive with a common constant <1, has a single fixed point. Therefore,

initial H-algebra $\equiv$ final H-coalgebra,

see [7]. Since each $H(\_) + X$ is also locally contractive, we again get

$$\mathbb{T} = \mathbb{F}.$$

Remark 2.23.

(1) The Kleisli category

$$\mathscr{A}_{\mathbb{T}} \to \mathscr{A}$$

of the completely iterative monad is the above category $\mathscr{K}$ of all H-algebras $\tau_X : HTX \to TX$ (with its forgetful functor $\mathscr{K} \to \mathscr{A}$). This follows from the Substitution Theorem.

(2) The Eilenberg–Moore category

$$\mathscr{A}^{\mathbb{T}} \to \mathscr{A}$$

of all $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms seems to be a new construct. As seen in 2.22, it is usually infinitary.

3. Solution theorem

3.1. Recall from the Introduction that a *solution* of an equation morphism $e : X \to T(X+Y)$ is a morphism $e^\dagger : X \to TY$ such that the following square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & TY \\
T(X+Y) \arrow[r, "T{[e^\dagger, \eta_Y]}"'] & TTY \arrow[u, "\mu_Y"']
\end{tikzcd}$$

commutes. Elgot used the language of algebraic theories, i.e., Kleisli categories, rather than monads. Both equations and solutions are morphisms of the Kleisli category, here:

$$e : X \to X + Y \quad \text{and} \quad e^\dagger : X \to Y.$$

If we denote by $*$ the composition of the Kleisli category (i.e., $g * f = \mu_Z \cdot Tg \cdot f$ for $f : X \to TY$ and $g : Y \to TZ$ in $\mathscr{A}$) then a solution $e^\dagger$ is defined by the equality

$$e^\dagger = [e^\dagger, 1] * e.$$

This is the definition used in [15,16]. We are not going to use this notation below.

Recall further from the Introduction that a *flat equation morphism*

$$e : X \to HX + Y$$

is just another name for a coalgebra of $H(\_) + Y$. However, we can also view e as a guarded equation morphism. More precisely, we denote by

$$\rho_{X,Y} : HX + Y \to T(X + Y)$$

the "natural connecting morphism" whose left-hand component is

$$HX \xrightarrow{H\eta X} HTX \xrightarrow{HT\mathsf{inl}} HT(X + Y) \xrightarrow{\tau_{X+Y}} T(X + Y)$$

and the right-hand one is

$$Y \xrightarrow{\mathsf{inr}} X + Y \xrightarrow{\eta_{X+Y}} T(X + Y).$$

Since $\rho_{X,Y}$ factors through $[\tau_{X+Y}, \eta_{X+Y}\mathsf{inr}]$, we see that

$$\rho_{X+Y}e : X \to T(X + Y)$$

is a guarded equation morphism. We denote, for short, by

$$e^\dagger : X \to TY$$

a solution of $\rho_{X,Y}e$ (whenever there is no danger of confusion). Explicitly, $e^\dagger$ is a morphism such that the following diagram

$$\begin{tikzcd}
X \arrow[rr, "e^\dagger"] \arrow[d, "e"'] & & TY \\
HX + Y \arrow[d, "\rho_{X,Y}"'] & & \\
T(X + Y) \arrow[rr, "T[e^\dagger, \eta_Y]"'] & & TTY \arrow[uu, "\mu_Y"']
\end{tikzcd}$$

commutes.

Examples 3.2.

(1) For polynomial functors solutions of flat equations are discussed in the Introduction.

(2) For the finite-power-set functor $\mathscr{P}_f : \mathsf{Set} \to \mathsf{Set}$ a flat system of equations without parameters has the following form

$$x_1 \approx A_1$$

$$x_2 \approx A_2$$

$$\vdots$$

for a set $X=\{x_1,x_2,\ldots\}$ of variables, where A_1, A_2, ... are finite subsets of X. This is the concept of a flat system of equations as used in non-well-founded set theory.

The functor $\mathscr{P}_f$ is iteratable, see Example 2.10. In non-well-founded set theory, a final coalgebra $T\emptyset$ is described as the coalgebra of all hereditarily finite sets, see [13]. Thus, every solution of equation systems as above is found in that coalgebra. In well-founded set theory, solutions will be extensional trees modulo the equivalence described in Example 2.10.

(3) The power-set functor $\mathscr{P}$ leads to flat systems of equations without parameters of the form above, except that here the subsets A_1, A_2, ... of X are arbitrary, not necessarily finite. The possibility of having a unique solution for every flat system of equations is (one of the formulations of) the anti-foundation axiom leading to non-well-founded set theory, see [1,13].

Notation 3.3. We denote by

$$\tau^* : H \to T$$

the composite

$$H \xrightarrow{H\eta} HT \xrightarrow{\tau} T.$$

Observe that the following triangle

$$\begin{tikzcd}
HTX \arrow[r, "\tau_X"] \arrow[dr, "\tau^*_{TX}"'] & TX \\
 & TTX \arrow[u, "\mu_X"']
\end{tikzcd}$$

commutes for every object. This follows from μ_X being a homomorphism of H-algebras and $\mu \cdot \eta T = id$:

$$\begin{tikzcd}
HTX \arrow[rr, "\tau_X"] \arrow[d, "H\eta_{TX}"'] \arrow[dr, equal] & & TX \\
 & HTX \arrow[ur, "\tau_X"] & \\
HTTX \arrow[ur, "H\mu_X"] \arrow[rr, "\tau_{TX}"'] & & TTX \arrow[uu, "\mu_X"']
\end{tikzcd}$$

Solution Lemma 3.4. *For flat equation morphisms we have*

$$solution = corecursion.$$

That is, a flat equation morphism $e : X \to HX + Y$ has a unique solution, viz, the unique homomorphism of the coalgebra e into the final coalgebra TY of $H(\_) + Y$.

Proof. For any morphism $x : X \to TY$, consider the following diagram

$$\begin{tikzcd}
X \arrow[rr, "x"] \arrow[d, "e"'] & & TY \\
HX+Y \arrow[r, "Hx+id"] \arrow[d, "\tau^*{}_X+\eta_Y"'] & HTY+Y \arrow[ur, "{[\tau_Y,\eta_Y]}"] \arrow[d, "\tau^*{}_{TY}+\eta_Y"] & \\
TX+TY \arrow[r, "Tx+id"] \arrow[d, "{[T\mathsf{inl},T\mathsf{inr}]}"'] & TTY+TY \arrow[dr, "{[TTY,T\eta_Y]}"'] & \\
T(X+Y) \arrow[rr, "{T[x,\eta_Y]}"'] & & TTY \arrow[uuu, "\mu_Y"']
\end{tikzcd}$$

The lower square and the middle one clearly commute. Also the right-hand square commutes by 3.3. Now suppose we put $e^\dagger$ in the place of x in the diagram. Then the outer square commutes, and therefore the upper square does, which shows that $e^\dagger$ is an $H(\_)+Y$ coalgebra homomorphism, and thus $e^\dagger = \tilde{e}$, where $\tilde{e}$ denotes the unique homomorphism into the final coalgebra TY.

Conversely, if $\tilde{e}$ is put in the place of x, then the upper square commutes and thus the whole diagram does, which shows that $\tilde{e}$ is a solution for e. □

Remark 3.5. In the Introduction we have mentioned that every guarded equation morphism $e : X \to T(X+Y)$ has a "flattening" by introducing additional variables, Z. That is, there is a flat equation morphism

$$g : X + Z \to H(X+Z) + Y$$

such that to solve e is "the same" as to solve g. This is, in fact, a general phenomenon:

Proposition 3.6. *For every guarded equation morphism*

$$e : X \to T(X+Y)$$

there exists a flat equation morphism

$$g : X + Z \to H(X+Z) + Y$$

such that the left-hand component of $g^\dagger : X + Z \to TY$ *is a solution of* e.

