Conformance Games for Graded Semantics

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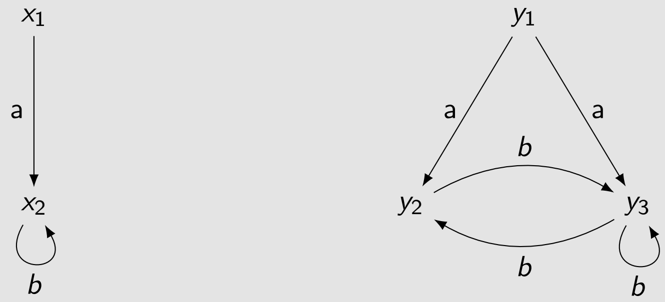
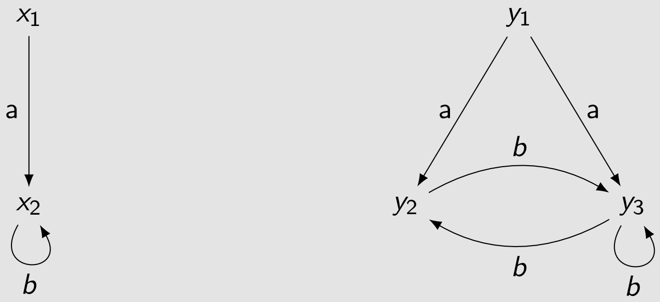


x_1
a
x_2
b
y_1
a
a
b
y_2
y_3
b
b



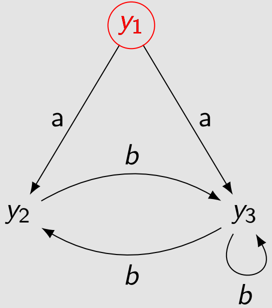
Bisimilarity

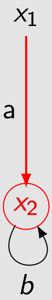
A bisimulation is a relation R such that for all xRy

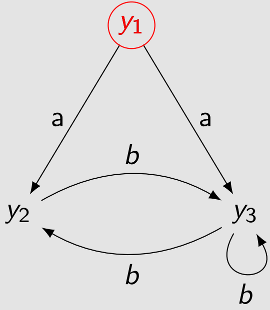
- $x \xrightarrow{a} x'$ implies that there is y' with $y \xrightarrow{a} y'$ and $x'Ry'$
- $y \xrightarrow{a} y'$ implies that there is x' with $x \xrightarrow{a} x'$ and $x'Ry'$

