Modal Characterization Theorems: from the classical to the quantitative coalgebraic

Paul Wild

October 25, 2022

Friedrich-Alexander-Universität Erlangen-Nürnberg

A van Benthem theorem for fuzzy modal logic (LICS'18)

Paul Wild, Lutz Schröder, Dirk Pattinson and Barbara König

The van Benthem Theorem

Theorem

Every bisimulation invariant first-order formula can be expressed by modal formulas.

Goal: formalize and prove this theorem in terms of fuzzy logic.

Fuzzy Modal Logic (1)

Theorem

*Every bisimulation invariant first-order formula can be expressed by **modal formulas**.*

Syntax:

$$\varphi, \psi ::= c \mid p \mid \neg\varphi \mid \varphi \ominus c \mid \varphi \wedge \psi \mid \Diamond\varphi$$

where $c \in \mathbb{Q} \cap [0,1]$ and $p \in \mathsf{At}$.

Semantics: given over *fuzzy relational models*

$$\mathcal{A} = (A, (p^{\mathcal{A}})_{p \in \mathsf{At}}, R^{\mathcal{A}})$$

- A is the set of states.
- $p^{\mathcal{A}} : A \to [0,1]$ is the interpretation for $p \in \mathsf{At}$.
- $R^{\mathcal{A}} : A \times A \to [0,1]$ is the transition relation.

Fuzzy Modal Logic (2)

Formulas are interpreted as functions $A \to [0,1]$:

- constants: $c(a) = c$
- propositions: $p(a) = p^{\mathcal{A}}(a)$
- negation: $(\neg\varphi)(a) = 1 - \varphi(a)$
- truncated subtraction: $(\varphi \ominus c)(a) = \max(\varphi(a) - c, 0)$
- conjunction: $(\varphi \wedge \psi)(a) = \min(\varphi(a), \psi(a))$
- modality: $(\Diamond\varphi)(a) = \sup_{a' \in A} \min(R^{\mathcal{A}}(a, a'), \varphi(a'))$

Notations: $a \wedge b = \min(a, b), a \vee b = \max(a, b), \bigvee = \sup, \bigwedge = \inf.$

Fuzzy First Order Logic

Theorem

*Every bisimulation invariant **first-order formula** can be expressed by modal formulas.*

Syntax:

$$\varphi, \psi ::= c \mid p(x) \mid R(x, y) \mid x = y \mid \neg\varphi \mid \varphi \ominus c \mid \varphi \wedge \psi \mid \exists x.\varphi$$

where $c \in \mathbb{Q} \cap [0, 1]$, $p \in \mathsf{At}$, x, y variables.

Semantics:

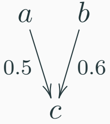
Let $\eta\colon \mathsf{Var} \to A$. $\varphi(\eta)$ is defined inductively:

- Boolean connectives and equality as expected
- $p(x)(\eta) = p^{\mathcal{A}}(\eta(x)), R(x, y)(\eta) = R^{\mathcal{A}}(\eta(x), \eta(y))$
- existential quantification: $(\exists x.\varphi)(\eta) = \bigvee_{a \in A} \varphi(\eta[x \mapsto a])$

Bisimulation Invariance

Theorem

*Every **bisimulation invariant** first-order formula can be expressed by modal formulas.*



$$d(a, b) = 0.1$$

- In fuzzy logic we can *quantify* how similarly two states behave.
- This gives rise to *behavioural distance* d.
- Bisimilar states have distance 0.
- φ *bisimulation invariant* $\iff$ φ non-expansive wrt. d:

$$|\varphi(a) - \varphi(b)| \leq d(a, b) \quad \text{for all states } a, b.$$

Modal Approximation

Theorem

Every bisimulation invariant first-order formula can be expressed by modal formulas.

- In classical modal logic, there are only finitely many modal formulas of fixed rank k (up to equivalence).
- In fuzzy modal logic, this is no longer true, because there are infinitely many truth constants $c \in \mathbb{Q} \cap [0,1]$.
- Thus, instead of showing that the bisimulation invariant formula φ is *equivalent* to some modal φ of rank k, we show that it can be *approximated* by such formulas:

$$\forall \varepsilon > 0 \;\; \exists \psi_\varepsilon \text{ modal of rank } k \;\; \|\varphi - \psi_\varepsilon\|_\infty \leq \varepsilon.$$

A Fuzzy van Benthem Theorem

Theorem

Every fuzzy first-order formula φ that is non-expansive wrt. behavioural distance d^G can be approximated by fuzzy modal formulas of some fixed rank k.

Next: define behavioural distance d^G via a bisimulation game.

