Generalized Kantorovich-Rubinstein Duality beyond Hausdorff and Kantorovich

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Kantorovich-Rubinstein Duality

discrete prob. distr. on X

Metric on X $\longrightarrow$ Metric on $\mathcal{D}X$

Method 1: Transport plans $\quad \rho \in \mathcal{D}(X \times X)$ s.t. $\sum_y \rho(x,y) = \mu(x)$

$$\sum_x \rho(x,y) = \nu(y)$$

$$\begin{array}{ccc|cl}
0.1 & 0 & 0 & 0.1 & \\
0.1 & 0.2 & 0 & 0.3 & \nu \\
0.1 & 0.2 & 0.3 & 0.6 & \\
\hline
0.3 & 0.4 & 0.3 & 1 & \\
 & \mu & & &
\end{array}$$

Cost of the plan ρ:

$$\sum_{x,y} \rho(x,y) \cdot d(x,y) = \mathbb{E}_\rho(d)$$

Distance: $\partial^W(d)(\mu,\nu) = \inf \{ \mathbb{E}_\rho(d) \mid \rho \text{ is a transport plan} \}$

Kantorovich-Rubinstein Duality

discrete prob. distr. on X

Metric on X $\longrightarrow$ Metric on $\mathcal{D}X$

Method 2: Price functions $f(x)$ = price at which a logistics firm buys
$g(y)$ = price at which it sells back

s.t. $g(y) - f(x) \leq d(x,y)$: not cheaper to transport yourself

Distance: $\delta^K(d)(\mu,\nu) = \sup\{\mathbb{E}_\nu(g) - \mathbb{E}_\mu(f) \mid (f,g) \text{ valid pair of price functions}\}$

<u>Theorem</u> The distances coincide: $\delta^K(d)(\mu,\nu) = \delta^W(d)(\mu,\nu)$

Predicate Liftings & Couplings

Expectation $\mathbb{E}: (X \to [0,1]) \longrightarrow (\mathcal{D}X \to [0,1])$

$$f \longmapsto (\mu \mapsto \mathbb{E}_\mu(f))$$

Predicate lifting $\lambda: (X \to [0,1]) \longrightarrow (FX \to [0,1])$

Transport plan: $\rho \in \mathcal{D}(X \times Y)$ s.t. $\mathcal{D}\pi_1(\rho) = \mu$, $\mathcal{D}\pi_2(\rho) = \nu$

Coupling: $t \in F(X \times Y)$ s.t. $F\pi_1(t) = t_1$, $F\pi_2(t) = t_2$

Generalized Constructions (Fuzzy Relations)

$r: X \times Y \to [0,1]$ $t_1 \in FX$ $t_2 \in FY$ λ pred. lifting for F

Wasserstein lax extension:

$$W_\lambda(r)(t_1,t_2) = \inf\{\lambda_{X\times Y}(r)(t) \mid t \text{ is coupling of } t_1, t_2\}$$

Kantorovich lax extension:

truncated subtraction: $y \ominus x = \max(y-x,0)$

$$K_\lambda^{rel}(r)(t_1,t_2) = \sup\{\lambda_Y(g)(t_2) \ominus \lambda_X(f)(t_1) \mid (f,g) \text{ r-nonexp.}\}$$

$(f: X \to [0,1], g: Y \to [0,1])$ r-nonexp. $\Leftrightarrow \forall x, y.\ g(y) - f(x) \leq r(x,y)$

Generalized Constructions (Pseudometrics)

d pseudometric on X $\quad t_1, t_2 \in FX \quad$ λ pred. lifting for F

Wasserstein lifting:

$$W_\lambda(d)(t_1,t_2) = \inf \{ \lambda_{X \times X}(d)(t) \mid t \text{ is coupling of } t_1, t_2 \}$$

Kantorovich lifting:

$$K_\lambda(d)(t_1,t_2) = \sup \{ |\lambda_X(f)(t_2) - \lambda_X(f)(t_1)| \mid f : (X,d) \overset{\text{nonexpansive}}{\xrightarrow[1]{}} (\mathbb{R}, \underbrace{d_e}_{\text{Euclidean distance}}) \}$$

Generalized Duality

For which F and λ do we have ... ?

