Trace Equivalence in Abstract GSOS

Oberseminar des Lehrstuhls für Theoretische Informatik

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Outline

1. Preliminaries

1.1 GSOS

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- **GSOS rules**

$$\frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \xrightarrow{b} u[x_1 \ldots x_n, y_{i,k} \ldots]} \quad \text{or} \quad \frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \downarrow}$$

with $\sigma \in \mathcal{O}, n = \text{ar } \sigma, u \in \Sigma^*, a_{i,k}, b \in A, I, J, K_i \subset [\![1, n]\!]$

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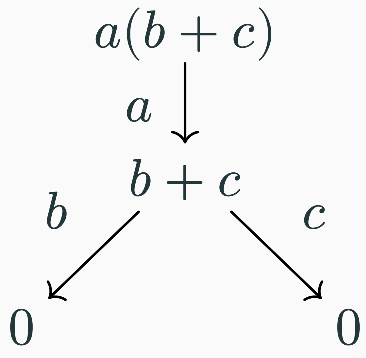

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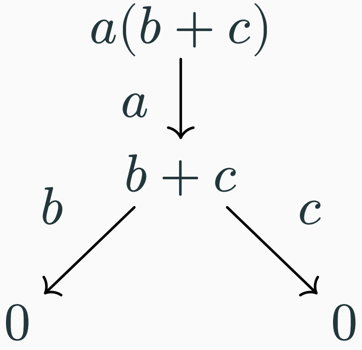

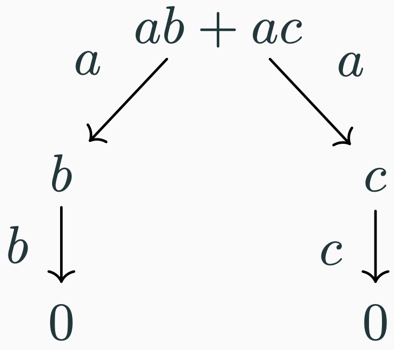

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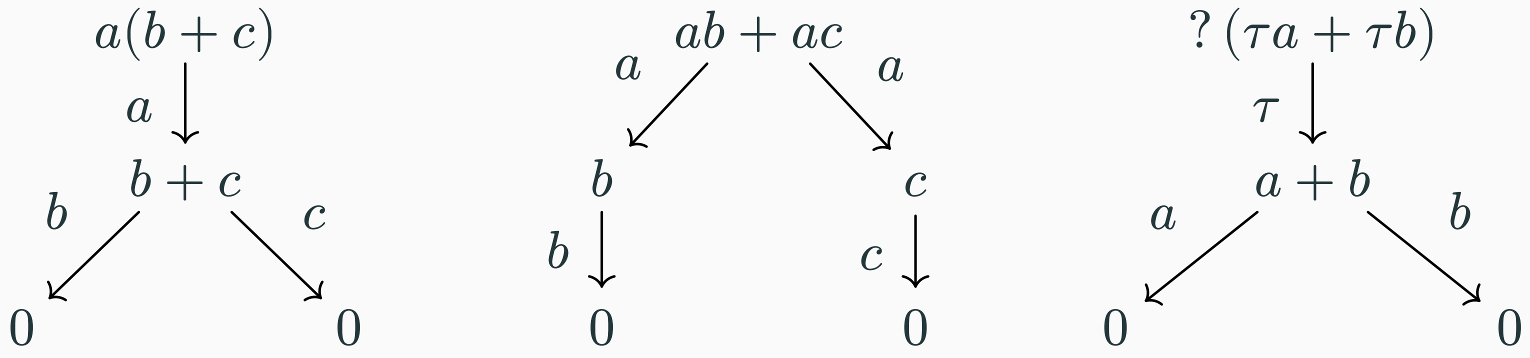

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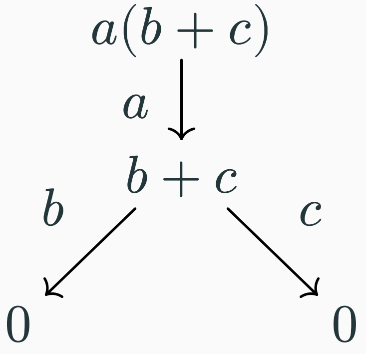

