Approximation Fixpoint Theory Meets Coalgebraic Modal Logic

Joint work in progress with Lutz Schröder and Daniel Hausmann

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Coalgebraic Modal Logic

Syntax

$\heartsuit \in \Lambda$ $\quad X \in \mathsf{Var}$

$$\psi, \phi := \bot \mid \top \mid \psi \wedge \phi \mid \psi \vee \phi \mid \heartsuit\psi \mid X \mid \mu X.\,\psi \mid \nu X.\,\psi$$

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Semantics

Models represented as *functor coalgebras*.

assigns structure

worlds — $(C, \xi \colon C \to \mathcal{F}(C))$

Interpret next-step modalities via *predicate liftings*

$$[\![\heartsuit]\!]_U(Y \colon \mathcal{P}(U)) \colon \mathcal{P}\mathcal{F}(U)$$

natural transformation $\mathcal{P} \Rightarrow \mathcal{P}\mathcal{F}$

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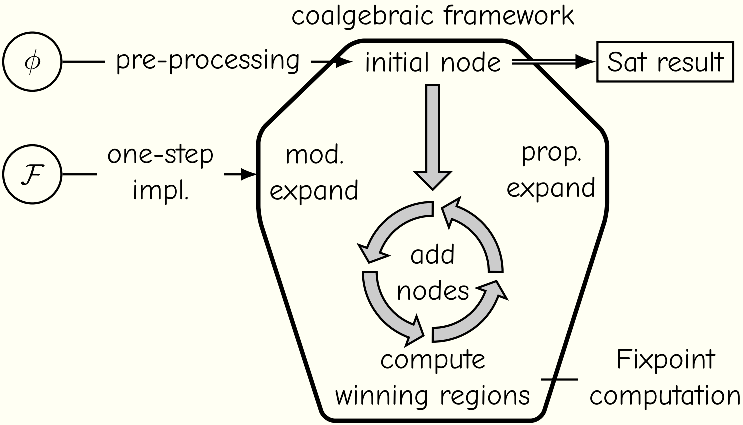
$$[\![\heartsuit\psi]\!] = \xi^{-1}[[\![\heartsuit]\!]_C([\![\psi]\!])]$$

Fixpoints in Coalgebraic ML



Fixpoint Theory Recap

Fixpoints

A fixpoint of a function $f : L \to L$ is a $l \in L$ where $l = f(l)$.

Knaster-Tarski

If $(L, \leq)$ is a complete lattice and $f : L \to L$ is monotone then the set F of fixpoints of f forms a complete lattice.

Approximation Fixpoint Theory

Idea

Go from crisp values to pairs of lower and upper bound.

Approximator

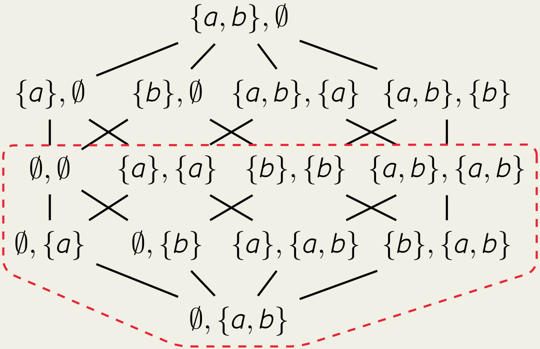
Given an operator $O\colon L \to L$ on (complete) lattice L.
$A\colon L \times L \to L \times L$ approximates O when

- $A(x, x) = (O(x), O(x))$
- $(x, y) \leq_i A(x, y)$

$\mathcal{P}$-Bilattice I

Information order $\leq_i$

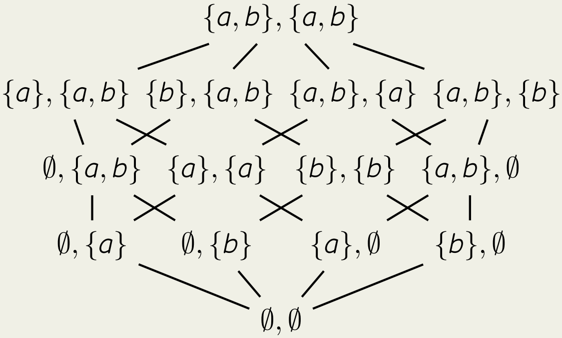
$$(S_1, P_1) \leq_i (S_2, P_2) \text{ iff } S_1 \subseteq S_2 \text{ and } P_1 \supseteq P_2$$



