Iterations and their Topology

Stefan Milius and Halimeh Moghbeli

The Chair of Theoretical Computer Science (INF8)

11 February 2025

Motivation

- Different views of objects make Mathematical Theories rich.

Topological spaces

Definition

A topological space is a pair (X, τ), where X is a set and τ (called a topology on X) is a family of subsets of X such that:

- τ is closed under finite intersections;
- τ is closed under arbitrary unions.

Definition

- Each element of τ is called open. A subset of X is called closed, if its complement is open.
- If both A and A^c are open, then A is called clopen.

Definition

Let (X, τ) be a topological space and $A \subseteq X$. The interior of A (denoted A^o) is defined to be the largest open subset of X contained in A.

Another view of Topological spaces

This induces a map

$$I \colon \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$$
$$A \mapsto A^o.$$

based on the theorem

$$A \in \tau \text{ iff } A^o = A$$

we have

$$\text{Img(I)} = \tau.$$

So:

$$A \in \mathrm{Img}(I) \text{ iff } I(A) = A$$

Consequently:

$$\text{for all } A \subseteq X,\ I(I(A)) = I(A).$$

Definition

A topological space is a set X with a function $I\colon \mathscr{P}(X) \to \mathscr{P}(X)$ which for all $A, B \subseteq X$ satisfies the following axioms:

- $I(A \cap B) = I(A) \cap I(B)$;
- $I(A) \subseteq A$;
- $I^2(A) = I(A)$;
- $I(X) = X$.

Direct image mapping

Let $f : X \to X$ be a function. Then we have the direct image mapping on $\mathscr{P}(X)$ which is:

$$f[-] : \mathscr{P}(X) \to \mathscr{P}(X), A \mapsto f[A],$$
$$f[A] = \{f(a) \mid a \in A\}.$$

Are direct image maps topologies?

To have this, we need

$$\text{for all } A \subseteq X,\ f[A] \overset{?}{\subseteq} A$$

Invariant subsets

Definition

Let $f : X \to X$ be a function on a set X. A subset A of X is called invariant under f, if $f(A) \subseteq A$.

- The set of all invariant subsets of X under f, will be denoted by $\mathfrak{L}(X, f)$.

Does this topology deserve a name!!!?

Theorem

For any set X *and each function* $f : X \to X$, $\mathfrak{L}(X, f)$ *is an Alexandroff topology on* X.

Orbits or Iterations of a function

Definition

Let $f\colon X \to X$ be a function and $x \in X$. The sequence $\{f^n(x)\}_{n\geq 0}$ is called an orbit of f at x. This set will be denoted by U_x, for each $x \in X$.

Iterations form a basis

Theorem

Let $f : X \to X$ be a function, then the set of all orbits of f forms a basis for $\mathfrak{L}(X, f)$. Moreover, it is the smallest basis for $\mathfrak{L}(X, f)$.

Indiscrete Topology and non-surjective Functions

Theorem

For each non-surjective $f\colon X \to X$, $\mathfrak{L}(X, f) \neq \{\emptyset, X\}$. *In other words, the indiscrete topology on a set* X *never induced by non-surjective functions.*

Identity map and its Topology

Theorem

For a function $f : X \to X$, the followings are equivalent:

- f *is identity;*
- $\mathfrak{L}(X, f) = \mathscr{P}(X)$;
- $\mathfrak{L}(X, f)$ *is Hausdorff.*

Theorem

Let $f : X \to X$ be a function. Then f is bijective if and only if for each $A \subseteq X$,

$$A \in \mathfrak{L}(X, f) \text{ if and only if } A^c \in \mathfrak{L}(X, f).$$

Theorem

If $f : X \to X$ is a bijective function (which is not the identity function), then $\mathfrak{L}(X, f)$ is not T_0.

Fixed points and open sets

Theorem

Let $f : X \to X$ be a function. Then the following statements are equivalent:

1. $f(x) = x$,
2. $\{x\}$ *is open,*

Moreover, if f is bijective, then they are equivalent with

3. $\{x\}$ *is closed.*

Continuity

Theorem

Let $h\colon X \to Y$, $f\colon X \to X$ *and* $g\colon Y \to Y$ *be functions such that* $vg \circ h = h \circ f$ *(or equivalently, the following diagram is commutative). Then* $h\colon X \to Y$ *is continuous with respect to the topologies* $\mathfrak{L}(X, f)$ *and* $\mathfrak{L}(Y, g)$ *considered on* X *and* Y*, respectively.*

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\;h\;]{} & Y \end{array}$$

Naturally Continuous

Theorem

Let $f : X \to X$ be a function, then f is continuous with respect to the $\mathfrak{L}(X, f)$.

Some conditions under which the converse it true

Theorem

If $h\colon X \to Y$ *is a continuous function with respect to the topologies* $\mathfrak{L}(X, f)$ *and* $\mathfrak{L}(Y, g)$ *on* X *and* Y*, respectively, and moreover,* $g \circ h = h$*. Then* $h \circ f = h = g \circ h$*.*

Theorem

Let $f: X \to X$ be a continuous function, where $g: X \to X$ is an arbitrary function on X and the topologies $\mathfrak{L}(X, g)$ and $\mathfrak{L}(X, f)$ are those ones considered on the domain and co-domain of f, respectively. Then $\mathfrak{L}(X, f) \subseteq \mathfrak{L}(X, f \circ g)$.

Subspace topology

Theorem

Let $f\colon X \to X$ be a function and $B \in \mathfrak{L}(X, f)$. Then the subspace topology on B, induced by $\mathfrak{L}(X, f)$ coincides $\mathfrak{L}(B, f\mid_B)$, where $f\mid_B\colon B \to B$ denotes the restriction function of f to B.

Compact spaces

Theorem

Consider the topological space $(X, \mathfrak{L}(X, f)))$. Then, X is compact if and only if $X \setminus f(X)$ is finite and $f(X)$ is compact.

Thank you!