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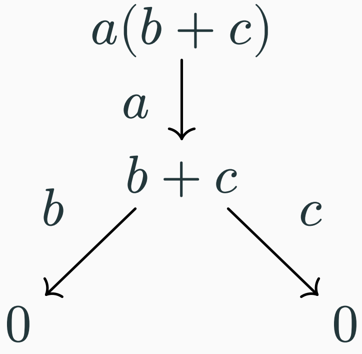

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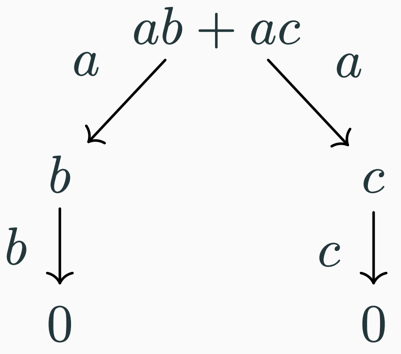

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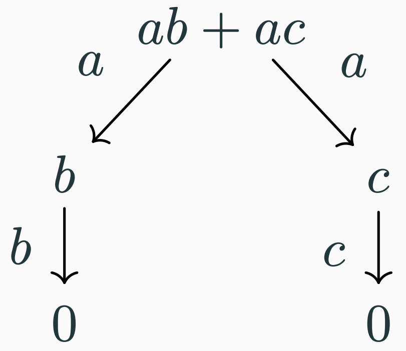

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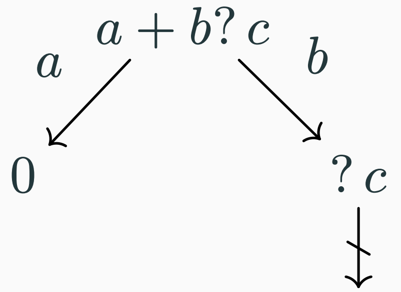

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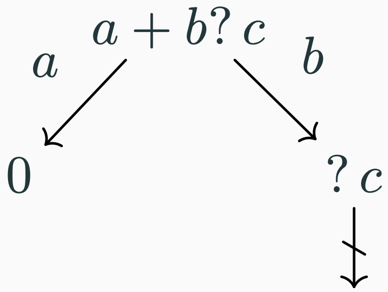

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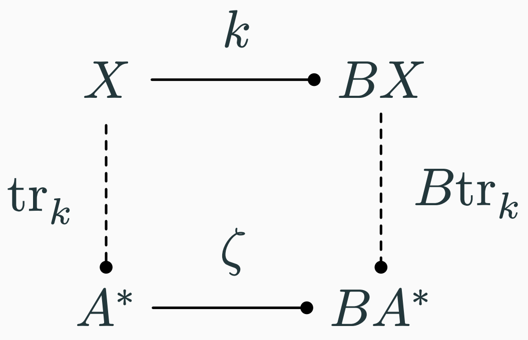
- for any $k : X \multimap BX$,



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Example:

$$\frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?\,t \xrightarrow{\tau} t' + t''} \qquad \frac{}{a.t \xrightarrow{a} t}\forall a \qquad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t}\forall a, b \qquad \frac{t \downarrow}{a.t \xrightarrow{a} t}\forall a$$

😕 😕 😎 😎

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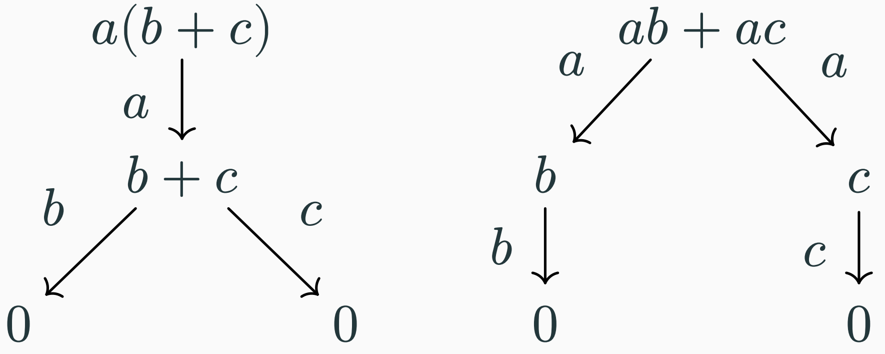

