Algebraic Language Theory with Effects

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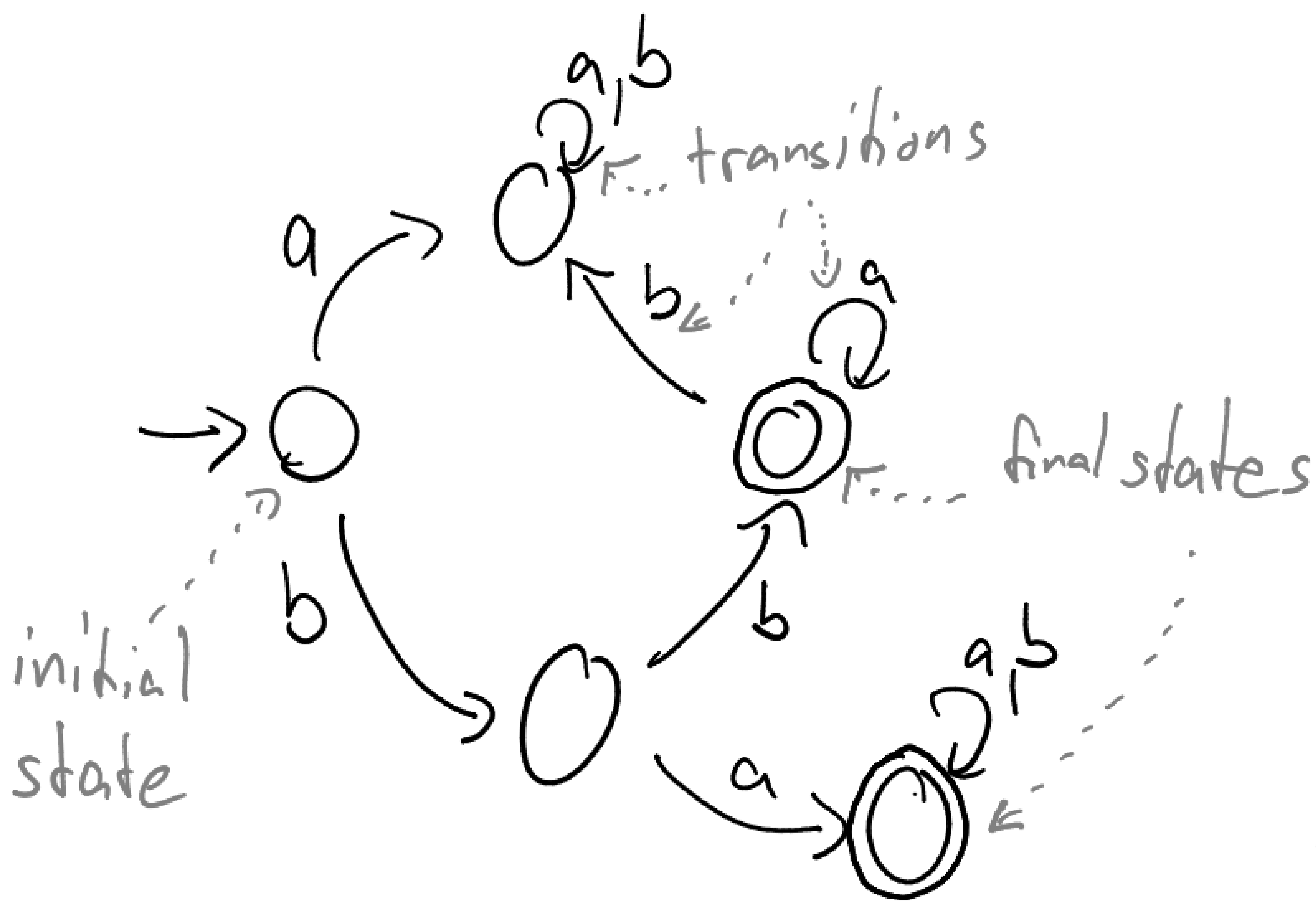
Thorsten Wißmann, Henning Urbat, Stefan Milius

Introduction: Automata, Languages and Monoids

DFAs

$\Sigma = \{a, b\}$



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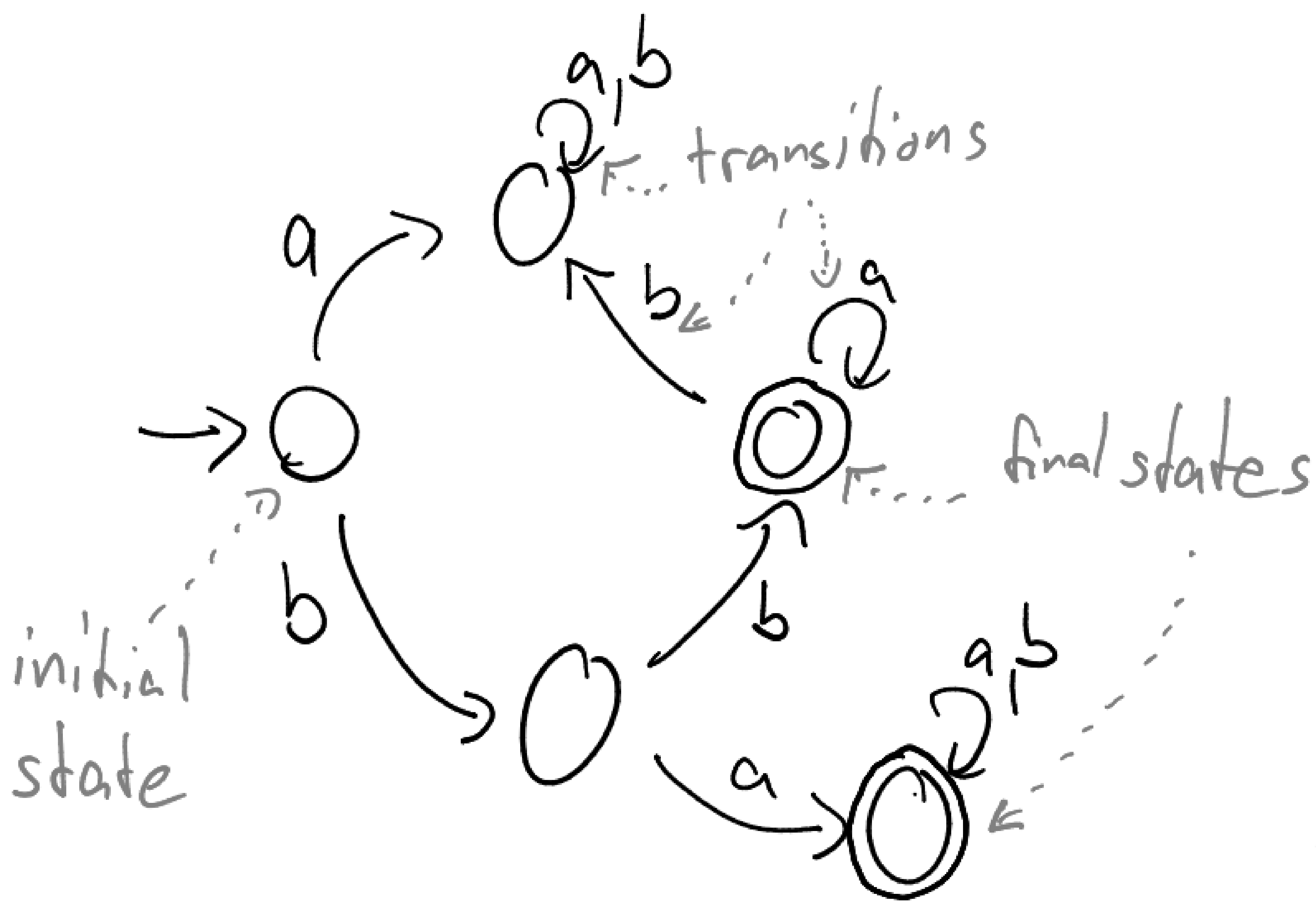

$\widehat{=}$

Finite state set Q + functions

$1 \xrightarrow{i} Q$, $\quad Q \times \Sigma \xrightarrow{\delta} Q$, $\quad Q \xrightarrow{f} 2$

initial state — transitions — final states $\subseteq Q$

curry/extend to

$\delta^* : \Sigma^* \to (Q \to Q)$

$\delta^*(\varepsilon) = id$

$\delta^*(aw) = Q \xrightarrow{\delta(-,a)} Q \xrightarrow{\delta^*(w)} Q$

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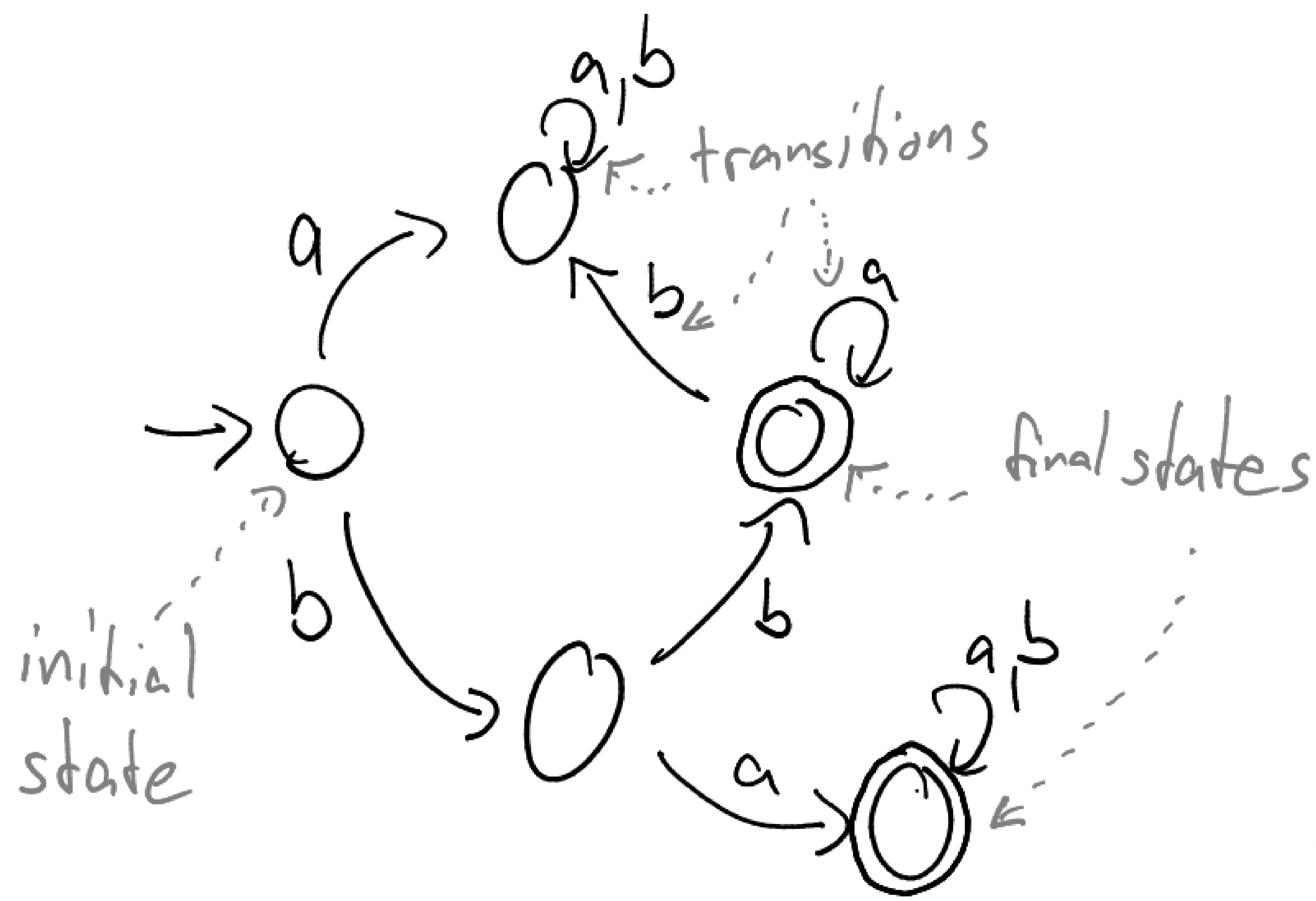

$\hat{=}$ Finite state set Q + functions

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initial state, transitions, final states $\subseteq Q$

curry/extend to

$\delta : \Sigma^* \to (Q \to Q)$

$\delta^*(\varepsilon) = id$

$\delta^*(aw) = Q \xrightarrow{\delta(-,a)} Q \xrightarrow{\delta^*(w)} Q$

$\rightsquigarrow$ computes a Language $L : \Sigma^* \to 2$

$w \mapsto (1 \xrightarrow{i} Q \xrightarrow{\delta^*(w)} Q \xrightarrow{f} 2)$

Introduction: Automata, Languages and Monoids

Monoid: Set M +

$1 \xrightarrow{e} M$, $M \times M \xrightarrow{\cdot} M$

$$\begin{array}{ccccc} M \times 1 & \xrightarrow{id \times e} & M \times M & \xleftarrow{e \times id} & 1 \times M \\ \cong & & \downarrow \cdot & & \cong \\ M & = & M & = & M \end{array} \quad \text{(unit)}$$

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{1 \times \cdot} & M \times M \\ \cdot \times 1 \downarrow & & \downarrow \cdot \\ M \times M & \xrightarrow{\cdot} & M \end{array} \quad \text{(assoc)}$$

homomorphism:

$$\begin{array}{ccc} M \times M & \xrightarrow{\cdot} & M \\ f \times f \downarrow & & \downarrow f \\ N \times N & \xrightarrow{\cdot} & N \end{array}$$

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Examples:

$$\Sigma^*: \quad 1 \xrightarrow{\varepsilon} \Sigma^*, \quad \Sigma^* \times \Sigma^* \xrightarrow{concat} \Sigma^*$$

(free monoid!)