Proof. Since e is guarded, we have a commutative triangle

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "f"'] & T(X+Y) \\
 & HT(X+Y)+Y \overset{\mathrm{def}}{=} Z \arrow[u, "{[\tau_{X+Y},\eta_{X+Y}\mathsf{inr}]}"']
\end{tikzcd}$$

The above object Z has the property that

$$X + Z = T(X+Y).$$

More precisely, $T(X+Y)$ is a coproduct of X and Z with injections

$$X \xrightarrow{\eta_{X+Y}\mathsf{inl}} T(X+Y)$$

and

$$Z = HT(X+Y)+Y \xrightarrow{id+\mathsf{inr}} HT(X+Y)+(X+Y) = T(X+Y)$$

respectively. The morphism g we are to define thus has the codomain $HT(X+Y)+Y = Z$. Put simply

$$g = [f, id] : X + Z \to Z.$$

The solution $g^\dagger : X + Z = X + HT(X+Y) + Y \to TY$ has components $h_1 : X \to TY$, $h_2 : HT(X+Y) \to TY$ and $h_3 : Y \to TY$. The property of being a solution means, by the Solution Lemma, precisely that $[h_1, h_2, h_3] : T(X+Y) \to TY$ is a homomorphism of coalgebras. That is, $g^\dagger$ is a solution if and only if the following square

$$\begin{tikzcd}
T(X+Y) = X + HT(X+Y) + Y \arrow[r, "{[f, HT(X+Y)+Y]}"] \arrow[d, "{[h_1,h_2,h_3]}"'] & HT(X+Y)+Y \arrow[d, "{H[h_1,h_2,h_3] = Hg^\dagger + id}"] \\
TY \arrow[r, equal, "{[\tau_Y, \eta_Y]}"'] & HTY + Y
\end{tikzcd}$$

commutes. Equivalently, iff the following hold:

$$h_3 = \eta_Y$$

$$h_2 = \tau_Y \cdot Hg^\dagger$$

$$h_1 = [\tau_Y, \eta_Y] \cdot (Hg^\dagger + id) \cdot f = [h_2, \eta_Y] \cdot f.$$

We prove that h_1 solves e. Since $g^\dagger \cdot \eta_{X+Y} = [h_1, h_3] = [h_1, \eta_Y]$ and $e = [\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \cdot f$ we are to prove the commutativity of the outward square in the following diagram

$$\begin{tikzcd}
X \arrow[rr, "h_1"] \arrow[d, "f"'] & & TY \\
HT(X+Y)+Y \arrow[urr, "{[h_2, \eta_Y]}"] \arrow[d, "{[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]}"'] & T(X+Y) \arrow[ur, "g^\dagger"'] & \\
T(X+Y) \arrow[ur, equal] \arrow[r, "T\eta_{X+Y}"'] & TT(X+Y) \arrow[u, "\mu_{X+Y}"'] \arrow[r, "Tg^\dagger"'] & TTY \arrow[uu, "\mu_Y"']
\end{tikzcd}$$

The right-hand inner square commutes because $g^\dagger$ is a homomorphism of H-algebras: $g^\dagger \cdot \tau_{X+Y} = h_2 = \tau_Y \cdot Hg^\dagger$ and thus, by Substitution Theorem it is enough to observe that

$$(g^\dagger \cdot \mu_{X+Y}) \cdot \eta_{T(X+Y)} = g^\dagger = \mu_Y \cdot \eta_{TY} \cdot g^\dagger = (\mu_Y \cdot Tg^\dagger) \cdot \eta_{T(X+Y)}.$$

All the other inner parts also commute (e.g., $g^\dagger \cdot [\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] = [h_2, h_3] = [h_2, \eta_Y]$). $\square$

Remark 3.7. The proof of the preceding proposition gives more than the statement: every solution $e^\dagger$ of the original equation morphism yields a solution of the flat one by the rule

$$g^\dagger \equiv X + Z = T(X+Y) \xrightarrow{T[e^\dagger,\eta_Y]} TTY \xrightarrow{\mu_Y} TY.$$

In fact, the morphism

$$\mu_Y \cdot T[e^\dagger,\eta_Y] : X + HT(X+Y) + Y \to TY$$

has the following components

$$h_3 = \mu_Y \cdot T[e^\dagger,\eta_Y] \cdot \eta_{X+Y} \cdot \mathsf{inr} = \mu_Y \cdot \eta_{TY} \cdot \eta_Y = \eta_Y$$

(by naturality: $T[e^\dagger,\eta_Y] \cdot \eta_{X+Y} = \eta_{TY} \cdot [e^\dagger,\eta_Y]$)

$$h_2 = \mu_Y \cdot T[e^\dagger,\eta_Y] \cdot \tau_{X+Y} = \mu_Y \cdot \tau_{TY} \cdot HT[e^\dagger,\eta_Y] = \tau_Y \cdot H\mu_Y \cdot HT[e^\dagger,\mu_Y] = \tau_Y \cdot Hg^\dagger$$

(since $T(\_)$ and μ_Y are homomorphisms of H-algebras), and

$$h_1 = \mu_Y \cdot T[e^\dagger,\eta_Y] \cdot \eta_{X+Y} \cdot \mathsf{inl} = \mu_Y \cdot \eta_{TY} \cdot e^\dagger = e^\dagger : X \to TY.$$

Moreover, by definition of $(\_)^\dagger$ for $e = [\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \cdot f$,

$$h_1 = e^\dagger = \mu_Y \cdot T[e^\dagger,\eta_Y] \cdot [\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \cdot f = [h_2, \eta_Y] \cdot f.$$

Thus, the three equations of the above proof hold, i.e., $g^\dagger$ is a homomorphism of $(H(\_) + Y)$-coalgebras.

Corollary 3.8 (**Solution Theorem**). *Given an iteratable functor, every guarded equation morphism has a unique solution.*

Remark. This is the result called Parametric Corecursion by Moss, see [24] We have proved it, independently, in [2].

Proof. In fact, the existence follows from 3.4 and 3.6. The uniqueness from 3.7: since $g^\dagger = \mu_Y \cdot T[e^\dagger,\eta_Y]$ implies $g^\dagger \cdot \eta_{X+Y} = \mu_Y \cdot \eta_{TY} \cdot [e^\dagger,\eta_Y] = [e^\dagger,\eta_Y]$ we have $e^\dagger = g^\dagger \cdot \eta_{X+Y} \cdot \mathsf{inr}$. Thus, the uniqueness of $g^\dagger$ (see 3.4) proves the uniqueness of $e^\dagger$. □

4. Completely iterative monads

Assumption 4.1. In the present section we assume that a category $\mathscr{A}$ with finite coproducts is given such that coproduct injections are monomorphisms. (One can work, more generally, with binary coproducts without further restriction, see Remark 4.16 below.)

We are going to introduce solutions of guarded equations in general monads, and obtain the concept of complete iterativity for monads. Our main result will be that the above monad $\mathbb{T}$ is a free completely iterative monad on the given functor H.

Elgot has introduced the concept of an *ideal algebraic theory* in order to speak about ideal equations and (completely) iterative theories. As we show below, his concept is the special case, for $\mathscr{A} = \mathsf{Set}$ and for finitary monads, of the following:

Definition 4.2. A monad $\mathbb{S} = (S, \eta, \mu)$ on $\mathscr{A}$ is called *ideal* provided that
(i) S is a coproduct of endofunctors, $S = S' + Id$, with $\eta = \mathsf{inr} : Id \to S$
and
(ii) $\mu : SS \to S$ restricts to $\mu' : S'S \to S'$.

Remark 4.3. More precisely, we should say that an ideal monad is a sixtuple $(S, \eta, \mu, S', \sigma, \mu')$ consisting of a monad (S, η, μ), a natural transformation $\sigma : S' \to S$ forming inl of the coproduct $S = S' + Id$ with $\eta = \mathsf{inr}$, and a natural transformation $\mu' : S'S \to S'$ such that the following square (expressing "a restriction of μ")

$$\begin{tikzcd} S'S \arrow[r, "\sigma S"] \arrow[d, "\mu'"'] & SS \arrow[d, "\mu"] \\ S' \arrow[r, "\sigma"'] & S \end{tikzcd}$$

commutes.

However, the above definition is precise enough since we assume that coproduct injections in $\mathscr{A}$ (and, thus, in $[\mathscr{A}, \mathscr{A}]$) are monomorphisms, which makes μ' unique.

Examples 4.4.
(1) The completely iterative monad $\mathbb{T}$ for a given iteratable endofunctor H, see Definition 2.21, is ideal. Here

$$T = HT + Id$$

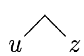
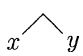
with coproduct injections τ and η. And for $\mu' = H\mu$ the relevant square commutes, because each $\mu_X : TTX \to TX$ is (by definition) a homomorphism of H-algebras:

$$\begin{tikzcd} HTTX \arrow[r, "\tau_{TX}"] \arrow[d, "H\mu_X"'] & TTX \arrow[d, "\mu_X"] \\ HTX \arrow[r, "\tau_X"'] & TX \end{tikzcd}$$

(2) Consider the variety of algebras on one binary operation given by the single equation

$$(xy)z = x.$$

The corresponding monad $\mathbb{S}$ is easily seen to be such that $\eta : Id \to S$ is a coproduct injection. However, this monad is not ideal: this follows from the fact that although none of the terms

$t =$  and $s =$ 

is congruent to a variable, the term $t[s/u]$ is congruent to x.