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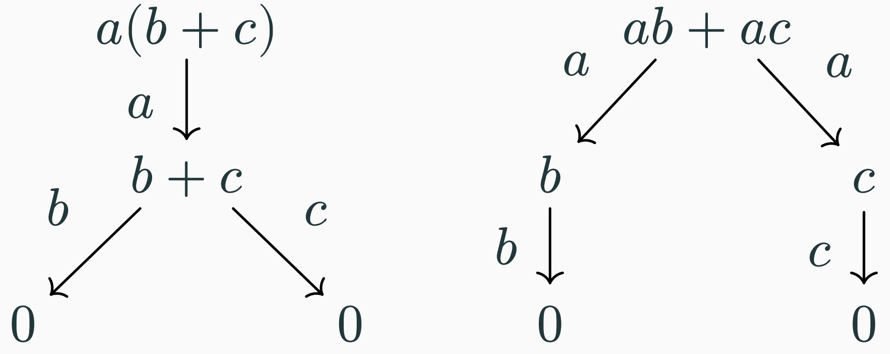

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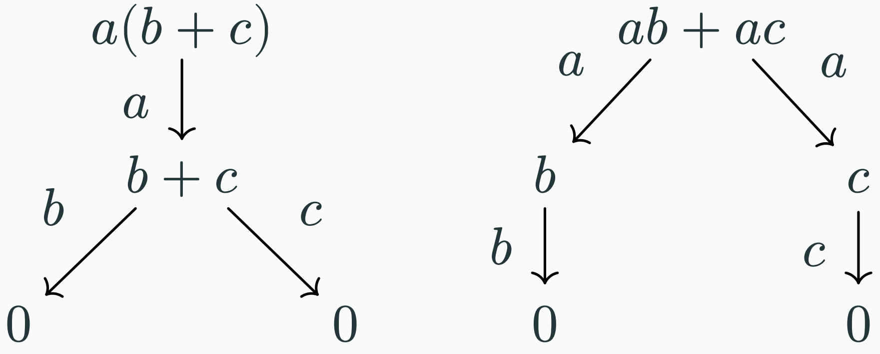

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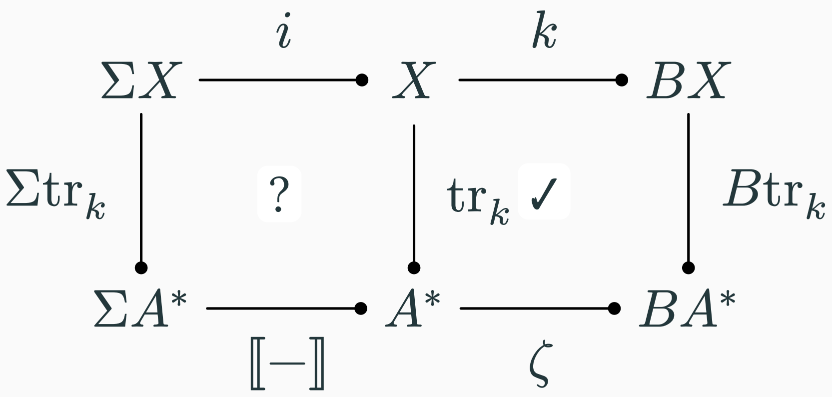

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2. Result

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- Recall

$$\begin{array}{ccc} \Sigma^* X & \xrightarrow{\;i\;} & X \\ {\scriptstyle \Sigma^*\mathrm{tr}_k}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tr}_k} \\ \Sigma^* Z & \xrightarrow[{[\![-]\!]}]{} & Z \end{array}$$

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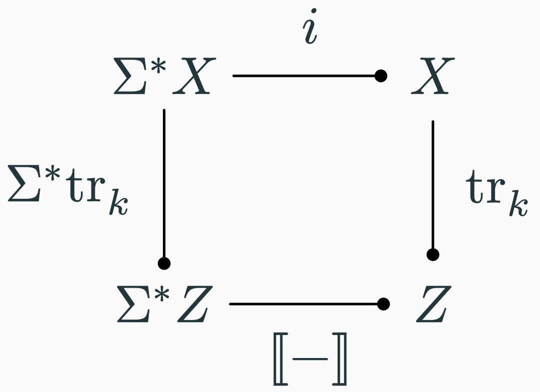

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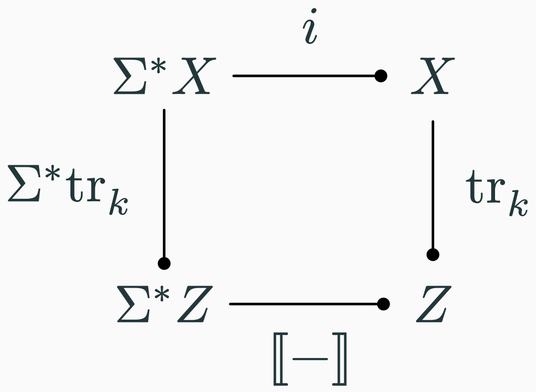

