

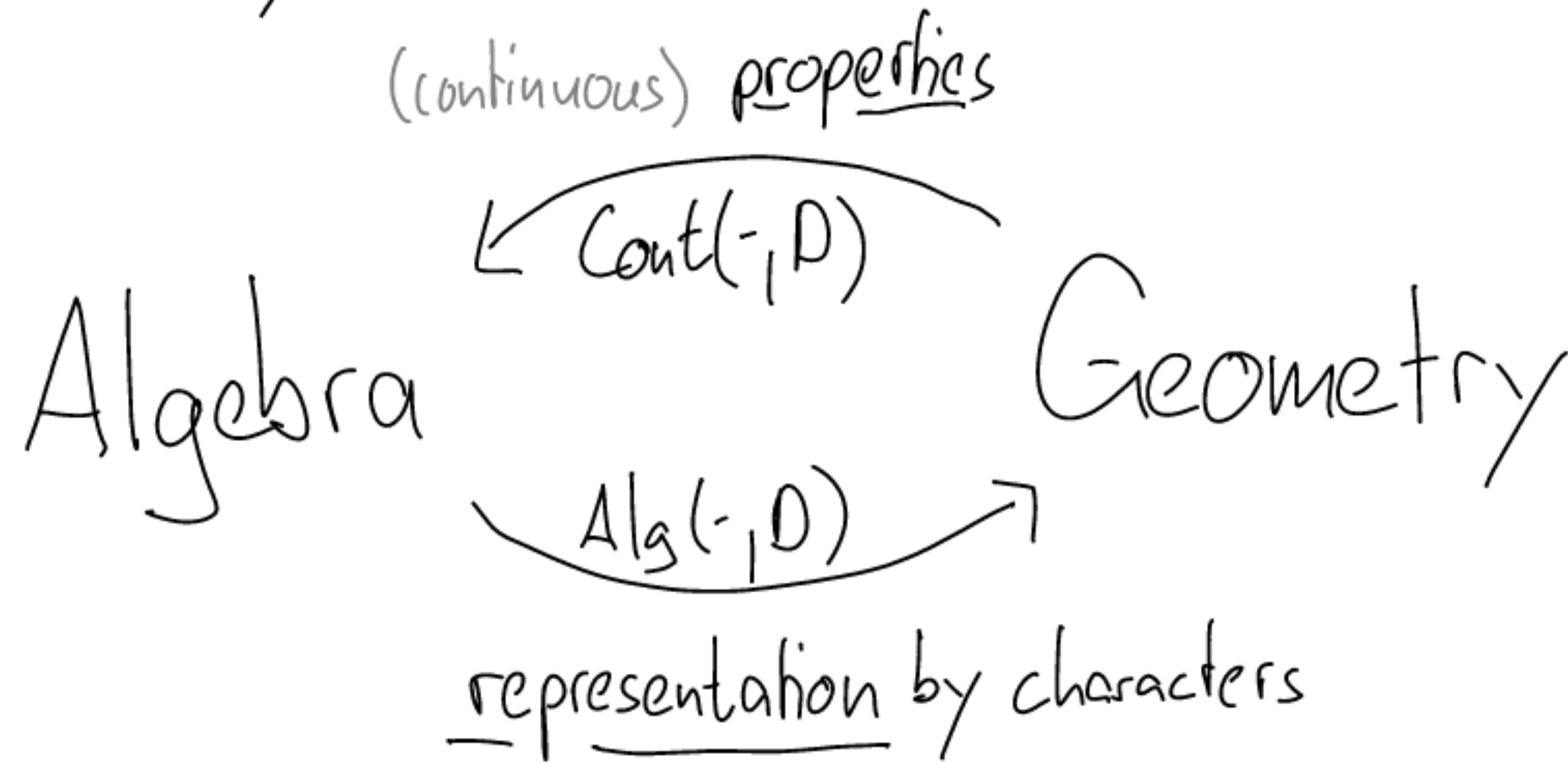
The Stone Duality of Stone Duality

Fabian Lenke, soon-to-be joined work w. Henning & Stefan?
(FAU Erlangen-Nürnberg)

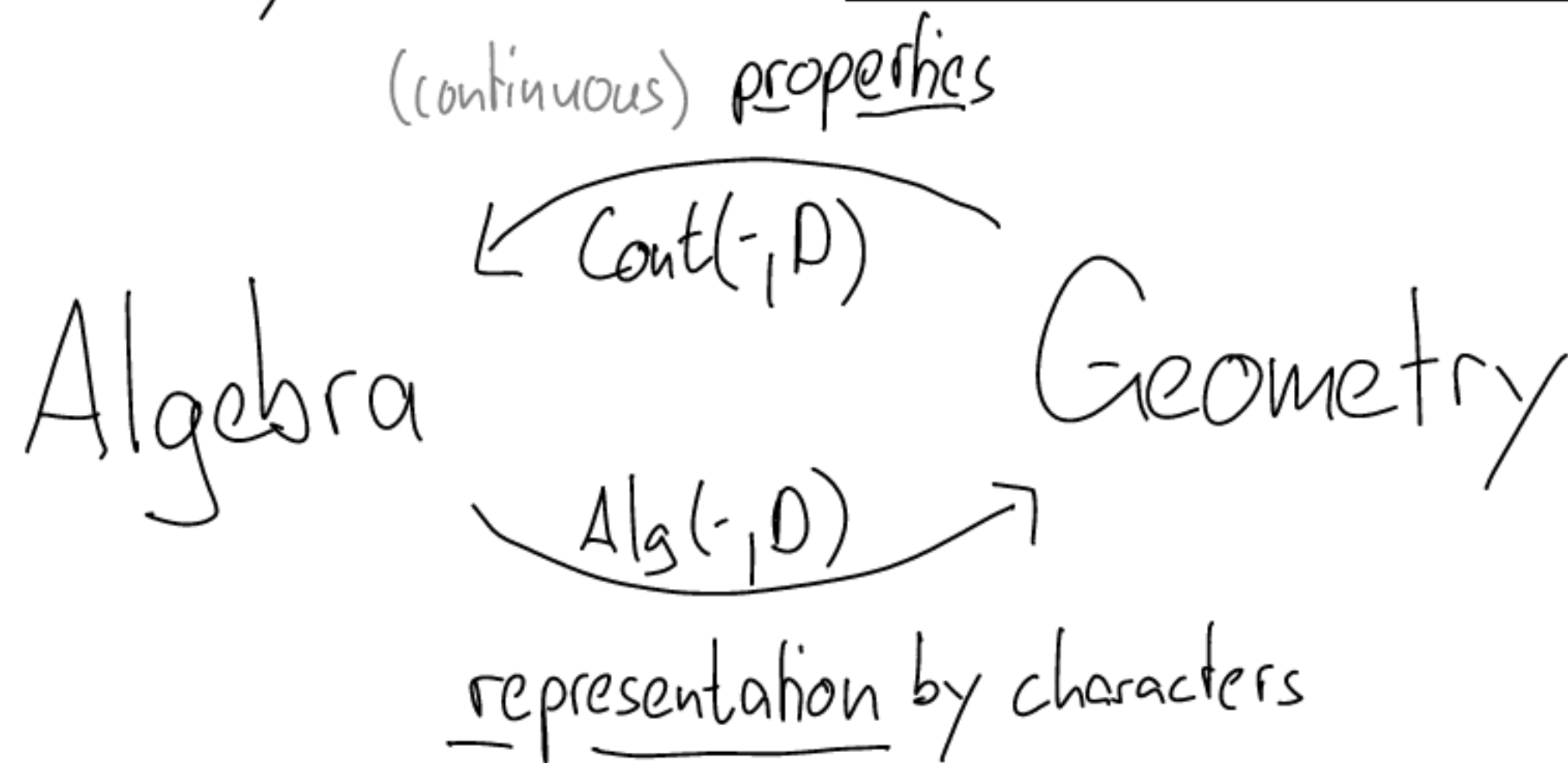
J.cs Oberseminar

20.10.26

Duality Theory



Duality Theory



For example:

Logic	:	Theory	vs.	Model	} ¹ Maruyama CSL'13
Algebra	:	Field/Ring	vs.	Variety / Scheme	
Rep. Theory	:	Ab. Group	vs.	Representation	
Prog. Lang.	:	Property	vs.	Denotation	
(Quantum) Mech.	:	Observable	vs.	State	
⋮		⋮		⋮	

Example: Stone Duality

Propositional logic: $\perp / \neg \varphi / \varphi \vee \psi / p \in V$ ↑ atomic props. (countable) + rules
+ axioms

⇒ Algebraic Model: Boolean Algebra of Formulas / provable equivalence = $\mathcal{FV}$

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Propositional logic: $\perp / \neg \varphi / \varphi \vee \psi / p \in V$ ^{atomic props. (countable)} + rules
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$\leadsto$ Algebraic Model: Boolean Algebra of Formulas / provable equivalence = $\mathcal{FV}$

But: syntactic (prop. $\hat{=}$ element of $\mathcal{FV}$), but proposition $\hat{=}$ statement about things

$\leadsto$ Other models: Propositions as observations about a space

Example: Stone Duality

Let $C = \mathbb{Z}^V \cong \mathbb{Z}^{\mathbb{N}}$ be the Cantor space
 $\uparrow$ all possible valuations

Level 1 : $p_1 \wedge p_2 \rightarrow p_4 \hat{=} \{b_0 b_1 \dots \in C \mid \text{if } b_0 = 1 \text{ and } b_2 = 1 \text{ then } b_4 = 1\} \subseteq C$

Example: Stone Duality

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Level 2: Assemble all such correspondences: $\mathcal{FV} \rightarrow \mathcal{PC}$
 $\varphi \mapsto \{b \in C \mid b \models \varphi\}$ $\left. \vphantom{\begin{matrix} \mathcal{FV} \rightarrow \mathcal{PC} \\ \varphi \mapsto \{b \in C \mid b \models \varphi\} \end{matrix}} \right\} \begin{array}{l} \text{BA-morphism,} \\ \text{image} \hat{=} \text{"decidable} \\ \text{subsets"} \end{array}$

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Level 3: Lvl. 2 works for all BAs! $\eta_B: B \rightarrow \text{Set}(\text{BA}(B, \mathbb{Z}))$ ($\eta_{\mathcal{FV}} = e$)
 $b \mapsto \{u \mid u \models b \text{ iff } b \in u\}$

$\leadsto$ Assemble into $\eta: \text{Id}_{\text{BA}} \rightarrow \text{Set}(\text{BA}(-, \mathbb{Z}), \mathbb{Z})$

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(Level 4: ???)