Duality:

- $K_\lambda(d) = W_\lambda(d)$ for all pseudometrics d
- $K_\lambda^{rel}(r) = W_\lambda(r)$ for all fuzzy relations r

Correspondence: Replace LHS with $K_\Lambda^{(rel)} = \sup_{\lambda' \in \Lambda} K_{\lambda'}^{(rel)}$

Generic Results

Theorem We always have $K_\lambda \leq W_\lambda$ and $K_\lambda^{rel} \leq W_\lambda$.

[Baldan, Bonchi, Kerstan, König 2018]

Theorem We always have correspondences

$$K_{\underline{\Lambda}} = W_\lambda \quad \text{and} \quad K_\Lambda^{rel} = W_\lambda .$$

[Wild, Schröder 2021]

Λ is based on a functor presentation of F and may thus be very large...

Overview of Results

✓ duality
~ correspondence
✗ correspondence, but no small Λ known

F	λ	New?	d	r	
$\mathcal{D}$	$\mathbb{E}$	no	✓	✓	classical KR duality
$\mathcal{P}$	sup	no	✓[1]	~[2]	duality for Hausdorff distance
$\mathrm{Poly}(\mathcal{D}, \mathcal{P})$	$\mathrm{poly}(\mathbb{E}, \mathrm{sup})$	no	~[3]	✗	
$\mathcal{D}$	$\mathbb{E}_p$	yes	✗	✗	p-Wasserstein distance
$\mathcal{D}$	"generally"	yes	✓	✓	Lévy-Prokhorov distance
$\mathcal{C}$	$\mathrm{sup} \circ \mathbb{E}$	yes	✓	✗	Convex Powerset functor

duality known to fail in general

1 Baldan, Bonchi, Kerstan, König 2018
2 Wild, Schröder 2021
3 Humeau, Petrisan, Rot 2025

p-Wasserstein distance

$p \geq 1$

$$\partial^{W,p}(d)(\mu,\upsilon) = \inf \left\{ \left(\mathbb{E}_{\varsigma}(d^{p}) \right)^{1/p} \mid \varsigma \text{ coupling of } \mu,\upsilon \right\}$$

pointwise

$\leadsto$ predicate lifting $\mathbb{E}_p(f)(\mu) = \left(\mathbb{E}_\mu(f^p) \right)^{1/p}$

__Proposition.__ For $p=2$, $K_{\mathbb{E}_p} \neq W_{\mathbb{E}_p}$.

Lévy-Prokhorov Distance

$$\delta^{LP}(d)(\mu,\nu) = \inf\{\varepsilon \mid \forall A \subseteq X.\ \mu(A) \leq \nu(A^d_\varepsilon) + \varepsilon\}$$

$$A^d_\varepsilon = \{x \mid d(A,x) \leq \varepsilon\}$$

Ky Fan representation: $\delta^{LP}(d) = W_\lambda(d)$, where

$$\lambda_X(f)(\mu) = \inf_{\varepsilon \geq 0} \max(\varepsilon, \mu(\{x \mid f(x) > \varepsilon\}))$$

"$f(x)$ is high with high probability"

"f is generally true"

Lévy-Prokhorov Distance

Theorem We have $K_\lambda = W_\lambda$ and $K_\lambda^{rel} = W_\lambda$.

Proof sketch

- $W_\lambda(d)(\mu,\nu) = \inf\limits_{\varepsilon \geq 0} \max(\varepsilon, W_E(r^\varepsilon)(\mu,\nu))$

$$r^\varepsilon(x,y) = \begin{cases} 0, & \text{if } r(x,y) < \varepsilon \\ 1, & \text{if } r(x,y) \geq \varepsilon \end{cases}$$

- Use classical duality. Costs are in $\{0,1\}$, so prices are too.
- Transform price functions for $K_E(r^\varepsilon)$ into ones for $K_\lambda(r)$. $\square$

Convex Powerset Functor

subfunctor of $\mathcal{P}\mathcal{D}$

$$\mathcal{C}X = \{\emptyset \neq D \subseteq \mathcal{D}X \mid D \text{ is convex}\}$$

