Heterogeneous Simulations

Historia Calamitatum

Florian Guthmann
`florian.guthmann@fau.de`

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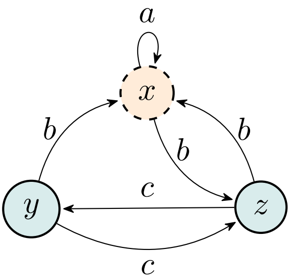
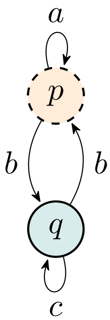
Motivation

What states are *behaviourally equivalent*?





$$r = \{(x, p), (y, q), (z, q)\}$$

Bisimulation of $\mathcal{A}$-LTSs

Let $(X, \gamma \colon X \to \mathcal{P}(\mathcal{A} \times X))$, $(Y, \delta \colon Y \to \mathcal{P}(\mathcal{A} \times Y))$

$$
\begin{array}{ccc}
x & \overset{r}{\rule{3em}{0.4pt}} & y \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\
\gamma(x) & \overset{\overline{F}r}{\rule{3em}{0.4pt}} & \delta(y)
\end{array}
$$

$$
\begin{aligned}
\gamma(x) \overset{\tilde{r}}{\rule{2em}{0.4pt}} \delta(y) \iff & (\forall\, x' \in \gamma(x).\, \exists\, y' \in \delta(y).\, x' \overset{r}{\rule{2em}{0.4pt}} y') \wedge \\
& (\forall\, y' \in \delta(y).\, \exists\, x' \in \gamma(x).\, x' \overset{r}{\rule{2em}{0.4pt}} y')
\end{aligned}
$$

Idea

Just lift r along the functor!

The Barr Extension*

Definition

Given a relation $r\colon X \nrightarrow Y$, take the span $X \xleftarrow{\pi_1} r \xrightarrow{\pi_2} Y$ then

$$\overline{F}r = \mathrm{gr}\,(F\pi_2) \circ (\mathrm{gr}\,(F\pi_1))^\circ$$

Example

For $F := \mathcal{P}$

$$A\ \overline{\mathcal{P}}r\ B \iff (\forall\, a \in A.\, \exists\, b \in B.\, a \xrightarrow{r} b) \wedge (\forall\, b \in B.\, \exists\, a \in A.\, a \xrightarrow{r} b)$$

Egli-Milner Extension

*Barr 1970.

No Barr Extension

Remark

$\overline{F}$-similarity captures behavioural equivalence *iff* F preserves weak pullbacks

Not all functors do, e.g. the monotone neighbourhood functor $\mathcal{M}$ does not

$\mathcal{P} \circ \mathcal{P}$ restricted to upsets

Idea

Need a laxer notion of extension!

Lax Extensions[†]

Definition

Let $F\colon \mathsf{Set} \to \mathsf{Set}$. A lax extension L of F consists of a map

$$(r\colon X \mathrel{+\!\!\!\rightarrow} Y) \mapsto (Lr\colon FX \mathrel{+\!\!\!\rightarrow} FY)$$

such that

$$r \subseteq r' \implies Lr \subseteq Lr' \qquad \text{(monotonicity)}$$

$$Ls \circ Lt \subseteq L(s \circ t) \qquad \text{(lax functoriality)}$$

$$\mathrm{gr}\,(Ff) \subseteq L(\mathrm{gr}\,(f)) \text{ and } \mathrm{gr}\,(Ff)^{\circ} \subseteq L(\mathrm{gr}\,(f)^{\circ}) \qquad \text{(lifting)}$$

[†] Marti and Venema 2012.

Example

Lemma

The monotone neighbourhood functor $\mathcal{M}$ has a lax extension given by

$$\tilde{\mathcal{M}}(r) = \mathrm{Forth}(\mathrm{Back}(r)) \cap \mathrm{Back}(\mathrm{Forth}(r))$$

$$\mathrm{Forth}(r) = \left\{ (U, V) \,\middle|\, \forall\, u \in U\,.\, \exists\, v \in V\,.\, u \xrightarrow{\;r\;} v \right\}$$

$$\mathrm{Back}(r) = \left\{ (U, V) \,\middle|\, \forall\, v \in V\,.\, \exists\, u \in U\,.\, u \xrightarrow{\;r\;} v \right\}$$