Example: Point-free Topology

Topology: spaces $(X \in \text{Set}, \mathcal{Q}_X \subseteq \mathcal{P}X)$ $\left\{ \begin{array}{l} \text{topology: closed under } \cup \text{ and } \cap \\ \text{Mor: } X \xrightarrow{f} Y \text{ s.t. } f^*[\mathcal{Q}_Y] \subseteq \mathcal{Q}_X \end{array} \right.$

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[Isbell/Johnstone]: Really we care about observables $U \in \mathcal{O}_X$, not points $x \in X$

💡: $\mathcal{O}_X$ is a frame: complete lattice s.t. $\bigvee_i U_i \wedge V = (\bigvee_i U_i) \wedge V$

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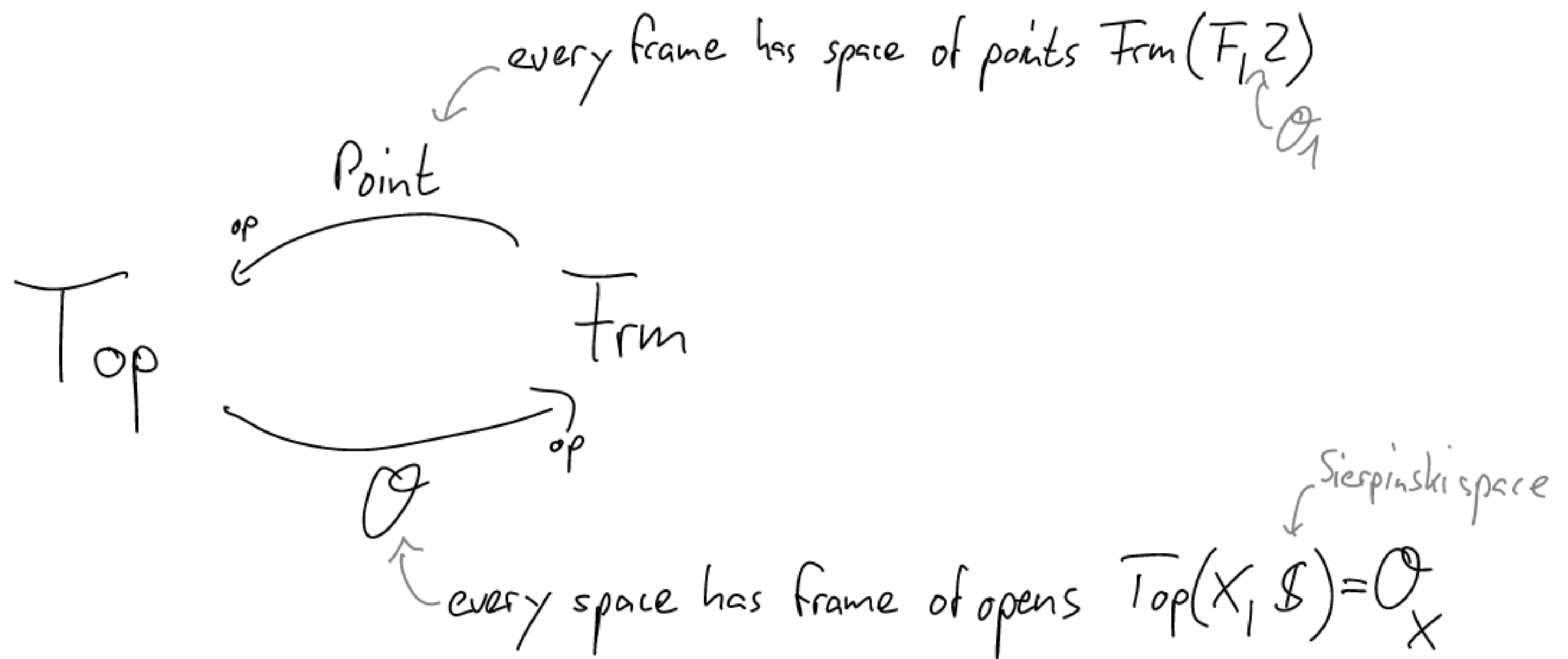
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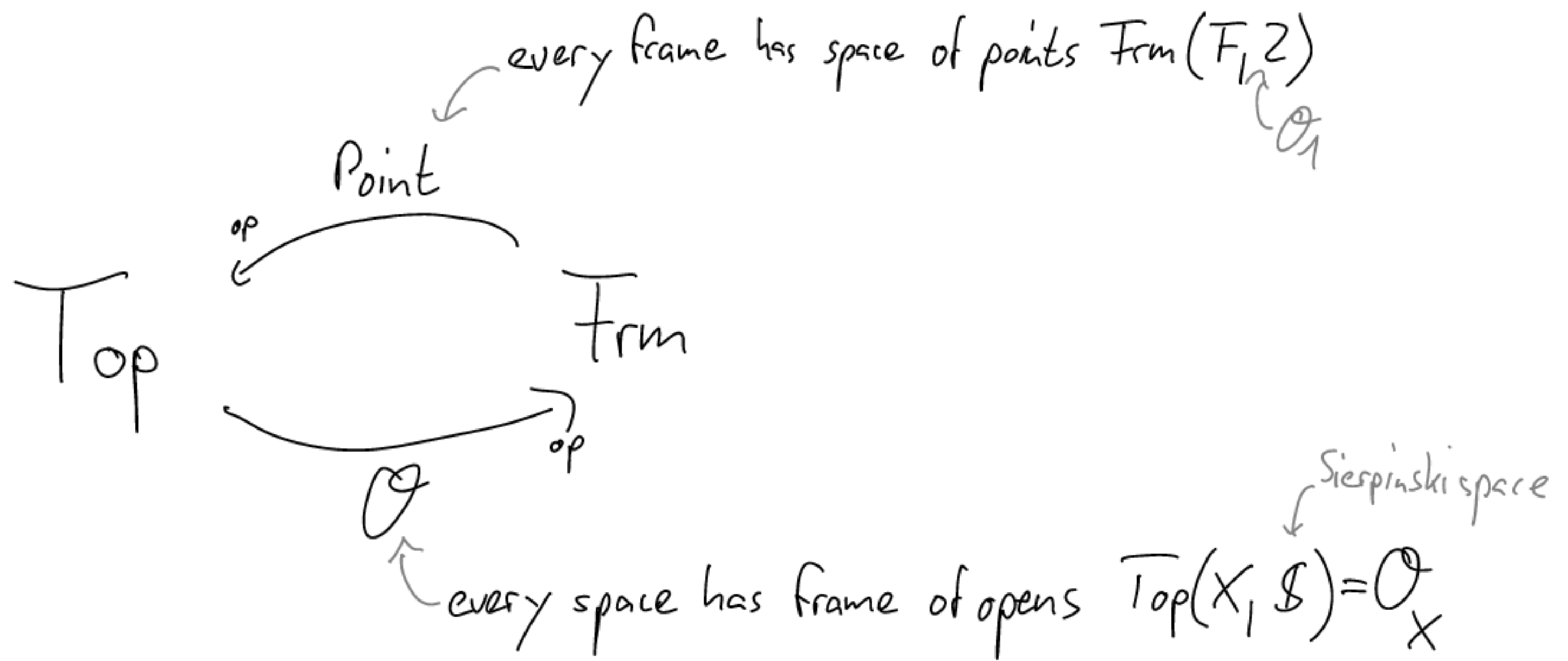
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md Algebraic Theory for topology!

Example: Point-free Topology



Example: Point-free Topology



⚡ Not an equivalence!

⚡₁ Spaces with too few opens, (e.g. $(2, \{\emptyset, 2\})$, $(10, \{\emptyset, 10\})$: $\mathcal{O}_2 \cong \mathcal{O}_{10}$)

⚡₂ Frames with too few points, e.g. $\text{RegOpens}(\mathbb{R})$

Equivalences from Adjunctions

Radical idea by Johnstone: "Real category of spaces" is $\mathbf{Frm}^{\text{op}}$ "locales"

Also: $\mathbf{Top} \xrightleftharpoons[\mathcal{O}^{\text{op}}]{\mathcal{P}\mathcal{L}} \mathbf{Frm}$ induces duality $\mathbf{SobSp} \simeq^{\text{op}} \mathbf{SpaFrm}$ s.t. $\mathcal{P}\mathcal{L}F \in \mathbf{SobSp}$
 $\mathcal{O}X \in \mathbf{SpaFrm}$
 $\uparrow$ enough opens $\uparrow$ enough points

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↑ enough opens ↑ enough points

Thm: Every adjunction $\mathcal{C} \xrightleftharpoons[\mathcal{F}]{\mathcal{G}} \mathcal{D}$ induces equivalence
 between fixed points $\mathbf{Fix}(\mathcal{G}\mathcal{F}) \simeq \mathbf{Fix}(\mathcal{F}\mathcal{G})$

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 $\text{Op } X \in \mathbf{Sp}^{\mathbf{Form}}$
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Thm: Every adjunction $\mathcal{C} \xrightleftharpoons[\text{F}]{\text{G}} \mathcal{D}$ induces equivalence between fixed points $\text{Fix}(GF) \simeq \text{Fix}(FG)$

• Equivalent are:

- (1) η_G is iso (2) ϵ_F is iso (3) GF idemp. monad (4) FG idemp. comonad
- (5) $\forall D \in \mathcal{D}: GD \in \text{Fix } GF$ (6) $\forall C \in \mathcal{C}: FC \in \text{Fix } FG \dots$

↳ $F+G$ is "idempotent" / "Galois" if contravariant

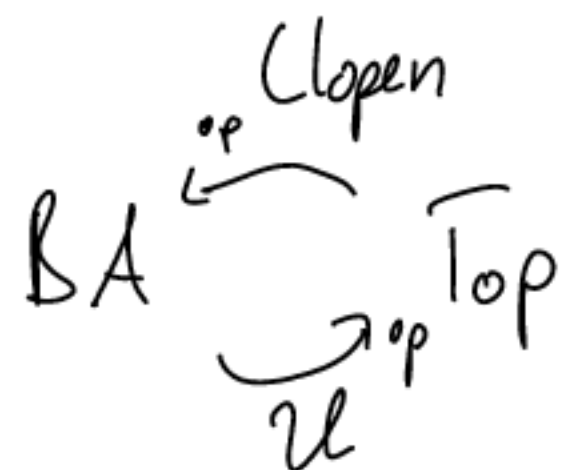
Equivalences from Adjunctions: Examples

Thm: Galois adjunction $\mathcal{C} \begin{matrix} \xrightarrow{G} \\ \xleftarrow{F} \end{matrix} \mathcal{D}$ $\leadsto$ equivalence $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{C}$

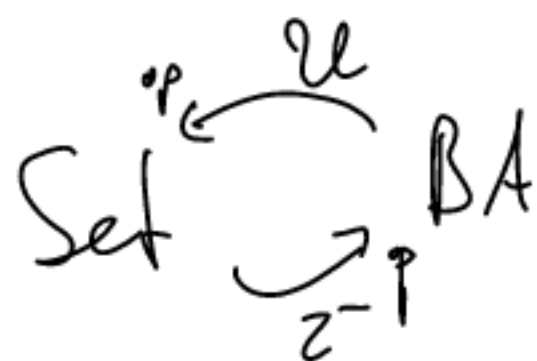
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Thm: Galois adjunction $\mathcal{C} \begin{matrix} \xleftarrow{\circ p} G \\ \xrightarrow{\circ p} F \end{matrix} \mathcal{D} \rightsquigarrow \text{equivalence } \mathcal{C} \begin{matrix} \xleftarrow{\circ p} G \\ \xrightarrow{\circ p} F \end{matrix} \mathcal{D} \simeq^{\circ p} \mathcal{D} \begin{matrix} \xleftarrow{\circ p} F \\ \xrightarrow{\circ p} G \end{matrix} \mathcal{C}$