$$Q \to Q: \quad 1 \xrightarrow{id} (Q \to Q), \quad (Q \to Q) \times (Q \to Q) \xrightarrow{;} (Q \to Q)$$

$$(f;g)(x) = g(f(x))$$

$$\Delta_Q = Q \times Q + \{0,1\}: \quad 1 \xrightarrow{1} \Delta_Q$$

$$\Delta_Q \times \Delta_Q \to \Delta_Q$$

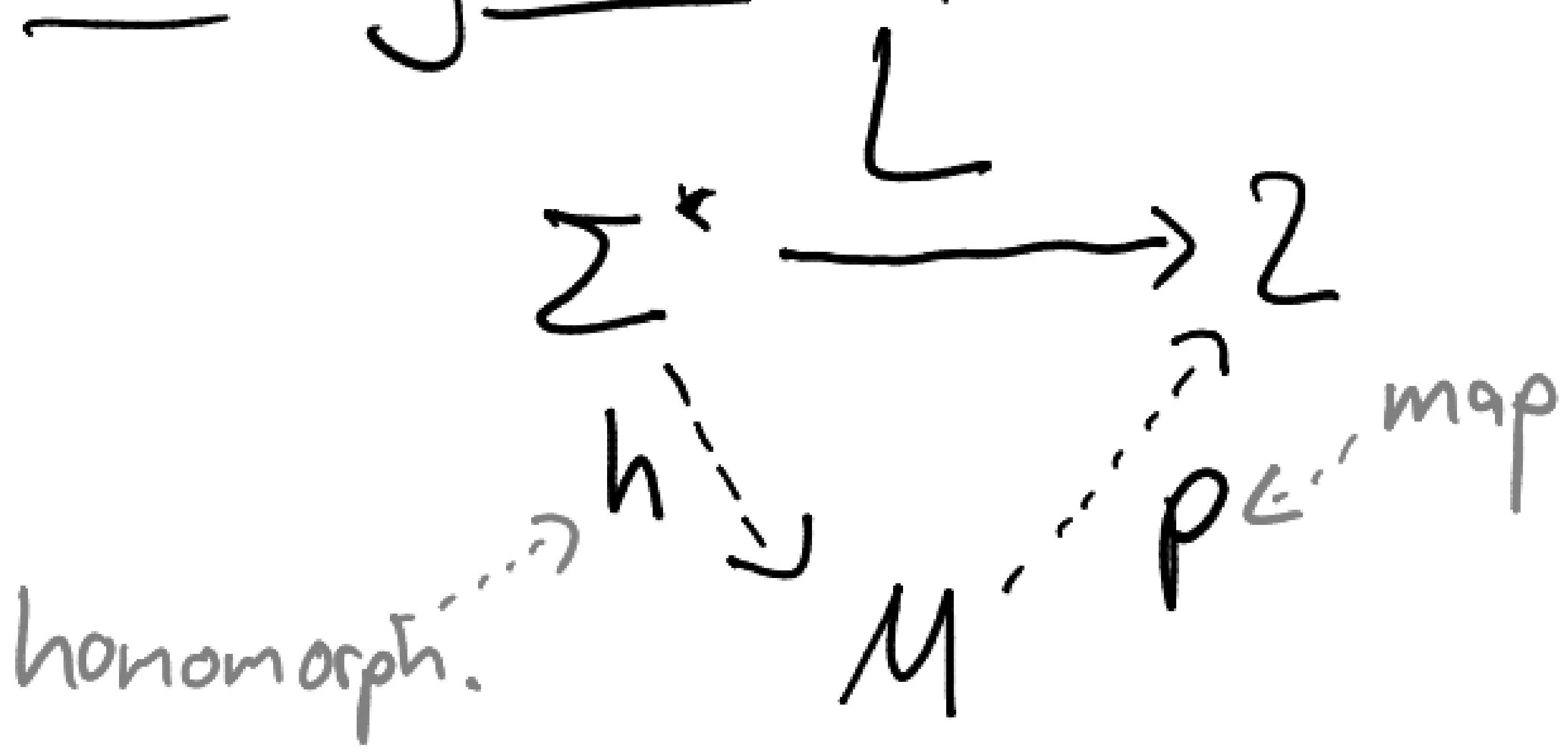
$$1 \cdot x = x \cdot 1 = x,$$

$$0 \cdot x = x \cdot 0 = 0$$

$$(s,t) \cdot (t', u) = \begin{cases} (s,u) & \text{if } t = t' \\ 0 & \text{else} \end{cases}$$

Introduction: Automata, Languages and Monoids

Recognition by homomorphisms



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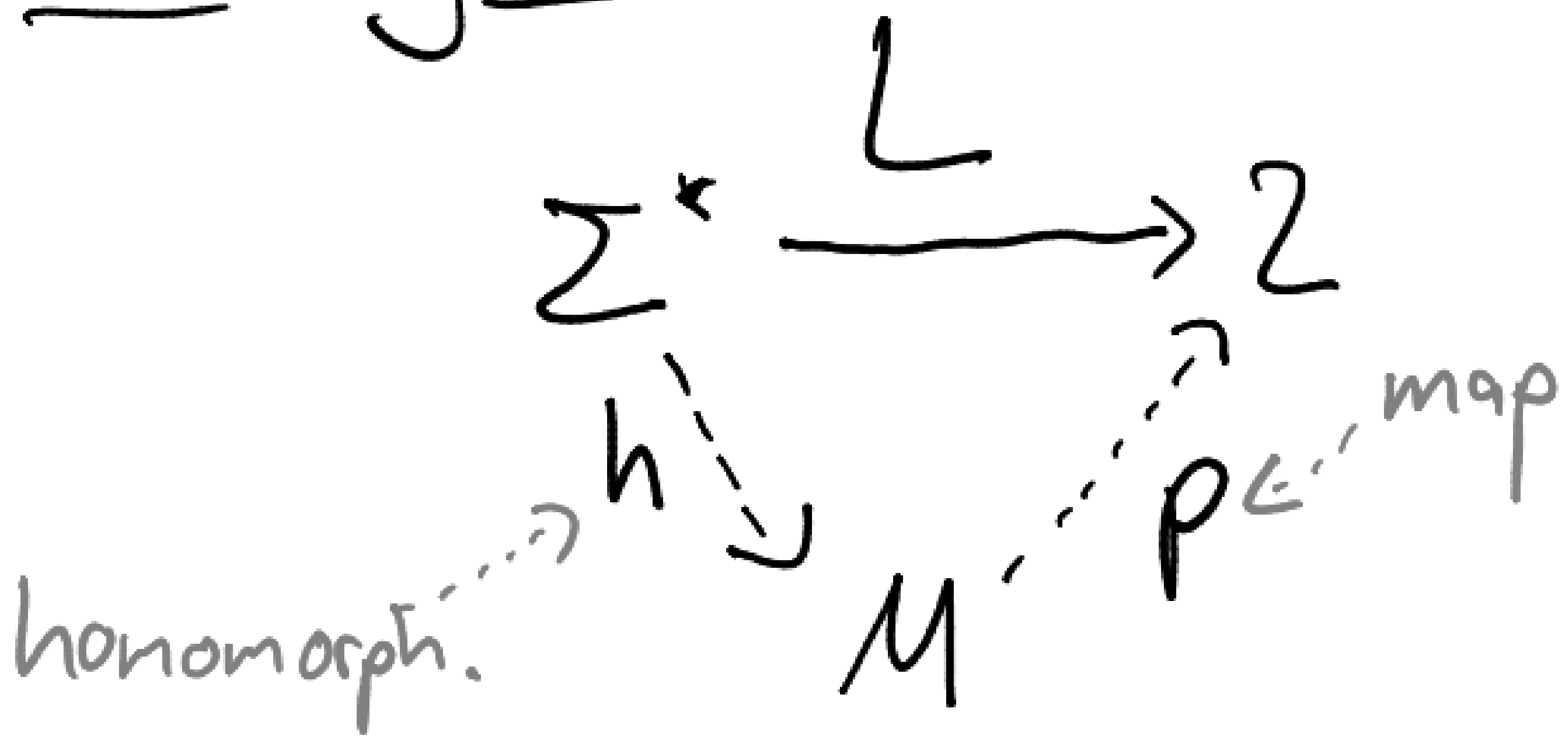
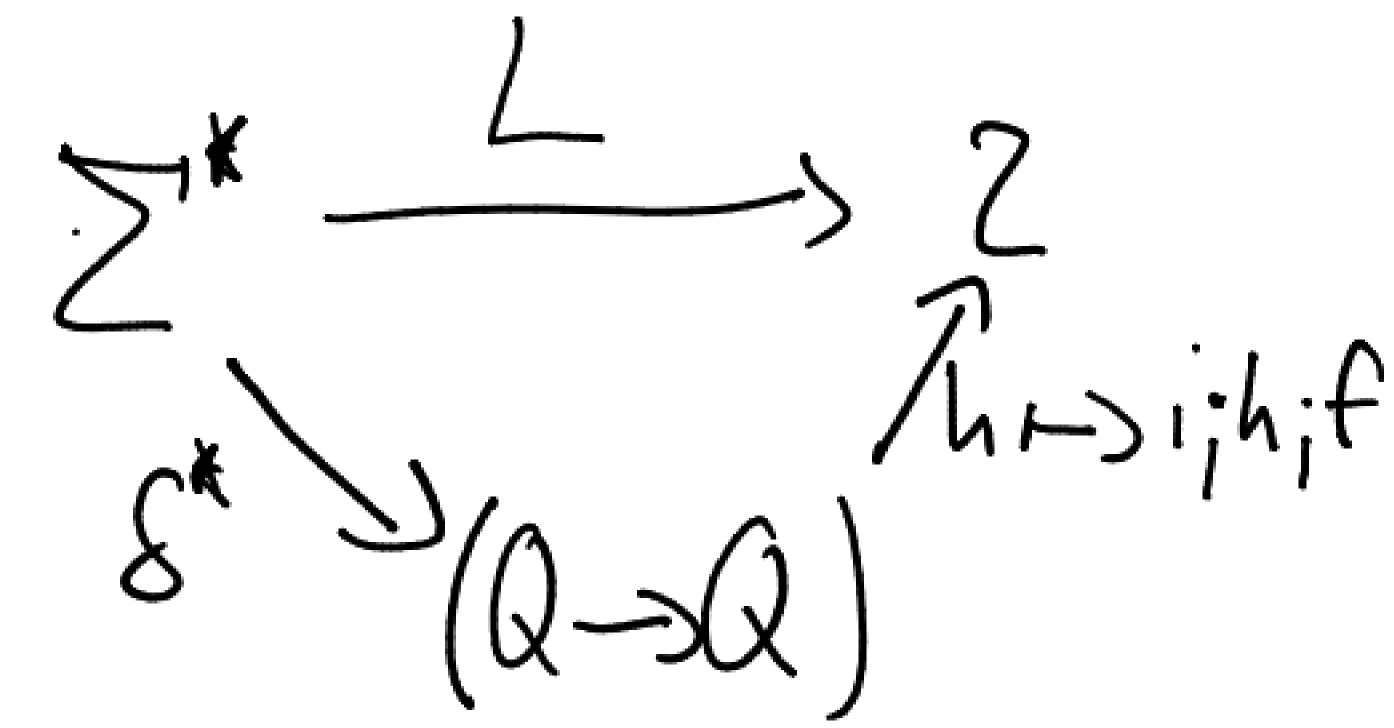
Recognition by homomorphisms



Finite Monoids ≙ DFAs!

DFA on Q ⇝ Monoid $(Q \to Q)$:



Monoid M ⇝ DFA on M:

$$1 \xrightarrow{e} M, \quad M \times \Sigma \xrightarrow{\delta} M, \quad M \xrightarrow{p} 2$$

$$m, a \mapsto m \cdot h(a)$$

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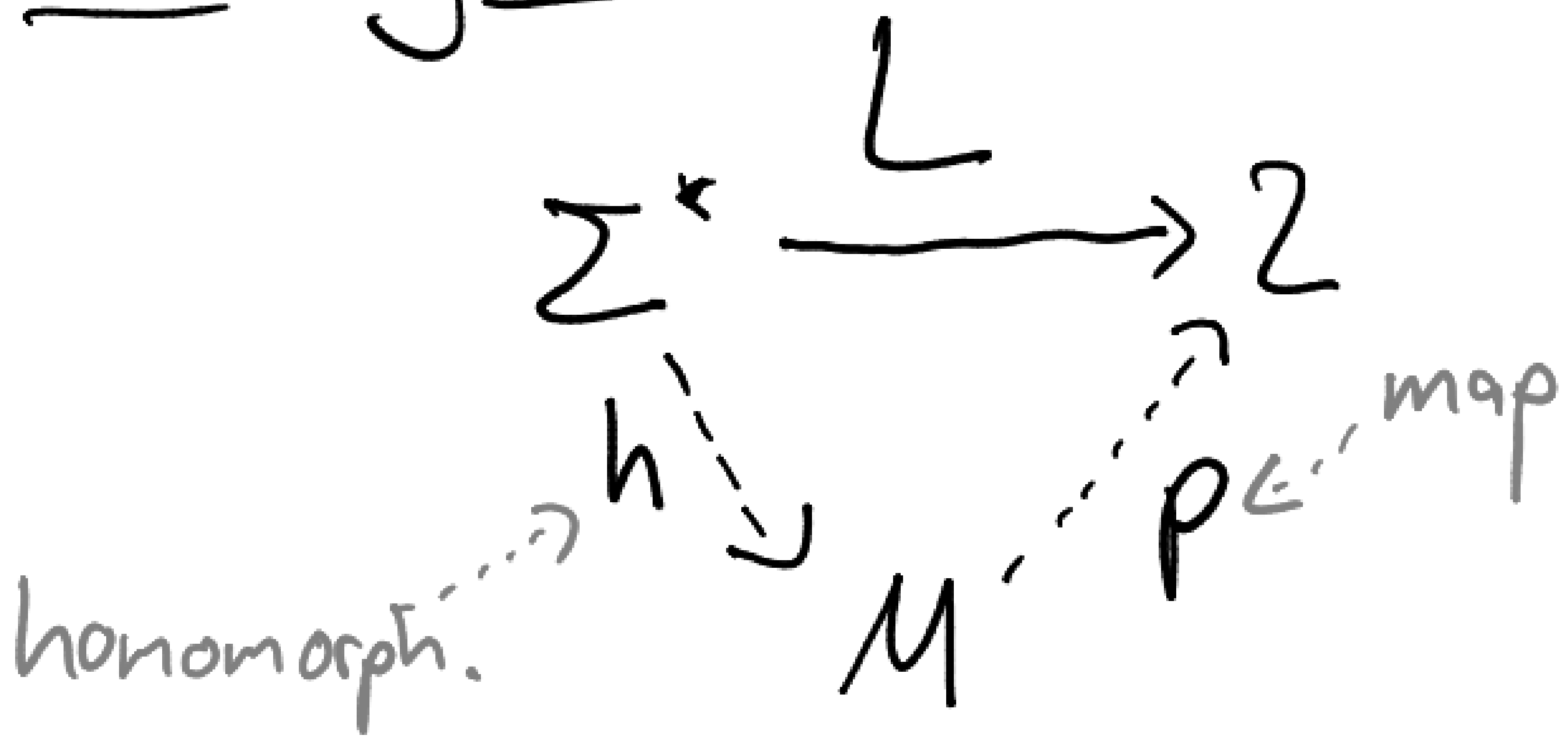
Recognition by homomorphisms