Remark 4.5. The definition of ideal theory used by Elgot is the following. An *algebraic theory* (in the sense of Lawvere) is a category whose objects are given by the set $\mathbb{N}$ of natural numbers and such that for each $n \geq 0$ there are so-called *distinguished morphisms*

$$i_1, \dots, i_n : 1 \to n$$

which form coproduct injections. Such a theory is called *ideal* whenever the following property holds: if $f : 1 \to n$ is not distinguished, then $g \cdot f : 1 \to m$ is not distinguished for every $g : n \to m$. Recall that every finitary variety gives rise to an algebraic theory as follows: an arrow

$$s : n \to m$$

is a substitution that gives for each of n variables $x_1, \dots, x_n$ a term $s(x_i)$ in m variables. The distinguished morphism

$$i_k : 1 \to n$$

substitutes x_k for the given variable.

Recall further that finitary varieties correspond to finitary monads on Set. Moreover, for every finitary variety, the notion of ideal monad as defined in 4.2 coincides with the notion of ideal theory:

Lemma 4.6. *The algebraic theory corresponding to a finitary variety $\mathscr{V}$ is ideal if and only if the finitary monad corresponding to $\mathscr{V}$ is ideal.*

Proof. Suppose the theory of a given finitary variety is ideal. Let $(S, \eta, s \mapsto \hat{s})$ be the corresponding finitary monad given by its Kleisli triple. Then for arbitrary finite sets X, Y and substitution $s : X \to SY$, the homomorphism $\hat{s} : SX \to SY$ satisfies the following property: if $t \in SX$ is not (congruent to) a variable, then neither is $\hat{s}(t) \in SY$. In particular, this is true for $Sf = \widehat{f \cdot \eta_Y}$ for any $f : X \to Y$. Since Sf preserves variables, we conclude that $S = S' + Id$ with coproduct injection $\eta : Id \to S$ (for infinite sets use that S is finitary). That μ restricts to μ' follows since $\mu_Y = \widehat{id_{SY}}$.

Conversely, suppose that the finitary monad (S, η, μ) of a given variety $\mathscr{V}$ is ideal in the sense of Definition 4.2. Let $s : X \to SY$ be any substitution where X and Y are finite, and let $t \in S'X$. Then $\hat{s}(t)$ is in $S'Y$ since $\hat{s} = \mu_Y \cdot Ss$, which on $S'X$ restricts to $\mu'_Y \cdot S's$. But this is equivalent to the theory of $\mathscr{V}$ being ideal in the sense of Elgot. $\square$

Definition 4.7. Let $\mathbb{S}$ be an ideal monad on $\mathscr{A}$.

(1) By an *equation morphism* we understand a morphism in $\mathscr{A}$ of the form

$$e : X \to S(X + Y), \quad X, Y \text{ are objects of } \mathscr{A}.$$

(2) By a *solution* of e is understood a morphism

$$e^\dagger : X \to SY$$

for which the following diagram

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & SY \\ S(X+Y) \arrow[r, "{S[e^\dagger,\eta_Y]}"'] & SSY \arrow[u, "\mu_Y"'] \end{tikzcd}$$

commutes.

(3) We call e *guarded* if it factors through $S'(X+Y)+Y$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & S(X+Y) \\ & S'(X+Y)+Y \arrow[u, "{[\sigma_{X+Y},\eta_{X+Y}\cdot\mathsf{inr}]}"'] \end{tikzcd}$$

Definition 4.8. An ideal monad is called *completely iterative* provided that every guarded equation morphism has a unique solution.

Example 4.9. The monad $\mathbb{T}$ associated with an iteratable functor H is completely iterative. This is the Solution Theorem.

We are going to prove that solutions are preserved by monad morphisms. Recall that for monads $\mathbb{S}=(S,\eta,\mu)$ and $\tilde{\mathbb{S}}=(\tilde{S},\tilde{\eta},\tilde{\mu})$ a *monad morphism* $\varphi:\mathbb{S}\to\tilde{\mathbb{S}}$ is a natural transformation $\varphi:S\to\tilde{S}$ such that the following diagrams

$$\begin{tikzcd} S \arrow[rr, "\varphi"] & & \tilde{S} \\ & Id \arrow[ul, "\eta"] \arrow[ur, "\tilde{\eta}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} S \arrow[r, "\varphi"] & \tilde{S} \\ SS \arrow[u, "\mu"] \arrow[r, "\varphi*\varphi"'] & \tilde{S}\tilde{S} \arrow[u, "\tilde{\mu}"'] \end{tikzcd}$$

commute. (Here, $\varphi*\varphi$ denotes the horizontal composition, i.e., $\varphi*\varphi=\varphi\tilde{S}\cdot S\varphi=\tilde{S}\varphi\cdot\varphi S$.)

Definition 4.10. If $\mathbb{S}$ and $\tilde{\mathbb{S}}$ are ideal monads, we call a morphism $\varphi:\mathbb{S}\to\tilde{\mathbb{S}}$ *ideal* if it has the form $\varphi=\varphi'+id$ for a natural transformation $\varphi':S'\to\tilde{S}'$.

Lemma 4.11. *Monad morphisms preserve solutions of equations. That is, given a monad morphism* $\varphi:\mathbb{S}\to\tilde{\mathbb{S}}$ *and given an equation morphism* $e:X\to S(X+Y)$ *with a solution* $e^\dagger:X\to SY$ (*w.r.t.* $\mathbb{S}$), *then the equation morphism*

$$X\xrightarrow{e}S(X+Y)\xrightarrow{\varphi_{X+Y}}\tilde{S}(X+Y)$$

has a solution

$$X\xrightarrow{e^\dagger}SY\xrightarrow{\varphi_Y}\tilde{S}Y$$

Proof. The following diagram

$$
\begin{tikzcd}
X \arrow[r, "e^{\dagger}"] \arrow[d, "e"] & SY \arrow[rr, "\varphi_Y"] & & \widetilde{S}Y \\
S(X+Y) \arrow[r, "{S[e^{\dagger},\eta_Y]}"] \arrow[drr, "{S[\varphi_Y\cdot e^{\dagger},\widetilde{\eta}_Y]}"'] \arrow[dd, "\varphi_{X+Y}"'] & SSY \arrow[u, "\mu_Y"'] \arrow[dr, "S\varphi_Y"] & & \\
 & & S\widetilde{S}Y \arrow[dr, "\varphi_{\widetilde{S}Y}"] & \\
\widetilde{S}(X+Y) \arrow[rrr, "{\widetilde{S}[\varphi_Y\cdot e^{\dagger},\widetilde{\eta}_Y]}"'] & & & \widetilde{S}\widetilde{S}Y \arrow[uuu, "\widetilde{\mu}_Y"']
\end{tikzcd}
$$

commutes: for the middle triangle notice that the following triangle

$$
\begin{tikzcd}
X+Y \arrow[r, "{[e^{\dagger},\eta_Y]}"] \arrow[dr, "{[\varphi_Y\cdot e^{\dagger},\widetilde{\eta}_Y]}"'] & SY \arrow[d, "\varphi_Y"] \\
 & \widetilde{S}Y
\end{tikzcd}
$$

commutes. $\square$

Remark 4.12.

(1) Elgot used a slightly more restrictive concept than guarded equation: his *ideal* equation morphism is an equation morphism $e : X \to S(X+Y)$ which factors through $\sigma_{X+Y} : S'(X+Y) \to S(X+Y)$. Note that all equations used in the main result, Theorem 4.14 below, are ideal, which shows that that result remains valid if complete iterativeness is defined by means of ideal, rather than guarded, equation morphisms.

(2) Given an ideal monad $\mathbb{S}$ with $S = S' + Id$ an *ideal transformation* from a functor H to $\mathbb{S}$ is a natural transformation $H \to S$ which factors through $\sigma : S' \to S$. Example: $\tau^* : H \to T$ of Notation 3.3 is ideal.

Lemma 4.13. *For every ideal equation morphism the solution is also ideal*, *i.e.*, *it factors through* σ_Y.

Proof. Given

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e'"'] & S(X+Y) \\
 & S'(X+Y) \arrow[u, "\sigma_{X+Y}"']
\end{tikzcd}
$$

consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "e^\dagger"] \arrow[dd, "e'"'] & & SY \\
& S'Y \arrow[ur, "\sigma_Y"] & \\
S'(X+Y) \arrow[r, "{S'[e^\dagger,\eta_Y]}"'] \arrow[d, "\sigma_{X+Y}"'] & S'SY \arrow[u, "\mu'_Y"] \arrow[dr, "\sigma_{SY}"'] & \\
S(X+Y) \arrow[rr, "{S[e^\dagger,\eta_Y]}"'] & & SSY \arrow[uuu, "\mu_Y"']
\end{tikzcd}$$

□

Theorem 4.14 (**Free completely iterative monads**). *For every iteratable endofunctor H the monad $\mathbb{T}$ of Corollary* 2.18 *is a free completely iterative monad on H.*

More precisely: the natural transformation $\tau^* : H \to T$ is ideal, and given a completely iterative monad $\mathbb{S} = (S, \eta^S, \mu^S)$ and an ideal transformation $\lambda : H \to S$ then there exists a unique ideal monad morphism $\bar{\lambda} : \mathbb{T} \to \mathbb{S}$ for which the following triangle

$$\begin{tikzcd}
H \arrow[r, "\tau^*"] \arrow[dr, "\lambda"'] & T \arrow[d, "\bar{\lambda}"] \\
& S
\end{tikzcd}$$

commutes.