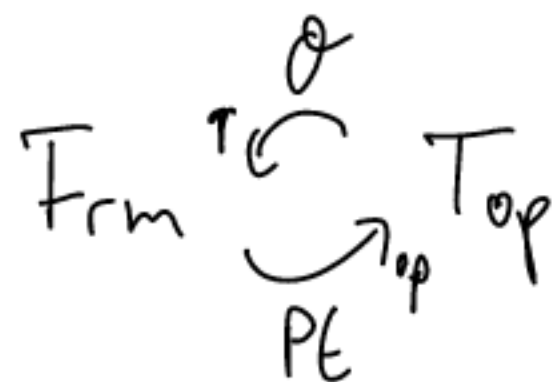
Examples:



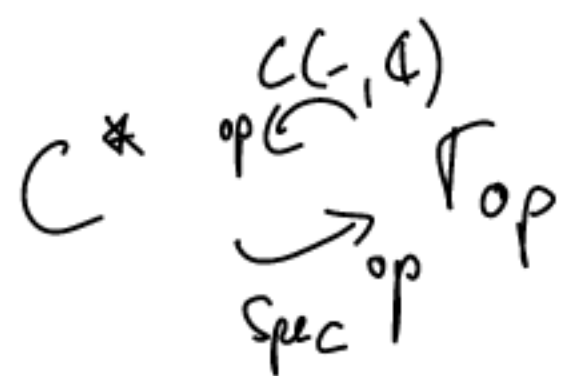
$$\rightsquigarrow \text{BA} \simeq^{\circ p} \text{Stone}^{\circ p}$$



$$\rightsquigarrow \text{Set} \simeq^{\circ p} \text{CABA}$$



$$\rightsquigarrow \text{SpFrm} \simeq^{\circ p} \text{SubTop}$$

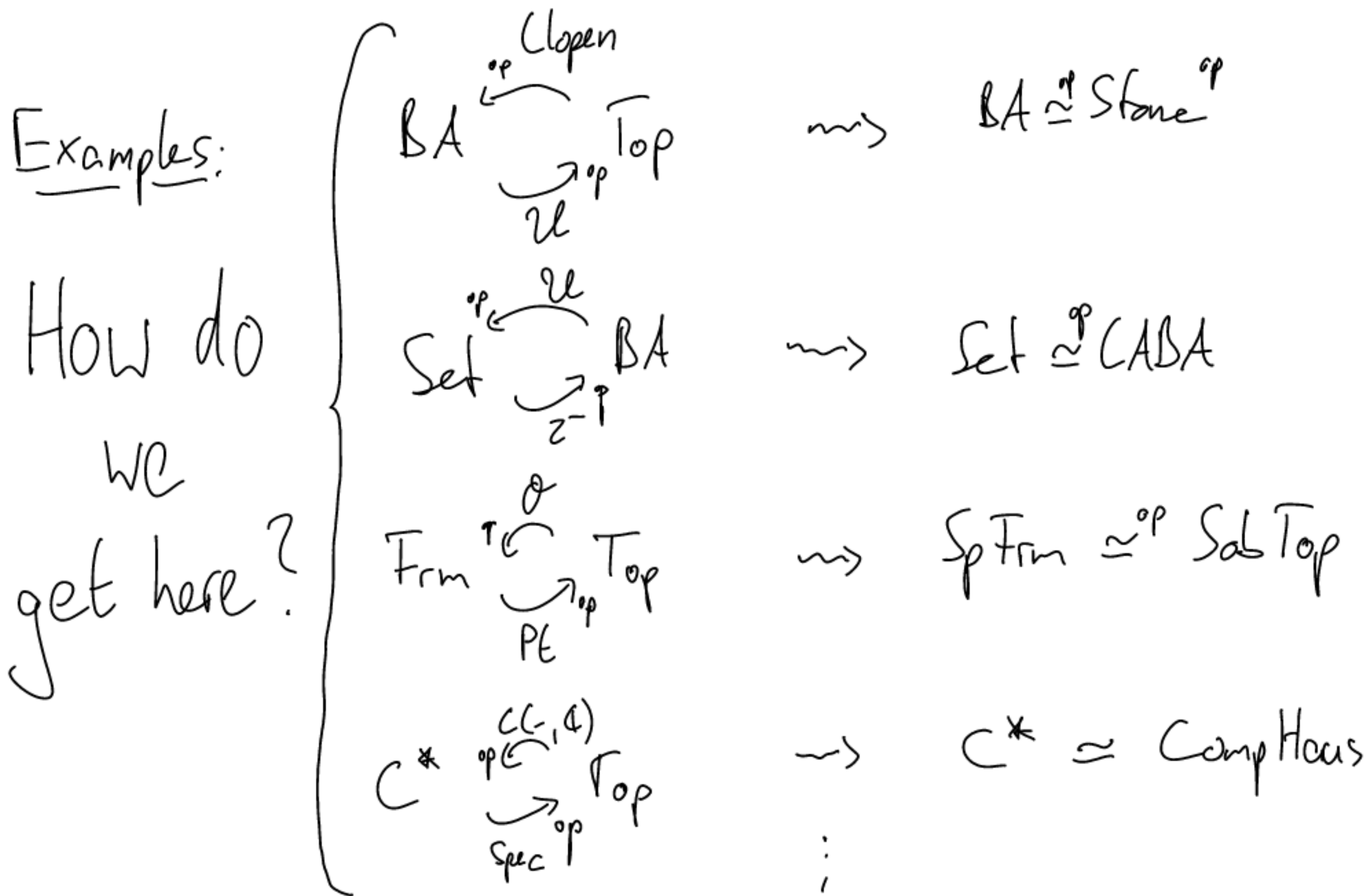


$$\rightsquigarrow \mathcal{C}^* \simeq \text{CompHaus}$$

;

Equivalences from Adjunctions: Examples

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Here: Constructive Setup

Given algebraic semantics $\leadsto$ · what are the spaces?
· can I recover the algebraic theory?
· is the spatial semantics complete?

Given spatial semantics $\leadsto$ · what are the algebras?
· can I recover spaces?
· is the algebraic semantics complete?
·

A General Duality of Algebra and Geometry

We try to answer these questions uniformly:

- (1) Given algebraic theory $\mathbb{T} \rightarrow$ build spaces $\mathrm{Sp}\mathbb{T}$
 - (1a) plus Galois adjunction $\hat{=}$ "Spatial representation theory of $\mathbb{T}$ "

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- (2) Given spaces $\mathcal{D} \rightarrow$ build alg. theory $\text{Clone}\mathcal{D}$
(2a) plus Galois adjunction $\cong$ "Algebraic representation theory of $\mathcal{D}$ "

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(2a) plus Galois adjunction $\hat{=}$ "Algebraic representation theory of $\mathcal{D}$ "

||

- (3) Assemble into Meta-Duality: Algebra vs. Space
 $\hookrightarrow$ "Level 4 of Stone Duality"

Spaces from Algebras

 Nishizawa-Komorida-Katsumata '20

Input: Algebraic Theory $\hat{=}$ monad $\mathbb{T}$ on $\mathbf{Set}$, algebra $\mathbb{T}D \xrightarrow{1} D$

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 $\rightarrow$ Mor: $f: X \rightarrow Y$ s.t. $\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{\quad} & \mathcal{O}_X \\ \downarrow & \circ^f & \downarrow \\ D^Y & \xrightarrow{\quad} & D^X \end{array}$ restricts

$\swarrow$ power: pointwise operations

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Ex: • $\mathbb{T} = \mathcal{F} \hat{=} \mathcal{UP}_f$: $\mathbf{Sp}(\mathcal{F}, 2) = \mathbf{Top}$ • $\mathbb{T} = \overset{\text{free countable BA}}{\mathcal{B}_{\omega_1}}$: $\mathbf{Sp}(\mathcal{B}_{\omega_1}, 2) = \mathbf{Meas}$

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power: pointwise operations

Ex: $\bullet \mathbb{T} = \mathcal{F} \hat{=} \mathcal{UP}_f: \mathbf{Sp}(\mathcal{F}, 2) = \mathbf{Top}$ free countable BA $\downarrow$
 $\bullet \mathbb{T} = \mathcal{B}: \mathbf{Sp}(\mathcal{F}, 2) = \mathbf{RingSet}$ $\bullet \mathbb{T} = \mathcal{B}_{\omega_1}: \mathbf{Sp}(\mathcal{B}_{\omega_1}, 2) = \mathbf{Meas}$
 $\bullet \mathbb{T} = 1: \mathbf{Sp}(1, 1) \hat{=} \mathbf{Set}$

Spaces from Algebras

¹ Nishizawa-Komorida-Katsumata '20

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• $\mathbb{T} = \mathcal{B}: \mathbf{Sp}(\mathcal{B}, 2) = \mathbf{RingSet}$

• $\mathbb{T} = 1: \mathbf{Sp}(1, 1) \hat{=} \mathbf{Set}$

• $\mathbb{T} = 2^{2^-}: \mathbf{Sp}(2^{2^-}, 2) \hat{=} \mathbf{Setoid}$

• $\mathbb{T} = (\mathcal{PX}, (2, 1))$: Intersection systems
 $\hat{=} \text{closure spaces}$

$\uparrow$ sets + equiv relation + New examples

$Sp(\pi, D)$ are S paces

Prop: $Sp(\pi, D)$ is topological over Set .  [NKK'20]

Interlude: What are Spaces?

Algebra

Monad $\mathbb{T}$ (on Set)

Categorical

Theory ("Syntax")

Interlude: What are Spaces?

Algebra

Monad Π (on Set)

$$EM(\pi) \xrightarrow{u^\pi} \text{Set}$$

↑ algebras: $\Pi A \xrightarrow{\eta} A$
 $\Pi h \downarrow \quad \downarrow h$ ← morphism
 $\Pi B \rightarrow B$

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Theory ("Syntax")

Models ("Semantics")

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monadic right adjoints $\downarrow$
 $\mathbf{Set}$

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Ext. Characterisation

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$$\begin{array}{ccc} A & \xrightarrow[k]{r. \text{ adj.}} & B \\ \swarrow \text{monadic} & & \searrow \\ \mathbf{Set} & & \mathbf{Set} \end{array} \quad \begin{array}{ccc} EM & & \\ \hookrightarrow & & \\ \hookrightarrow & & \end{array} \quad \begin{array}{ccc} \mathbb{T} & \xleftarrow{\kappa} & \mathbb{S} \\ & \text{monad morph.} & \end{array}$$

monad from adj.