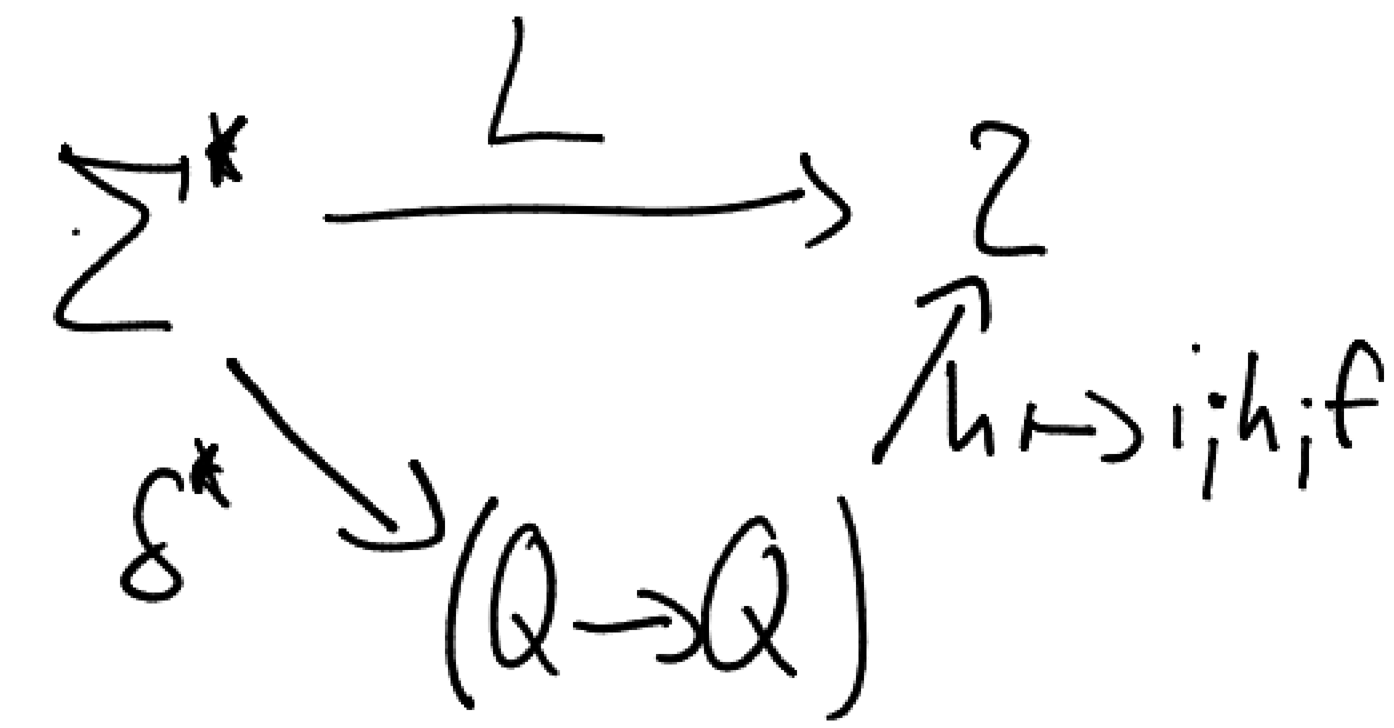
Algebraic Language Theory

⇝ Properties Monoids vs. Languages

e.g. when is a language commutative/star-free?

Finite Monoids ≙ DFAs!

DFA on Q ⇝ Monoid (Q→Q):



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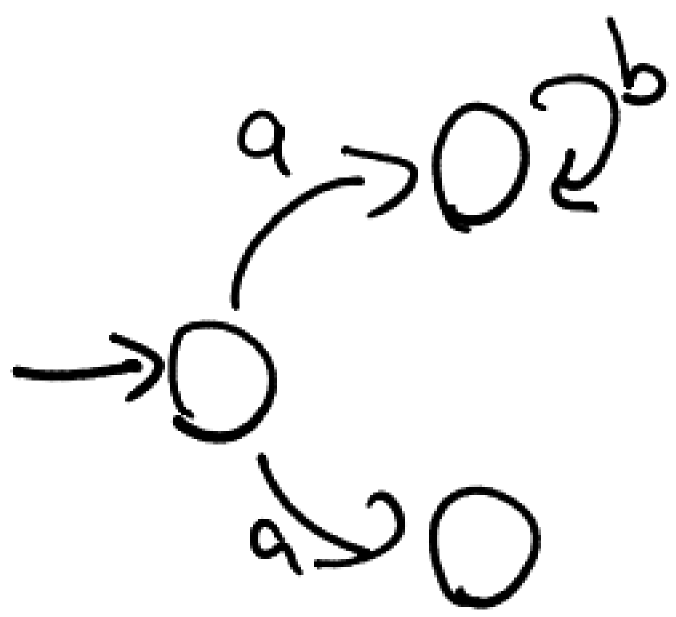
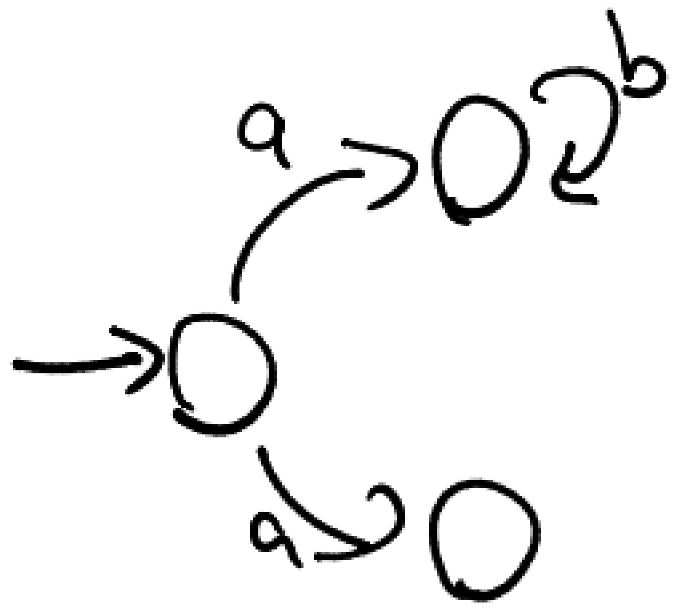
Adding Effects to Computation



Nondeterminism

$L : \Sigma^* \to 2$

Adding Effects to Computation



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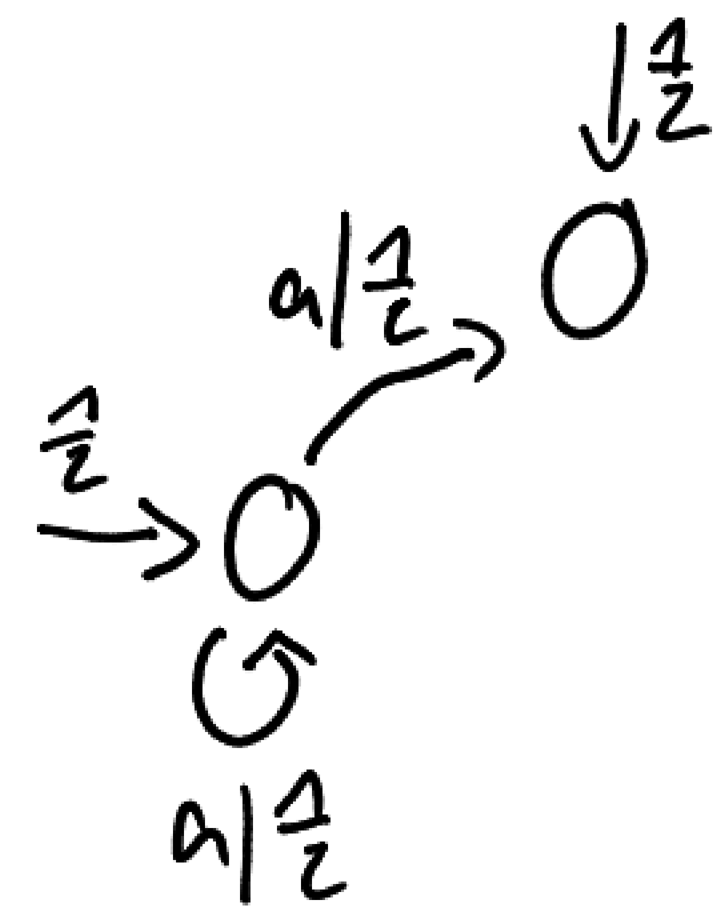

Probablities

$L : \Sigma^* \to [0,1]$

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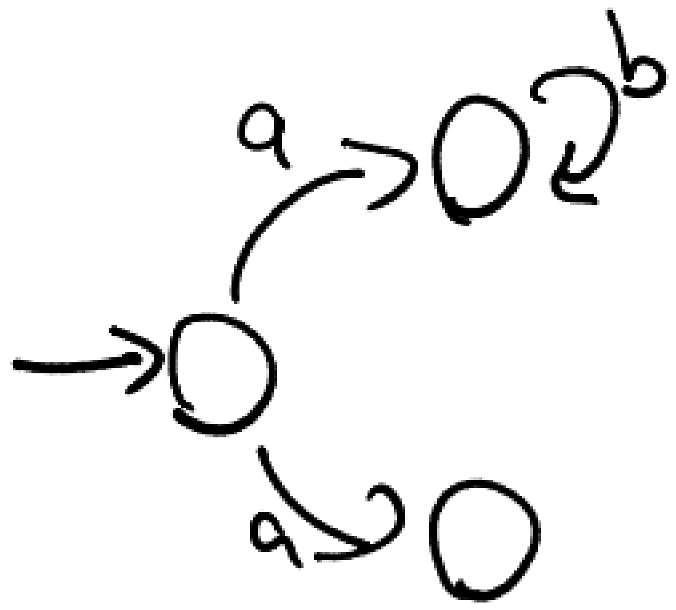

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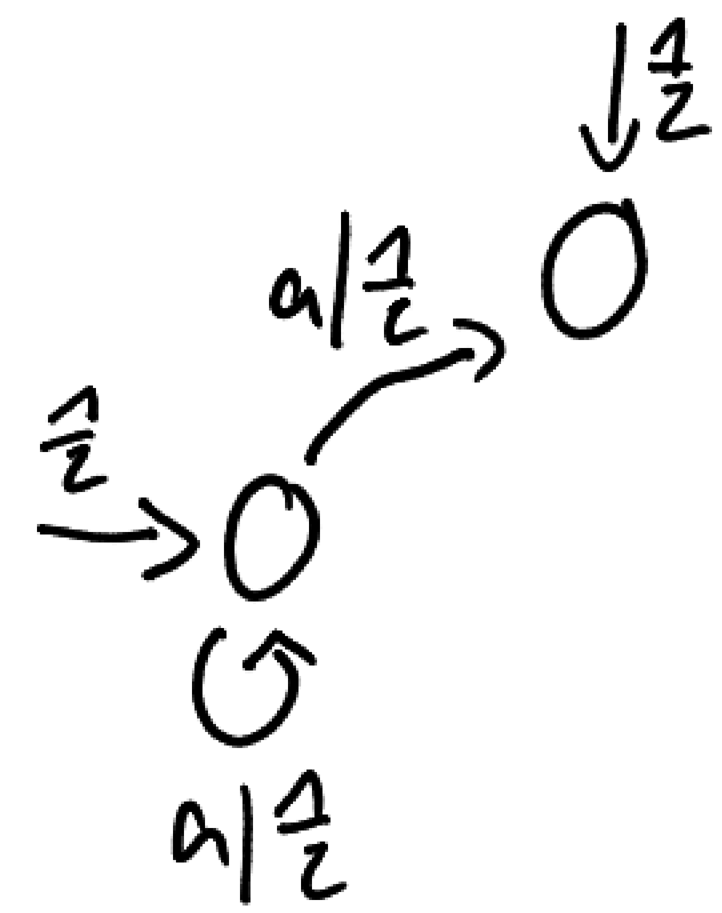