Predicate lifting: $\lambda = \sup \circ \mathbb{E}$ $\qquad \lambda_X(f)(A) = \sup_{\mu \in A} \mathbb{E}_\mu(f)$

Hausdorff distance

Known results:

- $W_\lambda = W_{\sup} \circ W_{\mathbb{E}} = \delta^H \circ \delta^K$ ← Kantorovich distance
- Duality fails for $\mathcal{P}\mathcal{D}$

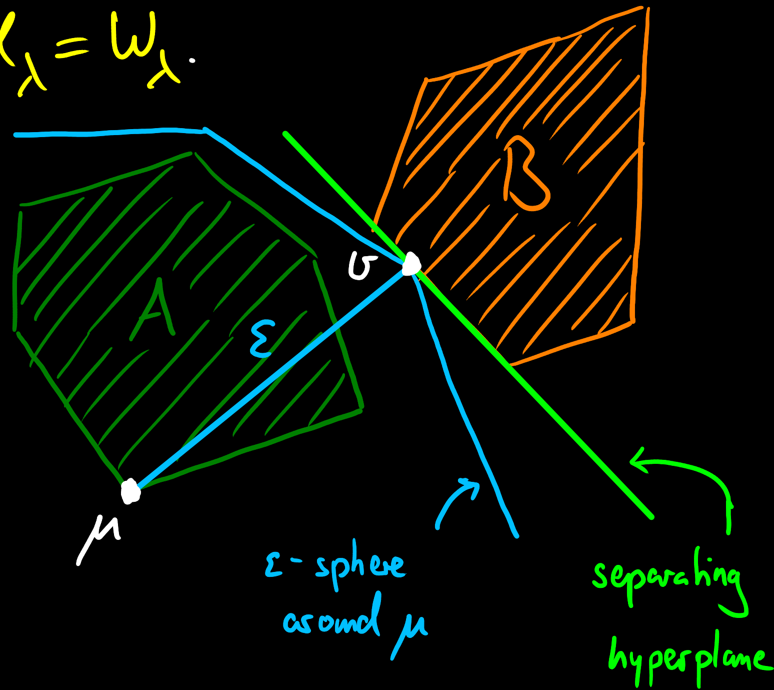
Convex Powerset Functor

Theorem We have $K_\lambda = W_\lambda$.

Proof sketch.

$$\delta^H(\delta^K(d))(A, B)$$
$$= \delta^K(d)(\mu, \upsilon) = \varepsilon$$

Extract price function from hyperplane



Computing Distances

finitely supported

Compute $W_\lambda(d)(\mathrm{conv}\{\mu_1, \ldots, \mu_n\}, \mathrm{conv}\{\nu_1, \ldots, \nu_n\})$

- Algorithm based on W_λ: need to solve exponentially many LPs
- Algorithm based on K_λ: need to solve polynomially many LPs

$$K_\lambda(d)(A,B) = \sup\left\{ \left| \sup_{\nu \in B} \mathbb{E}_\nu(f) - \sup_{\mu \in A} \mathbb{E}_\mu(f) \right| \;\middle|\; f : (X,d) \to \mathbb{R} \text{ nonexp.} \right\}$$

Idea: These suprema are reached at some μ_i and ν_j

$\rightsquigarrow$ Two LP for each i and j

Beyond Sets

More general dualities:

metric spaces

- Hausdorff functor $H: \mathrm{Met} \to \mathrm{Met}$, HX = closed subsets of X
- Kantorovich functor $K: \mathrm{Met} \to \mathrm{Met}$, KX = Radon prob. measures on X

pseudometric spaces

Theorem Duality holds also for $C: \mathrm{PMet} \to \mathrm{PMet}$,

CX = non-empty convex subsets of KX

Open: What about Lévy-Prokhorov?

Future Work

- Duality for Lévy-Prokhorov beyond sets
- Correspondence for $\mathcal{C}$ over fuzzy relations

  conjecture: $\max(K^{rel}_{\sup\circ\mathbb{E}}, K^{rel}_{\inf\circ\mathbb{E}}) = W_{\sup\circ\mathbb{E}}$
- Distinguishing formulae
- Generalized Lévy-Prokhorov

  WIP
- Dualities / correspondences involving lax couplings
- Fully categorical Kantorovich-Rubinstein duality