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Models $\begin{matrix} \hookleftarrow \\ \hookrightarrow \end{matrix}$ Syntax

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¹ Wyler 1971

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Models $\curvearrowright$ Syntax

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Functor $F: \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Cat}_1$ ¹

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Functor $F: \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Cat}_1$ ¹

$$\int F \xrightarrow{1-1} \mathbf{Set}$$

spaces: $(X \in \mathbf{Set}, \mathcal{Q}_X \in FX)$

$$\begin{array}{c} f \downarrow \\ (Y, \mathcal{Q}_Y) \end{array}$$

f continuous if

$$\mathcal{Q}_X \subseteq Ff(\mathcal{Q}_Y)$$

$f\mathcal{Q}_Y$

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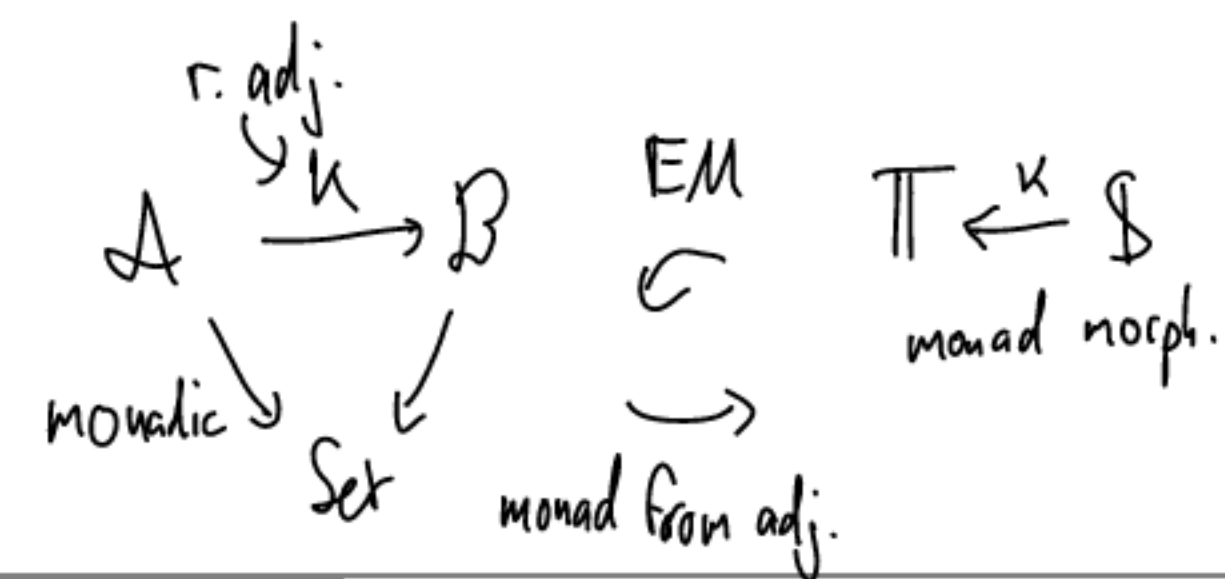
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lift (+prop)

$$\exists \text{ init } f_i \rightarrow D_i$$

topol. functor:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\text{s.t.}} & X \\ \downarrow & & \downarrow \\ \text{Set} & & \begin{array}{c} f_i \rightarrow p D_i \\ \vdots \\ f_j \rightarrow p D_j \end{array} \end{array}$$

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monadic right adjoints

$$\begin{array}{ccc}
 & \text{r. adj.} & \\
 & \swarrow \downarrow \searrow & \\
 A & \xrightarrow{\quad} & B \\
 \swarrow \quad \searrow & & \\
 \text{monadic} & & \text{Set}
 \end{array}$$

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Models ("Semantics")

Ext. Characterisation

Models $\leftrightarrow$ Syntax

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Functor $F: \text{Set}^{\text{op}} \rightarrow \text{Cat}$ 

$$\int F \xrightarrow{1-1} \text{Set}$$

spaces: $(X_{\text{clst}}, \mathcal{Q}_X \in \mathcal{F}X)$
 $f \downarrow$
 $(Y, \mathcal{Q}_Y)$

f continuous if

$$Q_x \subseteq \underbrace{ff(Q_y)}_{f^*Q_y}$$

lift (+prop)

↓

$\exists i \text{ in } I_i \rightarrow D_i$

$\rightarrow D_j$

topol. functor:

pres. lifts

$$D \xrightarrow{F} E$$

$\int$

$$\mathcal{D} \downarrow \text{Set}$$

s.t.

$$X \begin{matrix} \xrightarrow{f_i} p^{D_i} \\ \vdots \\ \xrightarrow{f_j} p^{D_j} \end{matrix}$$
$$\begin{array}{ccc}
 \mathcal{D} & \xrightarrow{F} & \mathcal{E} \\
 \downarrow \text{topolog.} & & \downarrow \\
 \text{Set} & & \text{Set}
 \end{array}
 \quad
 \begin{array}{ccc}
 & \int & \\
 \swarrow & & \searrow \\
 & \simeq & \\
 & \xrightarrow{\text{fibres}} &
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{D} & \xrightarrow{\text{op}} & \mathcal{C} \\
 \downarrow & & \downarrow \\
 \mathcal{E} & & \mathcal{F}
 \end{array}$$

Categories of Spaces

Expl: • Top, Meas, Unit, ...

• $\mathbf{PMet}_1$: $(X, d: X \times X \rightarrow [0, 1] \text{ pseudometric}) + \text{nondec. maps}$

• $\mathbf{Pred}_\lambda$: $(X, R \subseteq X^\lambda) + \text{rel-pres. maps}$

• $\mathbf{Preord}$: preordered sets

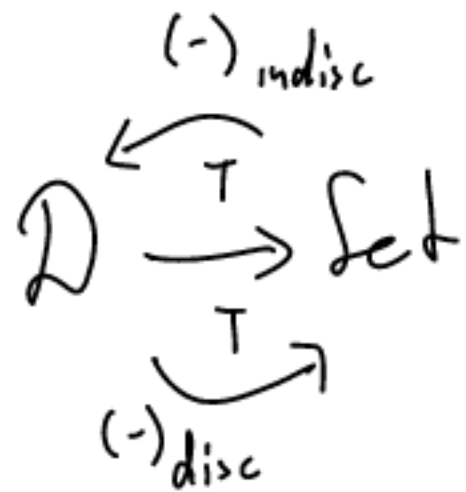
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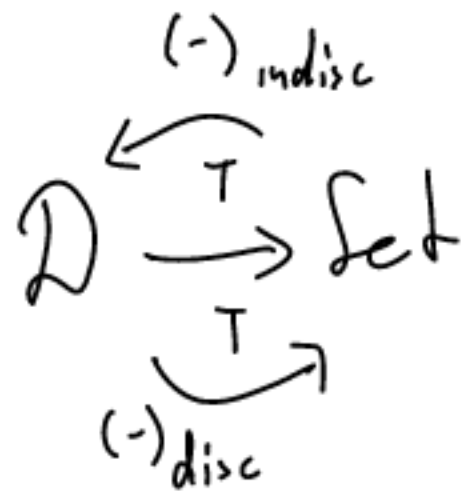


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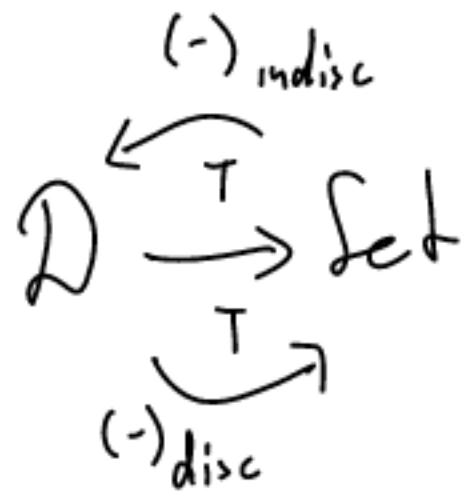


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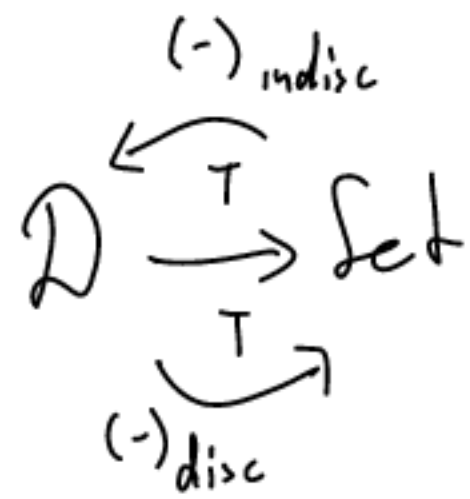
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$$\begin{array}{ccc} D & \xrightarrow{R} & E \\ \downarrow & \swarrow & \\ \text{Set} & & \end{array} \quad \begin{array}{c} R \text{ adj.} \\ \hline R \text{ preserves init. lifts} \end{array}$$

D topological (over Set)
 $\Rightarrow$ complete, cocomplete, ...

$Sp(\pi, D)$ are spaces

Prop: $Sp(\pi, D)$ is topological over Set .

- Initial lift of $X \xrightarrow{f_i} (x_i, \mathcal{O}_i \hookrightarrow_{\pi} D^Y_i)$ is generated sub- π -alg $\langle g_i \cdot f_i \mid f_i \in \mathcal{O}_i, i \in I \rangle_{\pi}$
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Prop: $\begin{array}{ccc} \mathcal{S}D & \xrightarrow{\pi} & TE \\ \downarrow D & \xrightarrow{F} & E \end{array}$ induces concrete functor $Sp(\mathcal{S}, D) \rightarrow Sp(\pi, E)$

Expl: $\begin{array}{ccc} Sp(\mathbb{F}, 2) & & Sp(\mathbb{R}, 2) \\ Top & \rightarrow & Meas \end{array}$
 $\begin{array}{cc} \mathbb{F}2 & \mathbb{R}2 \\ \downarrow 2 & \downarrow 2 \end{array}$

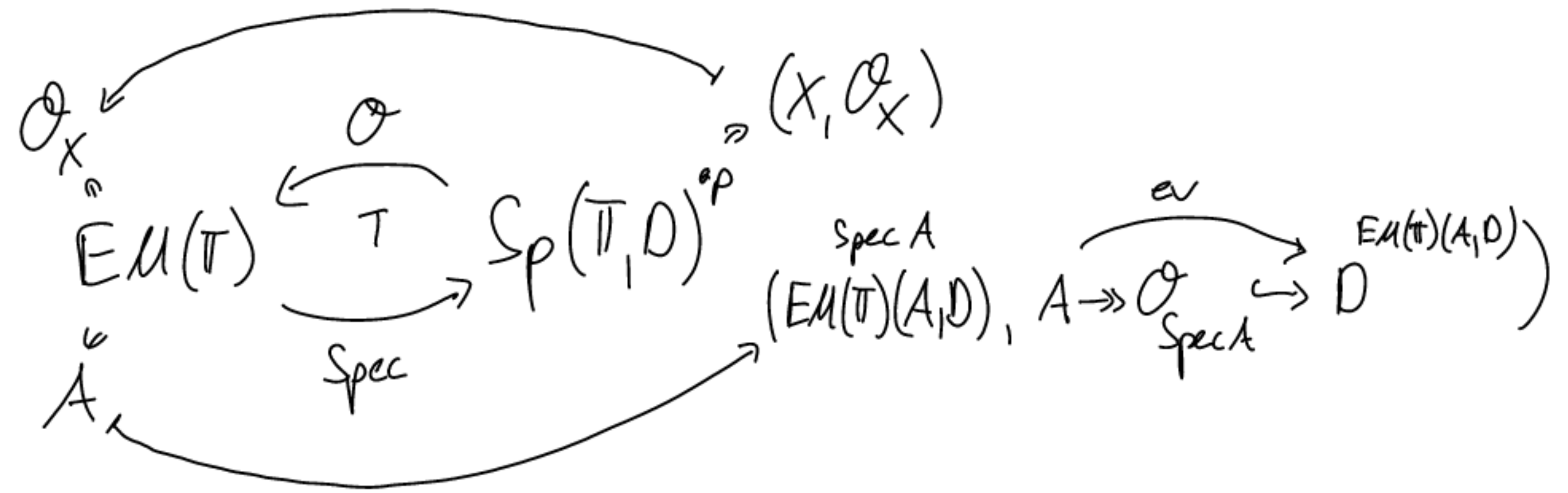
$\begin{array}{ccc} Sp(\mathbb{B}, 2) & & Sp(\mathbb{F}, 2) \\ RingSet & \rightarrow & Top \end{array}$
 $\begin{array}{cc} \mathbb{B}2 & \mathbb{F}2 \\ \downarrow 2 & \downarrow 2 \end{array}$