States x, y are bisimilar if there is a bisimulation with xRy

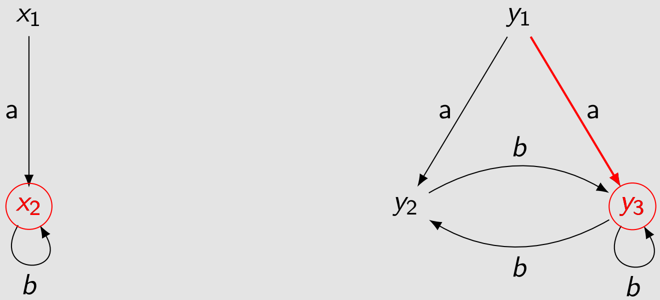

x_1
a
x_2
b


y_1
a
a
b
y_2
y_3
b
b

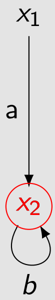


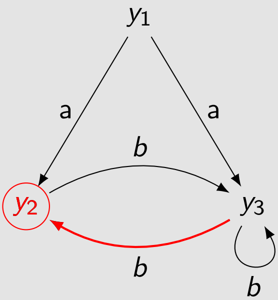


$\xrightarrow{a} x_2$

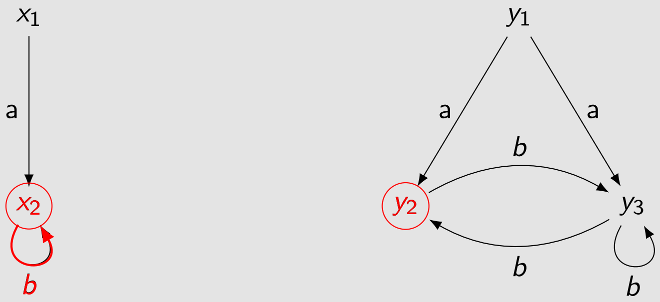


$\xrightarrow{a} y_3$

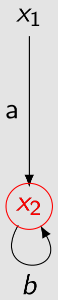


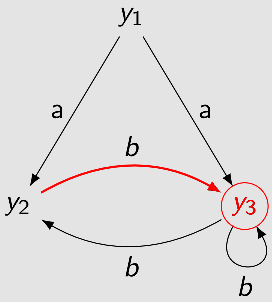


$\xrightarrow{b} y_2$

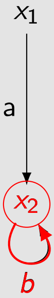


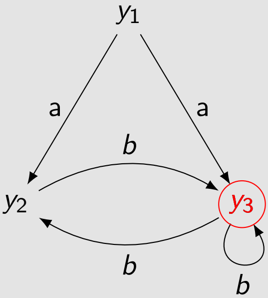
$\xrightarrow{b} x_2$





$\xrightarrow{b} y_3$





$\xrightarrow{b} x_2$

Contributions

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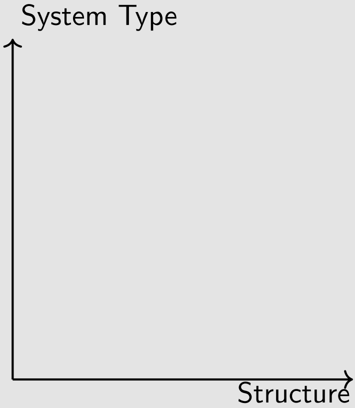
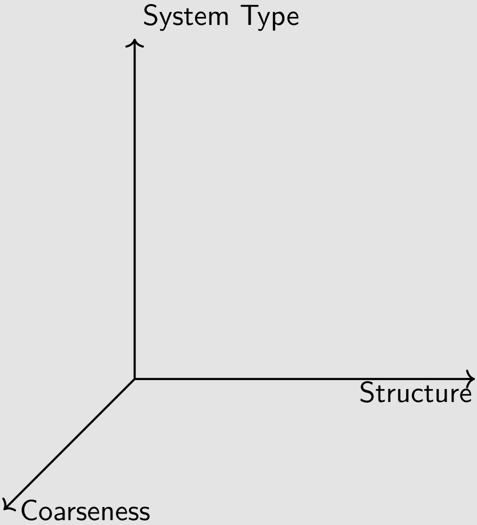
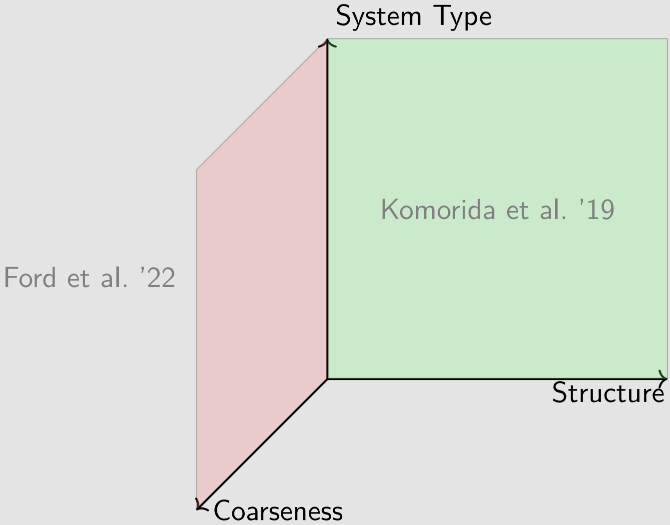