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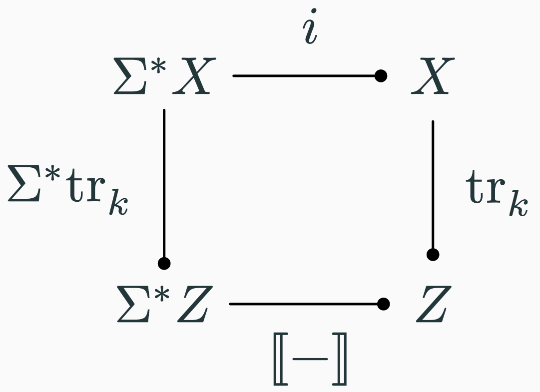

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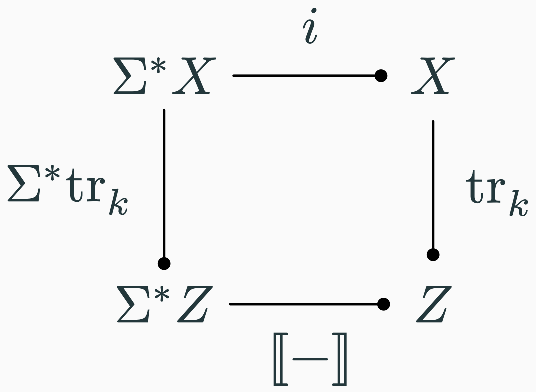

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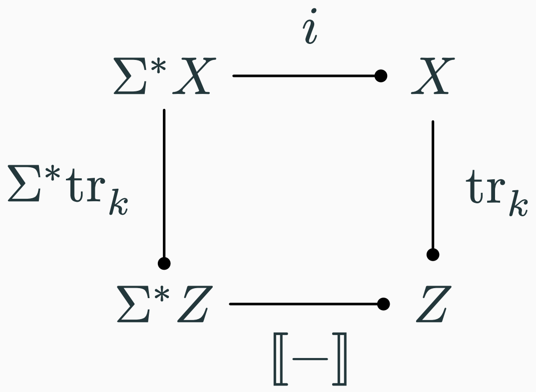

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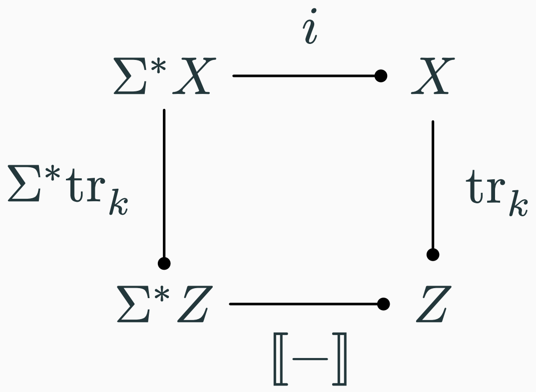

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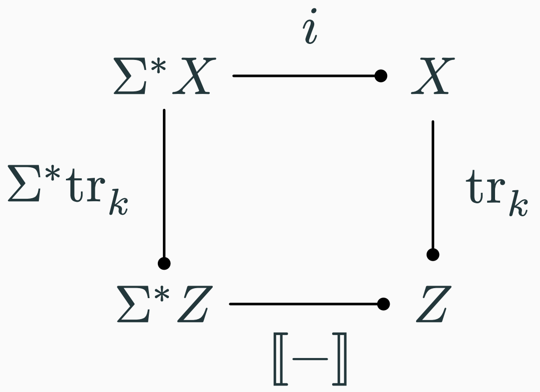