Remark 4.15.

(1) Since $\sigma : S' \to S$, being a coproduct injection, is a (pointwise) monomorphism, the last condition on the ideal morphism $\bar{\lambda} = \bar{\lambda}' + id$ is equivalent to stating that for $\bar{\lambda}' : HT \to S'$ the following triangle

$$\begin{tikzcd}
H \arrow[r, "H\eta"] \arrow[dr, "\lambda'"'] & HT \arrow[d, "\bar{\lambda}'"] \\
& S'
\end{tikzcd}$$

commutes.

(2) Categorically, the statement of the theorem says that every iteratable functor H in $[\mathscr{A}, \mathscr{A}]$ has a universal arrow w.r.t. the forgetful functor

$$U : \mathsf{CIM}(\mathscr{A}) \to [\mathscr{A}, \mathscr{A}]$$

of the category $\mathsf{CIM}(\mathscr{A})$ of all completely iterative monads and ideal morphisms. Beware! The functor U assigns to every completely iterative monad $\mathbb{S} = (S, \eta^S, \mu^S)$ the functor S', not S. This choice of U corresponds to the requirement that $\lambda : H \to S$ be an *ideal* transformation.

(3) The assumption that H be iteratable is fundamental: it has been proved in [23] that every endofunctor generating a free completely iterative monad is iteratable.

Proof. I. *Uniqueness of* $\bar{\lambda}$.

Observe that in our monad $\mathbb{T}$ the following equation morphism

$$HTX \xrightarrow{H\eta_{TX}} HTTX \xrightarrow{\tau_{TX}} TTX = T(HTX + X)$$

is guarded. Its solution is simply

$$\tau_X : HTX \to TX.$$

In fact, the following diagram

$$\begin{tikzcd}
HTX \arrow[rr, "\tau_X"] \arrow[d, "H\eta_{TX}"'] \arrow[dr, equal] & & TX \\
 & HTX \arrow[ur, "\tau_X"] & \\
HTTX \arrow[ur, "H\mu_X"] \arrow[d, "\tau_{TX}"'] & & \\
TTX \arrow[rr, "{T[\tau_X, \eta_X] = id}"'] & & TTX \arrow[uuu, "\mu_X"']
\end{tikzcd}$$

commutes.

Suppose a monad morphism $\bar{\lambda}: \mathbb{T} \to \mathbb{S}$ as above is given. By Lemma 4.11, the following equation morphism

$$\begin{tikzcd}
HTX \arrow[r, "H\eta_{TX}"] \arrow[drr, "\lambda_{TX}"'] & HTTX \arrow[r, "\tau_{TX}"] & TTX \arrow[d, "\bar{\lambda}_{TX}"] \\
 & & STX = S(HTX + X)
\end{tikzcd}$$

has the solution

$$\bar{\lambda}_X \cdot \tau_X : HTX \to SX,$$

and since λ_{TX} is ideal, the solution is unique. This determines the left-hand component of $\bar{\lambda}_X : HTX + X \to SX$, and the right-hand one is clear from $\bar{\lambda}_X \cdot \eta_X = \eta^S_X$.

Shorter: we have the formula

$$\bar{\lambda}_X = [(\lambda_{TX})^\dagger, \eta^S_X]. \tag{4.1}$$

II. *Existence of* $\bar{\lambda}$. Our task is to show that, given λ, formula (4.1) defines an ideal monad morphism $\bar{\lambda}: \mathbb{T} \to \mathbb{S}$ with $\lambda = \bar{\lambda} \cdot \tau^*$.

(a) Naturality of $\bar{\lambda}_X$: given a morphism $f : X \to Y$ we want to show the commutativity of the following square

$$\begin{tikzcd} TX = HTX + X \arrow[r, "Tf = HTf + f"] \arrow[d, "{[(\lambda_{TX})^\dagger, \eta^S_X]}"'] & TY = HTY + Y \arrow[d, "{[(\lambda_{TY})^\dagger, \eta^S_Y]}"] \\ SX \arrow[r, "Sf"'] & SY \end{tikzcd}$$

The right-hand components are clear. For the left-hand components we use the following, easily established, fact:

> Given a guarded equation morphism $e : Z \to T(Z + X)$ then also $e' = T(id + f) \cdot e : X \to T(Z + Y)$ is guarded, and $(e')^\dagger = Tf \cdot e^\dagger$, for every morphism $f : X \to Y$.

Apply this to $e = \lambda_{TX}$: we conclude that in the desired square

$$\begin{tikzcd} HTX \arrow[r, "HTf"] \arrow[d, "(\lambda_{TX})^\dagger"'] & HTY \arrow[d, "(\lambda_{TY})^\dagger"] \\ SX \arrow[r, "Sf"'] & SY \end{tikzcd}$$

the lower passage is a solution of $e' = S(id_{HTX} + f) \cdot \lambda_{TX}$. It suffices to show that the upper passage also solves e'. This is true because the following diagram

$$\begin{tikzcd} HTX \arrow[r, "HTf"] \arrow[d, "\lambda_{TX}"'] & HTY \arrow[r, "\tau_Y"] \arrow[d, "\lambda_{TY}"'] & TY \arrow[r, "\bar{\lambda}_Y"] & SY \\ STX \arrow[r, "STf"'] \arrow[d, equal] & STY = S(HTY + Y) \arrow[rr, "{S[\bar{\lambda}_Y \cdot \tau_Y, \eta^S_Y]}"'] & & SSY \arrow[u, "\mu^S_Y"'] \arrow[dd, equal] \\ S(HTX + X) \arrow[ur, "S(HTf + f)"'] \arrow[d, "S(id_{HTX} + f)"'] & & & \\ S(HTX + Y) \arrow[rrr, "{S[\bar{\lambda}_Y \cdot \tau_Y \cdot HTf, \eta^S_Y]}"'] & & & SSY \end{tikzcd}$$

commutes. In fact, the upper right-hand square commutes due to the fact that λ_{TY} has solution $\bar{\lambda}_Y \tau_Y$, see (4.1). To see that the lower square commutes, extract S and observe that the two components obviously commute.

(b) Equality $\lambda = \bar{\lambda} \cdot \tau^*$. This follows from the next commutative diagram (where we use $\bar{\lambda}_X \cdot \tau_X = (\lambda_{TX})^\dagger$):

$$
\begin{tikzcd}
HX \arrow[rr, "\tau^*_X"] \arrow[dr, "H\eta_X"'] \arrow[ddd, "\lambda_X"'] & & TX \arrow[r, "\bar{\lambda}_X"] & SX \\
& HTX \arrow[ur, "\tau_X"'] \arrow[d, "\lambda_{TX}"] & & \\
& STX \arrow[drr, "S[\bar{\lambda}_X \cdot \tau_X, \eta^S_X]"] & & \\
SX \arrow[ur, "S\eta_X"] \arrow[rrr, "S\eta^S_X"'] & & & SSX \arrow[uuu, "\mu^S_X"']
\end{tikzcd}
$$

From $\mu^S \cdot S\eta^S = id$ we conclude that $\lambda = \bar{\lambda} \cdot \tau^*$.