Remark

Egli-Milner extension $\overline{\mathcal{P}}(r) = \mathrm{Forth}(r) \cap \mathrm{Back}(r)$

Bisimulations

For coalgebras $(C, \gamma \colon C \to FC)$, $(D, \delta \colon D \to FD)$ a relation $r \colon C \nrightarrow D$ is a L-bisimulation if

$$\begin{array}{ccc} C & \xrightarrow{\quad r \quad} & D \\ {\scriptstyle \mathrm{gr}(\gamma)} \big\downarrow & \subseteq & \big\uparrow {\scriptstyle \mathrm{gr}(\delta)^{\circ}} \\ FC & \xrightarrow[\quad Lr \quad]{} & FD \end{array}$$

$$c \overset{r}{\text{——}} d \implies \gamma(c) \overset{Lr}{\text{——}} \delta(d)$$

Lax Extensions

Definition

A lax extension L is **identity preserving** if

$$L(\Delta_X) \subseteq \Delta_{FX}$$

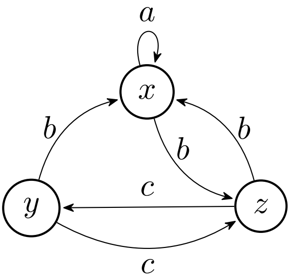
It is **converse preserving** if

$$L(r^\circ) = (Lr)^\circ$$

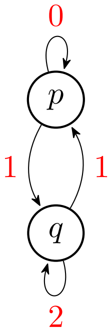
Result (Marti and Venema 2012)

If L preserves identities and converses then L-bisimulation captures behavioural equivalence

Systems of Different Type



$$FX = \mathcal{P}\left(\mathcal{A} \times X\right)$$



$$FX = \mathcal{P}\left(\mathbb{N} \times X\right)$$

Relational Connectors[‡]

Definition

Let $F, G \colon \mathsf{Set} \to \mathsf{Set}$. A relational connector $L \colon F \to G$ maps a relation $r \colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ to

$$Lr \colon FX \relbar\joinrel\mapstochar\joinrel\rightarrow GY$$

such that for $r' \colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y, f \colon X' \to X,\ g \colon Y' \to Y$

$$r \subseteq r' \implies Lr \subseteq Lr' \qquad \text{(monotonicity)}$$

$$L(\mathrm{gr}\,(g)^{\circ} \circ r \circ \mathrm{gr}\,(f)) = (\mathrm{gr}\,(Gg))^{\circ} \circ Lr \circ \mathrm{gr}\,(Ff) \qquad \text{(naturality)}$$

[‡] Nora et al. 2025.

LTSs over different Alphabets

Let $FX = \mathcal{P}\,(\mathcal{A} \times X)$, $GY = \mathcal{P}\,(\mathcal{B} \times Y)$
For $R \colon \mathcal{A} \nrightarrow \mathcal{B}$ let $L_R \colon F \to G$

$$S\ L_R r\ T \iff \forall\,(a,b) \in R.\,(\forall\,(a,x) \in S.\,\exists\,(b,y) \in T.\,x \overset{r}{\longrightarrow} y) \wedge$$
$$(\forall\,(b,y) \in T.\,\exists\,(a,x) \in S.\,x \overset{r}{\longrightarrow} y)$$

Constructions

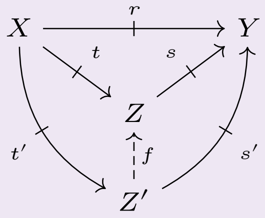
Definition

For $K\colon F \to G$, $L\colon G \to H$ the **composite** $L \circ K\colon F \to H$ is given on $r\colon X \nrightarrow Y$:

$$(L \circ K)r = \bigcup_{r = s \circ t} Ls \circ Kt$$

$$= Ls \circ Kt$$

couniversal factorization



Constructions

Definition

For $L\colon F \to G$, the **converse** $L^\circ\colon G \to F$ on $r\colon X \nrightarrow Y$ is given by

$$L^\circ r = L(r^\circ)^\circ\colon GX \nrightarrow FY$$

Lemma

For relational connectors $L\colon F \to G$, $K\colon G \to H$ *we have*

$$(L^\circ)^\circ = L$$

and

$$(K \circ L)^\circ = L^\circ \circ K^\circ$$

Constructions

Definition

The **identity relational connector** $\mathrm{Id}_F \colon F \to F$ is given on $r \colon X \nrightarrow Y$ by

$$x \,\mathrm{Id}_F r\, y \iff \forall G \colon \mathsf{Set} \to \mathsf{Set}, L \colon G \to F, s \colon Z \nrightarrow X, z \in GZ.$$
$$z\, Ls\, x \implies z\, L(r \circ s)\, y$$