$\begin{array}{ccc} Sp(C^*, \mathbb{C}) & & Sp(\mathbb{F}, 2) \\ C^*Space & \rightarrow & Top \end{array}$
 $\begin{array}{ccc} C^*\mathbb{C} & & \mathbb{F}2 \\ \downarrow & \xrightarrow{\neq 0} & \downarrow \\ \mathbb{C} & & 2 \end{array}$

A Canonical Natural Duality Theory

NRK'20

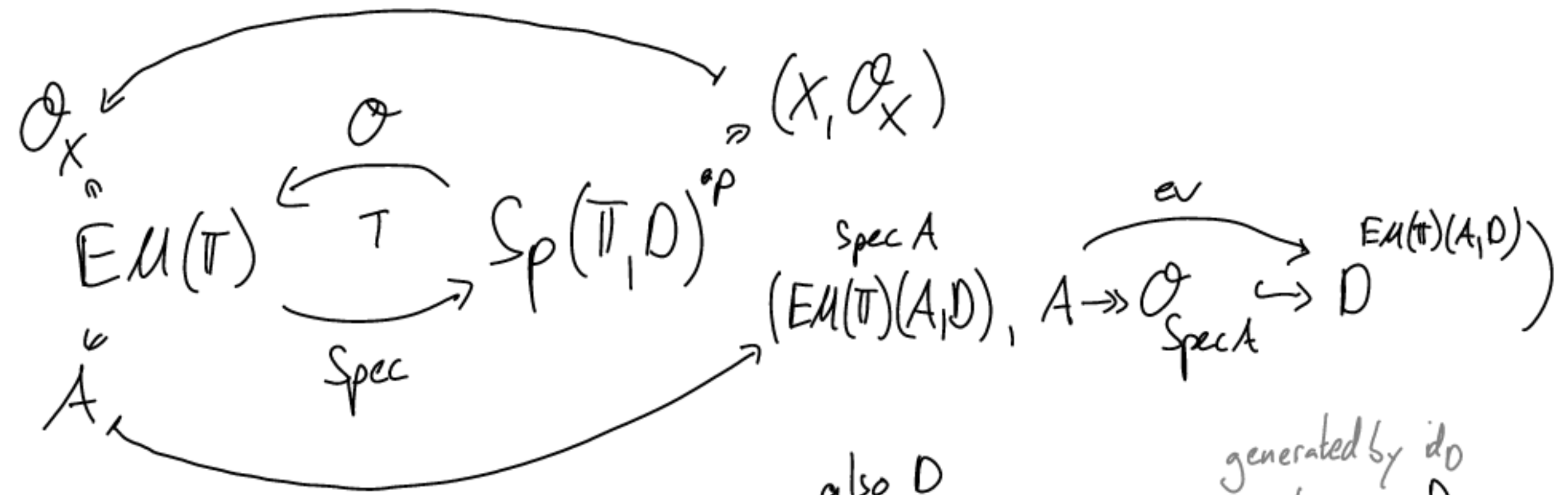
Thm: • Hom-functor $EM(\pi)(-, D)$ lifts to $Sp(\pi, D)$ and has right adjoint $\mathcal{O}$:



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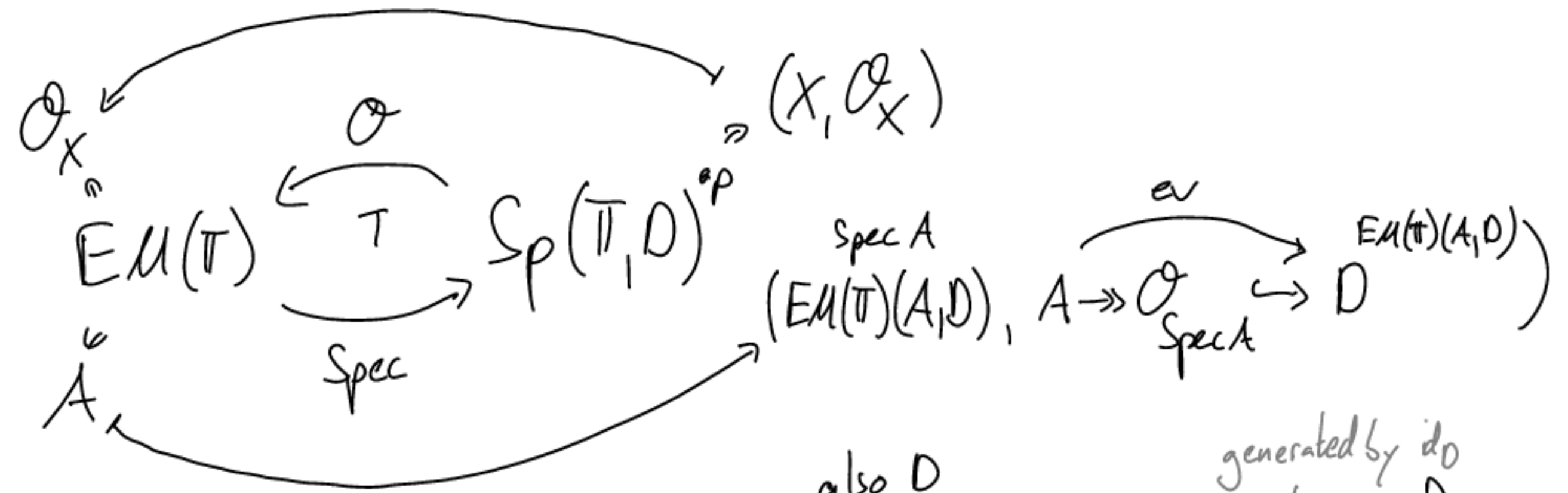


+ Adj. is representable: $D \in EM(\pi)$ is also space $Spec \pi 1 = (ID, \pi 1 \twoheadrightarrow \mathcal{O}_D \hookrightarrow D)$
 and $|\mathcal{O}| \cong Sp(\pi, D)(-, D)$, $|Spec| \cong EM(\pi)(-, D)$

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+ Adj. is Galois: $\eta_{\mathcal{O}}: \mathcal{O} \rightarrow \mathcal{O}_{Spec \mathcal{O}}$, $\eta_{Spec}: Spec \rightarrow Spec \mathcal{O}_{Spec}$ isos

A Canonical Natural Duality Theory

Galois adj. $\xrightarrow{GAN}$ Duality $\begin{matrix} EM\pi \hookrightarrow Sp(\pi, D) \\ \downarrow \quad \downarrow \\ \mathcal{O}[Sp(\pi, D)] \simeq Spec[EM\pi] \end{matrix}$

Def: Algebra A D -spatial if $A \cong \mathcal{O}Spec A$ is.

Lem: A spatial iff D coseparating for A : $\forall a \neq b \in A \exists s: A \rightarrow D$ s.t. $s(a) \neq s(b)$ ↑
 D cosep in $EM(\pi)$ iff all A spatial

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$\hookrightarrow$ Cor: Galois adj. $\mathcal{O} \dashv Spec$ restricts to duality $EM(\pi)_{D\text{-spa}} \simeq^{\circ} Sp(\pi, D)_{sob}$

Examples

Point-free top. duality: $(\mathcal{F}, 2)$

$$\begin{array}{ccc} \text{Frm} & \xrightarrow{\text{ur}} & \text{Top} \\ \downarrow & & \downarrow \\ \text{Frm}_{2\text{-spa}} & \xrightarrow{\text{ur}} & \text{Top}_{\text{sol}} \end{array}$$

Loomis-Sikorski: $(\mathcal{B}_+, 2)$

$$\begin{array}{ccc} \sigma\text{BA} & \xrightarrow{\text{ur}} & \text{Meas} \\ \downarrow & & \downarrow \\ \sigma\text{BA}_{2\text{-spa}} & \xrightarrow{\text{ur}} & \text{Meas}_{\text{sol}} \end{array}$$

"Stone": $(\mathcal{B}, 2)$

$$\begin{array}{ccc} \text{BA} & \xrightarrow{\text{ur}} & \text{RingSet} \rightarrow \text{Top} \\ \downarrow & & \downarrow \\ \text{BA}_{2\text{-spa}} & \xrightarrow{\text{ur}} & \text{CompDiff} \simeq \text{Stone} \end{array}$$

2 cosep : ultrafilter lemma $\nearrow$ compact, differentiated

Tarski: $(2^2, 2)$

$$\begin{array}{ccc} \text{CABA} & \xrightarrow{\text{ur}} & \text{Schoid} \\ \downarrow & & \downarrow \\ \text{CABA}_{2\text{-spa}} & \xrightarrow{\text{ur}} & \text{Set} \end{array}$$

Closure Operators: $(\mathcal{P}, 2)$

$$\begin{array}{ccc} \text{CLat} & \xrightarrow{\text{ur}} & \text{Sp}(\mathcal{P}, 2) \\ \downarrow & & \downarrow \\ \text{CLat}_{2\text{-spa}} & \xrightarrow{\text{ur}} & \text{Sp}(\mathcal{P}, 2)_{\text{sol}} \\ & & \downarrow \\ & & \text{CLat} \end{array}$$

Gelfand Duality?: $(\mathcal{C}^*, \mathbb{C})$

$$\begin{array}{ccc} \mathcal{C}^*\text{Alg} & \xrightarrow{\text{ur}} & \text{Sp}(\mathcal{C}^*, \mathbb{C}) \rightarrow \text{Top} \\ \downarrow & & \downarrow \\ \mathcal{C}^*\text{Alg}_{2\text{-spa}} & \xrightarrow{\text{ur}} & \text{Sp}(\mathcal{C}^*, \mathbb{C})_{\text{sol}} \simeq \text{CHaus} \end{array}$$

$\nearrow$ intrinsic char.?

Monads from Spaces

Powers $(-)^{(-)}$ in $\mathcal{C}$: $\mathcal{C}(A, B^X) \cong \text{Set}(X, \mathcal{C}(A, B))$, so $\text{Set} \begin{array}{c} \xleftarrow{\mathcal{C}(-, B)} \\ \tau \\ \xrightarrow{B^-} \end{array} \mathcal{C}^{\text{op}}$

Def: Let $\mathcal{D}$ have all powers of $D \in \mathcal{D}$. The $\mathcal{D}$ -clone is $\text{Clone}(\mathcal{D}, \mathcal{D}) := \mathcal{D}(0^-, \mathcal{D})$

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Expl:

Sierpinski

• $\text{Clone}(\overset{\text{top}}{\mathcal{D}}, \mathcal{D}_{\text{indix}}) = \mathcal{D}^{0^-}$

• $\text{Clone}(\overset{\text{Sierpinski}}{\text{Top}}, \mathcal{B}) = \text{Frm}$

• $\text{Clone}(\text{Top}, 2_{\text{disc}}) = \mathcal{B}$

• $\text{Clone}(\text{Top}, \mathbb{C}) = \mathbb{C}^*$

• $\text{Clone}(\text{Pred}_1, (2, [1] \leq 2)) X = \{U \in 2^{2^X} \mid X \in U\} \cong 2^{2^X}$

→ Prop: $\text{EM}^{\text{op}} \cong \text{Set}$

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Expl:

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Note: Every monad is a clone: $\Pi \in \text{Kl}(\Pi)^{\text{op}}(1^X, 1)$

Question: Is every monad $\cong \text{Clone}(\mathcal{D}, \mathcal{D})$ for top. $\mathcal{D}$?