(c) $\bar{\lambda}$ is an ideal monad homomorphism. In fact, since λ is an ideal transformation, say $\lambda = \sigma \cdot \lambda'$ (where λ' is unique and natural, since σ, being a coproduct injection, is pointwise monomorphic), we have for $(\lambda_{TX})^\dagger$ the following diagram

$$
\begin{tikzcd}
HTX \arrow[rrr, "(\lambda_{TX})^\dagger"] \arrow[dd, "\lambda'_{TX}"'] & & & SX \\
& & S'X \arrow[ur, "\sigma_X"] & \\
S'TX \arrow[rr, "S'[(\lambda_{TX})^\dagger, \eta^S_X]"'] \arrow[d, "\sigma_{TX}"'] & & S'SX \arrow[u, "\mu'^S_X"] \arrow[dr, "\sigma_{SX}"'] & \\
STX \arrow[rrr, "S[(\lambda_{TX})^\dagger, \eta^S_X]"'] & & & SSX \arrow[uuu, "\mu^S_X"']
\end{tikzcd}
$$

Put

$$\bar{\lambda}'_X = \mu'^S_X \cdot S'[(\lambda_{TX})^\dagger, \eta^S_{TX}] \cdot \lambda'_{TX} : HTX \to S'X$$

to obtain a natural transformation

$$\bar{\lambda}' : HT \to S' \quad \text{with } \bar{\lambda} = \bar{\lambda}' + id.$$

It remains to verify that $\bar{\lambda}$ is a monad morphism. Since $\eta : Id \to T$ is a coproduct injection, we have

$$\bar{\lambda}_X \cdot \eta_X = [(\lambda_{TX})^\dagger, \eta^S_X] \cdot \eta_X = \eta^S_X.$$

Next, we are to show that the following square

$$
\begin{tikzcd}
HTT + T = TT \arrow[r, "\bar{\lambda}T"] \arrow[d, "\mu"'] & ST \arrow[r, "S\bar{\lambda}"] & SS \arrow[d, "\mu^S"] \\
T \arrow[rr, "\bar{\lambda}"'] & & S
\end{tikzcd}
$$

commutes. The right-hand components are both equal to $\bar{\lambda}: T \to S$: for the lower passage this follows from $\mu \cdot \eta T = id$, for the upper one from

$$(\mu^S \cdot S\bar{\lambda} \cdot \bar{\lambda}T) \cdot \eta T = \mu^S \cdot S\bar{\lambda} \cdot \eta^S T = \mu^S \cdot \eta^S S \cdot \bar{\lambda} = \bar{\lambda}.$$

Thus, we are to establish the commutativity of the left-hand components:

$$\begin{tikzcd}
HTTZ \arrow[r, "(\lambda_{TTZ})^{\dagger}"] \arrow[d, "\tau_{TZ}"'] & STZ \arrow[r, "S\bar{\lambda}_Z"] & SSZ \arrow[dd, "\mu^S_Z"] \\
TTZ \arrow[d, "\mu_Z"'] & & \\
TZ \arrow[rr, "\bar{\lambda}_Z"'] & & SZ
\end{tikzcd} \tag{4.2}$$

In the following proof of (4.2) we put $\tilde{\lambda}_Z = \lambda_{TZ}{}^{\dagger} : HTZ \to SZ$ and

$$f \equiv HTTZ + HTZ \xrightarrow{[\lambda_{TTZ}, S\mathsf{inr} \cdot \tilde{\lambda}_Z]} S(HTTZ + HTZ + Z) = STTZ.$$

This is an equation morphism (with variables $X = HTTZ + HTZ$ and parameters Z) and it is guarded. In fact, use Lemma 4.13 on $e = \lambda_{TZ}$ to get a morphism e' with $\tilde{\lambda}_Z = \sigma_{TTZ} e'$, then the following triangle

$$\begin{tikzcd}
HTTZ + HTZ \arrow[r, "f"] \arrow[dr, "{[\lambda'_{TTZ}, S'\mathsf{inr} \cdot e']}"'] & STTZ \\
 & S'TTZ \arrow[u, "\sigma_{TTZ}"']
\end{tikzcd}$$

commutes. We are going to prove that the solution of f is given as follows

$$f^{\dagger} \equiv HTTZ + HTZ \xrightarrow{[\bar{\lambda}_Z \cdot \mu_Z \cdot \tau_{TZ}, \tilde{\lambda}_Z]} SZ. \tag{4.3}$$

That is, we will verify that the following square

$$\begin{tikzcd}
HTTZ + HTZ \arrow[r, "{[\bar{\lambda}_Z \cdot \mu_Z \cdot \tau_{TZ}, \tilde{\lambda}_Z]}"] \arrow[d, "{[\lambda_{TTZ}, S\mathsf{inr} \cdot \tilde{\lambda}_Z]}"'] & SZ \\
S(HTTZ + HTZ + Z) \arrow[r, "{S[\bar{\lambda}_Z \cdot \mu_Z \cdot \tau_{TZ}, \tilde{\lambda}_Z, \eta^S_Z]}"'] & SSZ \arrow[u, "\mu^S_Z"']
\end{tikzcd}$$

commutes. It is sufficient to concentrate on the left-hand components (the right-hand ones are both $\tilde{\lambda}_Z$ due to $\mu^S_Z \cdot S\eta^S_Z = id$). For this we consider the following diagram:

$$\begin{tikzcd}
HTTZ \arrow[r, "\tau_{TZ}"] \arrow[drr, "H\mu_Z"] \arrow[dddrr, "{H[\bar\lambda_Z\cdot\mu_Z\cdot\tau_{TZ},\tilde\lambda_Z,\eta^S_Z]}"] \arrow[ddd, "\lambda_{TTZ}"'] & TTZ \arrow[r, "\mu_Z"] & TZ \arrow[rr, "\bar\lambda_Z"] & & SZ \\
& & HTZ \arrow[u, "\tau_Z"'] \arrow[urr, "\tilde\lambda_Z=\lambda_{TZ}{}^\dagger"'] \arrow[dd, "H\bar\lambda_Z"'] \arrow[dr, "\lambda_{TZ}"] & & \\
& & & STZ \arrow[ddr, "{S[\tilde\lambda_Z,\eta^S_Z]}"] & \\
STTZ \arrow[d, equal] & & HSZ \arrow[drr, "\lambda_{SZ}"'] & & \\
S(HTTZ+HTZ+Z) \arrow[rrrr, "{S[\bar\lambda_Z\cdot\mu_Z\cdot\tau_{TZ},\tilde\lambda_Z,\eta^S_Z]}"'] & & & & SSZ \arrow[uuuu, "\mu^S_Z"']
\end{tikzcd}$$

All parts commute: this is obvious, except for the middle triangle. We show that this commutes even if we delete H. Use $TTZ = HTTZ + HTZ + Z$ with coproduct injections τ_{TZ}, $\eta_{TZ}\cdot\tau_Z$ and $\eta_{TZ}\cdot\eta_Z$ respectively: the left-hand components are $\bar\lambda_Z\cdot\mu_Z\cdot\tau_{TZ}$, the middle ones are $\tilde\lambda_Z = \bar\lambda_Z\cdot\mu_Z\cdot\eta_{TZ}\cdot\tau_Z = \bar\lambda_Z\cdot\tau_Z$, and the right-hand ones are $\eta^S_Z = \bar\lambda_Z\cdot\eta_Z$. This proves (4.3).

But the morphism f also has the following solution

$$f^\dagger \equiv HTTZ + HTZ \xrightarrow{[\mu^S_Z\cdot S\bar\lambda_Z\cdot\tilde\lambda_{TZ},\tilde\lambda_Z]} SZ. \tag{4.4}$$

In fact, the following square

$$\begin{tikzcd}
HTTZ+HTZ \arrow[r, "{[\mu^S_Z\cdot S\bar\lambda_Z\cdot\tilde\lambda_{TZ},\tilde\lambda_Z]}"] \arrow[d, "{f=[\lambda_{TTZ},S\mathsf{inr}\cdot\tilde\lambda_Z]}"'] & SZ \\
S(HTTZ+HTZ+Z) \arrow[r, "{S[\mu^S_Z\cdot S\bar\lambda_Z\cdot\tilde\lambda_{TZ},\tilde\lambda_Z,\eta^S_Z]}"'] & SSZ \arrow[u, "\mu^S_Z"']
\end{tikzcd}$$

commutes: the right-hand components commute trivially (as above) and for the left-hand ones consider the following diagram:

$$\begin{tikzcd}
HTTZ \arrow[rr, "\tilde\lambda_{TZ}=\lambda_{TTZ}{}^\dagger"] \arrow[dd, "\lambda_{TTZ}"'] & & STZ \arrow[r, "S\bar\lambda_Z"] & SSZ \arrow[r, "\mu^S_Z"] & SZ \\
& & SSTZ \arrow[u, "\mu^S_{TZ}"] \arrow[r, "SS\bar\lambda_Z"] & SSSZ \arrow[u, "\mu^S_{SZ}"'] \arrow[ddr, "S\mu^S_Z"] & \\
STTZ \arrow[d, equal] & S(HTTZ+TZ) \arrow[ur, "{S[\tilde\lambda_{TZ},\eta^S_{TZ}]}"] & & & \\
S(HTTZ+HTZ+Z) \arrow[ur, equal] \arrow[rrrr, "{S[\mu^S_Z\cdot S\bar\lambda_Z\cdot\tilde\lambda_{TZ},\tilde\lambda_Z,\eta^S_Z]}"'] & & & & SSZ \arrow[uuu, "\mu^S_Z"']
\end{tikzcd}$$

It commutes: this is obvious for all parts except the lower part, for which we delete S to obtain

$$\begin{tikzcd} HTTZ + HTZ + Z \arrow[r, equal] \arrow[ddr, "{[\mu^S_Z \cdot S\bar{\lambda}_Z \cdot \bar{\lambda}_{TZ}, \tilde{\lambda}_Z, \eta^S_Z]}"'] & HTTZ + TZ \arrow[r, "{[\tilde{\lambda}_{TZ}, \eta^S_{TZ}]}"] \arrow[dd, "{[\mu^S_Z \cdot S\bar{\lambda}_Z \cdot \tilde{\lambda}_{TZ}, \bar{\lambda}_Z]}"] & STZ \arrow[d, "S\bar{\lambda}_Z"] \\ & & SSZ \arrow[dl, "\mu^S_Z"] \\ & SZ & \end{tikzcd}$$

which commutes since $\mu^S \cdot \eta^S S = id$.