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Remark: need dst

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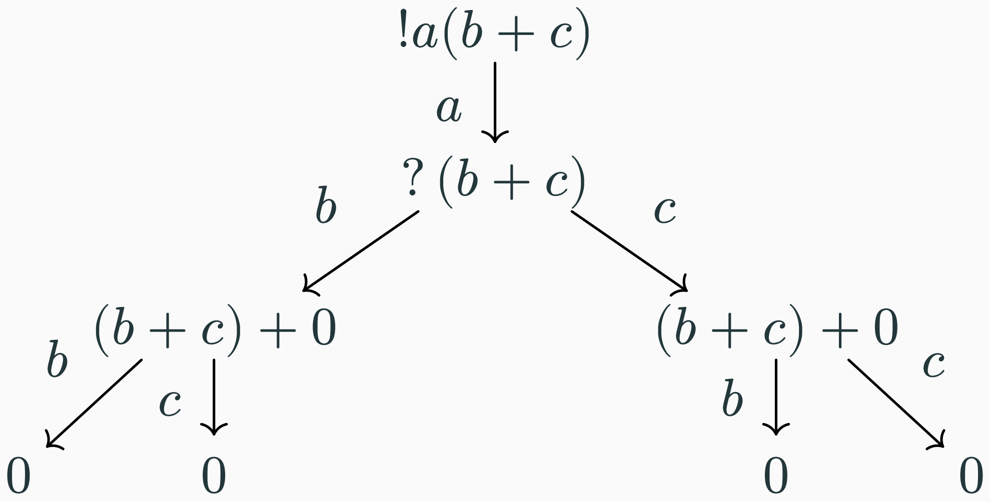

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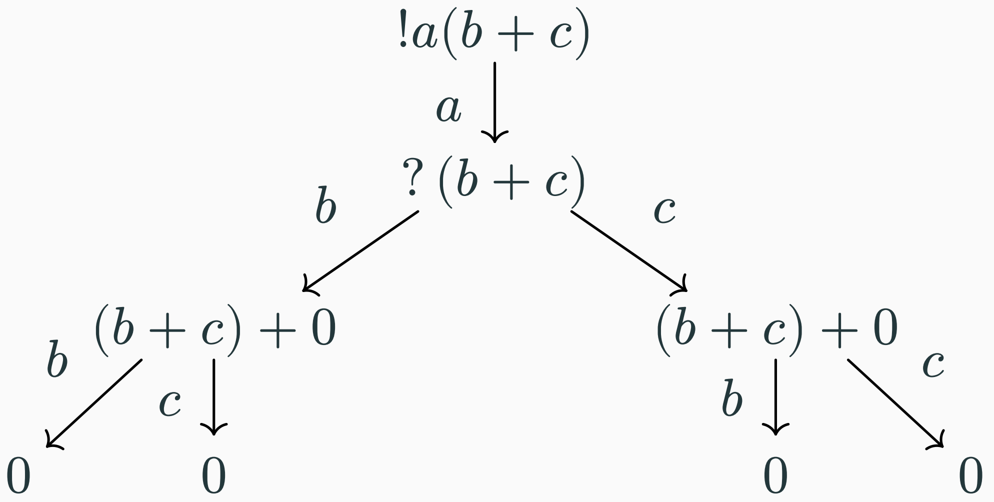

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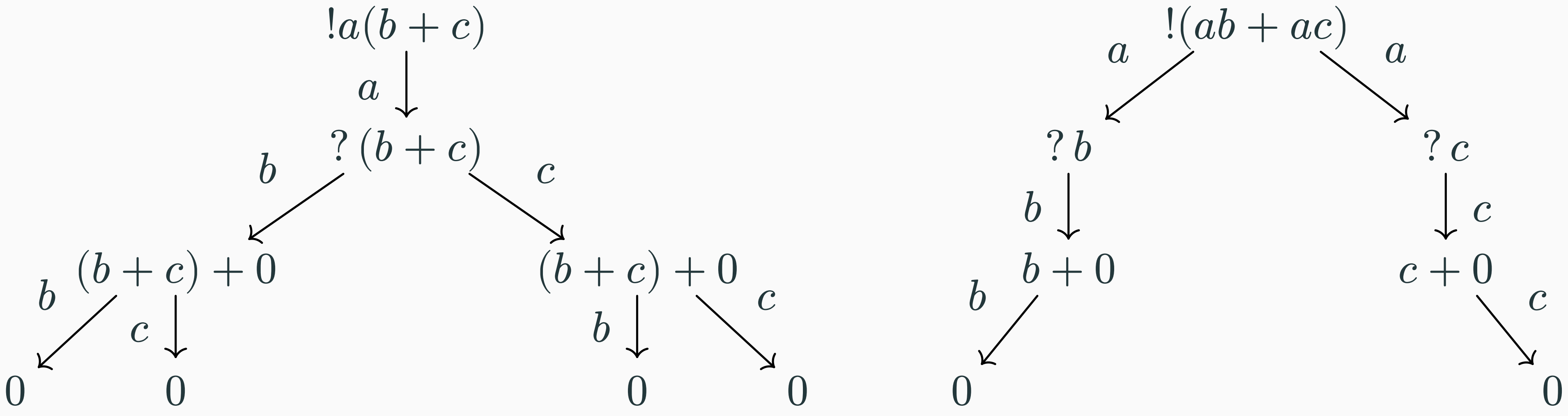