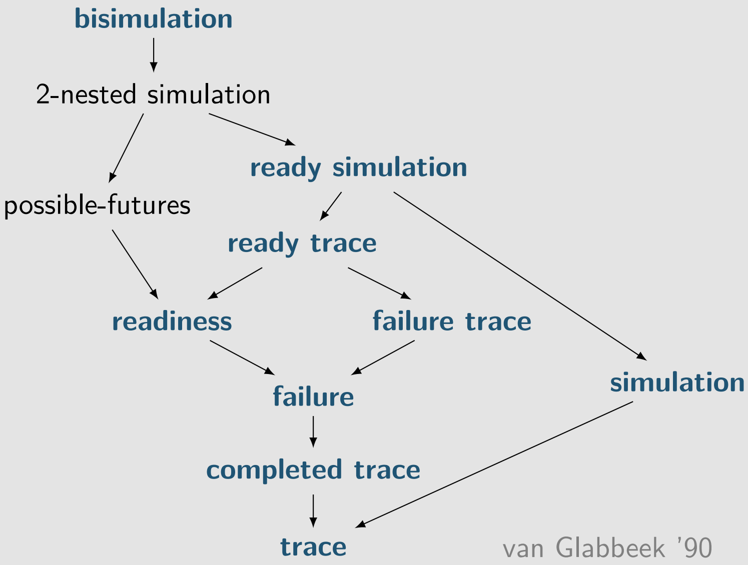
Contributions



Contributions



The LT/BT Spectrum



van Glabbeek '90

Graded Monads

Graded Monads

A *graded monad* $\mathbb{M}$ consists of

- $M_n \colon \mathbf{C} \to \mathbf{C}$ for $n \in \mathbb{N}$
- $\mu^{ij} \colon M_i M_j \Rightarrow M_{i+j}$
- $\eta \colon Id \Rightarrow M_0$

Subject to the usual monad laws (+ indices)

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Subject to the usual monad laws (+ indices)

Graded Algebras

A *graded M_n-algebra A* consists of

- $\mathbf{C}$-objects A_k for $k \leq n$
- $a^{ij} \colon M_i A_j \Rightarrow A_{i+j}$ for $i + j \leq n$

Subject to the usual algebra laws (+ indices)

Graded Semantics

Graded Semantics

A *graded semantics* for G-coalgebras consists of a graded monad $\mathbb{M}$ and a natural transformation $\alpha\colon G \Rightarrow M_1$

For $\gamma\colon X \to GX$ define inductively $\gamma^{(k)}\colon X \to M_k 1$:

$$\gamma^{(0)}\colon X \xrightarrow{\eta} M_0 X \xrightarrow{M_0 !} M_0 1$$

$$\gamma^{(k+1)}\colon X \xrightarrow{\alpha\cdot\gamma} M_1 X \xrightarrow{M_1\gamma^{(k)}} M_1 M_k 1 \xrightarrow{\mu^{1k}} M_{k+1} 1$$

Example: Probabilistic Traces

PTS

Probabilistic Transition Systems are coalgebras $\gamma \colon X \to \mathcal{D}(A \times X)$

Probabilistic Trace Monad

- $M_n = \mathcal{D}(A^n \times X)$
- Multiplication $\mu_X^{i,j} \colon \mathcal{D}(A^i \times \mathcal{D}(A^j \times X)) \to \mathcal{D}(A^{i+j} \times X)$ multiplies (and adds) probabilities
- $\eta_X \colon X \to \mathcal{D}X$, unit of $\mathcal{D}$
- $\alpha = id$

Then $\gamma^{(n)}(x) \in D(A^n)$ probabilities of traces of x after n steps

Depth-1 Monads

Depth-1 Graded Monads

A graded monad is *depth-1* if the following diagram is a coequalizer:

$$M_1 M_0 M_0 \underset{\mu^{10} M_0}{\overset{M_1 \mu^{00}}{\rightrightarrows}} M_1 M_0 \xrightarrow{\mu^{10}} M_1$$

Canonical M_1 Algebras

Canonical Algebra

An M_1-algebra A is *canonical* if it is free over its 0-part

$$A_0 \xrightarrow{f_0} B_0$$

$$\begin{array}{ccc} M_1 A_0 & \xrightarrow{M_1 f_0} & M_1 B_0 \\ \downarrow a^{10} & & \downarrow b^{10} \\ A_1 & \dashrightarrow_{f_1} & B_1 \end{array}$$

Lemma

If $\mathbb{M}$ is depth-1, then the M_1-algebra $(M_0X, M_1X, \mu^{0,0}, \mu^{0,1}, \mu^{1,0})$ is canonical

(Pre)determinization

$\bar{M}_1$

Let $E\colon \mathbf{Alg}_0(\mathbb{M}) \to \mathbf{Alg}_1(\mathbb{M})$ be the functor extending M_0-algebras to their canonical M_1-algebra

$$\bar{M}_1\colon (\mathbf{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathbf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathbf{Alg}_0(\mathbb{M}))$$

It is immediate that $M_1 = U\bar{M}_1 F$

$$\frac{X \xrightarrow{\alpha\cdot\gamma} M_1 X = U\bar{M}_1 FX}{FX \xrightarrow{\gamma^{\#}} \bar{M}_1 FX}$$

Categories with Structure

Topological Categories / $\mathbf{CLat}_\sqcap$-fibrations

Categories that admit initial liftings

$$\begin{array}{ccc} \mathcal{E} & \quad & A \overset{f_i}{\dashrightarrow} B_i \\ \downarrow U & & \\ \mathbf{Set} & & X \xrightarrow{f_i} UB_i \end{array}$$