Since solutions are unique, the two solutions of f above are equal. The equality of the right-hand components in (4.3) and (4.4) is precisely the fact that (4.2) above commutes. This concludes the proof of (c). □

Remark 4.16. The above theorem holds, more generally, in categories $\mathscr{A}$ with binary coproducts also when we do not assume that coproduct injections are monomorphisms. However, we have to define ideal equations and solutions differently, then. In the present approach, a guarded equation morphism $e : X \to S(X + Y)$ is one that factors as

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & S(X+Y) \\ & S'(X+Y)+Y \arrow[u, "{[\sigma_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]}"'] \end{tikzcd}$$

and, as long as coproduct injections are monomorphisms, we do not need a name for the factorizing arrow. Now generally, we can introduce guarded equation morphisms as arrows $f : X \to S'(X+Y)+Y$. And a solution of f is, then, defined as a morphism $f^{\dagger'} : X \to S'Y + Y$ such that the following diagram

$$\begin{tikzcd} X \arrow[r, "f^{\dagger'}"] \arrow[d, "f"'] & S'Y + Y \\ S'(X+Y)+Y \arrow[r, "{S'[\sigma_Y \cdot f^{\dagger'}, \eta_Y]+Y}"'] & S'SY + Y \arrow[u, "\mu'_Y + Y"'] \end{tikzcd}$$

commutes. An ideal monad $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ is called completely iterative if every guarded equation arrow f has a unique solution $f^{\dagger'}$.

In this greater generality it remains true that for every iteratable functor H

(i) the monad $\mathbb{T}$ is completely iterative,

and

(ii) $\mathbb{T}$ is a free completely iterative monad on H.

The latter means, now, that for every completely iterative monad $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ and every natural transformation $\lambda' : H \to S'$ there exists a unique monad morphism

$$\bar{\lambda} : \mathbb{T} \to \mathbb{S}$$

such that
(a) $\bar{\lambda}$ is ideal, i.e., has the form $\bar{\lambda} = \bar{\lambda}' + id$ for $\bar{\lambda}' : HT \to S'$,
and
(b) the triangle of Remark 4.15

$$\begin{tikzcd} H \arrow[r, "H\eta"] \arrow[dr, "\lambda'"'] & HT \arrow[d, "\bar{\lambda}'"] \\ & S' \end{tikzcd}$$

commutes.

In other words, the functor U of Remark 4.15 has a universal arrow for every iteratable H. The proof is the same as the proof of Theorem 4.14 above.

5. A completely iterative monoid of an object

We can view the procedure of forming the monad $\mathbb{T}$ of Section 2 globally by working, instead of in the given category $\mathscr{A}$, in the endofunctor category $[\mathscr{A}, \mathscr{A}]$. Here H is an object. If H is iteratable, then 2.21 defines another object, T, together with a morphism (natural transformation)

$$\alpha : T \to HT + Id.$$

This is a coalgebra of the functor

$$\hat{H} : [\mathscr{A}, \mathscr{A}] \to [\mathscr{A}, \mathscr{A}]$$

defined on objects by

$$\hat{H}(S) = H \cdot S + Id \quad (\text{for all } S : \mathscr{A} \to \mathscr{A})$$

and analogously on morphisms. We prove below that T is a final $\hat{H}$-coalgebra.

Within the realm of locally small categories (i.e., with small hom-sets) with coproducts this global approach is equivalent to that of Section 2:

Proposition 5.1. *Let $\mathscr{A}$ be a locally small category with coproducts. For every endofunctor H, the following are equivalent:*
(1) *H is an iteratable object of $[\mathscr{A}, \mathscr{A}]$, i.e., a final $\hat{H}$-coalgebra exists.*
(2) *H is an iteratable endofunctor, i.e., all final $(H(\_) + X)$-coalgebras exist.*

Remark.
(i) More detailed: if T is a final $\hat{H}$-coalgebra, we prove that TX is a final coalgebra of $H(\_) + X$ for all objects X. And vice versa.
(ii) The proof that 2 implies 1 holds for all categories $\mathscr{A}$ with binary coproducts.

For the proof that 1 implies 2, only copowers indexed by hom-sets of the category $\mathscr{A}$ are used. Thus the proposition also holds e.g. for the category $\mathscr{A} = \mathsf{Set}_{fin}$ of finite sets, and for any poset $\mathscr{A}$ with binary joins.

Proof. 1 implies 2: For every pair X, Y of objects in $\mathscr{A}$ denote by $K_{X,Y}$ the following endofunctor

$$K_{X,Y}A = \coprod_{\mathscr{A}(X,A)} Y$$

for objects A, analogously for morphisms. This is just a left Kan extension of Y, considered as a functor $\mathbf{1} \to \mathscr{A}$, along the functor $X : \mathbf{1} \to \mathscr{A}$. In fact, for every functor $P : \mathscr{A} \to \mathscr{A}$ we have a bijection

$$\frac{K_{X,Y} \to P}{Y \to PX}$$

natural in P, which to every natural transformation $\varphi : K_{X,Y} \to P$ assigns the composite

$$Y \xrightarrow{u} \coprod_{\mathscr{A}(X,X)} Y \xrightarrow{\varphi_X} PX,$$

where u is the id_X-injection. Conversely, given a morphism $f : Y \to PX$, the corresponding natural transformation $f^@ : K_{X,Y} \to P$ has components

$$f^@_A : \left(\coprod_{h : X \to A} Y \right) \to PA$$

determined by $Y \xrightarrow{f} PX \xrightarrow{Ph} PA$.

Let $\alpha : T \to HT + Id$ be a final $\hat{H}$-coalgebra. We will show that

$$\alpha_X : TX \to HTX + X$$

is a final $(H(\_) + X)$-coalgebra for every X.

In fact, for every $(H(\_) + X)$-coalgebra

$$b : Y \to HY + X$$

when composing b with

$$Hu + id : HY + X \to H \left(\coprod_{\mathscr{A}(X,X)} Y \right) + X = (\hat{H}K_{X,Y})X$$

we obtain a morphism

$$\bar{b} : Y \to (\hat{H}K_{X,Y})X$$

which by the above adjointness yields an $\hat{H}$-coalgebra

$$\bar{b}^@ : K_{X,Y} \to \hat{H}K_{X,Y}.$$

Let φ be the unique homomorphism of $\hat{H}$-coalgebras

$$\begin{tikzcd} K_{X,Y} \arrow[r, "\bar{b}^@"] \arrow[d, "\varphi"'] & \widehat{H}K_{X,Y} \arrow[d, "\widehat{H}\varphi"] \\ T \arrow[r, "\alpha"'] & \widehat{H}T \end{tikzcd}$$

Then $\varphi = f^{@}$ for a unique $f : Y \to TX$, and the commutativity of the above square yields the commutativity of

$$\begin{tikzcd} Y \arrow[r, "b"] \arrow[d, "f"'] & HY + X \arrow[d, "Hf + id_X"] \\ TX \arrow[r, "\alpha_X"'] & HTX + X \end{tikzcd}$$

2 implies 1: It has been noted above (see Corollary 2.18) that if $\alpha_X : TX \to HTX + X$ denotes a final coalgebra for $H(\_) + X$, then the assignment $X \mapsto TX$ can be extended to a functor $T : \mathscr{A} \to \mathscr{A}$.

Analogously one can show that the collection of all α_X's constitutes a natural transformation $\alpha : T \to H \cdot T + Id$. Thus, α makes T an $\hat{H}$-coalgebra.