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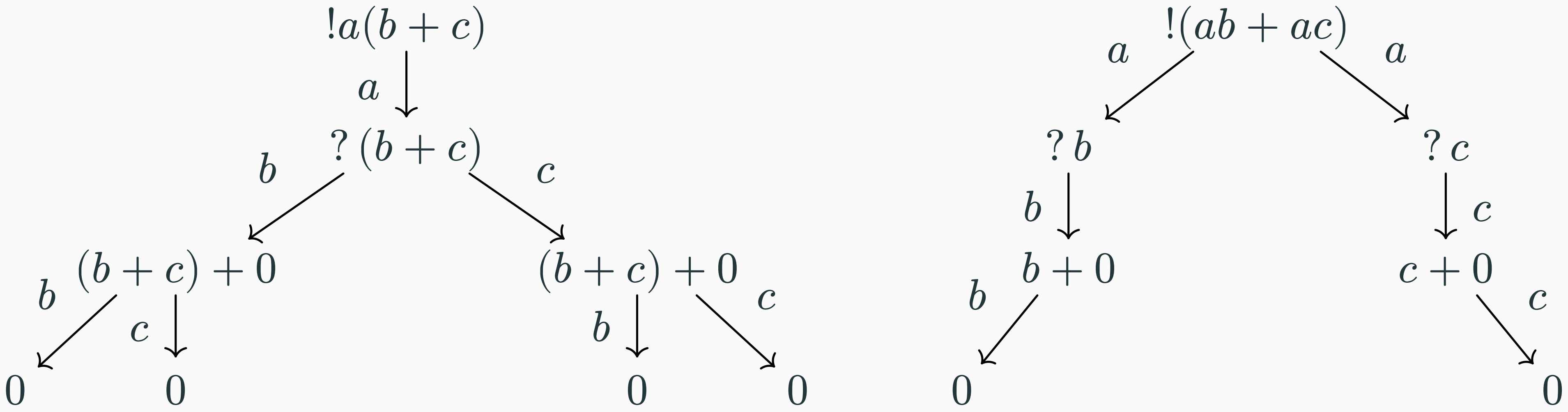

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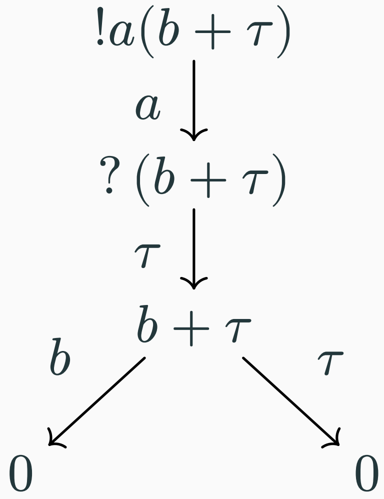

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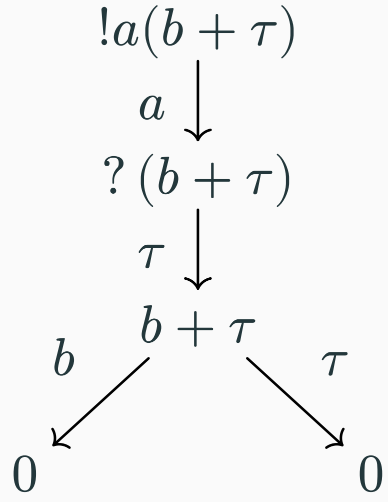

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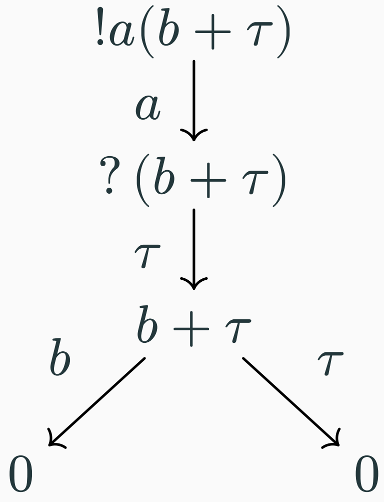