Game-based Distance (1)

Bisimulation game for fuzzy logic:

- The game is parametrised by some $\varepsilon \geq 0$
- Two players, spoiler S and duplicator D
- Configurations: pairs of states (a, b)
- Moves:
  - S picks a' such that $R(a, a') > \varepsilon$
  - D picks b' such that $R(b, b') \geq R(a, a') - \varepsilon$
  - New configuration: (a', b')

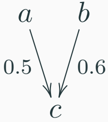
  S may also swap the two sides before his move
- Whoever is unable to move, loses
- Winning condition for D before every round:

$$|p(a) - p(b)| \leq \varepsilon \text{ for all } p \in \mathsf{At}.$$

Game-based Distance (2)

The corresponding distances are:

$$d^G(a,b) = \bigwedge\{\varepsilon \mid D \text{ wins the } \varepsilon\text{-game for } (a,b)\}$$
$$d^G_n(a,b) = \bigwedge\{\varepsilon \mid D \text{ wins the } n\text{-round } \varepsilon\text{-game for } (a,b)\}$$



$$d^G(a,b) = 1$$

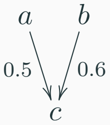
D wins for $\varepsilon = 0.1$, but loses for $\varepsilon < 0.1$.

Logic-based Distance

Using modal formulas, we can define:

$$d^L(a,b) = \bigvee_{\varphi \text{ modal}} |\varphi(a) - \varphi(b)|$$

$$d^L_n(a,b) = \bigvee_{\varphi \text{ modal, rk}\varphi \leq n} |\varphi(a) - \varphi(b)|$$



$$d^L_1(a,b) = 0.1 \text{ with } \varphi = \Diamond 1$$

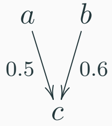
Function-based Distance

Behavioural distance via a *Kantorovich* construction:

$$d_0^K(a,b) = 0$$

$$d_{n+1}^K(a,b) = \bigvee_{p \in \mathsf{At}} |p(a) - p(b)| \vee \bigvee_{f\colon (A, d_n^K) \to [0,1] \text{ nonexp.}} |(\Diamond f)(a) - (\Diamond f)(b)|$$

$$(\Diamond f)(a) = \bigvee_{a' \in A} R(a,a') \wedge f(a')$$



$d_1^K(a,b) = 0.1$ with $f = x \mapsto 1$.

Equivalence of Distances

Theorem

Let $\mathcal{A}$ be a model and $n \geq 0$. Then

1. $d_n^G = d_n^K = d_n^L =: d_n$ *on $\mathcal{A}$.*
2. (A, d_n) *is a totally bounded pseudometric space.*
3. *The rank n formulas are a dense subset of the space of non-expansive maps $(A, d_n) \to [0, 1]$.*

Coalgebraic View

Consider the set functors F and G:

$$\mathsf{F}X = [0,1]^X, \quad \mathsf{F}f(g)(y) = \bigvee_{f(x)=y} g(x)$$

where $f\colon X \to Y, g \in [0,1]^X, y \in Y$.

$$\mathsf{G}X = [0,1]^{\mathsf{At}} \times \mathsf{F}X$$

Models $\mathcal{A} = (A, (p^{\mathcal{A}})_{p\in\mathsf{At}}, R^{\mathcal{A}})$ are *coalgebras* $\alpha\colon A \to \mathsf{G}A$:

$$\alpha(a) = (\lambda p.p^{\mathcal{A}}(a),\ \lambda a'.R^{\mathcal{A}}(a,a')).$$

Uniform Approximation

- $F_n := \mathsf{G}^n(\{*\})$ is the set of all n-step behaviours.
- We can construct a model $\mathcal{F}$ on the set $F := \bigcup_{n \geq 0} F_n$:

$$p^{\mathcal{F}}(h, g) = h(p), \quad R^{\mathcal{F}}((h, g), y) = \begin{cases} g(y), & \text{if } y \in F_n, \\ 0, & \text{otherwise.} \end{cases}$$

- For every model $\mathcal{A}$, there is a map $\pi_n \colon A \to F$ such that

$$d_n(a, \pi_n(a)) = 0.$$

- Thus:

$$\|\varphi - \psi\|_\infty \leq \varepsilon \text{ on } \mathcal{F} \implies \|\varphi - \psi\|_\infty \leq \varepsilon \text{ on all models } \mathcal{A}.$$

A characterization theorem for a modal description logic (IJCAI'19)

Paul Wild, Lutz Schröder, Dirk Pattinson and Barbara König

The logic $\mathcal{ALC}(\mathbf{P})$ - Syntax

Quantitative Probabilistic $\mathcal{ALC}$:

$$C, D ::= q \mid A \mid C \ominus q \mid \neg C \mid C \sqcap D \mid \mathbf{P}\, r.\, C$$