To verify that α is indeed a final $\hat{H}$-coalgebra, consider any coalgebra $\beta : S \to H \cdot S + Id$. For each X in $\mathscr{A}$ there exists a unique morphism $f_X : SX \to TX$ such that the following square

$$\begin{tikzcd} SX \arrow[r, "\beta_X"] \arrow[d, "f_X"'] & HSX + X \arrow[d, "Hf_X + id"] \\ TX \arrow[r, "\alpha_X"'] & HTX + X \end{tikzcd}$$

commutes. It is easy to show that the collection of f_X's is natural in X and that it defines a unique natural transformation $f : S \to T$ for which the following square

$$\begin{tikzcd} S \arrow[r, "\beta"] \arrow[d, "f"'] & HS + Id \arrow[d, "Hf + id"] \\ T \arrow[r, "\alpha"'] & HT + Id \end{tikzcd}$$

commutes. □

Remark 5.2. In Example 2.15 we have formulated properties of a category $\mathscr{A}$ so that every continuous endofunctor H be iteratable. Let us observe that the corresponding completely iterative monad, T, is also continuous: by Proposition 5.1, T is a final $\hat{H}$-coalgebra. Now $\hat{H}$ is an endofunctor of the category $[\mathscr{A}, \mathscr{A}]$ which also satisfies 1.–3, of Example 2.15. Consequently, we have the formula

$$T = \lim_{n < \omega} \hat{H}^n(C_1),$$

where C_1 (the constant endofunctor of $\mathscr{A}$ with value 1) is a terminal object of $[\mathscr{A}, \mathscr{A}]$. Since each $\hat{H}(C_1)$ is easily seen to be continuous, we obtain T as a limit of continuous functors—thus, T is continuous.

Remark 5.3. For every category $\mathscr{A}$ the endofunctor category $[\mathscr{A}, \mathscr{A}]$ is monoidal with composition as a tensor product and Id as a unit. Moreover composition distributes over coproducts on the left: $(H+K)\cdot L=(H\cdot L)+(K\cdot L)$. This leads us to consider an arbitrary monoidal category

$$(\mathscr{B},\otimes,I)$$

with coherence isomorphisms (for all H, K, L in $\mathscr{B}$):

$$l_H : I\otimes H \to H, \quad r_H : H\otimes I \to H$$

and

$$a_{H,K,L} : H\otimes(K\otimes L)\to(H\otimes K)\otimes L$$

satisfying the usual laws, and which is left-distributive in the following sense:

Definition 5.4.
(1) A monoidal category is called *left-distributive* if it has binary coproducts and the canonical morphisms

$$d_{H,K,L} : (H\otimes L)+(K\otimes L)\to(H+K)\otimes L$$

are all isomorphisms.
(2) An object H of a monoidal category $\mathscr{B}$ is said to be *iteratable* provided that the endofunctor $\hat{H}:\mathscr{B}\to\mathscr{B}$ defined by

$$\hat{H}(B)=H\otimes B+I$$

has a final coalgebra.
(3) A left distributive monoidal category with each object iteratable is called an *iteratable category*.

Examples 5.5.
(1) The category

$$\mathit{Cont}[\mathsf{Set},\mathsf{Set}]$$

of continuous endofunctors (i.e., those preserving ω^{op}-limits) of Set is iteratable: we know that continuous functors are closed under
(a) composition (here: a tensor product)
(b) identity functor (here: unit I)
and
(c) finite coproducts,
thus $\mathit{Cont}[\mathsf{Set},\mathsf{Set}]$ is a distributive monoidal subcategory of $[\mathsf{Set},\mathsf{Set}]$. Now, every continuous functor is iteratable, and by Remark 5.2 the completely iterative monad is also continuous; therefore $\mathit{Cont}[\mathsf{Set},\mathsf{Set}]$ is an iteratable category.
(2) More in general, $\mathit{Cont}[\mathscr{A},\mathscr{A}]$ is an iteratable category for every locally small category $\mathscr{A}$ satisfying conditions 1.–3, of Example 2.15.
(3) The category

$$\mathit{Fin}[\mathsf{Set},\mathsf{Set}]$$

of all finitary endofunctors of Set (i.e., those preserving filtered colimits) is iteratable. In fact, finitary functors are closed under composition, identity functor, and finite coproducts, thus, $\mathit{Fin}[\mathsf{Set},\mathsf{Set}]$ is a distributive monoidal subcategory of $[\mathsf{Set},\mathsf{Set}]$.

A completely iterative monad $\mathbb{T}$ of a finitary functor H exists, since finitary functors always have final coalgebras, see [11], Theorem 1.2, and each $H(\_)+X$ is clearly finitary. However, this monad is seldom finitary, see Example 2.22(2).

We can form a finitary part $\mathbb{T}_{\mathit{fin}}$ of every monad $\mathbb{T}$ on Set (see [21]): it is obtained by restricting the underlying functor T to the full subcategory $\mathsf{Set}_{\mathit{fin}}$ of finite sets, and then forming a left Kan extension of $T/\mathsf{Set}_{\mathit{fin}}$ along the embedding of $\mathsf{Set}_{\mathit{fin}}$ in Set.

It is easy to verify that $\mathbb{T}_{\mathit{fin}}$ is a final coalgebra of the endofunctor $H\cdot(\_)+\mathit{Id}$ of $\mathit{Fin}[\mathsf{Set},\mathsf{Set}]$. In fact, given any coalgebra

$$S \to H\cdot S + \mathit{Id}$$

(with S finitary, of course) the unique $\hat{H}$-homomorphism $f: S\to T$ is easily seen to have a factorization through the canonical morphism $m: T_{\mathit{fin}}\to T$. That is, we have a unique $f': S\to T_{\mathit{fin}}$ with $f=m\cdot f'$. And f' is the unique homomorphism of coalgebras of the functor $H\cdot(\_)+\mathit{Id}$, considered as an endofunctor of $\mathit{Fin}[\mathsf{Set},\mathsf{Set}]$.

Example: the functor

$$H: \mathsf{Set}\to\mathsf{Set} \quad \text{with } HZ = Z\times Z$$

has the completely iterative monad $\mathbb{T}$ where TX are all binary trees with leaves indexed in X. And $\mathbb{T}_{\mathit{fin}}$ is the finitary monad where $T_{\mathit{fin}}X$ are all binary trees with leaves indexed in a finite subset of X.

(4) More generally, if $\mathscr{A}$ is a locally finitely presentable category (see [8]) then $\mathit{Fin}[\mathscr{A},\mathscr{A}]$, the category of finitary endofunctors of $\mathscr{A}$, is iteratable. The argument is the same: we form a completely iterative monad $\mathbb{T}$ in $[\mathscr{A},\mathscr{A}]$, which exists by Theorem 1.2 in [11] (although formulated for Set, it holds in all locally presentable categories) and then take a finitary part $\mathbb{T}_{\mathit{fin}}$ just as in (3) above.

(5) Let $\mathscr{B}$ be a left distributive monoidal category having a terminal object 1 and limits of ω^{op}-chains which commute with both the tensor product and the binary coproduct. Then every object H is iteratable and T is a limit of the following countable chain:

$$1 \xleftarrow{!} H\otimes 1 + I \xleftarrow{H\otimes ! + id} H\otimes(H\otimes 1 + I) + I \xleftarrow{H\otimes(H\otimes ! + id)+id} \cdots$$

For example: the category of sets with a binary product as $\otimes$ and a terminal object I as a unit is an iteratable category: the (polynomial) functor

$$\hat{H}(Z) = H\times Z + I$$

has a final coalgebra

$$T = H^{\infty}$$

for every set H.

And the cartesian closed category Cat of all small categories is an iteratable category. Every small category H is iterable with

$$T = 1 + H + (H \times H) + \cdots + H^{\omega}$$

(6) Let H be an iteratable Abelian group (where we consider the category Ab of all Abelian groups with the usual tensor product). Then a final coalgebra of $\hat{H}$ is, as we show below in 5.8, a monoid in the given monoidal category—thus, in the present case

T is a ring.

Notation 5.6. For every iteratable object H we denote by T and $\alpha : T \to H \otimes T + I$ a final coalgebra of $\hat{H}$. By Lambek's Lemma, T is a coproduct of $H \otimes T$ and I. We denote the injections by

$$\tau : H \otimes T \to T \quad \text{and} \quad \eta : I \to T$$

where $\alpha^{-1} = [\tau, \eta]$.