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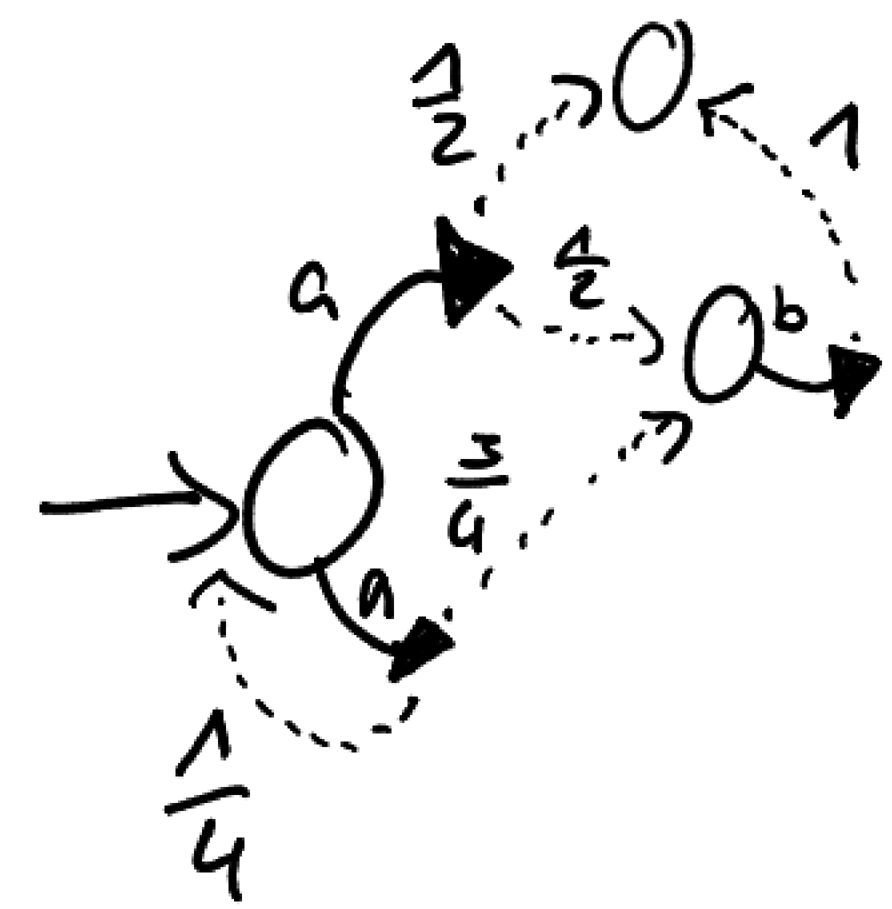
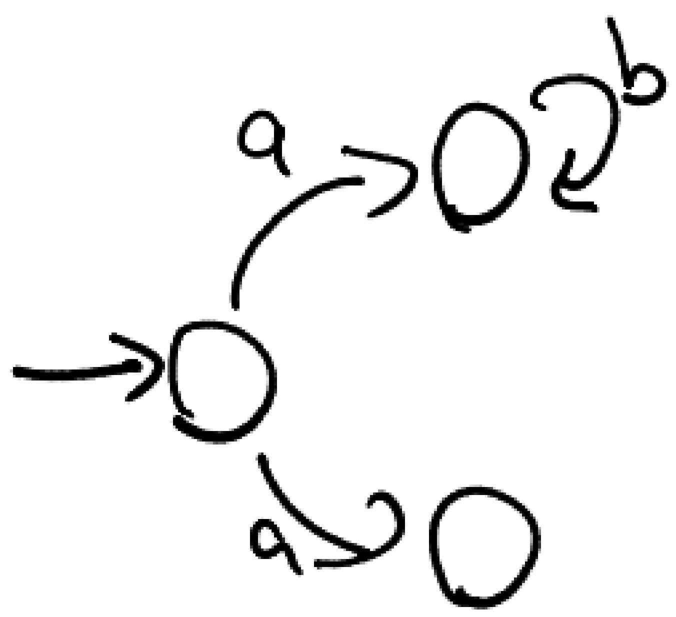


Nondet + Probabilities

$L : \Sigma^* \to [0,1]$ (?)

Adding Effects to Computation



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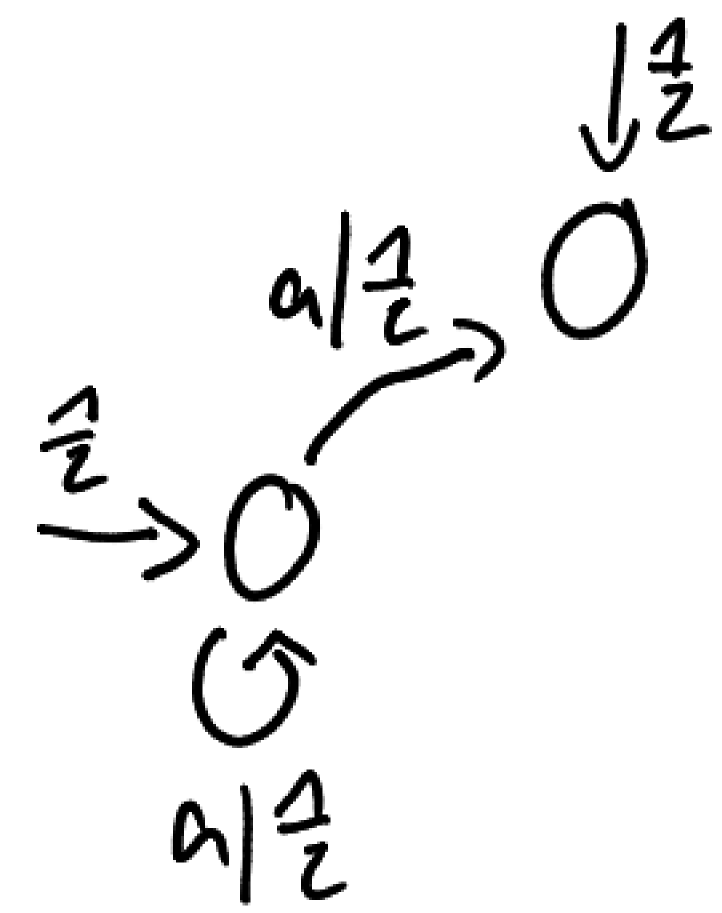

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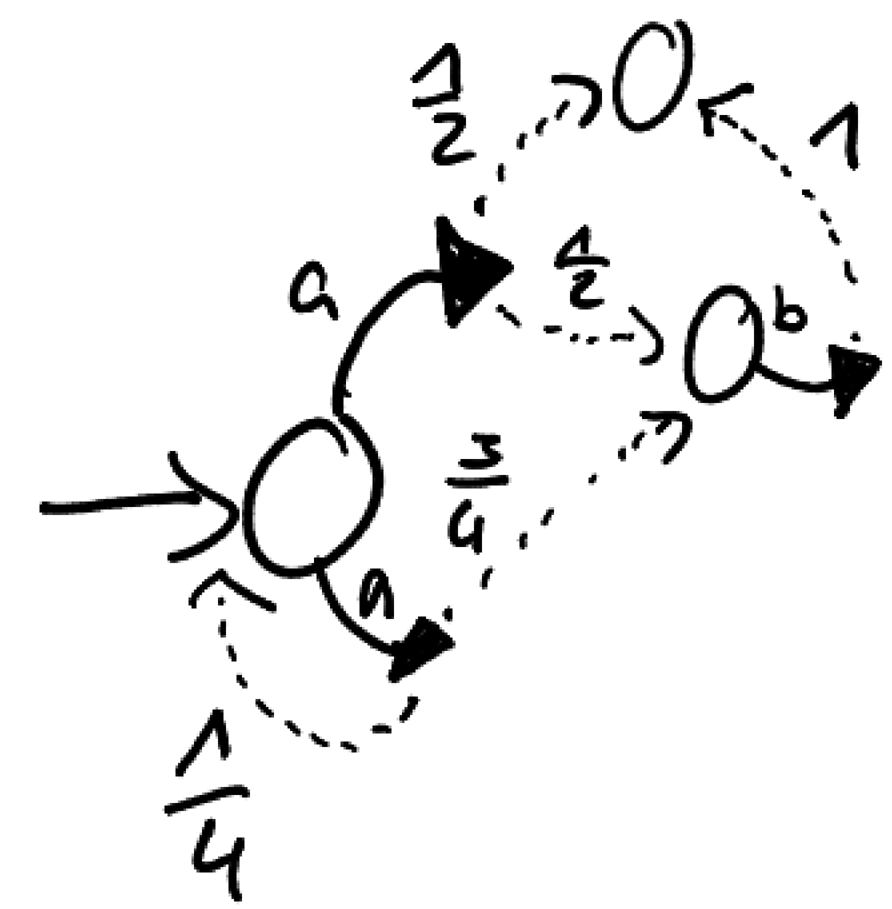

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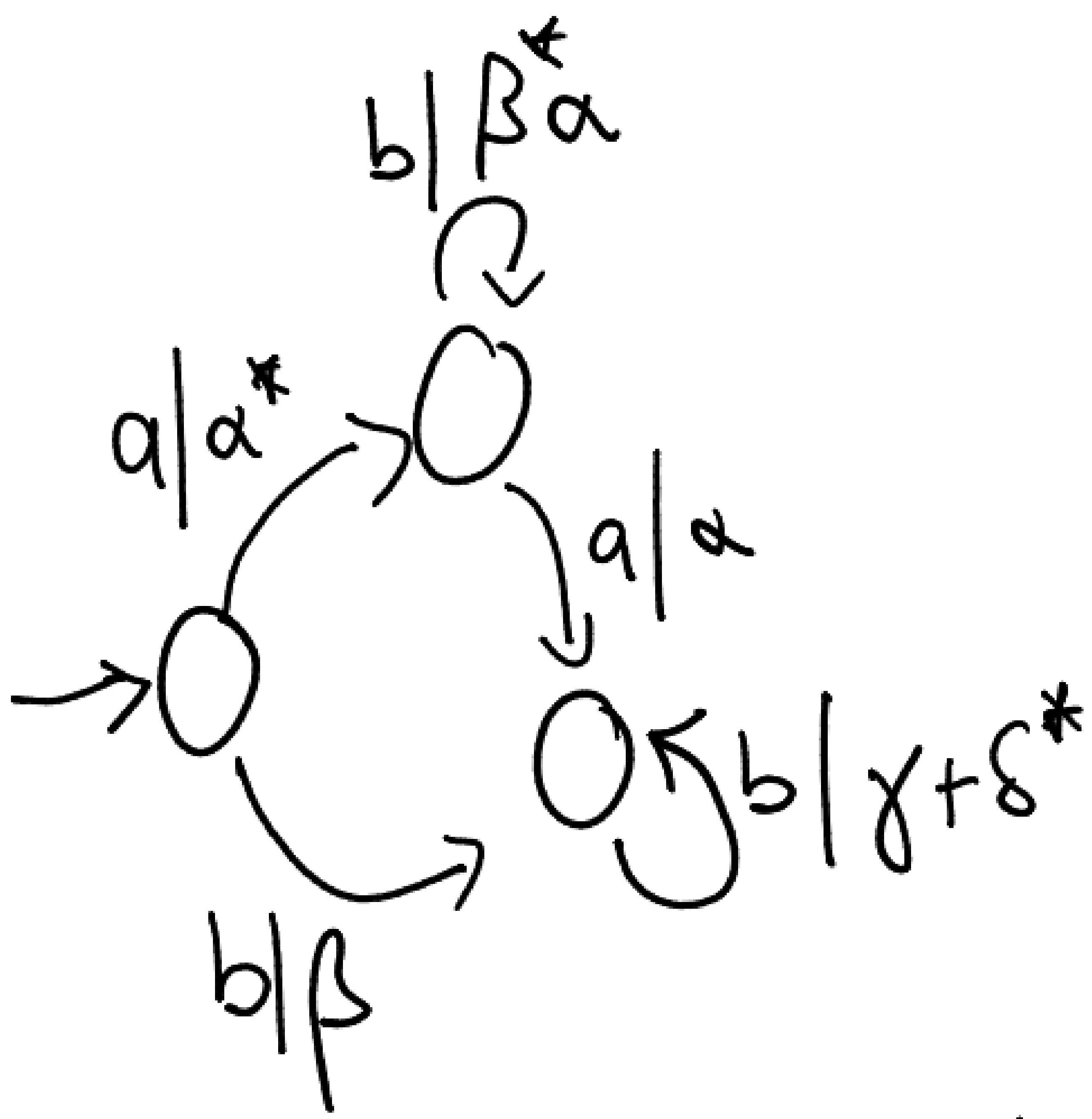
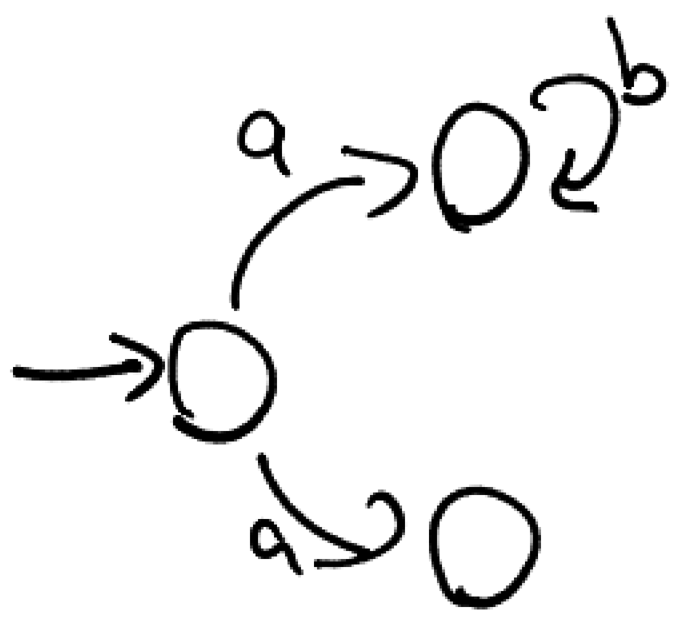


Weights in Semiring (here: Reg(α,β,γ,δ))

$L : \Sigma^* \to \Gamma^*$

Adding Effects to Computation



Nondeterminism

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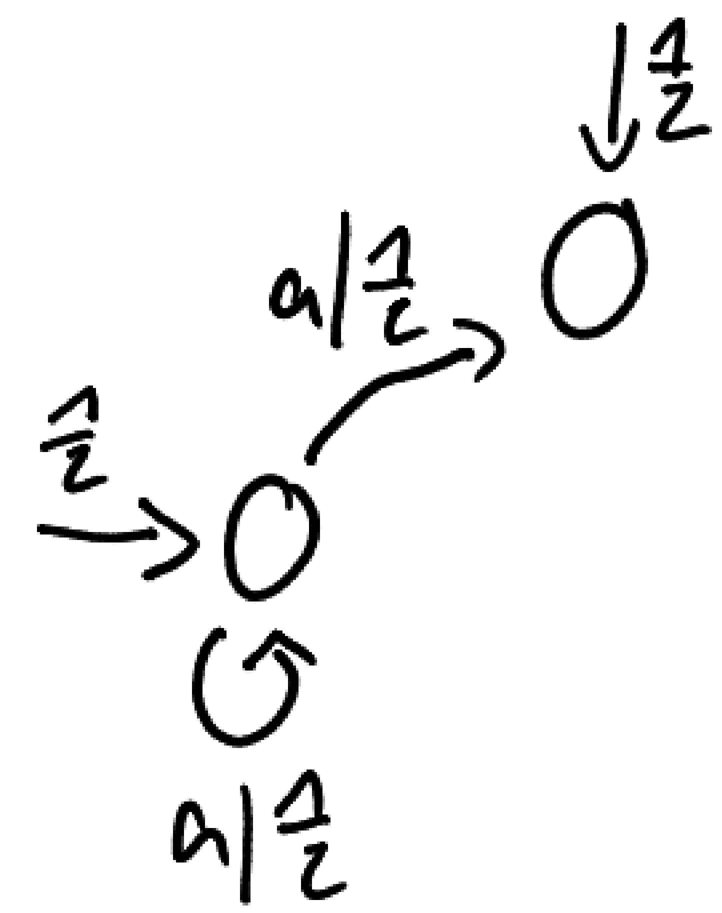