- rational constants $q \in \mathbb{Q} \cap [0, 1]$
- basic concept names $A \in \mathsf{N_C}$
- subtraction of constants $\ominus$
- expected value over r-successors $\mathbf{P}$ ($r \in \mathsf{N_R}$)

$$\mathsf{Loud} \sqcap \mathbf{P}\, \mathsf{hasSource}.\, (\mathsf{Large} \sqcap \mathbf{P}\, \mathsf{hasMood}.\, \mathsf{Angry})$$

The logic $\mathcal{ALC}(\mathsf{P})$ - Semantics

Models: $\mathcal{I} = (\Delta^\mathcal{I}, (A^\mathcal{I})_{A \in \mathsf{N_C}}, (r^\mathcal{I})_{r \in \mathsf{N_R}})$, where

- $\Delta^\mathcal{I}$ is a set (the *domain*)
- $A^\mathcal{I} \colon \Delta^\mathcal{I} \to [0,1]$
- $r^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$

$$\text{such that} \quad \sum_{a' \in \Delta^\mathcal{I}} r^\mathcal{I}(a,a') \in \{0,1\} \quad \text{for each } a \in \Delta^\mathcal{I}.$$

In other words, for role r each state a is either

- r-*blocking* – $r_a := r^\mathcal{I}(a,\cdot)$ is zero; or
- r-*transient* – r_a is a discrete probability distribution on $\Delta^\mathcal{I}$.

The logic $\mathcal{ALC}(\mathbf{P})$ - Semantics

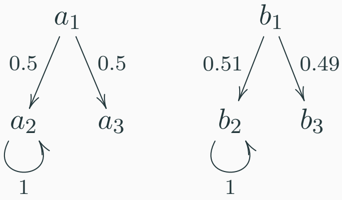
Interpretations: $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$, where

$$
\begin{aligned}
q^{\mathcal{I}}(a) &= q \\
(C \ominus q)^{\mathcal{I}}(a) &= \max(C^{\mathcal{I}}(a) - q, 0) \\
(\neg C)^{\mathcal{I}}(a) &= 1 - C^{\mathcal{I}}(a) \\
(C \sqcap D)^{\mathcal{I}}(a) &= \min(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) \\
(\mathbf{P}\, r.\, C)^{\mathcal{I}}(a) &= \mathbb{E}_{r_a}(C^{\mathcal{I}}) = \textstyle\sum_{a' \in \Delta^{\mathcal{I}}} r_a(a') \cdot C^{\mathcal{I}}(a')
\end{aligned}
$$

Bisimulation Invariance

From now on, restrict to a single role π.

- Classically, bisimulations are used to tell whether two states exhibit the same behaviour.
- However, consider the following states:



$$d(a_1, b_1) = 0.01$$

- With a *behavioural distance* d we can *quantify* how similarly two states behave. Bisimilar states have distance 0.

Towards a Characterization Theorem

- Defer the precise definition of *bisimulation distance* d for now.
- φ *bisimulation invariant* $\iff$ φ non-expansive wrt. d:

$$|\varphi(a) - \varphi(b)| \leq d(a, b) \quad \text{for all states } a, b.$$

- Characterize $\mathcal{ALC}(\mathbf{P})$ using bisimulation invariance:
  - All $\mathcal{ALC}(\mathbf{P})$-concepts are bisimulation invariant.
  - Every bisimulation invariant property can be approximated by $\mathcal{ALC}(\mathbf{P})$-concepts.

Correspondence Language

Quantitative probabilistic first-order logic (FO(**P**)):

$$\varphi, \psi ::= q \mid A(x) \mid x = y \mid \varphi \ominus q \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x.\, \varphi \mid x\mathbf{P}\lceil y : \varphi \rceil$$

Semantics:

$$A(x_i)(\bar{a}) = A^{\mathcal{I}}(a_i)$$

$$(\exists x_0.\, \varphi(x_0, x_1, \ldots, x_n))(\bar{a}) = \sup\{\varphi(a_0, \bar{a}) \mid a_0 \in \Delta^{\mathcal{I}}\}$$

$$(x_i\mathbf{P}\lceil y : \varphi(y, x_1, \ldots, x_n)\rceil)(\bar{a}) = \mathbb{E}_{r_{a_i}}(\varphi(\,\cdot\,, \bar{a}))$$