This makes T into an algebra for the functor $H \otimes \_$. More generally, every object S of $\mathscr{B}$ yields an algebra

$$\tau_S \equiv H \otimes (T \otimes S) \xrightarrow{a_{H,T,S}} (H \otimes T) \otimes S \xrightarrow{\tau \otimes id_S} T \otimes S$$

(where $a_{H,T,S}$ is the associativity isomorphism). Put

$$\eta_S \equiv S \xrightarrow{r_S} I \otimes S \xrightarrow{\eta \otimes id_S} T \otimes S.$$

Substitution Theorem 5.7. *Let H be an iteratable object in a monoidal category $\mathscr{B}$. For every morphism*

$$s : S \to T$$

in $\mathscr{B}$ there is a unique homomorphism

$$\hat{s} : T \otimes S \to T$$

of algebras of type $H \otimes \_$ with

$$s = \hat{s} \cdot \eta_S.$$

Proof. This is quite analogous to the proof of Theorem 2.17. We turn the object $T \otimes S + T$ into an $\hat{H}$-coalgebra as follows:

$$T \otimes S + T \cong H \otimes T \otimes S + S + T \xrightarrow{id + [s, id]} H \otimes T \otimes S + T \cong$$

$$\cong H \otimes T \otimes S + H \otimes T + I \xrightarrow{[H \otimes id, H \otimes \mathsf{inr}] + id} H \otimes (T \otimes S + T) + I.$$

The unique homomorphism

$$f = [f_1, f_2] : T \otimes S + T \to T$$

of $\hat{H}$-coalgebras is the unique morphism of $\mathscr{B}$ which has the second component, f_2, an endomorphism of the final $\hat{H}$-coalgebra $\alpha: T \to H \otimes T + I$, thus,

$$f_2 = id,$$

and for the first component we get two commutative diagrams: one tells us that f_1 is a homomorphism of $(H \otimes \_)$-algebras, and the other one is as follows:

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[d, "s"'] & T \otimes S \arrow[ddd, "f_1"] \\
T \arrow[d, "\alpha"'] & \\
H \otimes T + I \arrow[d, "H \otimes f_2 + id"'] & \\
H \otimes T + I \arrow[r, "{[\tau,\eta]}"'] & T
\end{tikzcd}$$

Since $f_2 = id$, this diagram tells us that $f_1 \cdot \eta_S = s$, which proves the Substitution Theorem. $\square$

Corollary 5.8. *For every iteratable object H, a final $\hat{H}$-coalgebra T is a monoid with respect to*

$$\eta : I \to T$$

and

$$\mu = \widehat{id_T} : T \otimes T \to I.$$

Proof. In fact, the equality $\mu \cdot \eta_T = id$ follows from the definition of μ and the other two equalities defining monoids in $(\mathscr{B}, \otimes, I)$ easily follow from the uniqueness of $\hat{s}$. $\square$

Definition 5.9. The monoid of the above corollary is called a *completely iterative monoid generated by* an iteratable object H.

We now prove a remarkable property of iteratable categories $\mathscr{B}$: denote by

$$\mathscr{T} : \mathscr{B} \to \mathscr{B}$$

the functor assigning to every object H a completely iterative monoid generated by H. Then $\mathscr{T}$, as an object of $[\mathscr{B}, \mathscr{B}]$, is itself a completely iterative monoid: it is generated by $Id_{\mathscr{B}}$. Example: Set is an iteratable category, see Example 5.5(5), and the assignment $H \mapsto H^\infty$ is, as an object of $[\mathsf{Set}, \mathsf{Set}]$, itself a completely iterative monoid generated by Id.

For every monoidal category $\mathscr{B}$ we consider $[\mathscr{B}, \mathscr{B}]$ as a monoidal category (with the "pointwise" tensor product $P \otimes Q : H \mapsto P(H) \otimes Q(H)$ and the "pointwise" unit $C_I : H \mapsto I$).

Theorem 5.10. *Suppose that $(\mathscr{B}, \otimes, I)$ is an iteratable category. Then the following hold:*

(1) *The functor category $[\mathscr{B}, \mathscr{B}]$ is iteratable.*
(2) *The assignment of a completely iterative monoid to every object is an endofunctor of $\mathscr{B}$ which, as an object of $[\mathscr{B}, \mathscr{B}]$, is itself a completely iterative monoid generated by $Id_{\mathscr{B}}$.*

Proof. 1. First observe that $[\mathscr{B}, \mathscr{B}]$ is indeed a distributive monoidal category, since the required structure is transported pointwise from $\mathscr{B}$.

Consider now any functor $H : \mathscr{B} \to \mathscr{B}$. To show that the derived functor

$$\hat{H} = H \otimes (\_) + C_I : [\mathscr{B}, \mathscr{B}] \to [\mathscr{B}, \mathscr{B}]$$

has a final coalgebra, form, for each B in $\mathscr{B}$, a final coalgebra of the functor $H(B) \otimes (\_) + I$:

$$a_B : T(B) \to H(B) \otimes T(B) + I.$$

It is clear that there is a unique canonical way of making the assignment $B \mapsto T(B)$ functorial: consider any morphism $f : B \to C$ in $\mathscr{B}$ and define $T(f) : T(B) \to T(C)$ to be the unique morphism such that the following diagram

$$\begin{tikzcd} T(B) \arrow[r, "a_B"] \arrow[d, "T(f)"'] & H(B) \otimes T(B) + I \arrow[r, "H(f)\otimes T(B)+id"] & H(C) \otimes T(B) + I \arrow[d, "H(C)\otimes T(f)+id"] \\ T(C) \arrow[rr, "a_C"'] & & H(C) \otimes T(C) + I \end{tikzcd}$$

commutes. It is easy to show that this indeed defines a functor $T : \mathscr{B} \to \mathscr{B}$.

The collection of morphisms $a_B : T(B) \to H(B) \otimes T(B) + I$ is natural in B and thus defines a coalgebra for $H \otimes (\_) + C_I$:

$$a : T \to H \otimes T + C_I.$$

To show that a is a final coalgebra, consider any coalgebra

$$b : S \to H \otimes S + C_I.$$

For every B in $\mathscr{B}$ there exists a unique morphism $\lambda_B : S(B) \to T(B)$ such that the following square

$$\begin{tikzcd} S(B) \arrow[r, "b_B"] \arrow[d, "\lambda_B"'] & H(B) \otimes S(B) + I \arrow[d, "H(B)\otimes \lambda_B + id"] \\ T(B) \arrow[r, "a_B"'] & H(B) \otimes T(B) + I \end{tikzcd}$$

commutes. To show that the collection (λ_B) constitutes a natural transformation, observe that, for every $f: B \to C$, both

$$\lambda_C \cdot S(f): S(B) \to T(C) \quad \text{and} \quad T(f) \cdot \lambda_B : S(B) \to T(C)$$

are homomorphisms of $(H(C) \otimes (\_) + I)$-coalgebras from

$$(H(f) \otimes S(B) + id) \cdot b_B : S(B) \to H(C) \otimes S(B) + I$$

to

$$a_C : T(C) \to H(C) \otimes T(C) + I$$

and therefore they are equal.

We have formed a final coalgebra

$$a: T \to H \otimes T + C_I.$$

2. Put $\Phi(B) = T_B$ for every object B, where T_B denotes a completely iterative monoid generated by B, and extend the assignment $B \mapsto \Phi(B)$ to a functor $\Phi: \mathscr{B} \to \mathscr{B}$ as in the first part of the proof.

Let us now consider the functor

$$Id \otimes (\_) + C_I : [\mathscr{B}, \mathscr{B}] \to [\mathscr{B}, \mathscr{B}].$$

The collection of morphisms $a_B : \Phi(B) \to B \otimes \Phi(B) + I$ defines a coalgebra for $Id \otimes (\_) + C_I$:

$$a : \Phi \to Id \otimes \Phi + C_I$$

and it follows from the first part of the proof that this coalgebra is final.

To conclude the proof use the monoidal version of the existence of a completely iterative monad from Corollary 5.8. □

Finally, we show that if H is an iteratable object (with the corresponding monoid T) of a left distributive monoidal category $\mathscr{B}$, then guarded equation morphisms have unique solutions.

Definition 5.11. Let H be an iteratable object of a left distributive category $\mathscr{B}$ with a completely iterative monoid T. Every morphism of the form

$$e: S \to T \otimes (S + I) \qquad S \text{ an object of } \mathscr{B}$$

is called an *equation morphism*. It is called *guarded* if it factors through $[\tau \otimes (S+I), (\eta \otimes (S+I)) \cdot \mathsf{inr}]$:

$$\begin{tikzcd}
S \arrow[r, "e"] \arrow[dr, dotted] & T \otimes (S+I) \\
& H \otimes T \otimes (S+I) + I \arrow[u, "{[\tau\otimes(S+I),(\eta\otimes(S+I))\cdot \mathsf{inr}]}"']
\end{tikzcd}$$

Solution Theorem 5.12. *For every iteratable object H every guarded equation morphism $e: S \to T \otimes (S+I)$ has a unique solution, i.e., there exists a unique morphism $e^\dagger : S \to T$ such that the following diagram*

$$\begin{tikzcd} S \arrow[r, "e^\dagger"] \arrow[d, "e"'] & T \\ T \otimes (S+I) \arrow[r, "T\otimes[e^\dagger,\eta]"'] & T \otimes T \arrow[u, "\mu"'] \end{tikzcd}$$

commutes.

Proof. The proof is analogous to the proof of Corollary 3.8. □

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