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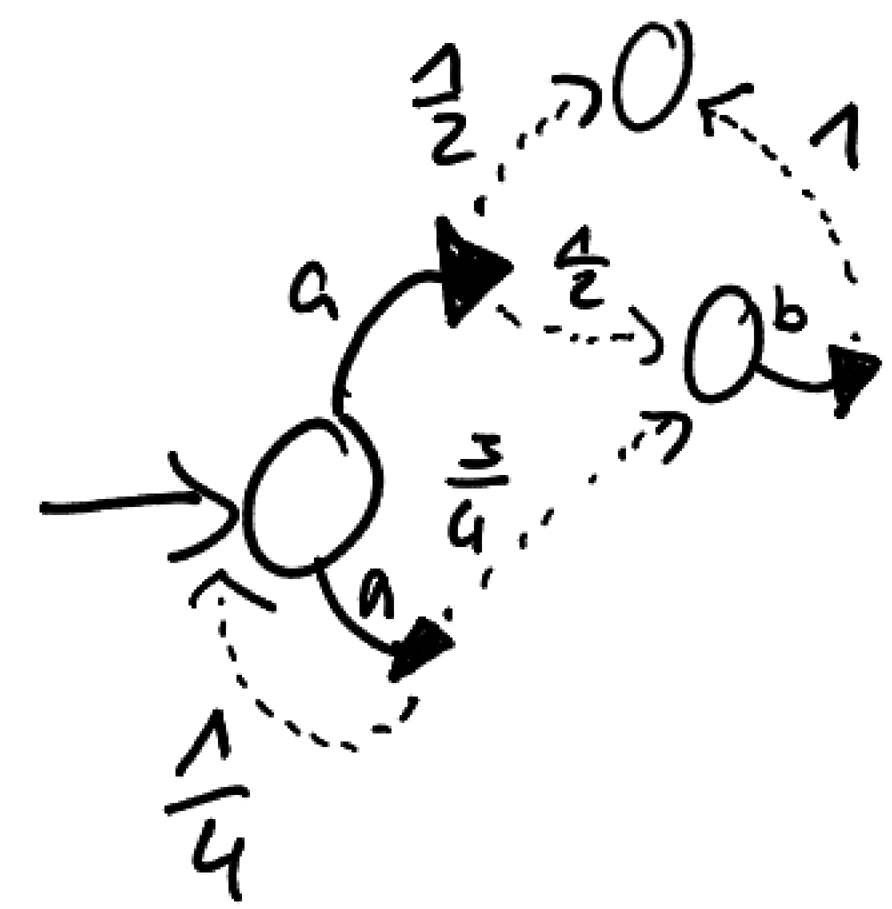

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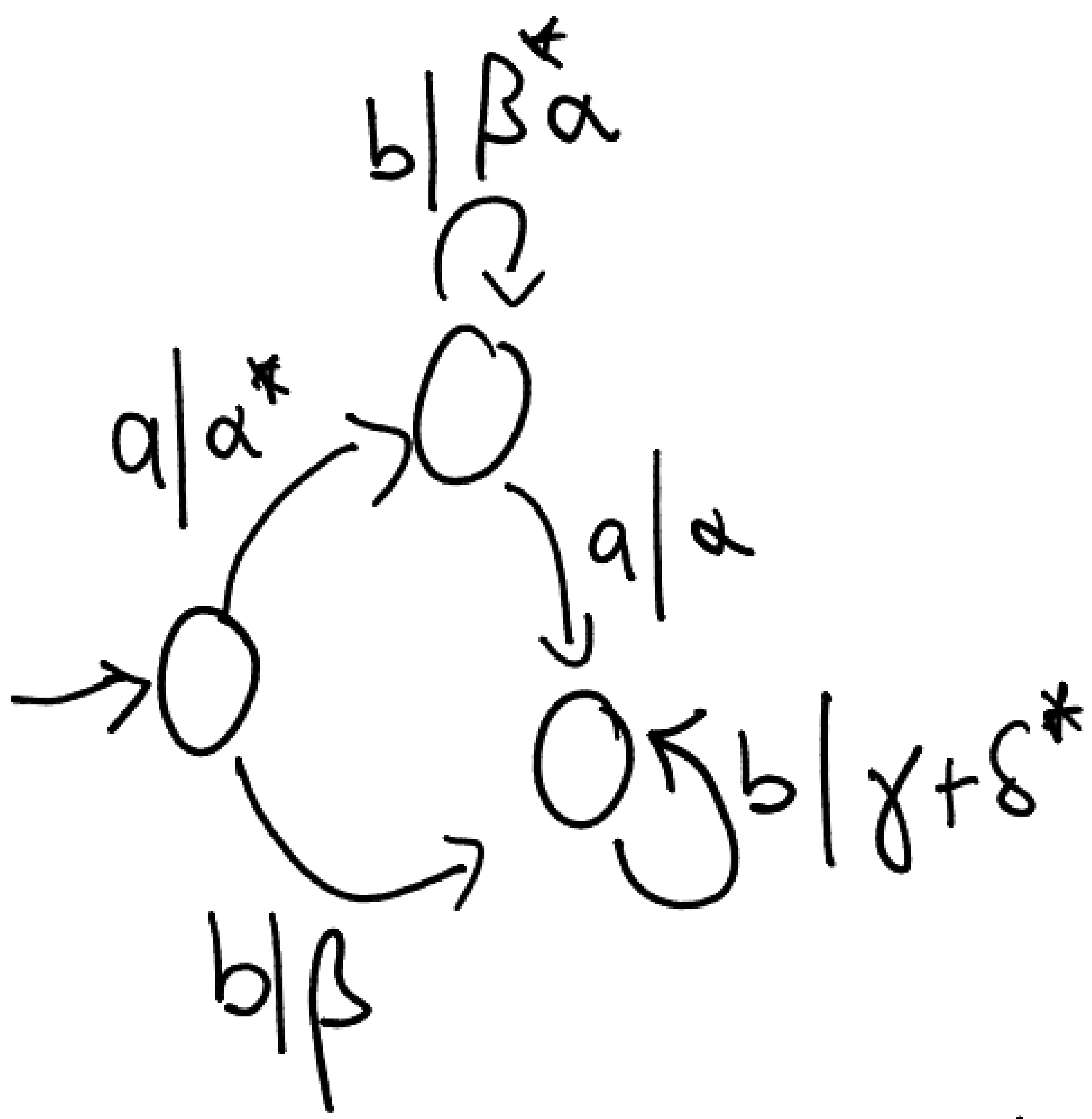

Weights in Semiring (here: $Reg(\alpha,\beta,\gamma,\delta)$)

$L : \Sigma^* \to \Gamma^*$

⇝ Uniform Description?

Modelling Effects: Monads

Monad: Functor $T : Set \to Set$

$$Id \xrightarrow{\eta} T, \quad TT \xrightarrow{\mu} T$$

$$\begin{array}{ccccc}
Id\,T & \xrightarrow{\eta T} & TT & \xleftarrow{T\eta} & T\,Id \\
 & = \searrow & \downarrow \mu & \swarrow = & \\
 & & T & &
\end{array}$$

$$\begin{array}{ccc}
TTT & \xrightarrow{\mu T} & TT \\
T\mu \downarrow & & \downarrow \mu \\
TT & \xrightarrow{\mu} & T
\end{array}$$

Modelling Effects: Monads

Monad: Functor $T: \mathbf{Set} \to \mathbf{Set}$

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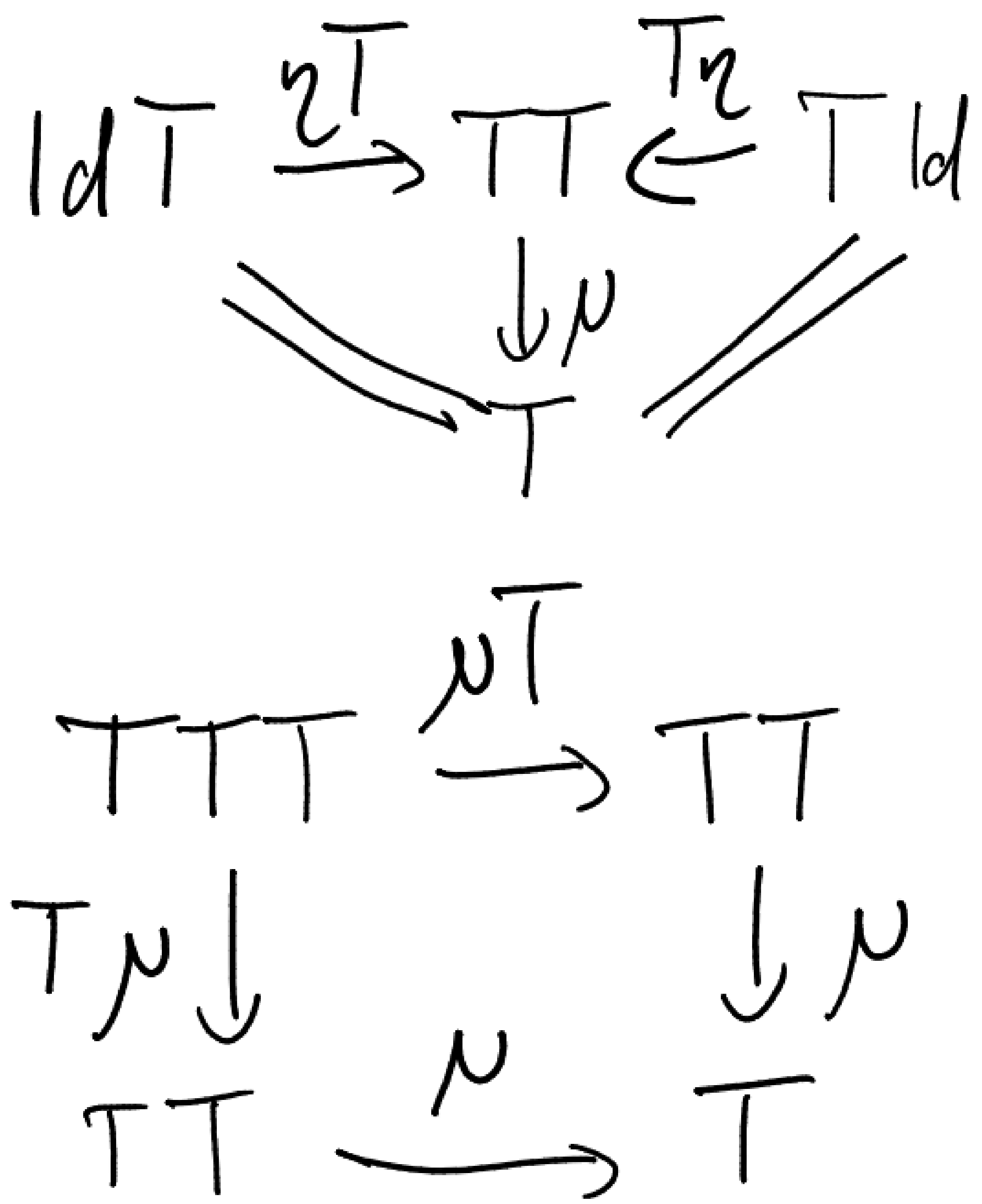

Examples:

- $\mathcal{P}_f X = \{U \subseteq_f X\}$, $X \to \mathcal{P}_f X$, $x \mapsto \{x\}$, $\mathcal{P}_f\mathcal{P}_f X \to \mathcal{P}_f X$, $\mathcal{A} \mapsto \bigcup \mathcal{A}$
- $\mathcal{D}X = \{\text{formal sums } \sum_i r_i x_i \mid \sum_i r_i = 1\}$

  $X \to \mathcal{D}X$, $x \mapsto 1 \cdot x$, $\mathcal{D}\mathcal{D}X \to \mathcal{D}X$, $\sum_I r_i (\sum_{J_i} s_{ij} x_{ij}) \mapsto \sum_I \sum_{j \in J_i} r_i s_{ij} \cdot x_{ij}$
- $\mathcal{C}X = \{\text{finite polytopes of } \mathcal{D}X\}$

  $X \to \mathcal{C}X$, $x \mapsto \{1 \cdot x\}$, $\mathcal{C}\mathcal{C}X \to \mathcal{C}X$ [union of polytopes]
- $SX = \{\text{formal sums } \sum_i s_i x_i \text{ over semiring } S\}$
- $\mathcal{M}X = X^* = \{\text{lists over } X\}$

Modelling Effects: Monads and their Categories

Kleisli Category over T

objects Sets + morphisms

$$X \multimap Y := X \to TY$$

with composition $X \overset{f}{\multimap} Y \overset{g}{\multimap} Z$

$$\left(X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ\right)$$

left and right strengths

$X \times TY \multimap X \times Y \quad TX \times Y \multimap X \times Y$

$(x, t) \mapsto T(y \mapsto (x, y))(t)$

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$$(x, t) \mapsto T(y \mapsto (x, y))(t)$$

Algebras over T

Set X, map $TX \xrightarrow{x} X$ s.t.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ \| & & \downarrow x \\ & & X \end{array} \quad , \quad \begin{array}{ccc} TTX & \xrightarrow{Tx} & TX \\ \mu \downarrow & & \downarrow x \\ TX & \xrightarrow{x} & X \end{array} \quad , \quad \text{morphisms: } \begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ x \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