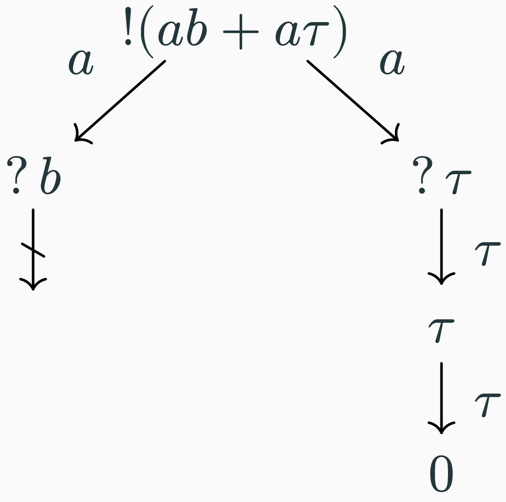

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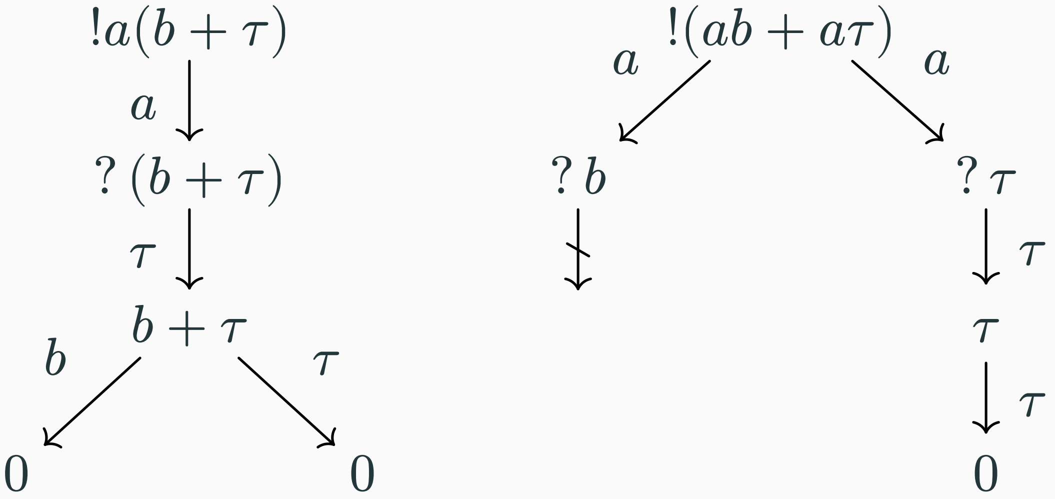

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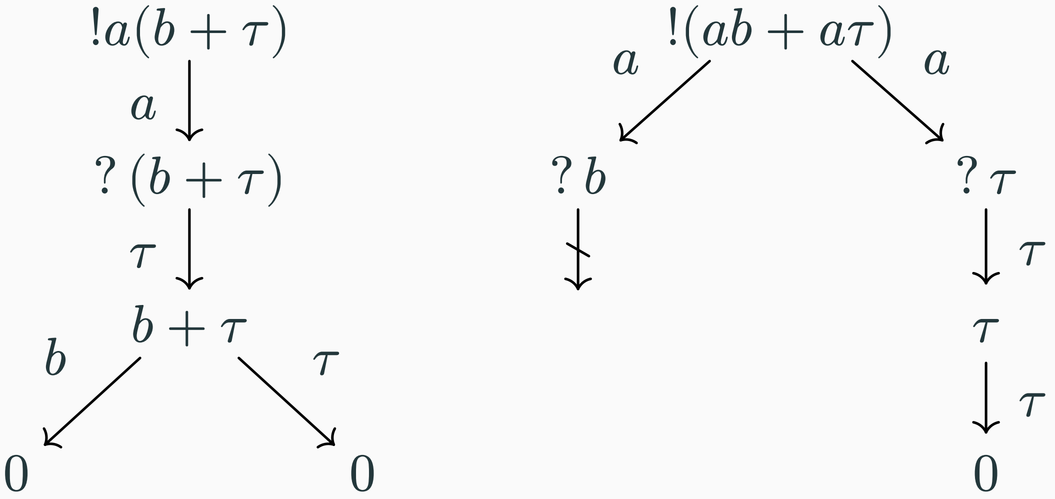

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$\longrightarrow$ observations that are “not used” 😕