Pullback

$\mathcal{E}_X$: Lattice of spaces above X

For $f : X \to Y$, get functor $f^\bullet : \mathcal{E}_Y \to \mathcal{E}_X$

Topological Categories

Examples

Category	Initial Lift
Topological spaces	Initial topology
Measurable spaces	(Basis of) Preimages of measurable sets
Equivalence relation	Relation reflection
Preorders	Order reflection
Pseudometric spaces	"Map and measure"

Setting

Ingredients

- Topological functor $U\colon \mathcal{E} \to \mathbf{Set}$
- Graded monad M_n on $\mathcal{E}$
- Functor $G\colon \mathcal{E} \to \mathcal{E}$ and depth-1 graded semantics $\alpha\colon G \Rightarrow M_1$

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Behavioural Conformance

Define $P^{\omega}_{\alpha} := \prod_{i \in \mathbb{N}} (\gamma^{(i)})^{\bullet} M_i 1$

Define P^{∞}_{α} as the initial lift w.r.t. $h \circ \eta$, where h ranges over coalgebra homs with domain $\gamma^{\#} \colon M_0 X \to \overline{M}_1 M_0 X$

Setting

Ingredients

- Topological functor $U\colon \mathcal{E} \to \mathbf{Set}$
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Behavioural Conformance

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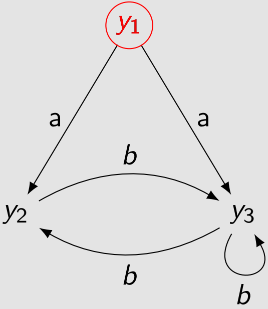
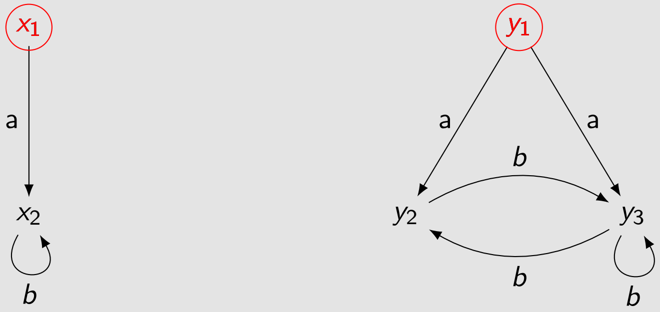
Define P_α^∞ as the initial lift w.r.t. $h \circ \eta$, where h ranges over coalgebra homs with domain $\gamma^\#\colon M_0 X \to \overline{M}_1 M_0 X$

Example

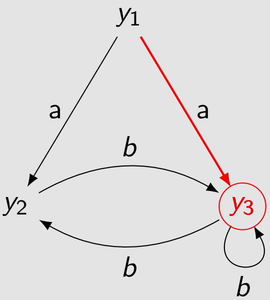
Equip $M_n X = \mathcal{D}(A^n \times X)$ with Wasserstein/total var. distance


x1
a
x2
b


y1
a
a
b
y2
y3
b
b



$$\xrightarrow{a} x_2 \quad \Leftrightarrow \quad \xrightarrow{a} y_2$$
$$\xrightarrow{a} x_2 \quad \Leftrightarrow \quad \xrightarrow{a} y_3$$



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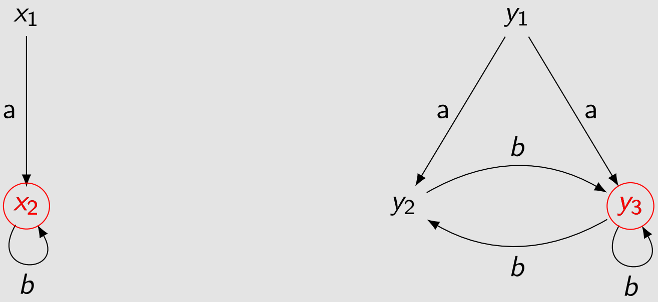

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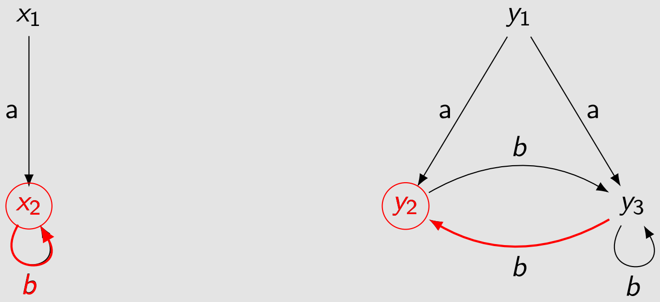