Example:

$$x\mathbf{P}\lceil z : z = y \rceil = \text{‘the successor of } x \text{ is probably } y\text{’}$$

$$= \text{probability of reaching } y \text{ from } x \text{ in one step}$$

Modal Approximation

- In classical $\mathcal{ALC}$, there are only finitely many modal formulas of fixed rank k (up to equivalence).
- In fuzzy modal logic, this is no longer true, because there are infinitely many truth constants $c \in \mathbb{Q} \cap [0,1]$.
- Thus, instead of showing that the bisimulation invariant formula φ is *equivalent* to some modal φ of rank k, we show that it can be *approximated* by such formulas:

$$\forall \varepsilon > 0 \;\; \exists \psi_\varepsilon \text{ modal of rank } k \;\; \|\varphi - \psi_\varepsilon\|_\infty \leq \varepsilon.$$

Bisimulation game

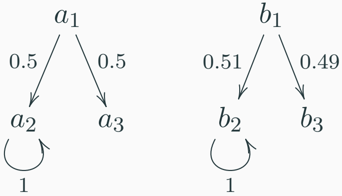
Game on models $\mathcal{I}, \mathcal{J}$ played by *Spoiler* (S) and *Duplicator* (D):

- *Configurations*: triples (a, b, ε), $a \in \Delta^{\mathcal{I}}$, $b \in \Delta^{\mathcal{J}}$, $\varepsilon \in [0, 1]$.
- *Moves*:
  - D picks $\mu \in \mathsf{Cpl}(\pi_a, \pi_b)$
  - D picks a function $\varepsilon' \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \to [0, 1]$ such that $\mathbb{E}_\mu(\varepsilon') \leq \varepsilon$
  - S picks (a', b') with $\mu(a', b') > 0$
  - New configuration: $(a', b', \varepsilon'(a', b'))$
- D wins if both states are blocking or $\varepsilon = 1$.
- S wins if exactly one state is blocking and $\varepsilon < 1$.
- Otherwise, D wins if she maintains the *winning condition*: $|A^{\mathcal{I}}(a) - A^{\mathcal{J}}(b)| \leq \varepsilon$ for all $A \in \mathsf{N_C}$.

$\mathsf{Cpl}(\pi_a, \pi_b)$: set of $\mu \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \to [0, 1]$ with marginals π_a and π_b:

$$\pi_a(a') = \textstyle\sum_{b'} \mu(a', b') \qquad \pi_b(b') = \textstyle\sum_{a'} \mu(a', b')$$

Example game



- Initial configuration: $(a_1, b_1, 0.01)$.
- First turn: D picks μ and ε' as follows:

μ	b_2	b_3	
a_2	0.5	0	0.5
a_3	0.01	0.49	0.5
	0.51	0.49	

ε'	b_2	b_3
a_2	0	1
a_3	1	0

Game-based distance

$$d^G(a,b) = \inf\{\varepsilon \mid D \text{ wins the game for } (a,b,\varepsilon)\}$$
$$d^G_n(a,b) = \inf\{\varepsilon \mid D \text{ wins the } n\text{-round game for } (a,b,\varepsilon)\}$$

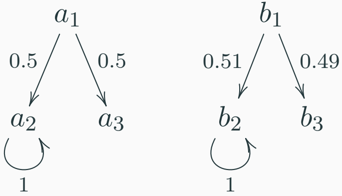
Lemma

Each $\mathcal{ALC}(\boldsymbol{P})$-concept of rank n is depth-n bisimulation-invariant, that is

$$|C(a) - C(b)| \leq d^G_n(a,b).$$

Using modal formulas, we can define:

$$d^L(a,b) = \sup\{|C(a) - C(b)| \mid C \in \mathcal{ALC}(\mathbf{P})\}$$
$$d^L_n(a,b) = \sup\{|C(a) - C(b)| \mid C \in \mathcal{ALC}(\mathbf{P}), \mathsf{rk}C \leq n\}$$



$$d^L_2(a,b) = 0.01 \text{ with } C = \mathbf{PP}1$$

Pseudometric Liftings

Let (X, d) be a pseudometric space. We define two pseudometrics on the space $\mathrm{D}X$ of discrete probability measures on X.

Definition (Kantorovich distance)

$$d^{\uparrow}(\pi_1, \pi_2) = \sup\{|\,\mathbb{E}_{\pi_1}(f) - \mathbb{E}_{\pi_2}(f)|\mid f \in \mathrm{Pred}(X, d)\}$$

where $\mathrm{Pred}(X, d)$ is the set of nonexpansive maps $(X, d) \to [0, 1]$.

Definition (Wasserstein distance)

$$d^{\downarrow}(\pi_1, \pi_2) = \inf\{\mathbb{E}_{\mu}(d) \mid \mu \in \mathsf{Cpl}(\pi_1, \pi_2)\}$$

These two *pseudometrics liftings* coincide:

Theorem (Kantorovich-Rubinstein duality)

For all π_1, π_2, $d^{\uparrow}(\pi_1, \pi_2) = d^{\downarrow}(\pi_1, \pi_2)$.