$\mathcal{P}$-Bilattice II

Truth order $\leq_t$

$$(S_1, P_1) \leq_t (S_2, P_2) \text{ iff } S_1 \subseteq S_2 \text{ and } P_1 \subseteq P_2$$



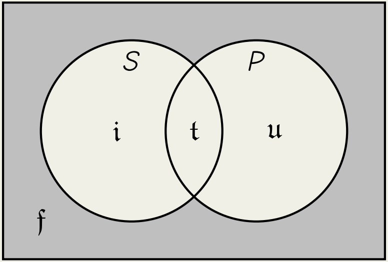
$\mathcal{P}$-Bilattice III

Lattice

If the underlying lattice was complete the bilattice inherits a complete lattice structure w.r.t. $\leq_i$.

3/4-valued view

Given (S, P) we can read off a 4-valued truth degree.



Argumentation in AFT

Abstract Dialectical Framework

Abstract dialectical framework (ADF) is a tuple $(S, C = \{C_s\}_{s \in S})$

- S is a set of *nodes*
- C is a set of prop. formulas with atoms S encoding *acceptance conditions*

Argumentation in AFT

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Intuition

- S is the set of (abstract) arguments
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Reasoning Task

Finding *model*: Set $M \subseteq S$ s.t. for all $s \in S$, $s \in M$ iff $M \models C_s$

Argumentation in AFT

Semantic Operator

$$G(M) := \{s \in S \mid M \models C_s\}$$

Approximator

$$aG(S,P) := \Big(\bigcap_{S \subseteq X \subseteq P} G(X), \bigcup_{S \subseteq X \subseteq P} G(X)\Big)$$

Logic Programming in AFT

Logic Program

A logic programm L is a set of horn clauses of the form $a_1, \ldots, a_n, \neg a_{n+1}, \ldots \neg a_m \to a_0$.

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Semantics

$$\begin{aligned} L(S) := \{a_0 \mid & a_1, \ldots, a_n, \neg a_{n+1}, \ldots \neg a_m \rightarrow a_0 \in P, \\ & \{a_1, \ldots, a_n\} \subseteq S, \\ & \{a_{n+1}, \ldots, a_m\} \cap S = \emptyset\} \end{aligned}$$

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$$L(S) := \{a_0 \mid a_1, \ldots, a_n, \neg a_{n+1}, \ldots \neg a_m \to a_0 \in P, \{a_1, \ldots, a_n\} \subseteq S, \{a_{n+1}, \ldots, a_m\} \cap S = \emptyset\}$$

Approximator

$$aL'(S, P) := \{a_0 \mid a_1, \ldots, a_n, \neg a_{n+1}, \ldots \neg a_m \to a_0 \in P, \{a_1, \ldots, a_n\} \subseteq S, \{a_{n+1}, \ldots, a_m\} \cap P = \emptyset\}$$

$$aL(S, P) := (aL'(S, P), aL'(P, S))$$

Autoepistemic ML I

Syntax

$$\psi, \phi := \bot \mid p \mid \psi \wedge \phi \mid K\psi$$

Semantics

$$W, w \not\models \bot$$
$$W, w \models p \text{ iff } p \in w$$
$$W, w \models \psi \wedge \phi \text{ iff } W, w \models \psi \text{ and } W, w \models \phi$$
$$W, w \models K\psi \text{ iff } W, v \models \psi \text{ for all } v \in W$$

Autoepistemic ML II

Theories

An *autoepistemic model* of a set of formulas T is a set W s.t.