Modelling Effects: Monads and their Categories

Kleisli Category over T

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$$X \mathrel{\circ\!\!\!\rightarrow} Y := X \rightarrow TY$$

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Examples:

- $\mathcal{P}_f$-algebras = join-semilattices
- $\mathcal{D}$-algebras = convex sets
- $\mathcal{C}$-algebras = convex semilattices
- $\mathcal{S}$-algebras = S-semimodules
- $\mathcal{M}$-algebras = monoids

$\rightsquigarrow$ Monads on Set $\triangleq$ algebraic theories

Modelling Effects: Monads and Computation

Example: $T = \mathcal{D}$

- algebras: Convex sets

↳ operations $+_r : X \times X \to X$, $r \in [0,1]$

$x +_r x = x$, $x +_r y = y +_{1-r} x$,

$x +_r (y +_s z) = (x +_{r'} y) +_{s'} z$ $\left(\begin{array}{l} s' = r + s - rs \\ r' = \frac{r}{s'} \end{array} \right)$

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Computation

① → **Syntax**

given by operations:

- $f : X \to X$, $x \mapsto x +_{\frac{1}{2}} y$, $y \mapsto x$, $z \mapsto x +_{\frac{1}{4}} z$
- $g : X \to X$, $x \mapsto y$, $y \mapsto y +_{\frac{1}{2}} z$, $z \mapsto x$

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Computation

① **Syntax**

given by operations:

- $f : X \multimap X$, $\quad x \mapsto x +_{\frac{1}{2}} y$, $\quad y \mapsto x$, $\quad z \mapsto x +_{\frac{1}{4}} z$
- $g : X \multimap X$, $\quad x \mapsto y$, $\quad y \mapsto y +_{\frac{1}{2}} z$, $\quad z \mapsto x$

② **Composition**: $f ; g$

Reasoning by equations

$X \overset{f}{\multimap} X \overset{g}{\multimap} X$

$x \mapsto x +_{\frac{1}{2}} y \mapsto y +_{\frac{1}{2}} (y +_{\frac{1}{2}} z) = (y +_{r'} y) +_{\frac{3}{4}} z = y +_{\frac{3}{4}} z$

$y \mapsto x \mapsto y +_{\frac{1}{2}} z$

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Modelling Effects: Monads and Computation

Example: $T = \mathcal{D}$

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Interpretation in an algebra

e.g. $p : X \to \mathcal{D}2 \cong [0,1]$ extends to $p^{\#} = \mathcal{D}X \xrightarrow{\mathcal{D}p} \mathcal{D}\mathcal{D}2 \xrightarrow{\mu_2} \mathcal{D}2$

$x \mapsto 0$,
$y \mapsto \frac{1}{2}$
$z \mapsto 1$

$\leadsto X \xrightarrow{f;g} \mathcal{D}X \xrightarrow{p^{\#}} \mathcal{D}2 \cong [0,1]$

$x \longmapsto \frac{7}{8}$

$y \longmapsto \frac{3}{4}$

$z \longmapsto \frac{3}{8}$

Computation

(1) **Syntax**

given by operations:

- $f : X \to X$, $\quad x \mapsto x +_{\frac{1}{2}} y$, $\quad y \mapsto x$, $\quad z \mapsto x +_{\frac{1}{4}} z$
- $g : X \to X$, $\quad x \mapsto y$, $\quad y \mapsto y +_{\frac{1}{2}} z$, $\quad z \mapsto x$

(2) **Composition**: $f ; g$

$X \xrightarrow{f} X \xrightarrow{g} X$

Reasoning by equations

$x \mapsto x +_{\frac{1}{2}} y \mapsto y +_{\frac{1}{2}} (y +_{\frac{1}{2}} z) = (y +_{r'} y) +_{\frac{3}{4}} z = y +_{\frac{3}{4}} z$

$y \mapsto x \mapsto y +_{\frac{1}{2}} z$

$z \mapsto x +_{\frac{1}{4}} z \mapsto y +_{\frac{1}{4}} x = x +_{\frac{3}{4}} y$

Modelling Effects: Monads and Computation

For general T: Kleisli-morphisms $(X \to TY) \hat{=} (X \,-\!\!\circ\!\!\to Y)$ represent T-effectful computations! e.g.

$id_{TX} : TX \,-\!\!\circ\!\!\to X$ $\hat{=}$ "evaluate effect"

$l_S : X \times TY \,-\!\!\circ\!\!\to X \times Y$ $\hat{=}$ "evaluate effect"

Modelling Effects: Monads and Computation

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↳ commutativity:

$$\begin{array}{ccc} TX \times TY & \overset{ls}{\mathrel{\circ\!\!\!\rightarrow}} & TX \times Y \\ rs \downarrow & & \downarrow rs \\ X \times TY & \overset{ls}{\mathrel{\circ\!\!\!\rightarrow}} & X \times Y \end{array}$$

$\forall X, Y, m \in TX, n \in TY$:

```
⌜do                 ⌜do
   x ← m      (*)      y ← n
   y ← n       =       x ← m
   return (x,y)⌟       return (x,y)⌟
```

⇝ $\mathcal{P}_f$, $\mathcal{D}$, $(-)+1$, $\mathcal{S}$ for commutative S

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$\forall X, Y, m \in TX, n \in TY$:

```
do           (*)   do
  x <- m      =      y <- n
  y <- n             x <- m
  return (x,y)       return (x,y)
```

⇝ $\mathcal{P}_f$, $\mathcal{D}$, $(-)+1$, S for commutative S

Fine-Grained:

- $m \in TX$ central if it satisfies $(*)$ $\forall Y, n \in TY$

  ⇝ center of T

- $f : X \mathrel{\circ\!\!\to} X'$ central if $\forall Y, Y', g : Y \mathrel{\circ\!\!\to} Y'$:

$$\begin{array}{ccccc} X \times Y & \overset{id \times g}{\longrightarrow} & X \times TY' & \overset{ls}{\mathrel{\circ\!\!\to}} & X \times Y' \\ f \times id \downarrow & & & & \downarrow f \times id \\ TX' \times Y & & & & TX' \times Y' \\ rs \downarrow\!\circ & & & & \circ\!\downarrow rs \\ X' \times Y & \overset{id \times g}{\longrightarrow} & X' \times TY' & \overset{ls}{\mathrel{\circ\!\!\to}} & X' \times Y' \end{array}$$

Effectful Automata for a Monad T

Fix: Monad T, output algebra O

T-FA: $1 \overset{i}{\multimap} Q, \quad Q \times \Sigma \overset{\delta}{\multimap} Q, \quad Q \overset{o}{\longrightarrow} O$

pure map!

can be chosen pure: $1 \overset{i}{\to} Q$

extends to $\delta^*: \Sigma^* \to (Q \multimap Q)$

Why not $Q \overset{o}{\multimap} O$?

⇝ sometimes not the "right" semantics, e.g. $T = \mathcal{P}_f^+$

Effectful Automata for a Monad T

Fix: Monad T, output algebra O

> Why not $Q \overset{o}{-\!\!\circ\!\!\to} O$?
>
> $\leadsto$ sometimes not the "right" semantics, e.g. $T = \mathcal{P}_f^+$

T-FA: $1 \overset{i}{-\!\!\circ\!\!\to} Q,\quad Q \times \Sigma \overset{\delta}{-\!\!\circ\!\!\to} Q,\quad Q \overset{o}{\longrightarrow} O$ (pure map!)

- i: can be chosen pure: $1 \overset{i}{\to} Q$
- δ: extends to $\delta^*: \Sigma^* \to (Q -\!\!\circ\!\!\to Q)$

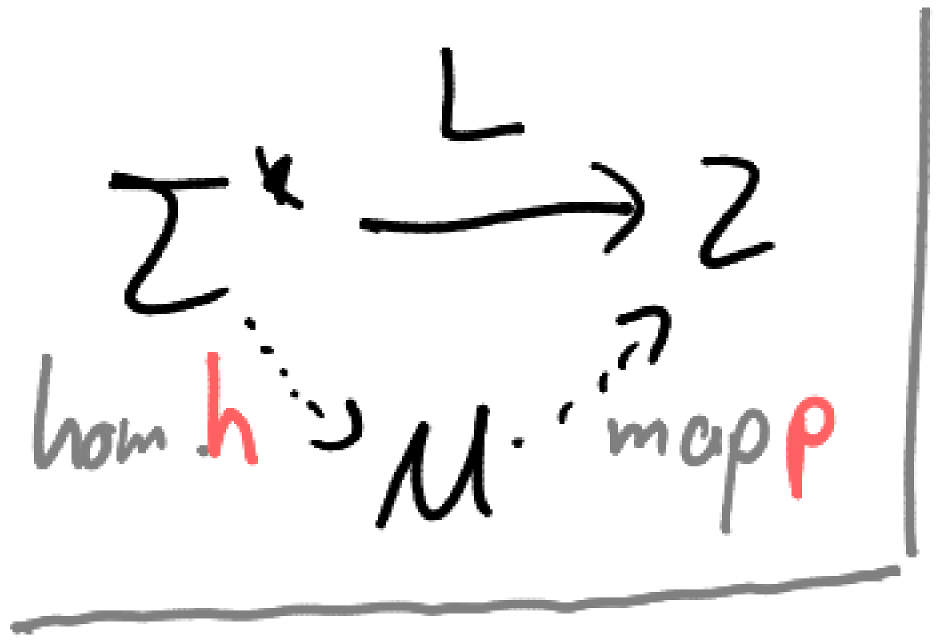
accepts $L: \Sigma^* \to O$

$$w \mapsto \left(1 \overset{i}{-\!\!\circ\!\!\to} Q \overset{\delta^*(w)}{-\!\!\circ\!\!\to} Q \overset{o}{\to} O\right)$$

$$\|$$

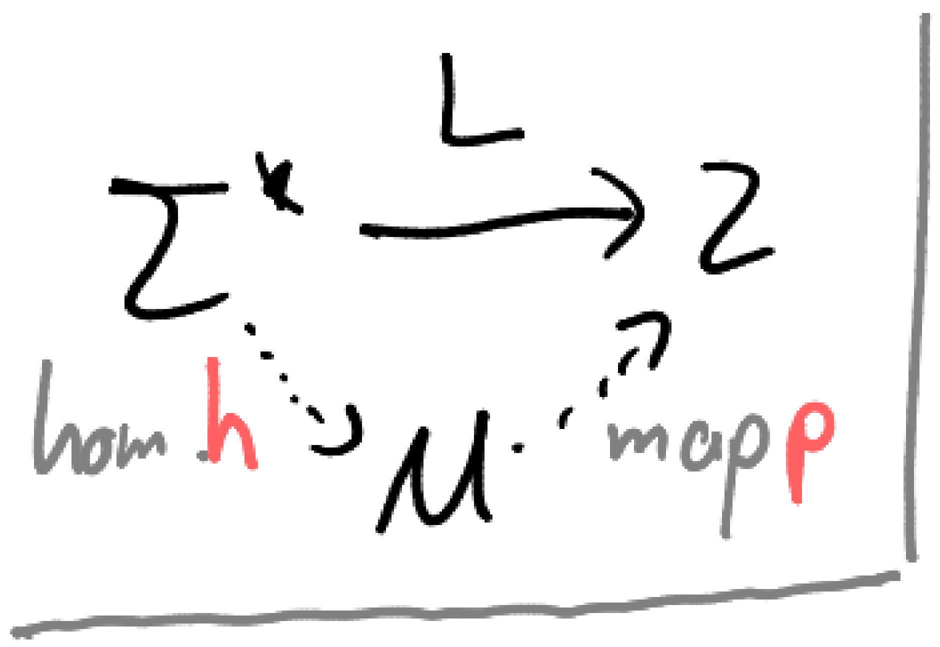
$$\left(1 \overset{i}{\to} Q \overset{w_1}{-\!\!\circ\!\!\to} Q \overset{w_2}{-\!\!\circ\!\!\to} Q \cdots \overset{w_n}{-\!\!\circ\!\!\to} Q \to O\right)$$

Algebras for Effectful Recognitition



Where to put the effects of T?

Algebras for Effectful Recognitition



Where to put the effects of T?

1.

Into the structure $\mathcal{M}$

⇝ effectful monoids:

$(T, \mathcal{M})$-bialgebras / T-monoids

Algebras for Effectful Recognitition

$\Sigma^* \xrightarrow{L} 2$, hom $h: \Sigma^* \to \mathcal{M}$, map $p: \mathcal{M} \to 2$

Where to put the effects of T?

1. Into the structure $\mathcal{M}$
   ⇝ effectful monoids:
   ⌈ $(T, \mathcal{M})$-bialgebras / T-monoids ⌋

2. Into the morphism h
   ⇝ effectful homomorphisms:
   ⌈ Kleisli homomorphisms ⌋

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Recall: DFA Q

$\rightsquigarrow$ monoid $Q \to Q$

recognizes same lang.

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\ L\ } & 2 \\ & {\scriptstyle \delta^*} \searrow & \uparrow {\scriptstyle i;-;f} \\ & & (Q \to Q) \end{array}$$

+ effects

$\rightsquigarrow$

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Recall: DFA Q

$\rightsquigarrow$ monoid $Q \to Q$

recognizes same lang.

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+ effects $\rightsquigarrow$

T-FA $\quad 1 \mathrel{-\!\!\circ\!\!\to} Q, \quad Q \times \Sigma \mathrel{-\!\!\circ\!\!\to} Q, \quad Q \xrightarrow{o} O$

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\ L\ } & O \\ {\scriptstyle \delta^*} \searrow & & \nearrow {\scriptstyle i;-;o} \\ & (Q \mathrel{-\!\!\circ\!\!\to} Q) & \end{array}$$

if i pure: T-hom.

Effectful Algebraic Recognition: (T, M)-bialgebras / T-monoids

Recall: DFA Q

⇝ monoid $Q \to Q$

recognizes same lang.

$$\Sigma^* \xrightarrow{L} 2, \qquad \Sigma^* \xrightarrow{\delta^*} (Q \to Q) \xrightarrow{i;-;f} 2$$

+ effects ⇝

T-FA $1 \nrightarrow Q$, $Q \times \Sigma \nrightarrow Q$, $Q \xrightarrow{o} O$

$$\Sigma^* \xrightarrow{L} O, \qquad \Sigma^* \xrightarrow{\delta^*} (Q \nrightarrow Q) \xrightarrow{i;-;o} O$$

if i pure: T-hom.

$$(Q \to TQ) \cong (TQ)^Q$$

<u>monoid</u> under kleisli comp.