Kantorovich and Wasserstein Distances

Behavioural distance via fixed point iteration:

$$d_0^K(a,b) = d_0^W(a,b) = 0$$
$$d_{n+1}^K(a,b) = \max(\sup_{A \in \mathsf{N_C}} |A^{\mathcal{I}}(a) - A^{\mathcal{I}}(b)|, (d_n^K)^{\uparrow}(\pi_a, \pi_b))$$
$$d_{n+1}^W(a,b) = \max(\sup_{A \in \mathsf{N_C}} |A^{\mathcal{I}}(a) - A^{\mathcal{I}}(b)|, (d_n^W)^{\downarrow}(\pi_a, \pi_b))$$

By Kantorovich-Rubinstein duality, $d_n^K = d_n^W$ for all n.

Equivalence of Distances and Density

Theorem

Let $\mathcal{I}$ be a model. Then for all $n \geq 0$:

- $d_n^G = d_n^W = d_n^K = d_n^L =: d_n$ *on* $\mathcal{A}$.
- *The rank-n $\mathcal{ALC}(\boldsymbol{P})$-concepts form a dense subset of the space* $\mathrm{Pred}(\Delta^\mathcal{I}, d_n)$ *of non-expansive maps* $(\Delta^\mathcal{I}, d_n) \to [0,1]$.

This is proven by induction on n. Some intuition:

- $d_n^G = d_n^W$ because the game is built to model W. distance.
- $d_n^W = d_n^K$ by Kantorovich-Rubinstein duality.
- $d_n^K = d_n^L$ follows from the density claim for $n-1$.

The Characterization Theorem

Theorem

Every bisimulation-invariant FO($\boldsymbol{P}$)*-formula of rank at most* n *can be approximated by* $\mathcal{ALC}(\boldsymbol{P})$*-concepts of rank at most* 3^n.

Characteristic logics for behavioural metrics via fuzzy lax extensions (CONCUR'20)

Paul Wild and Lutz Schröder

Introduction

Goal

Analyse the behaviour of transition systems involving quantitative data.

- Various system types can be modelled as coalgebras:
  - Labelled transition systems $\alpha \colon A \to \mathcal{P}(L \times A)$
  - Markov chains $\alpha \colon A \to \mathrm{D}A$
  - ...

In general: $\alpha \colon A \to TA$ for some set functor T

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  - ...

  In general: $\alpha\colon A \to TA$ for some set functor T
- Behavioural distances allow for a quantitative measure of process equivalence:

$d(a_1, b_1) = \varepsilon$

Introduction

Goal

Analyse the behaviour of transition systems involving quantitative data.

- To define behavioural distances, we make use of lax extensions:
  - Lax extensions give a coalgebraic account of *bisimulation*.
  - Using a lax extension, *lift* the set functor T to a functor on pseudometrics.
  - Behavioural distance arises from a coalgebraic *fixpoint* construction.

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    - Lax extensions give a coalgebraic account of *bisimulation*.
    - Using a lax extension, *lift* the set functor T to a functor on pseudometrics.
    - Behavioural distance arises from a coalgebraic *fixpoint* construction.
- We extract characteristic logics for these behavioural distances:
    - Coalgebraic modal logics with *modalities* defined using L.
    - *Real-valued semantics* give rise to logical distance.
    - Logical distance $=$ behavioural distance, amounting to a *Hennessy-Milner theorem*.

Fuzzy Relations

Definition

A fuzzy relation is a map $R\colon A \times B \to [0,1]$, also written $R\colon A \nrightarrow B$.

Convention: a, b are related by $R \iff R(a,b) = 0$.

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Let $R\colon A \nrightarrow B, S\colon B \nrightarrow C$ and $f\colon A \to B$.

- *Converse* relation: $R^\circ(b,a) = R(a,b)$.
- *Graph* of a function: $\mathrm{Gr}_f(a,b) = \begin{cases} 0, & \text{if } f(a) = b, \\ 1, & \text{otherwise.} \end{cases}$

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- *Composition* of relations: $(R;S)(a,c) = \inf_{b \in B} R(a,b) \oplus S(b,c)$.