$$W = \{w \mid W, w \models \phi \text{ for any } \phi \in T\}.$$

Given a model W, the *theory* of W

$$Th(W) := \{\phi \mid W, w \models \phi \text{ for all } w \in W\}.$$

Reasoning Task

Given sets of formulas T, E, decide if E is a *consistent stable expansion* of T i.e. whether there exists an autoepistemic model W of T such that $E = Th(W)$

Alternative Autoepistemic ML

3-valued pair semantics

Move to 3-valued interpretation and lower and upper bound on worlds in the model.

$$[\![K\psi]\!]_{(S,P)} = w \mapsto \begin{cases} \mathfrak{t} & \text{if for all } v \in P, [\![\psi]\!]_{(S,P)}(v) = \mathfrak{t} \\ \mathfrak{f} & \text{if for any } v \in S, [\![\psi]\!]_{(S,P)}(v) = \mathfrak{f} \\ \mathfrak{u} & \text{otherwise} \end{cases}$$

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Derivation Operator

Given a set of formulas T

$$\begin{aligned} \mathcal{D}_T(S,P) := (&\{w \mid [\![\phi]\!]_{(S,P)}(w) = \mathfrak{t} \text{ for any } \phi \in T\}, \\ &\{w \mid [\![\phi]\!]_{(S,P)}(w) \neq \mathfrak{f} \text{ for any } \phi \in T\}) \end{aligned}$$

Approximation Coalgebraic Modal Logic

Semantics

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assigns structure

worlds — $(C, \xi \colon \overbrace{C \to \mathcal{F}(C)})$

Interpret next-step modalities via *approximate predicate liftings*

$$[\![\heartsuit]\!] \colon \mathcal{P} \times \mathcal{P} \Rightarrow \mathcal{P}\mathcal{F} \times \mathcal{P}\mathcal{F}$$

$$[\![\heartsuit\psi]\!] = (\xi^{-1} \times \xi^{-1})[\![\heartsuit]\!]_C([\![\psi]\!])$$

Approximation Coalgebraic Modal Logic

Semantics

Models represented as *functor coalgebras*.

$$(C, \xi\colon \overbrace{C \to \mathcal{F}(C)})$$

worlds — $(C, \xi\colon C \to \mathcal{F}(C))$ — assigns structure

Interpret next-step modalities via *approximate predicate liftings*

$$[\![\heartsuit]\!] : \mathcal{P} \times \mathcal{P} \Rightarrow \mathcal{P}\mathcal{F} \times \mathcal{P}\mathcal{F}$$

$$[\![\heartsuit\psi]\!] = (\xi^{-1} \times \xi^{-1})[\![\heartsuit]\!]_C([\![\psi]\!])$$

Satisfaction

$M, w \models \phi$ iff $[\![\phi]\!]_M = (S, P)$ and $w \in S$ and $w \in P$

$M, w \overset{?}{\models} \phi$ iff $[\![\phi]\!]_M = (S, P)$ and $w \in P$

$M, w \not\models \phi$ iff $[\![\phi]\!]_M = (S, P)$ and $w \notin S$ and $w \notin P$

Coalgebraic Autoepistemic ML

Lifting AEL

Identity functor $\mathcal{I}$

$$[\![K]\!]_W(S_f, P_f) := (S_f \cap P_f, S_f \cup P_f)$$

Coalgebraic Autoepistemic ML

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Probabilistic AEL

Subdistribution functor $\mathcal{D}_s$

$$[\![K_{\geq n}]\!]_W(S_f, P_f) := (\{\mu \mid \underbrace{\mu(S_f \cap P_f)}_{\mu(X) = \sum_{x \in X} \mu(x)} \geq n\}, \{\mu \mid \mu(S_f \cup P_f) \geq n\})$$

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- From $\mathcal{P}$-lattice to $\mathcal{P}$-bilattice

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- $\subseteq$ monotonicity requirement relaxed to $\leq_i$ monotonicity
- 3/4-valued semantics
- Do you know any logics that can be supported by this change?