<u>T-algebra</u> product:

$$T(TQ)^Q \xrightarrow{\langle T\pi_q \rangle_q} (TTQ)^Q \xrightarrow{\mu^Q} (TQ)^Q$$

+

combine!

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Def <u>$(T, \mathcal{M})$-bialgebra</u> $X : TX \xrightarrow{x} X \xleftarrow{\bullet} \mathcal{M}X \;\hat{=}\; T$-alg + monoid structures

⌜algebras for $T + \mathcal{M}$⌟

⌜if all $- \bullet a$, $a \bullet -$ are T-morphisms: <u>T-monoid</u>⌟

⌜If T comm: monoids in $(\mathrm{Alg}\,T, \otimes, T1)$⌟

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Def $(T, \mathcal{M})$-bialgebra $X : TX \xrightarrow{\times} X \xleftarrow{\bullet} \mathcal{M}X$ $\hat{=}$ T-alg + monoid structures

⌜algebras for $T + \mathcal{M}$⌟

⌜if all $- \cdot a$, $a \cdot -$ are T-morphisms : T-monoid⌟

⌜If T comm: monoids in $(\mathrm{Alg}\,T, \otimes, T1)$⌟

⌜

Examples:

- $(\mathcal{D}, \mathcal{M})$-bialgebra: convex set that is also monoid
- $\mathbb{R}$-monoid: $\mathbb{R}$-algebra (algebraic sense)

⌟

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Def $(T, \mathcal{M})$-bialgebra $X : TX \xrightarrow{x} X \xleftarrow{\cdot} \mathcal{M}X$ ≙ T-alg + monoid structures

[algebras for $T + \mathcal{M}$]

[if all $- \cdot a$, $a \cdot -$ are T-morphisms: T-monoid]

[if T comm: monoids in $(\mathrm{Alg}\,T, \otimes, T1)$]

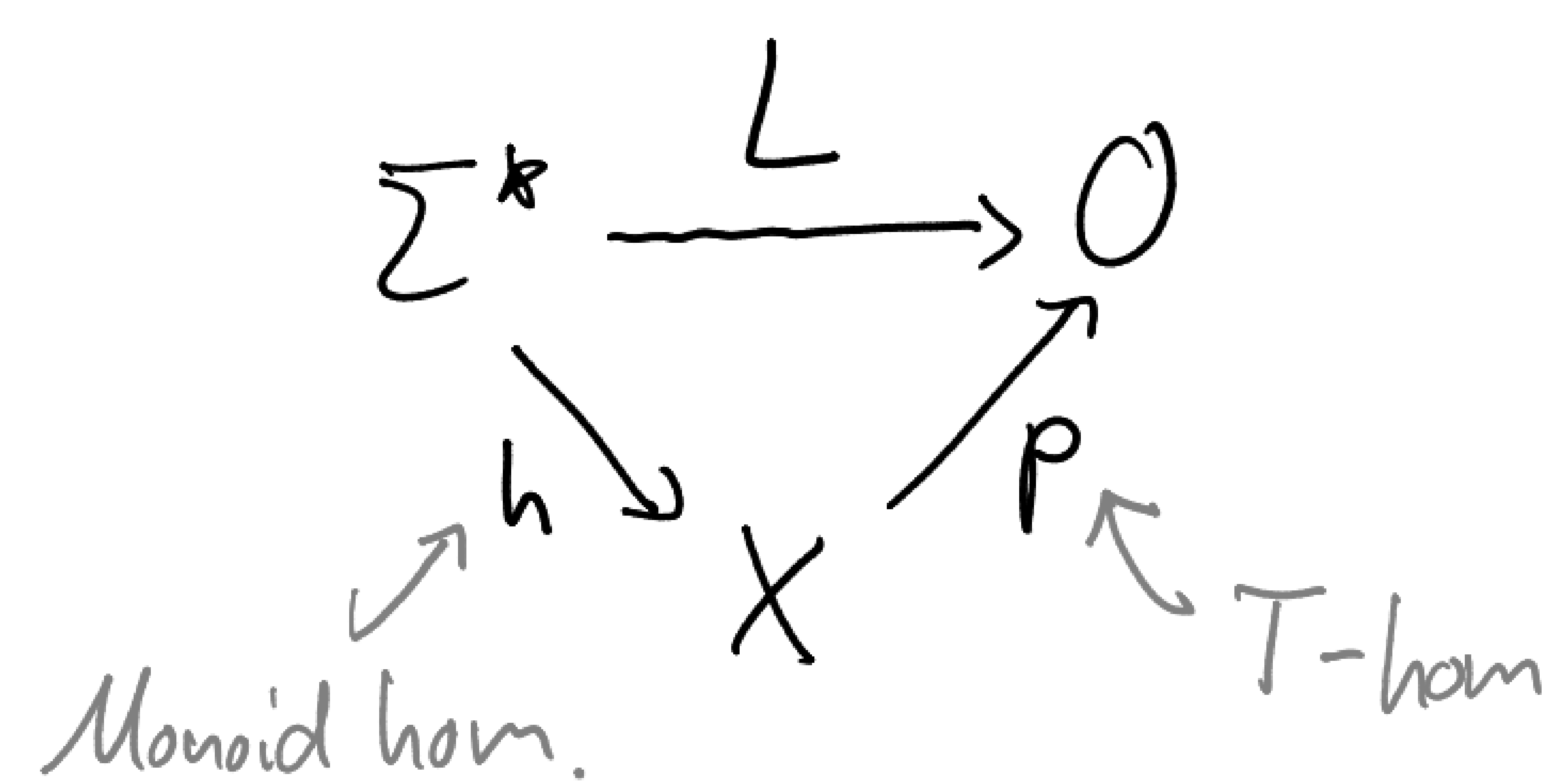
Examples:

- $(\mathcal{D}, \mathcal{M})$-bialgebra: convex set that is also monoid
- $\mathbb{R}$-monoid: $\mathbb{R}$-algebra (algebraic sense)

bialgebra X is fg-carried if $\exists$ finite G, $e: G \to X$ s.t.

$$TG \xrightarrow{Te} TX \xrightarrow{x} X \text{ surjective}$$

Recognition by bialgebras X:



Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Assume: X finite, then
$(X \circledast X)$ finitely generated

Theorem:

L is T-FA computable

iff

L is recognizable by fg-carried bialgebra

iff (T comm.)

L is recognizable by fg-carried T-monoid

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Assume: X finite, then $(X \circledast X)$ finitely generated

Theorem:

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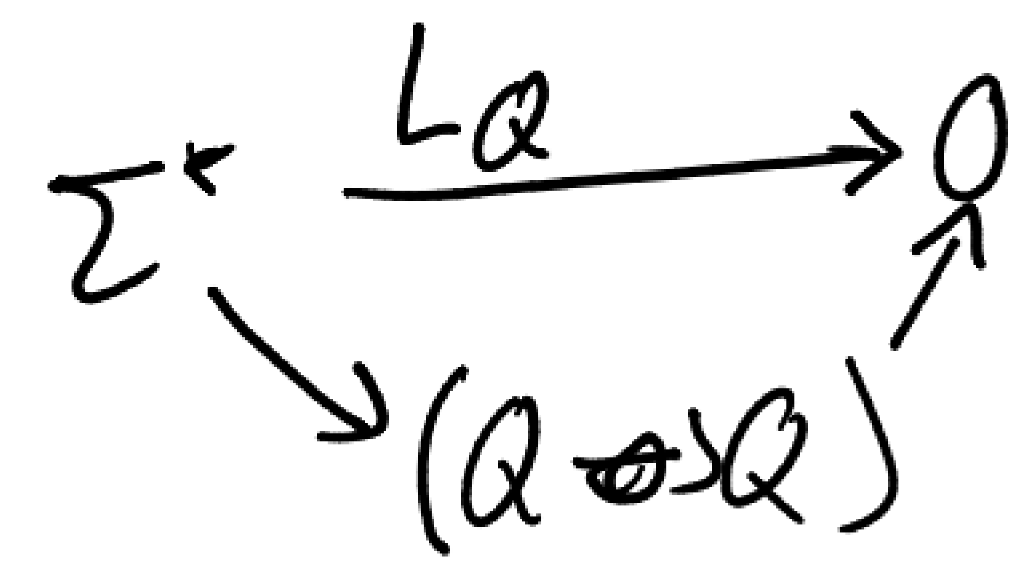
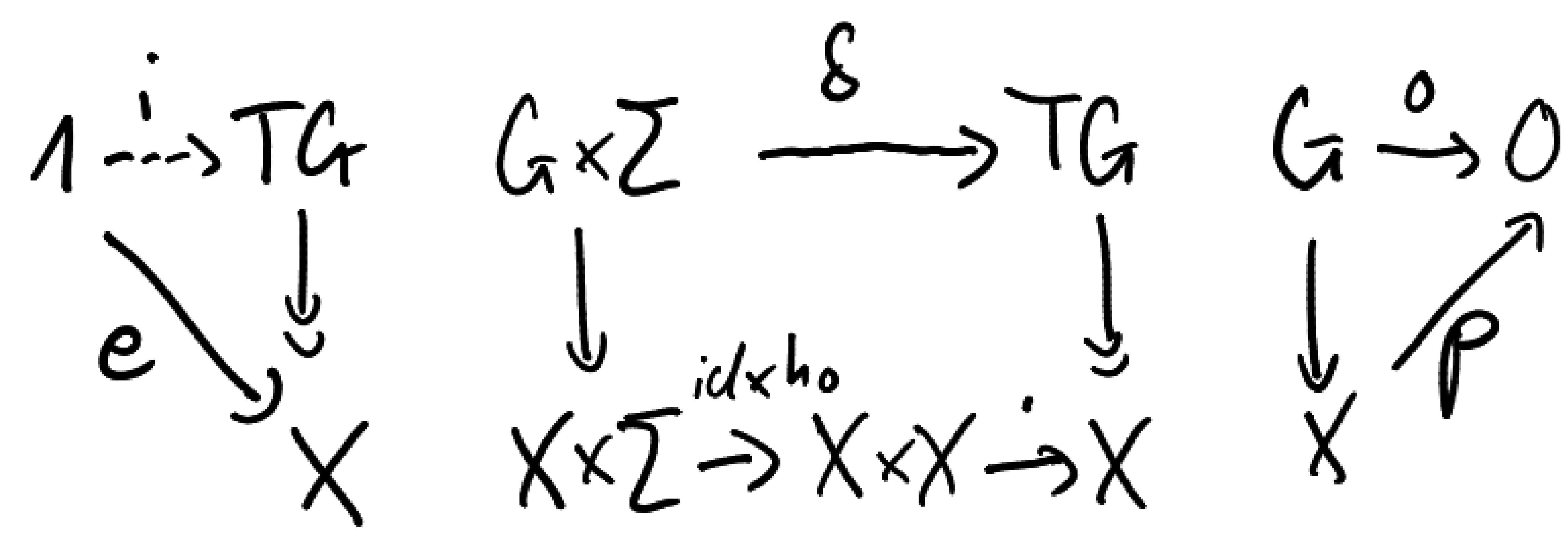
L is recognizable by fg-carried T-monoid

Proof Sketch:

- T-FA Q $\rightsquigarrow$ bialgebra $(Q \circledast Q)$



- fg-carried bialgebra X $\rightsquigarrow$ $\exists$ generators $TG \overset{c}{\twoheadrightarrow} X$



Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Our examples satisfy the assumption that $(X \twoheadrightarrow X)$ is fg-carried

$TG \overset{e}{\twoheadrightarrow} (TX)^X$

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Our examples satisfy the assumption that $(X \to\!\!\!\!\circ\; X)$ is fg-carried

$TG \overset{e}{\twoheadrightarrow} (TX)^X$

Idea: Represent effectful functions $X \overset{f}{\to\!\!\!\!\circ\;} X$ as T-combination of deterministic functions

e.g.: $T = \mathbb{R}^{(-)}$: $(X \overset{f}{\to\!\!\!\!\circ\;} X) \mathrel{\hat{=}} |X|^2$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1|X|} \\ \vdots & & \vdots \\ a_{|X|1} & \cdots & a_{|X||X|} \end{pmatrix} = a_{11} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & 0 & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ \vdots & & 0 & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + \cdots + a_{|X||X|} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots 0 & 1 \end{pmatrix}$

Partial functions $\delta_{ab} : |X| \to |X|$

$a \mapsto b$

$\bot$ else

Effectful Algebraic Recognition: (T, M)-bialgebras / T-monoids

Our examples satisfy the assumption that $(X \to X)$ is fg-carried

$TG \xrightarrow{e} (TX)^X$

Idea: Represent effectful functions $X \xrightarrow{f} X$ as T-combination of deterministic functions

e.g.: $T = \mathbb{R}^{(-)}$: $(X \xrightarrow{f} X) \mathrel{\hat{=}} |X|^2$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1|X|} \\ \vdots & & \vdots \\ a_{|X|1} & \cdots & a_{|X||X|} \end{pmatrix} = a_{11} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & 0 & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ \vdots & & 0 & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + \cdots + a_{|X||X|} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots 0 & 1 \end{pmatrix}$

Partial functions $\delta_{ab} : |X| \to |X|$, $a \mapsto b$, $\bot$ else

T	T-FA	M	h: $M \to (X \to X)$
$\mathcal{P}_f$	NFA	$(X \to X)$	partial fn ↪ relations
$\mathcal{D}$	PFA	$(X \to X)$	pure fns ↪ stoch mat.
$\mathcal{C}$	NPFA	$(X \to X)$	similar
$\mathcal{S}$	WFA	$(X \to X)$	δ_{ij} matrices ↪ all matrices
$\mathcal{M}$	?	$(X \to X)$	partial fn ↪ "order-relations"

Effectful Algebraic Recognition: $(T, \mathcal{M})$-bialgebras / T-monoids

Our examples satisfy the assumption that $(X \rightsquigarrow X)$ is fg-carried

$TG \overset{e}{\twoheadrightarrow} (TX)^X$

Idea: Represent effectful functions $X \overset{f}{\rightsquigarrow} X$ as T-combination of deterministic functions

e.g.: $T = \mathbb{R}^{(-)}$: $(X \overset{f}{\rightsquigarrow} X) \hat{=} |X|^2$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1|X|} \\ \vdots & & \vdots \\ a_{|X|1} & \cdots & a_{|X||X|} \end{pmatrix} = a_{11} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & 0 & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & & \vdots \\ 0 & \cdots & & & 0 \end{pmatrix} + \cdots + a_{|X||X|} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0\ 1 \end{pmatrix}$

Partial functions $\delta_{ab} : |X| \to |X|$
$a \mapsto b$
$\bot$ else

T	T-FA	$\mathcal{M}$	$h : \mathcal{M} \to (X \rightsquigarrow X)$
$\mathcal{P}_f$	NFA	$(X \to X)$	partial fn $\hookrightarrow$ relations
$\mathcal{D}$	PFA	$(X \to X)$	pure fns $\hookrightarrow$ stoch mat.
$\mathcal{C}$	NPFA	$(X \to X)$	similar
S	WFA	$(X \to X)$	δ_{ij} matrices $\hookrightarrow$ all matrices
$\mathcal{M}$	?	$(X \to X)$	partial fn $\hookrightarrow$ "order-relations"

Counter-example:

$T = \mathbb{N} \times (-)$ "writer monad"

problem: $\mathbb{N} \times \mathbb{N}$ not f.g.