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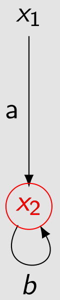

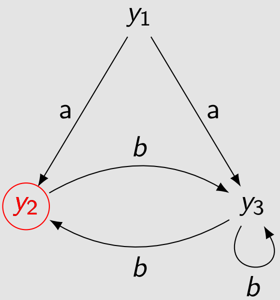

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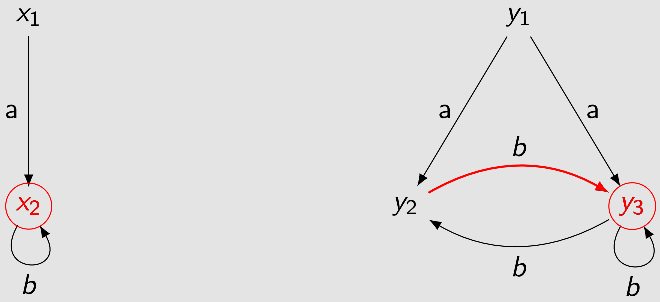
x_1
a
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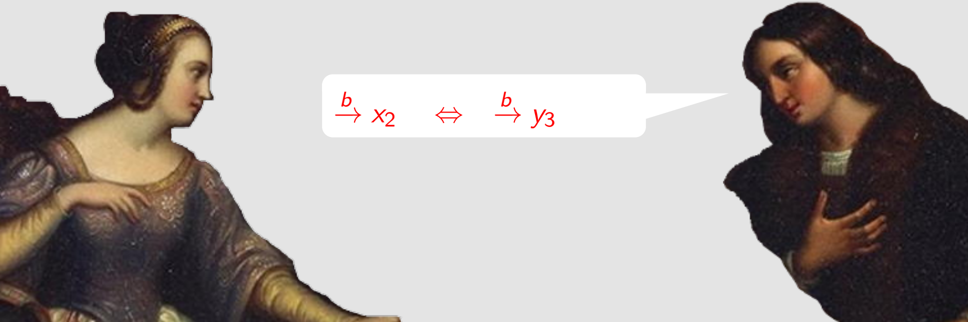
a
b
y_2
y_3
b
b


$\xrightarrow{b} x_2 \quad \Leftrightarrow \quad \xrightarrow{b} y_3$

Local bisimulation

A *local bisimulation* at (x, y) is a relation $R \subseteq X \times X$ such that

- $x \xrightarrow{a} x'$ implies that there is y' with $y \xrightarrow{a} y'$ and $x'Ry'$
- $y \xrightarrow{a} y'$ implies that there is x' with $x \xrightarrow{a} x'$ and $x'Ry'$

Game variant

To prove bisimilarity of (x, y)

1. Duplicator plays a local bisimulation R at (x, y)
2. Spoiler picks an element $(x', y') \in R$ as a new position.
3. Goto step 1.

A player that can not move loses, infinite plays are won by Duplicator.

Setting up the Game

How to Play

Basis B of $\mathcal{E}_{M_0X}$

Duplicator wants to show that conformance P holds in the behaviour of $(M_0X, \gamma^{\#})$.

Player	Position	Move
Duplicator	$P \in B$	$Z \subseteq B$ s.t. $P \sqsubseteq (\overline{M}_1\iota \cdot \gamma^{\#})^{\bullet}\overline{M}_1 C(\bigsqcup Z)$
Spoiler	$Z \subseteq B$	$P \in Z$

Setting up the Game

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Spoiler	$Z \subseteq B$	$P \in Z$

Additional Condition

Let P be the position after n rounds. Duplicator wins the n-round game if

$$P \sqsubseteq (M_0!_X)^{\bullet}M_01$$

Theorems

Assumptions

- Depth-1 graded semantics, $\bar{M}_1$ preserves initial arrows.
- M_0 is a lifting of a set functor.

Theorem

Let $P \in \mathcal{E}_{UX}$. Duplicator wins in all positions $P' \in B$ with $P' \sqsubseteq \eta_\bullet P$

- in the graded n-round conformance game iff $P \sqsubseteq (\gamma^{(n)})^\bullet M_n 1$.
- in the graded infinite conformance game iff $P \sqsubseteq P_\alpha^\infty$.