$x \oplus y = \min(x+y, 1)$

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- *Composition* of relations: $(R;S)(a,c) = \inf_{b \in B} R(a,b) \oplus S(b,c)$.
  ($x \oplus y = \min(x+y, 1)$)
- ε*-diagonal* on a set: $\Delta_{\varepsilon,A}(a_1,a_2) = \begin{cases} \varepsilon, & \text{if } a_1 = a_2, \\ 1, & \text{otherwise.} \end{cases}$

Fuzzy Lax Extensions

Definition

A fuzzy lax extension maps $R\colon A \nrightarrow B$ to $LR\colon TA \nrightarrow TB$ such that:

$$
\begin{aligned}
&\text{(L0)} \quad L(R^\circ) = (LR)^\circ \\
&\text{(L1)} \quad R_1 \leq R_2 \Rightarrow LR_1 \leq LR_2 \\
&\text{(L2)} \quad L(R;S) \leq LR;LS \\
&\text{(L3)} \quad L\mathsf{Gr}_f \leq \mathsf{Gr}_{Tf}
\end{aligned}
$$

We say that L is non-expansive, if additionally

$$
\text{(L4)} \quad L\Delta_{\varepsilon,A} \leq \Delta_{\varepsilon,TA}
$$

where A, B, C are sets, $R, R_1, R_2\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, $f\colon A \to B$, $\varepsilon > 0$.

Properties of Fuzzy Lax Extensions

Lemma

L satisfies Axiom (L4) $\iff$ $R \mapsto LR$ is non-expansive w.r.t. the supremum metric.

Properties of Fuzzy Lax Extensions

Lemma

L satisfies Axiom (L4) $\iff$ $R \mapsto LR$ is non-expansive w.r.t. the supremum metric.

Lemma

If $d\colon X \nrightarrow X$ is a pseudometric, then so is $Ld\colon TX \nrightarrow TX$.

Thus, L gives rise to a functor lifting of $T\colon \mathsf{Set} \to \mathsf{Set}$ to a functor $\overline{T}\colon \mathsf{PMet} \to \mathsf{PMet}$.

category of pseudometric spaces and non-expansive maps

The Hausdorff Lifting

Classically, bisimulations on Kripke frames arise via the *Egli-Milner extension*:

$$(U,V) \in \overline{\mathcal{P}}(R) \iff (\forall a \in U.\, \exists b \in V.\, (a,b) \in R) \wedge (\forall b \in V.\, \exists a \in U.\, (a,b) \in R).$$

$\overline{\mathcal{P}}$ is a two-valued lax extension of the powerset functor $\mathcal{P}$.

The Hausdorff Lifting

Classically, bisimulations on Kripke frames arise via the *Egli-Milner extension*:

$$(U, V) \in \overline{\mathcal{P}}(R) \iff (\forall a \in U. \exists b \in V. (a, b) \in R) \wedge (\forall b \in V. \exists a \in U. (a, b) \in R).$$

$\overline{\mathcal{P}}$ is a two-valued lax extension of the powerset functor $\mathcal{P}$.

Replacing $\forall$ with $\sup$, $\exists$ with $\inf$, $\wedge$ with $\max$ gives the Hausdorff lifting H:

$$HR(U, V) = \max(\sup_{a \in U} \inf_{b \in V} R(a, b), \sup_{b \in V} \inf_{a \in U} R(a, b)).$$

H is a non-expansive fuzzy lax extension of $\mathcal{P}$.

Quantitative Bisimulations

Definition

Let L be a lax extension of T, and let $\alpha\colon A \to TA$ and $\beta\colon B \to TB$ be coalgebras.

1. $R\colon A \nrightarrow B$ is an L-bisimulation if $LR \circ (\alpha \times \beta) \leq R$.

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Equivalently, $d^L_{\alpha,\beta}$ is the least fixed point of $R \mapsto LR \circ (\alpha \times \beta)$.

$\Longrightarrow$ L-bisimulations can be used to prove upper bounds for behavioural distance.

Fuzzy Predicate Liftings

Definition

An *n-ary (fuzzy) predicate lifting* is a natural transformation

$$\lambda\colon \mathsf{Q}^n \Rightarrow \mathsf{Q} \circ T,$$

where $\mathsf{Q}X = [0,1]^X$ is the contravariant fuzzy powerset functor.

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- *Dual* of λ: $\bar{\lambda}(f_1, \dots, f_n) = 1 - \lambda(1 - f_1, \dots, 1 - f_n)$.
- λ is *monotone* if $f_1 \le g_1, \dots, f_n \le g_n \implies \lambda(f_1, \dots, f_n) \le \lambda(g_1, \dots, g_n)$.
- λ is *nonexpansive* if

$$\|\lambda_X(f_1, \dots, f_n) - \lambda_X(g_1, \dots, g_n)\|_\infty \le \max(\|f_1 - g_1\|_\infty, \dots, \|f_n - g_n\|_\infty).$$

The Kantorovich Lifting

For $\mu_1, \mu_2 \in \mathsf{D}X$ and $d\colon X \nrightarrow X$ a metric,

$$Kd(\mu_1, \mu_2) = \sup\{\mathbb{E}\,\mu_1(f) - \mathbb{E}\,\mu_2(f) \mid f\colon (X, d) \to ([0,1], d_E) \text{ nonexpansive}\}.$$