Effectful Algebraic Recognition: Effectful Morphisms

$\Sigma^* \xrightarrow{L} 2$, $\quad \Sigma^* \xrightarrow{h} M \xrightarrow{p} 2$

h recognizes L $\overset{\text{(Thorsten)}}{\rightsquigarrow}$ don't change the type of M but of h!

Effectful Algebraic Recognition: Effectful Morphisms

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\;L\;} & 2 \\ & {\scriptstyle h}\searrow \quad \nearrow{\scriptstyle p} & \\ & M & \end{array}$$

h recognizes L $\overset{\text{(Thorsten)}}{\rightsquigarrow}$ **don't change the type of M but of h!**

Def: <u>Kleisli</u>-<u>hom</u> $h : M \rightsquigarrow N$ of ordinary monoids:

$$\begin{array}{ccc} M \times M & \xrightarrow{\;\cdot\;} & M \\ {\scriptstyle h \bar{\times} h}\downarrow & & \downarrow{\scriptstyle h} \\ N \times N & \xrightarrow{\;\cdot\;} & N \end{array}$$

$f : X \to TX'$, $g : Y \to TY'$

$$f \bar{\times} g : X \times Y \xrightarrow{f \times g} TX' \times TY' \xrightarrow{rs} X' \times TY' \xrightarrow{ls} X' \times Y'$$

Effectful Algebraic Recognition: Effectful Morphisms

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{L} & 2 \\ & {\scriptstyle h}\searrow \quad \nearrow{\scriptstyle p} & \\ & M & \end{array}$$

h recognizes L $\overset{\text{(Thorsten)}}{\rightsquigarrow}$ don't change the type of M but of h!

Def: Kleisli-hom $h: M \circ\!\!\rightarrow N$ of ordinary monoids:

$$\begin{array}{ccc} M \times M & \xrightarrow{\ \cdot\ } & M \\ {\scriptstyle h \,\bar{\times}\, h}\downarrow & & \downarrow{\scriptstyle h} \\ N \times N & \xrightarrow{\ \cdot\ } & N \end{array}$$

$f: X \to TX'$, $g: Y \to TY'$

$f \bar{\times} g: X \times Y \xrightarrow{f \times g} TX' \times TY' \overset{rs}{\circ\!\!\rightarrow} X' \times TY' \overset{ls}{\circ\!\!\rightarrow} X' \times Y'$

Def: M effectfully recognizes $L: \Sigma^* \to O$

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{L} & O \\ & {\scriptstyle h}\searrow \quad \nearrow{\scriptstyle p} & \\ & TM & \end{array}$$

Kleisli hom → h

p ← T-hom

Effectful Algebraic Recognition: Effectful Morphisms

Ass: For every finite X the bialgebra $(X \multimap X)$ is <u>monoidally</u> <u>fg</u>-<u>carried</u>:

(1) fg-carried:

$$\begin{array}{ccc} M & \overset{\exists e_0}{\dashrightarrow} & (X \multimap X) \\ \eta \downarrow & \nearrow e & \\ TM & & \end{array}$$

(2) e_0 is monoid homomorphism

(3) e_0 is <u>central</u>: $M \times X \overset{\bar{e}_0}{\multimap} X$ central

every $e_0(m) : X \multimap X$ central

Effectful Algebraic Recognition: Effectful Morphisms

Ass: For every finite X the bialgebra $(X \circ\!\!\to X)$ is monoidally fg-carried:

Theorem:

$L : \Sigma^* \to O$ is T-FA computable

iff

L is effectfully recognizable by finite monoid

(1) fg-carried:

$$\begin{array}{ccc} M & \overset{\exists e_0}{\dashrightarrow} & (X \circ\!\!\to X) \\ \eta \downarrow & \nearrow_{e} & \\ TM & & \end{array}$$

(2) e_0 is monoid homomorphism

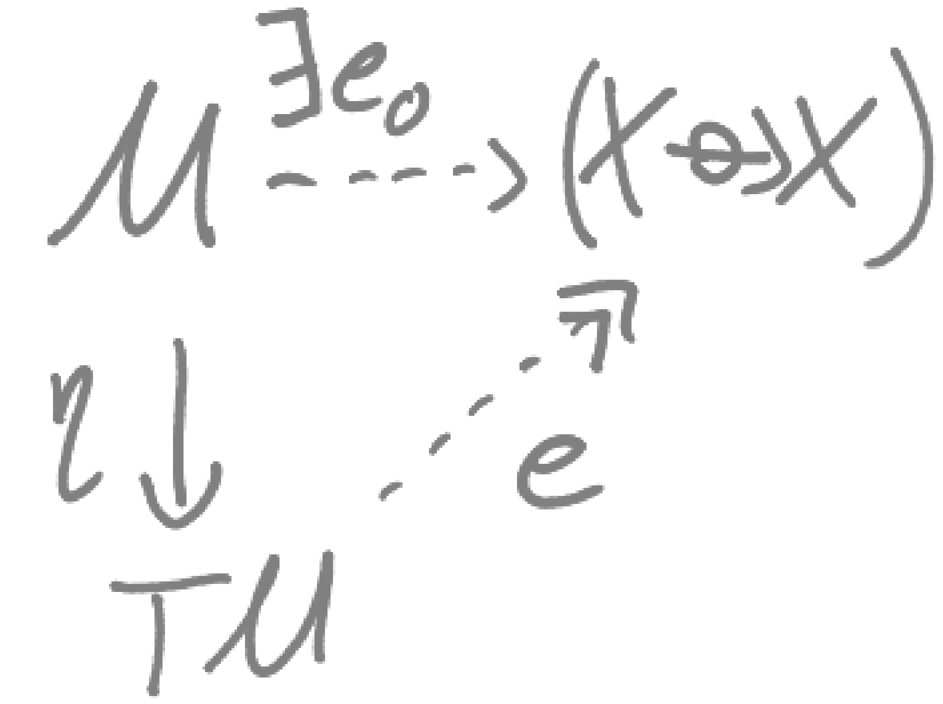
(3) e_0 is central: $M \times X \xrightarrow{\bar{e}_0} X$ ~~central~~

every $e_0(m) : X \circ\!\!\to X$ central

Effectful Algebraic Recognition: Effectful Morphisms

Ass: For every finite X the bialgebra $(X \rightarrowtriangle X)$ is <u>monoidally fg-carried</u>:

(1) fg-carried:



(2) e_0 is monoid homomorphism

(3) e_0 is <u>central</u>: $M \times X \overset{\bar{e}_0}{\rightarrowtriangle} X$ central / every $e_0(m): X \rightarrowtriangle X$ central

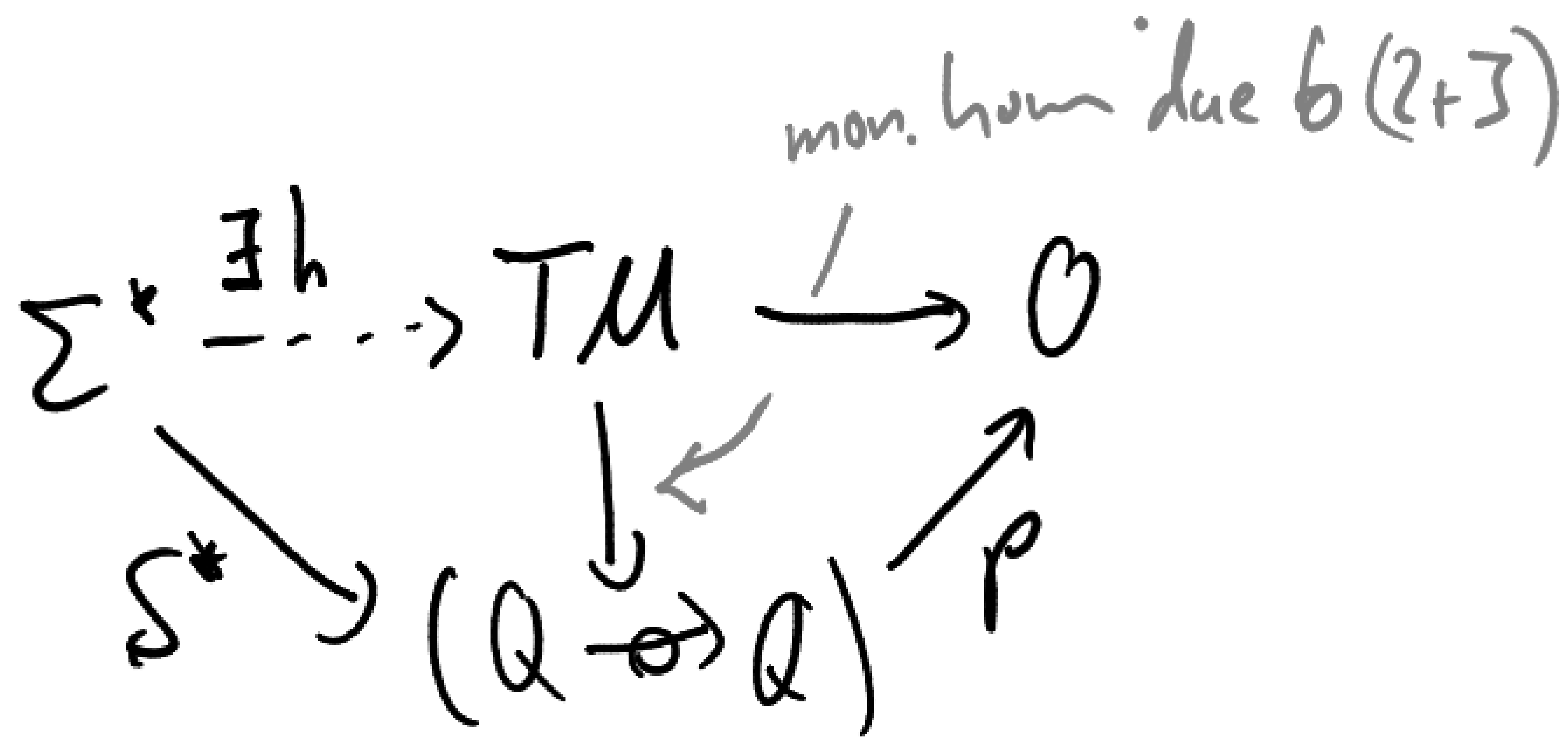
Theorem:

$L: \Sigma^* \to O$ is T-FA computable

iff

L is effectfully recognizable by finite monoid

Proof sketch: T-FA $\leadsto$ eff. hom



eff. hom $\leadsto$ T-FA

$\Sigma^* \overset{h}{\rightarrowtriangle} M$ rec L

$\Rightarrow$ TM is a bialgebra rec. L

Effectful Algebraic Recognition: Effectful morphisms

Our Examples all work: $(X \multimap X)$ is monoidally fg-carried

(1) fg-carried: $M \overset{\exists e_0}{\dashrightarrow} (X \multimap X)$, $\eta : M \downarrow TM$, $TM \overset{e}{\dashrightarrow} (X \multimap X)$

(2) e_0 is monoid homomorphism

(3) e_0 is central: $M \times X \overset{\bar{e}_0}{\multimap} X$ cenral

every $e_0(m) : X \multimap X$ central

Effectful Algebraic Recognition: Effectful morphisms

Our Examples all work: $(X \multimap X)$ is monoidally fg-carried

(1) fg-carried: $\mathcal{M} \overset{\exists e_0}{\dashrightarrow} (X \multimap X)$, $\mathcal{M} \xrightarrow{\eta} T\mathcal{M}$, e

(2) e_0 is monoid homomorphism

(3) e_0 is central: $\mathcal{M} \times X \overset{\overline{e_0}}{\multimap} X$ central / every $e_0(m) : X \multimap X$ central

Idea: $\mathcal{M}$ = deterministic functions,

i.e., $\mathcal{M} = (X \to X), (X \to X), \Delta_X$

are (2) a monoid

(2') s.t. embedding is a homomorphism

(3) only use central effects:

$$(X \to X) \overset{T = Id}{=} (X \multimap X)$$

$$(X \to X) \overset{T = (-)+1}{=} (X \multimap X)$$

} central submonads

$Id \hookrightarrow T$

$(-)+1 \hookrightarrow \mathcal{P}_f, S, List$

Effectful Algebraic Recognition:

T	T-FA	(T,M)-bialgebra recognizable	T-monoid recognizable	fin. monoid + effectful morphism recognizable
$\mathcal{P}_f$	NFA	✓	✓	✓
$\mathcal{D}$	PFA	✓	✓	✓
$\mathcal{C}$	NPFA	✓	X	✓
$\mathcal{S}$	WFA	✓	(if S comm.)	✓

(The last three columns are bracketed together under the heading "T-FA computable iff:")