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- in the graded infinite conformance game iff $P \sqsubseteq P^\infty_\alpha$.

Syntactic Perspective

$\Rightarrow$ Categories of Relational Structures

Relational Structures

Relational Signature

Category $\mathbf{Str}(\Pi)$

Objects are pairs (X, E), where X is a set and E consists of tuples $\pi(x_1, \ldots, x_{\mathrm{ar}(\pi)})$ with $\pi \in \Pi$ and $x_i \in X$

Morphisms $g\colon (X, E) \to (Y, E')$ are maps $g\colon X \to Y$ such that $\pi(x_1, \ldots, x_n) \in E$ implies $\pi(g(x_1), \ldots, g(x_n)) \in E'$

Horn Structures

Horn Axioms

Let $\mathcal{A}$ be a set of axioms of the form

$$\Phi \Rightarrow \psi$$

where ψ is a Π-edge in Var and Φ is a set of Π-edges in Var

Category $\mathbf{Str}(\Pi, \mathcal{A})$: Subcategory of $\mathbf{Str}(\Pi)$, closed under $\mathcal{A}$.

Examples of Horn Theories

Preorders

Signature $\Pi = \{\leq\}$, Axioms

$$x \leq x \qquad \{x \leq y, y \leq z\} \Rightarrow x \leq z$$

Pseudometric Spaces

Signature $\Pi = \{=_\epsilon \mid \epsilon \in [0,1] \cap \mathbb{Q}\}$, Axioms

$$x =_0 x \qquad x =_\epsilon y \Rightarrow y =_\epsilon x$$

$$\{x =_\epsilon y, y =_{\epsilon'} z\} \Rightarrow x =_{\epsilon+\epsilon'} z$$

$$x =_\epsilon y \Rightarrow x =_{\epsilon+\epsilon'} y$$

$$\{x =_{\epsilon'} y \mid [0,1] \cap \mathbb{Q} \ni \epsilon' > \epsilon\} \Rightarrow x =_\epsilon y$$

Relational Theories

Relations in Context

Set Σ of symbols σ, each with arity $\mathrm{ar}(\sigma) \in \mathbb{N}$ and depth $d(\sigma) \in \mathbb{N}$.

Relational theories $(\Sigma, \mathcal{E})$ are parametric over a set of Axioms $\mathcal{E}$ of the form

$$X \vdash_k \pi(t_1, \ldots, t_n)$$

where

- t_i are uniform depth k terms over Var
- $\pi \in \Pi \cup \{=\}$
- X is a Π-structure over Var

Varieties of Σ-Algebras

Calculus

Judgments of the form $X \vdash_k \pi(t_1, \ldots, t_n)$ over some context $X \in \mathbf{Str}(\mathcal{H})$ Monad $\mathbb{M}$, where $M_n X$ are depth-n terms in Σ modulo $=$, with provable edges.

Example: Probabilistic Trace Distance

$\Sigma = \{+_q \mid q \in [0,1]\} \cup \{a \mid a \in A\}$

$$\vdash_0 x +_1 y = x \qquad \vdash_0 x +_\epsilon x = x$$

$$\vdash_0 x +_q y = y +_{1-q} x \qquad \vdash_0 (x +_q y) +_v z = x +_{qv} (y +_{\frac{q-qv}{1-qv}} z)$$

$$\vdash_1 a(x +_q y) = a(x) +_q a(y)$$

$$\{x =_\epsilon y, x' =_{\epsilon'} y'\} \vdash_0 x +_q x' =_\delta y +_q y'$$

with $\delta = \epsilon q + \epsilon'(1-q)$

Advantages of Relational Structures

Basis for free!

Basis B consists of individual edges on M_0X

Admissibility $\cong$ Syntactic Proof

A set of edges Z is admissible at $\pi(x_1, \ldots, x_n)$ if $Z \vdash_1 \pi(\gamma^{\#}(x_1), \ldots, \gamma^{\#}(x_n))$

Monad Conditions

$(\Sigma, \mathcal{E})$ depth-1 $\Rightarrow$ $\mathbb{M}$ depth-1

M_0 lifting if all depth-0 axioms for $=$ have empty context

Conclusion

What we did

- Generalized graded games to topological categories
- Provided Syntactic Underpinnings

Future Work

- Relation to codensity games?
- Syntax for arbitrary topological categories?
- Strategies $\Leftrightarrow$ Formulae?