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Definition (Kantorovich Lifting) cf. Baldan et al. 2018

Let Λ be a set of monotone predicate liftings that is closed under duals.
For $R\colon A \nrightarrow B$, $K_\Lambda R\colon TA \nrightarrow TB$ is given by

$$K_\Lambda R(t_1, t_2) = \sup\{\lambda_A(f)(t_1) - \lambda_B(g)(t_2) \mid \lambda \in \Lambda, (f, g) \text{ is } R\text{-nonexpansive}\},$$

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Theorem

K_Λ is a lax extension. If all $\lambda \in \Lambda$ are nonexpansive, then K_Λ is nonexpansive.

The Wasserstein Lifting

Definition (Wasserstein lifting) cf. Baldan et al. 2018, Hofmann 2007

Let Λ be a set of monotone predicate liftings.

For $R\colon A \nrightarrow B$, $W_\Lambda R\colon TA \nrightarrow TB$ is given by

$$W_\Lambda R(t_1, t_2) = \sup_{\lambda \in \Lambda} \inf\{\lambda_{A \times B}(R)(t) \mid t \in T(A \times B), T\pi_1(t) = t_1, T\pi_2(t) = t_2\}.$$

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Suppose T preserves weak pullbacks and for each $\lambda \in \Lambda$,

$$\lambda_X(0_X) = 0_{TX} \quad \text{and} \quad \lambda_X(f \oplus g) \leq \lambda_X(f) \oplus \lambda_X(g).$$

Theorem

W_Λ is a lax extension. If all $\lambda \in \Lambda$ are nonexpansive, then W_Λ is nonexpansive.

Wasserstein Examples

Example (Wasserstein for distributions)

D has a nonexpansive fuzzy lax extension $W = W_{\{\lambda\}}$, where $\lambda_X(f)(\mu) = \mathbb{E}\,\mu(f)$.

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Example (Convex powersets)

$\mathcal{C}X =$ nonempty convex subsets of DX.

$\mathcal{C}$ has a nonexpansive fuzzy lax extension $L = W_{\{\lambda\}}$, where $\lambda_X(f)(A) = \sup_{\mu\in A} \mathbb{E}\,\mu(f)$.

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One can show that in fact $L = H \circ W = H \circ K$. ← Mio/Vignudelli 2020

Lax Extensions as Kantorovich Liftings

Goal

Given a fuzzy lax extension L, find a set Λ such that $L = K_\Lambda$.

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Idea

If the functor T is *finitary*, is has a finitary presentation:

- a *signature* Σ of operations with given finite arities
- for each $\sigma \in \Sigma$ of arity n a natural transformation $\sigma\colon (-)^n \Rightarrow T$

such that every element of TX has the form $\sigma_X(x_1, \ldots, x_n)$ for some $\sigma \in \Sigma$.

Moss Liftings

Definition

Let $\sigma \in \Sigma$ be n-ary. The Moss lifting $\mu^\sigma \colon \mathsf{Q}^n \Rightarrow \mathsf{Q} \circ T$ is defined as follows:

$$\mu^\sigma_X(f_1, \ldots, f_n)(t) = L\mathsf{ev}_X(\sigma_{\mathsf{Q}X}(f_1, \ldots, f_n), t),$$

where $\mathsf{ev}_X \colon \mathsf{Q}X \nrightarrow X$ is given by $\mathsf{ev}_X(f, x) = f(x)$.

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Theorem

We have $L = K_\Lambda$, where $\Lambda = \{\mu^\sigma \mid \sigma \in \Sigma\} \cup \{\overline{\mu^\sigma} \mid \sigma \in \Sigma\}$ is the set of all Moss liftings and their duals.

Moreover, L is nonexpansive iff all Moss liftings are nonexpansive.

Finitary Separability

What about non-finitary functors?

Note that every set functor T has a finitary part T_ω given by

$$T_\omega X = \bigcup \{Ti[TY] \mid Y \subseteq X \text{ finite}, i \colon Y \to X \text{ inclusion}\}.$$

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Theorem

If L is finitarily separable, then the Moss liftings for T_ω extend to a set Λ of predicate liftings for T such that $L = K_\Lambda$.

Real-valued Coalgebraic Modal Logic

Syntax of $\mathcal{L}_\Lambda$

$$\varphi, \psi ::= c \mid \varphi \ominus c \mid \neg\varphi \mid \varphi \wedge \psi \mid \lambda(\varphi_1, \ldots, \varphi_n) \qquad (c \in [0,1], \lambda \in \Lambda)$$

Semantics over a coalgebra $\alpha\colon A \to TA$

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Definition

Λ-logical distance: $d^\Lambda(a,b) = \sup\{|[\![\varphi]\!](a) - [\![\varphi]\!](b)| \mid \varphi \in \mathcal{L}_\Lambda\}$.