A Canonical Duality Theory (for spaces)

Dualizer: Let $\mathcal{D}$ be topological and $D \in \mathcal{D}$

$\Rightarrow |\mathcal{D}|$ is algebra: $\text{Clone}(\mathcal{D}, \mathcal{D}) \mathcal{D} = \mathcal{D}(\mathcal{D}^{\mathcal{D}}, \mathcal{D}) \rightarrow \mathcal{D}$
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Thm: The comparison functor $\mathcal{D}(-, \mathcal{D})_{\text{cmp}}: \mathcal{D}^{\mathcal{D}} \rightarrow \text{EM}(\text{Clone}(\mathcal{D}, \mathcal{D}))$
has a concrete Galois left adjoint: $A \mapsto \text{init EM}(A, \mathcal{D})$

$$\mathcal{D}^{\mathcal{D}} \begin{array}{c} \xleftarrow{\text{EM}(-, \mathcal{D})_{\text{cmp}}} \\ \perp \\ \xrightarrow{\mathcal{D}(-, \mathcal{D})_{\text{cmp}}} \end{array} \text{EM}(\text{Clone}(\mathcal{D}, \mathcal{D}))$$

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Cor: Duality by GAN

A Canonical Duality Theory (for spaces)

Examples:

$$D = \mathcal{S}$$

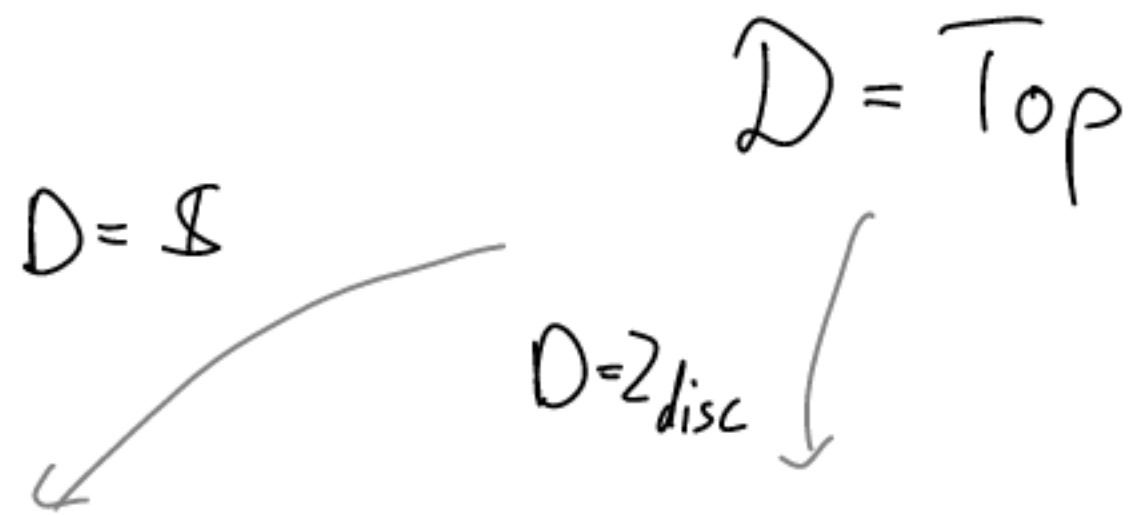
$$D = \text{Top}$$

$$\text{Clone}(\mathcal{S}) \cong \mathcal{F}$$

$$\begin{array}{ccc} & \text{Opens} & \\ \text{Frm}^{\text{op}} & \xleftarrow{\quad} & \text{Top} \\ & \xrightarrow{\quad} & \\ \uparrow & \text{Pt} & \uparrow \\ \text{SpFrm}^{\text{op}} & \simeq & \text{SoSTop} \end{array}$$

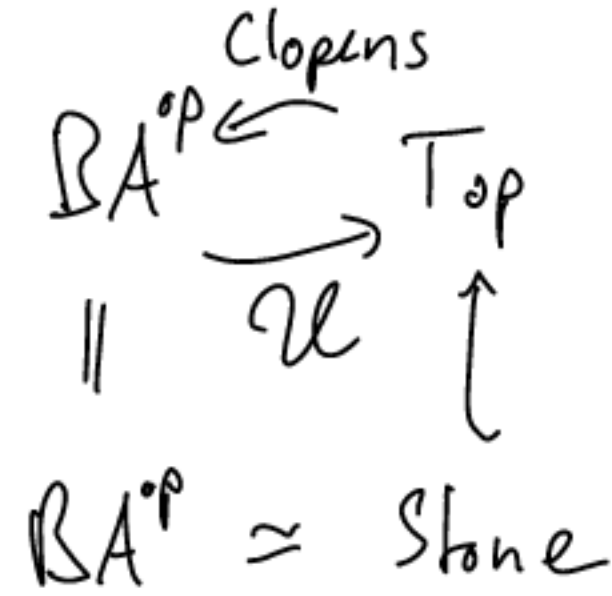
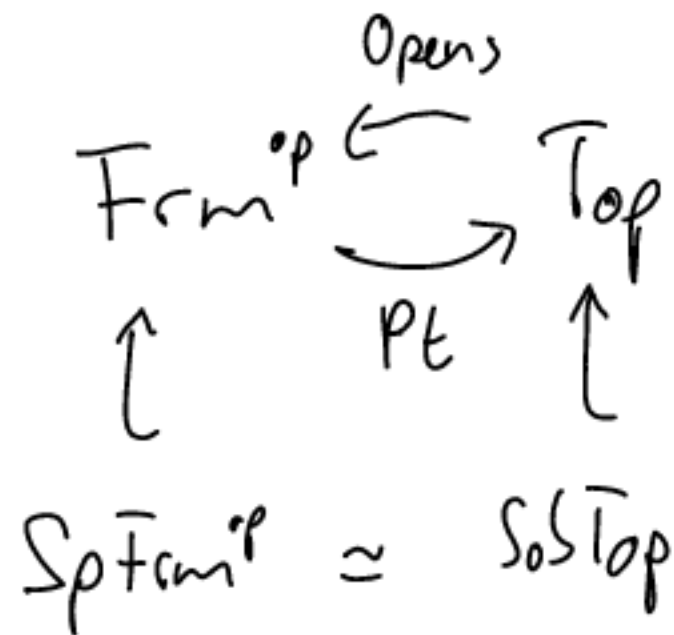
A Canonical Duality Theory (for spaces)

Examples:



$$\text{Clone}(S) \cong \mathcal{F}$$

$$\text{Clone}(2_{disc}) \cong \mathcal{B}$$



A Canonical Duality Theory (for spaces)

Examples:

$$D = \text{Top}$$

$$D = \mathbb{S}$$

$$D = 2_{\text{disc}}$$

$$D = 2_{\text{indisc}}$$

$$\text{Clone}(\mathbb{S}) \cong \mathbb{F}$$

$$\text{Clone}(2_{\text{disc}}) \cong \mathcal{B}$$

$$\text{Clone}(2_{\text{indisc}}) \cong 2^{\mathbb{Z}}$$

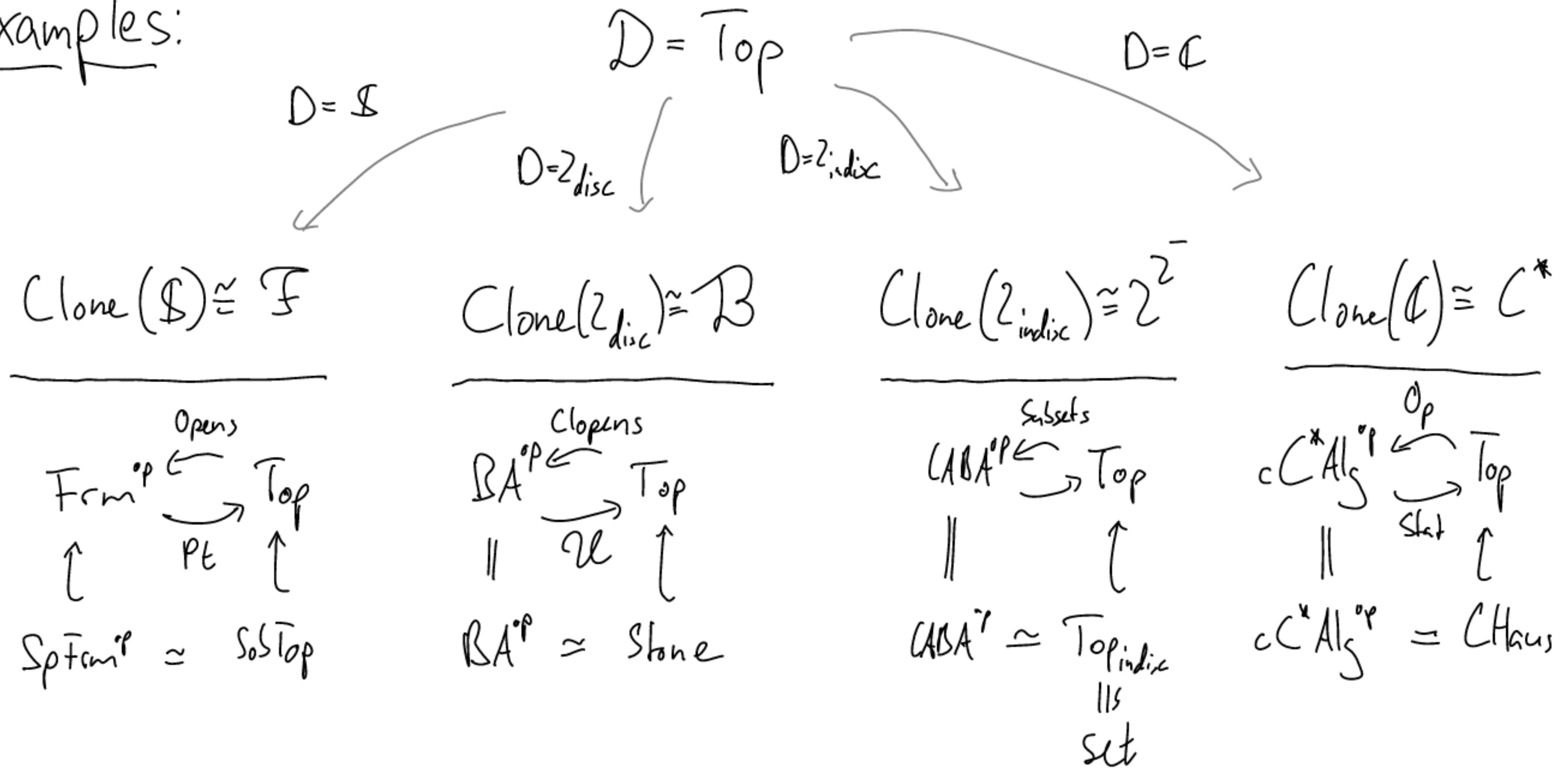
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$$\begin{array}{ccc} & \text{Clopens} & \\ \mathcal{B}\mathcal{A}^{\text{op}} & \xleftarrow{\quad} & \text{Top} \\ \parallel & \mathcal{U} & \uparrow \\ \mathcal{B}\mathcal{A}^{\text{op}} & \simeq & \text{Stone} \end{array}$$

$$\begin{array}{ccc} & \text{Subsets} & \\ \text{CABA}^{\text{op}} & \xleftarrow{\quad} & \text{Top} \\ \parallel & & \uparrow \\ \text{CABA}^{\text{op}} & \simeq & \text{Top}_{\text{indisc}} \\ & & \parallel \\ & & \text{Set} \end{array}$$