Theorem

As expected, for $L \colon F \to G$ *we have*

$$L = \mathrm{Id}_G \circ L = L \circ \mathrm{Id}_F$$

Lax Extensions are Connectors

Definition

A relational connector $L\colon F \to F$ is **transitive** if

$$L \circ L \leq L$$

lax functorality

L **extends** F if for all X,

$$\Delta_{FX} \subseteq L(\Delta_X)$$

lifting

Theorem

Lax extensions of F are precisely transitive relational connectors that extend F.

Heterogeneous Simulations

Definition

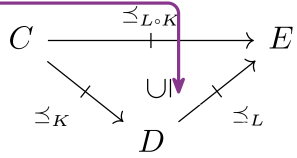
Let $(C, \gamma\colon C \to FC)$, $(D, \delta\colon D \to GD)$ be coalgebras. A relation $r\colon C \nrightarrow D$ is an **L-simulation** for a relational connector $L\colon F \to G$ if

$$\begin{array}{ccc} C & \xrightarrow{\;r\;} & D \\ {\scriptstyle \mathrm{gr}(\gamma)} \big\downarrow & \subseteq & \big\uparrow {\scriptstyle \mathrm{gr}(\delta)^{\circ}} \\ FC & \xrightarrow[\;Lr\;]{} & GD \end{array}$$

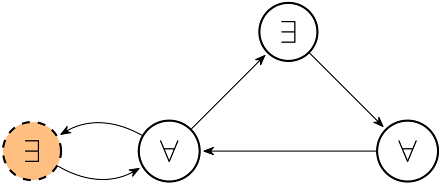
$c \in C$ and $d \in D$ are **L-similar** $(c \preceq_L d)$, if there exists an L-simulation r such that $c \overset{r}{\text{---}} d$

Composites of Simulations

Does the converse hold?



Safety Games



●-positions are unsafe!

Safety Games

Safety {∀, ∃}

Games

$$GX = 2 \times 2 \times X^{+}$$

Move by ∃ Move by ∀

Strategies

$$FY = Y \times \mathbb{N} + Y^{\mathbb{N}}$$

Define $L \colon G \to F$ on $r \colon C \nrightarrow D$

$$(b, o, (c_0, \dots, c_n))\; Lr\; (i, d) \iff b = \top \wedge o = \exists \wedge i \leq n \wedge c_i \overset{r}{\text{———}} d$$

$$(b, o, (c_0, \dots, c_n))\; Lr\; u \iff b = \top \wedge o = \forall \wedge \forall\, i \leq n.\, c_i \overset{r}{\text{———}} u(i)$$

Safety Games

Lemma

A strategy state d wins a game position c iff $c \preceq_L d$

Lemma

Similarity of L and L° composes, i.e.

$$\preceq_{L \circ L^\circ} = \preceq_L \circ \preceq_{L^\circ}$$

$x \preceq_{L \circ L^\circ} y \iff$ there is a strategy that wins both positions with the same (by index) moves

Safety Games, Anew

Games

$$GX = 2 \times 2 \times \mathcal{P}_{\mathrm{ne}}(X)$$

Strategies

$$FY = Y + \mathcal{P}_{\mathrm{ne}}(Y)$$

Define $L\colon G \to F$ on $r\colon C \nrightarrow D$

$$(b, o, T)\ Lr\ x \iff b = \top \wedge o = \exists \wedge \exists\, y \in T.\, x \xrightarrow{r} y$$

$$(b, o, T)\ Lr\ S \iff b = \top \wedge o = \forall \wedge \forall\, y \in T.\, \exists\, x \in S.\, x \xrightarrow{r} y$$

References

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- Marti, Johannes and Yde Venema (2012). "Lax Extensions of Coalgebra Functors". In: *Coalgebraic Methods in Computer Science*. Ed. by Dirk Pattinson and Lutz Schröder. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 150–169. ISBN: 978-3-642-32784-1. URL: https://coalg.org/cmcs12/papers/00010149.pdf.

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