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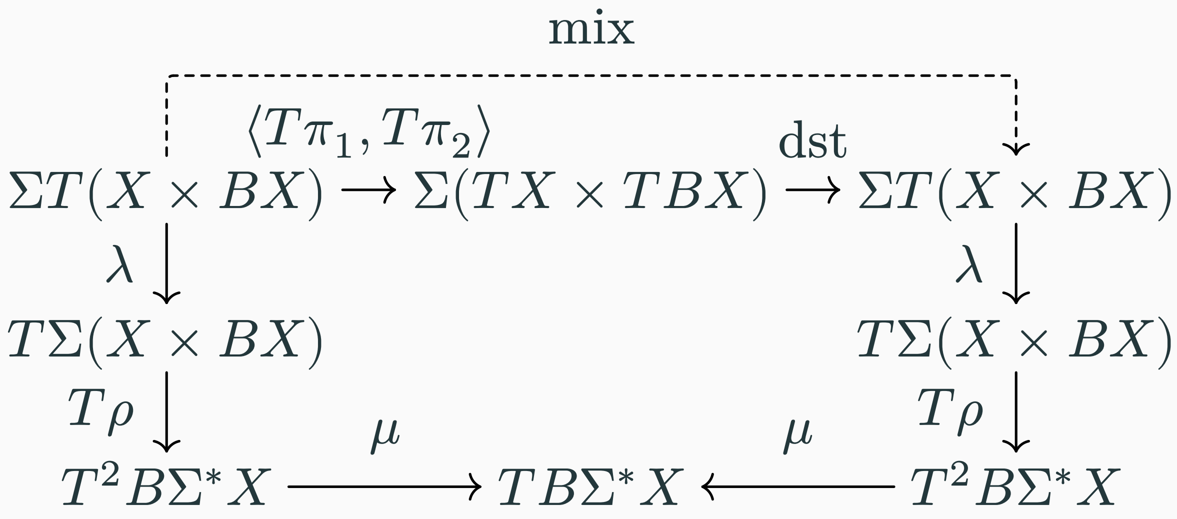

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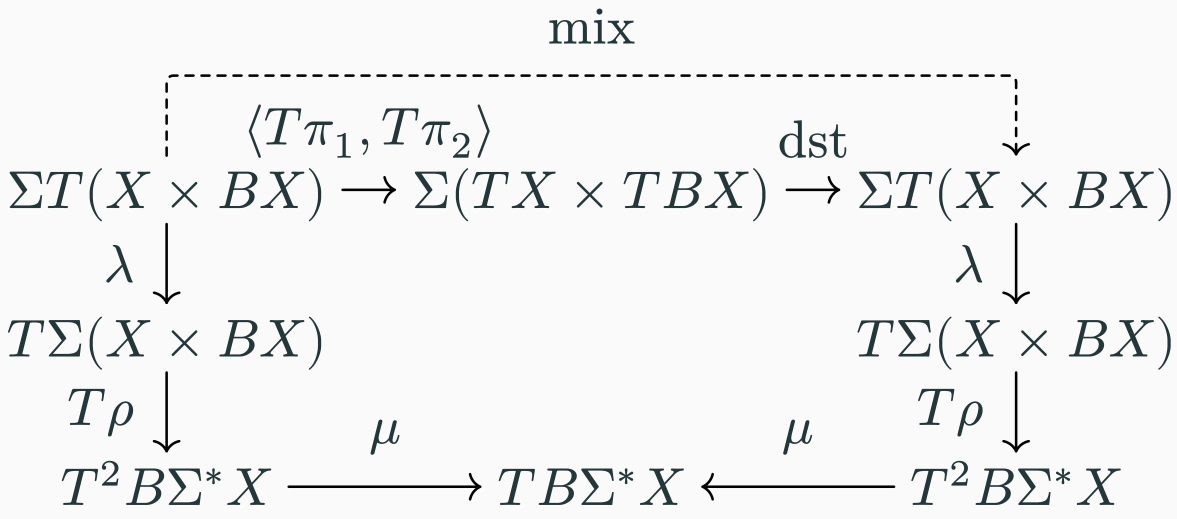

$$\Phi(\rho)(\sigma)(\text{mix } X_1 \dots \text{ mix } X_n) = \Phi(\rho)(\sigma)(X_1 \dots X_n)$$

where $X_i \subset X \times BX$

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$\longrightarrow$ Can we get back information on the original system ?

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- Thank you all for welcoming me in the chair ❤️

~ The End ~

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