A Canonical Duality Theory (for spaces)

Examples:



Topological Roundabout

Let $\mathcal{D}$ be a topological cat., $D \in \mathcal{D}$

$$\mathcal{D} \begin{array}{c} \xleftarrow{\text{EM}(-, D)_{\text{comp}}} \\ \tau \\ \xrightarrow{\text{EM}(-, D)_{\text{comp}}} \end{array} \text{EM}(\text{Clone}(\mathcal{D}, D))^{\text{op}}$$

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Topological Roundabout

Let $\mathcal{D}$ be a topological cat., $D \in \mathcal{D}$

Thm: $\Delta + \Gamma$ concrete idemp. adj

$$\cdot \Gamma(X, \mathcal{O}_X \hookrightarrow \mathcal{O}^X) = (X, \text{init } \mathcal{O}_X)$$

$$\cdot \Delta Y = (Y, \mathcal{D}(Y, D) \hookrightarrow \mathcal{O}^Y)$$

with $\mathcal{O} \cdot \Delta \simeq \mathcal{D}(-, D)_{\text{cmp}}$

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Def: • $\mathcal{D}$ opens classifier if $\eta: Id \xrightarrow{\sim} \Gamma \Delta$ iso: $\forall Y \in \mathcal{D}: Y \cong \text{init } \mathcal{D}(Y, D)$

$\mathcal{G}_{\text{dual}}$: edge classifier

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Topological Roundabout: Examples

[D opens classifier if structure is det. by morphisms into D]

Top. Category	Top	Meas	Ringlet	Pred	Pred ₂	PMet ₁	Set
Opens classifier D:	$\mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}$	$(\mathbb{Z}, \{1\})$	$\left(\begin{array}{c} \bullet \leftrightarrow \bullet^{\mathbb{Q}} + \begin{array}{c} \bullet \rightarrow \bullet \\ \downarrow \nearrow \end{array} \end{array} \right)$	$[0, 1]$	1

Topological Roundabout: Examples

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Question: Top. cat. without opens classifier? (most top. cat. don't have edge classifier)

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[D-tidyness: $\forall \mathcal{O}_X \subseteq \mathcal{O}^X [\forall q \in D(D^{\mathcal{O}_X}, D): q \circ \text{ev} \in \mathcal{O}_X] \Rightarrow [D(\text{init } \mathcal{O}_X, D) \subseteq \mathcal{O}_X]$

Intuition ??? compactness / Stone-Weierstr.

- Later: $\text{Sp}(\Pi, D)$ always D-tidy (Top, Meas, RingSet)
- PMet_1 is $[0, 1]$ -tidy / Setoid is $\mathbb{Z}$ -tidy $\leadsto \mathbb{K}_{\text{cat}}$ is $\mathbb{K}$ -tidy?
- Pred_1 is not $(\mathbb{Z}, \{1\})$ -tidy

Algebraic Roundabout

let Π be a monad, $D \in \mathbf{EM}\Pi$ algebra

$$\begin{array}{c} \mathbf{EM}\Pi \\ u^\top \downarrow \uparrow F^\top \\ \Pi \hookrightarrow \mathbf{Set} \end{array}$$

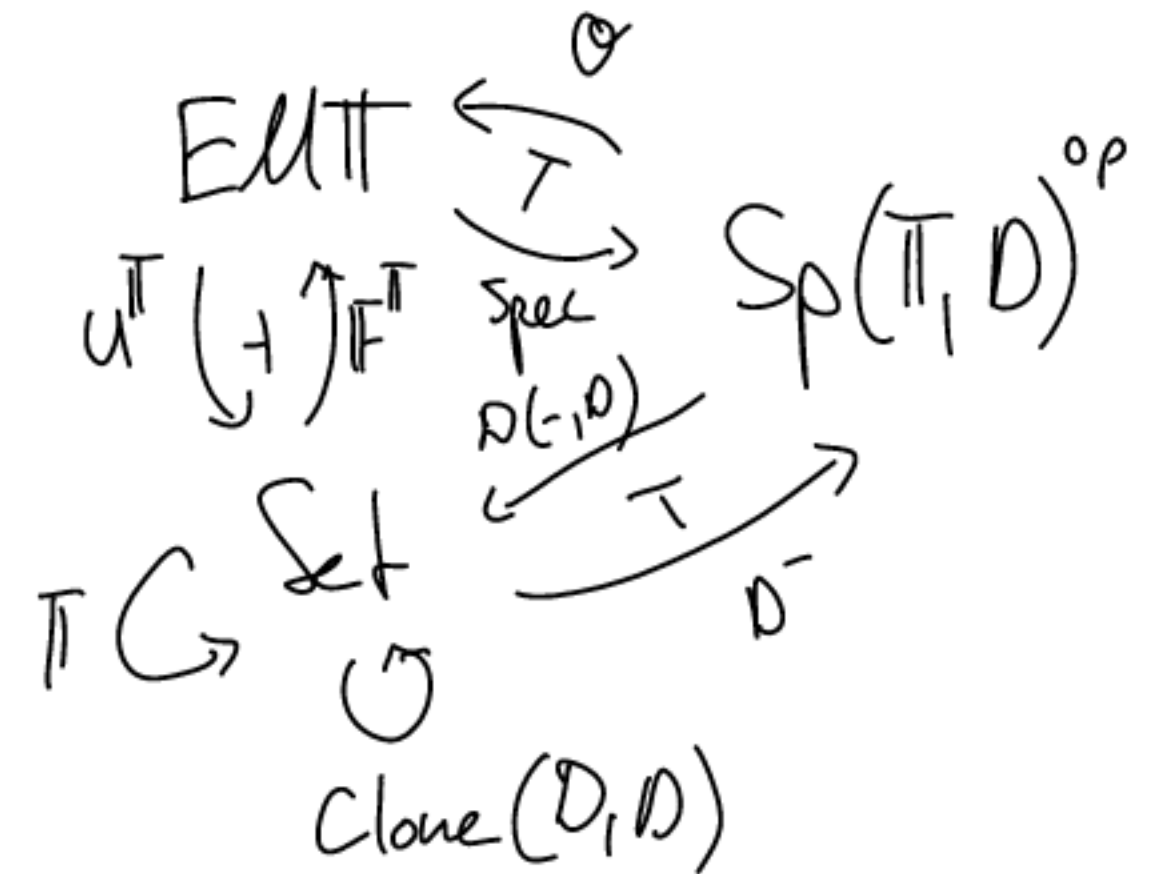
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Thm: • $\text{Clone}(\text{Sp}(\pi, D), D) \cong U^\pi \circ \text{Spec } F^\pi$

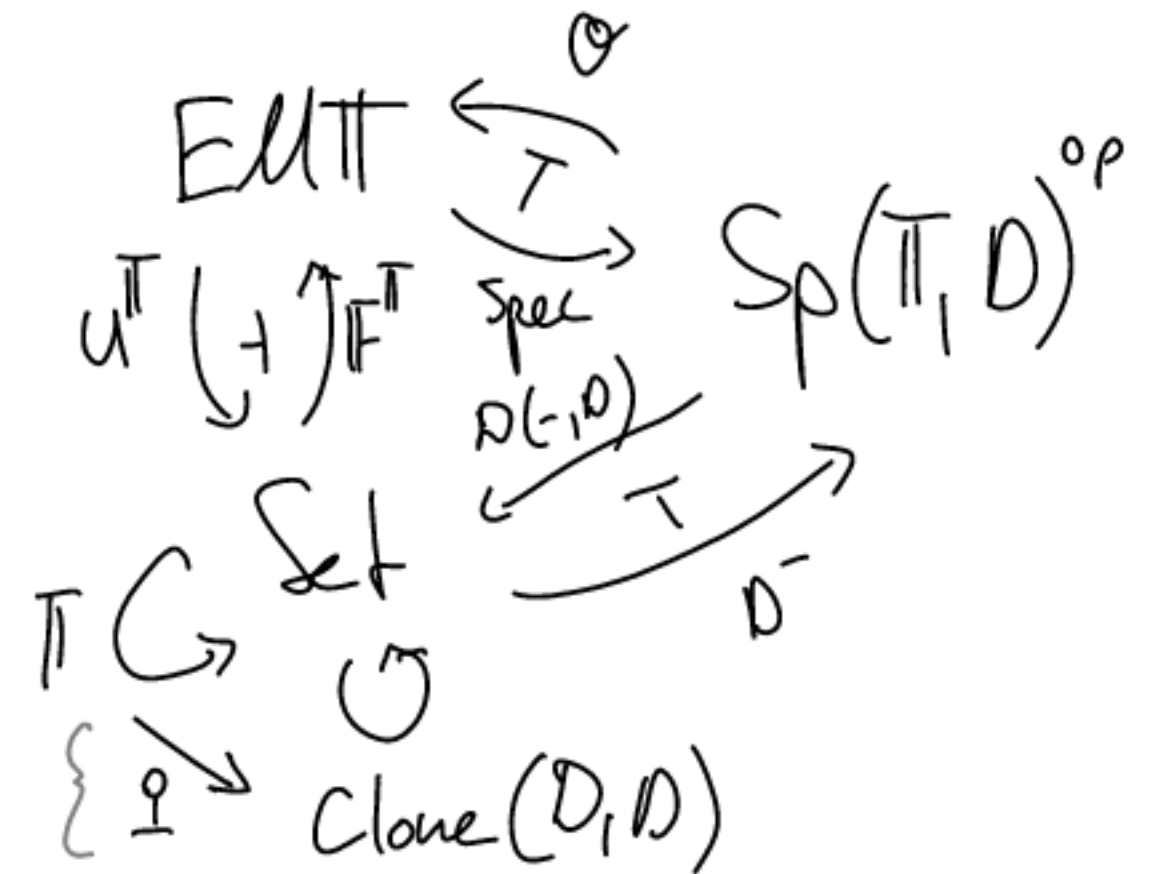
$\hookrightarrow$ Monad morphism (epi!)

$$\varphi = U^\pi \eta F^\pi: \pi X \twoheadrightarrow \text{Sp}(\pi, D)(D^X, D)$$

$$t \mapsto f \mapsto t[f] = \hat{f}(t)$$

such that

$$\begin{array}{ccc} \pi D & \xrightarrow{\varphi_D} & \text{Sp}(D)(D, D) \\ d \searrow_D & & \swarrow \text{ev}(\text{id}) \end{array}$$



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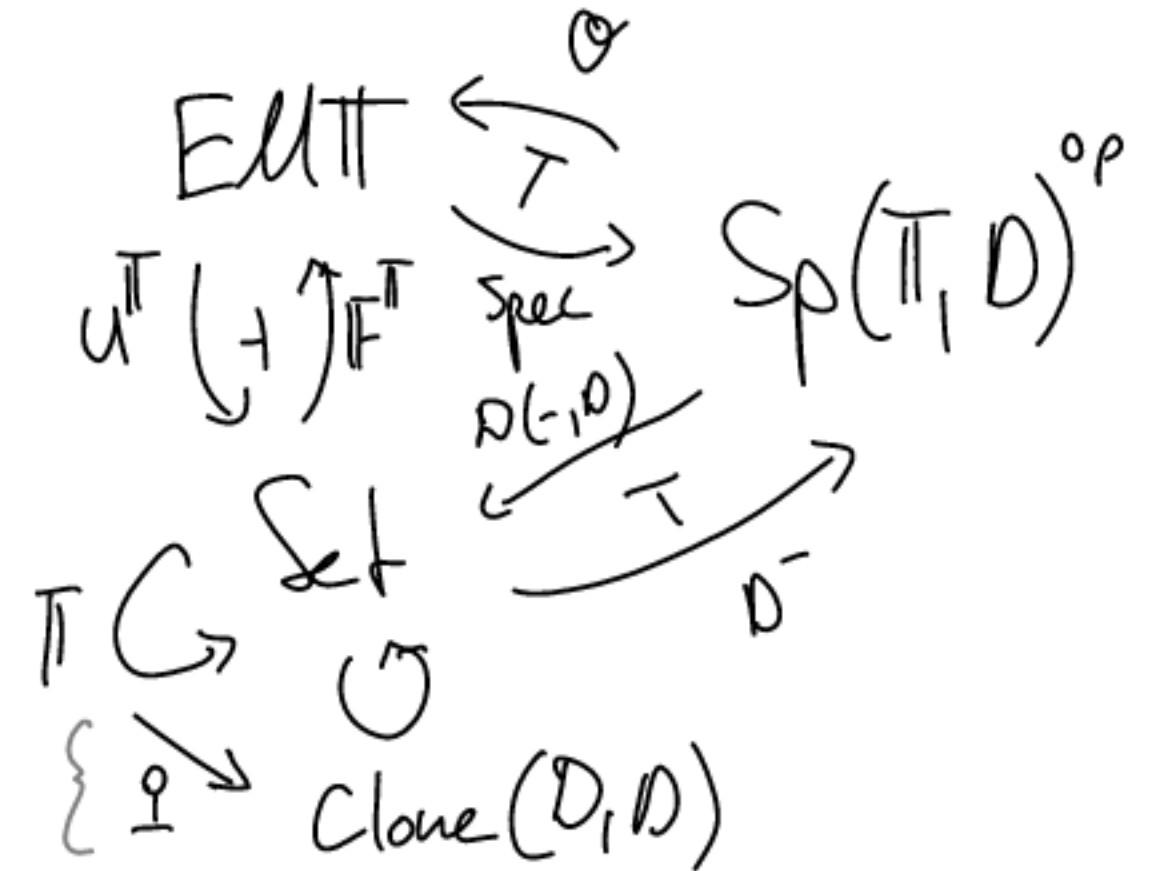
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Lemma-Def: π is D-spatial ($\begin{smallmatrix} \pi D \\ \downarrow_D \\ D \end{smallmatrix} \in \text{EM}\pi$) if φ is iso

iff D is free-coseparable $\forall s, t \in \pi X \exists f: X \rightarrow D$ s.t. $s[f] \neq t[f]$



Algebraic Roundabout, Examples

Lemma-Def: Π is D-spatial ($\begin{smallmatrix} \top D \\ \downarrow \\ D \end{smallmatrix} \in \text{EM } \Pi$) if $\underline{Q}$ is al "terms" can be separated in D
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Examples:

	$\text{EM}(\Pi)$	free coseparator	coseparator?
Set		$\mathbb{Z}$	✓
Ab		$\mathbb{Q}/\mathbb{Z}, \mathbb{Z}$	✓, ✗
Mon		$\{0, 1\}^*$	✗
Frm		$\mathbb{Z}$	✗

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$EM(\Pi)$ cocomplete

Lemma: All accessible monads have free-coseparator $D = \bigsqcup_{i < \lambda} F_i^\top$

Question: Is there a monad without free-coseparator?

The Duality of Duality Theory, Step 1: Categories

Idea: $\mathbf{Sp}$ and $\mathbf{Clone}$ form dual adjunction

↳ But: Inputs are "pointed monads" / "pointed top. cat."

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Obj: $(\pi \in \text{Mnd}, D \in \text{EM}(\pi))$
 $\uparrow_{\text{syntax}} \quad \uparrow_{\text{interpr.}}$

TOPCAT

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 $\uparrow_{\text{syntax}}$

$(x, \alpha_x) \in \int F$
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 $\mathbb{S} \xrightarrow{\varphi} \pi, \quad h: \varphi^* D \rightarrow E \text{ in } \text{EM} \pi$

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Both are [Semantics!]

$$\text{ALG} = \int \text{Mnd} \xrightarrow{\text{EM}} \text{Cat/Set} \xrightarrow{\text{down}} \text{Cat}$$

$\pi \longmapsto \text{EM} \pi$

$$\text{TOPCAT} = \int ([\text{Set}^{\text{op}}, \text{Cat}_1] \xrightarrow{\int} \text{Cat}_{\text{Set}} \rightarrow \text{Cat})$$

$F \longmapsto \int F$

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Prop: $S_p: \text{ALG} \rightarrow \text{TOPCAT}$ is a functor

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Fix: Restrict ALG and TOPCAT to cartesians

$(\varphi, \text{id}): (\mathcal{S}, \mathcal{D}) \rightarrow (\pi, \mathcal{D})$ $(\psi, \text{id}): (\mathbb{F}, (X, \mathcal{Q}_X)) \rightarrow (\mathcal{G}, (X, \mathcal{Y}_X))$

s.t. $\pi \mathcal{D} \xrightarrow{\varphi} \mathcal{S} \mathcal{D}$ s.t. $\psi \mathcal{Q}_X = \mathcal{Y}_X$

$\searrow \quad \swarrow$
 $\mathcal{D}$

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$$(\psi, id): (\mathbb{F}, D) \rightarrow (\Pi, D)$$

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$$(\psi, id): (\mathbb{F}, (X, \mathcal{Q}_X)) \rightarrow (\mathbb{G}, (X, \mathcal{P}_X))$$

$$s.t. \quad \psi \mathcal{Q}_X = \mathcal{P}_X$$

Prop: Clone: $\text{TOPCAT}_{\text{cart}} \rightarrow \text{ALG}_{\text{cart}}$ is a functor
(and S_p restricts to cartesian)

The Duality of Duality Theory, Step 3: Adjunction

Thm: Adjunction

$$\text{ALG}_{\text{cart}}^{\text{op}} \begin{matrix} \xleftarrow{\text{op}} \\ \xrightarrow{\text{top}} \end{matrix} \text{TOPCAT}_{\text{cart}}$$

- unit $\Pi \rightarrow \text{Clone}(\text{Sp}(\Pi, D), D)$

- counit induced by $\text{Sp}(\text{Clone}(D, D), D) \rightarrow D$
 $(x, \alpha_x) \mapsto \text{init } \alpha_x$ $\swarrow \text{IF}$

- adjunction is Galois!

Cor: Equivalence between Monads of the form $\text{Clone}(D, D)$
and top. categories of the form $\text{Sp}(\Pi, D)$

Relation to Predicate Liftings

$\leadsto$ Hom-Set bijection: $\varphi: \overline{\Pi} \rightarrow \text{Clone}(D, D) = \mathcal{D}(D, D)$
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Relation to Predicate Liftings

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Recall: "Simple" Codensity Lifting of Π on Set to D along $\Pi D \rightarrow D$

$$\hat{\Pi}: (X \in D) \mapsto \text{init}[\Pi X \xrightarrow{\Pi f} \Pi D \rightarrow D \mid f \in D(X, D)]$$

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Factors via $\text{Sp} \dashv \text{Clone}$:

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(2): extend to D : $\hat{\Pi} = \Gamma_{\#} \Pi_{\text{Sp}}$

What We Learned today

 NKU'20

(1) Every  monad $\mathbb{T}$ on $\mathbf{Set}$ ($\hat{=}$ algebraic theory) has canonical "dual" topological category (param. in $D \in \mathbf{EM}$)

$$\left(\begin{array}{c} \mathbf{EM}(\mathbb{T}) \\ \text{UI} \end{array} \begin{array}{c} \xleftarrow{\mathcal{P}} \\ \xrightarrow{\text{Spec} \circ \mathcal{P}} \end{array} \begin{array}{c} \mathbf{Sp}(\mathbb{T}, D) \\ \text{UI} \end{array} \right) \quad \mathcal{O}[\mathbf{Sp}] \simeq^{\mathcal{P}} \mathbf{Spec}[\mathbf{EM}]$$

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- $$\left\{ \begin{array}{c} \mathcal{D} \xrightleftharpoons[\mathcal{O} \circ q]{q \circ \text{Spec}} EM(\text{Clone}(\mathcal{D}, D)) \\ \text{UI} \qquad \qquad \qquad \text{UI} \\ Spec[EM] \qquad \qquad \mathcal{O}[D] \end{array} \right.$$
- (3) The induced representations assemble into the "duality of monads and spaces"
- $$\left\{ \begin{array}{c} \mathbb{T} \rightarrow \text{Clone}(Sp(\mathbb{T}, D), D) \\ + \\ Sp(\text{Clone}(D, D), D) \xrightleftharpoons[\text{r}]{\text{r} \circ \perp} \mathcal{D} \end{array} \right.$$
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Future Work

NKK'20

Dualizing Algebra/Space $D: 1 \rightarrow EM(\mathbb{T}) \rightsquigarrow$ parametrize to $D: \mathcal{C} \rightarrow EM(\mathbb{T})$

codensity lifting:

$$\begin{array}{ccc} \mathcal{C} & \rightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ EM(\mathbb{T}) & \rightarrow & Set \end{array}$$

algebraic geometry:

Zariski spectrum of ring R is scheme w. base

$$Z(R) \cong \operatorname{colim}_{D \in \text{Domain}} \operatorname{Ring}(R, ID) \quad \left(\operatorname{Spec} R = \operatorname{colim}_{D \in 1} \operatorname{Ring}(R, ID) \right)$$

What is the right generalization?

$$Set^{\mathbb{T}} \rightsquigarrow SMCC \mathcal{S}^{\mathbb{T}} \leftarrow \text{strong!}$$

✓ Sp and $Clone$ should work

$$\begin{aligned} \exists \Gamma: Sp(Clone(D, D), D) &\rightarrow D \\ (X, \mathcal{O}_X \hookrightarrow X \nabla D) &\mapsto (X, \text{init } \mathcal{O}_X) \text{ needs enrichment?} \end{aligned}$$

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Zariski Spectrum of $R \in Ring$ is scheme over

$$\text{colim}_{D \in \text{Domain}} Ring(R, D) \quad (\text{Spec } A = \text{colim}_{D \in 1} EM(A, \neq 0))$$

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Base: $Set^{\mathbb{T}} \rightsquigarrow SMCC \mathcal{V}^{\mathbb{T}}$ $\mathcal{V}$ -strong

✓ Sp and $Clone$ should work

? Γ takes initial lift of $\mathcal{Q}_X \hookrightarrow D^X$
 $\rightsquigarrow$ enriched top. categories

General setting 

$$\begin{array}{ccc} Sp(R, p)^{op} & \rightarrow & \mathcal{E} \quad \text{s.t. } p \text{ fibration} \\ \downarrow & \lrcorner & \downarrow \uparrow \downarrow \\ B^op & \xrightleftharpoons[\mathbb{T}]{R} & A \end{array}$$

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