A Hennessy-Milner Theorem

Theorem (Fixpoint approximation)

Let L be a non-expansive and finitarily separable lax extension of T and let α and β be T-coalgebras.

least fixpoint of $R \mapsto LR \circ (\alpha \times \beta)$

Put $d_0 = 0$ and $d_{n+1} = Ld_n \circ (\alpha \times \beta)$ for $n < \omega$. Then $d^L_{\alpha,\beta} = \sup_{n<\omega} d_n$.

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König/Mika-Michalski 2018

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König/Mika-Michalski 2018

As a corollary, we get:

Theorem (Hennessy-Milner Theorem for Lax Extensions)

Let L be a non-expansive finitarily separable fuzzy lax extension. Then there exists a set Λ of monotone non-expansive predicate liftings such that $L = K_\Lambda$ and $d^\Lambda = d^L$.

$\Longrightarrow$ $\mathcal{L}_\Lambda$ is a characteristic logic for L.

A Quantified Coalgebraic van Benthem Theorem (FoSSaCS'21)

Paul Wild and Lutz Schröder

A ~~Quantified~~ Quantitative Coalgebraic van Benthem Theorem (FoSSaCS'21)

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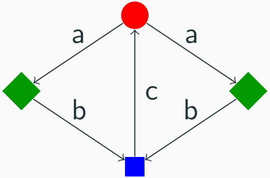
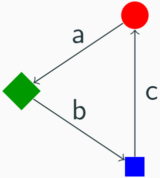
Introduction – Bisimulation invariance





Bisimilar states: indistinguishable in terms of successor behaviour.

Bisimulation invariant properties:

$$\Diamond_a \varphi = \text{there exists an } a\text{-successor satisfying } \varphi$$

$$\Box_b \varphi = \text{all } b\text{-successors satisfy } \varphi$$

Introduction – Modal logic

A *syntax* for bisimulation-invariant properties:

$$\varphi, \psi ::= \top \mid \varphi \wedge \psi \mid \neg\varphi \mid \Diamond_a \varphi \mid \Box_a \varphi \qquad (a \text{ label})$$

Lemma

Every modal formula is bisimulation-invariant.

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Theorem (Hennessy-Milner Theorem)

In finitely branching systems, two states agreeing on all modal formulae are bisimilar.

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Lemma

Every modal formula is bisimulation-invariant.

Theorem (Hennessy-Milner Theorem)

In finitely branching systems, two states agreeing on all modal formulae are bisimilar.

Theorem (van Benthem Theorem)

If a first-order property is bisimulation-invariant, it is equivalent to a modal formula.

Introduction – Markov chains

Behavioural distance d with $d(\bullet, \blacktriangle) = 0.01$

Real-valued probabilistic modal logic with $[\![\varphi]\!](x) \in [0,1]$:

- $\mathbb{E}\,\varphi =$ expected truth value of φ over successors
- Modal formulae are *non-expansive* wrt. d: $[\![\varphi]\!](x) - [\![\varphi]\!](y) \leq d(x,y)$

Probabilistic Hennessy-Milner Theorem: [van Breugel/Worrell 2005]
Probabilistic van Benthem Theorem: [Wild/Schröder/Pattinson/König 2019]

Introduction – Simulations

■ simulates ●.

Syntax for properties *preserved under simulation*:

$$\varphi, \psi ::= \bot \mid \top \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond_a \varphi \qquad (a \text{ label})$$

Hennessy-Milner Theorem for simulations: [van Glabbeek 2001]
van Benthem Theorem for simulations: [Lutz/Piro/Wolter 2010]

Our Contribution – Overview

Goal

General versions of the Hennessy-Milner and van Benthem Theorems that have all the previous examples as instances.

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Key Ingredients

- an algebra of truth values $\rightsquigarrow$ *value co-quantale* $\mathcal{V}$
- abstraction over system types $\rightsquigarrow$ T*-coalgebras of a functor* T
- a representation of the modalities $\rightsquigarrow$ *set of predicate liftings* Λ

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Idea

Modal logic $\mathcal{L}_\Lambda$ *characterizes non-expansiveness* wrt. behavioural distance d^K.

Value co-quantales

Value co-quantale $\mathcal{V}$ [Flagg, 1997]

- Completely distributive lattice $(V, \leq)$
- Monoid structure $\oplus$ that distributes over meets: $a \oplus \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} a \oplus b_i$.
- Subtraction $a \ominus b \leq c \iff a \leq b \oplus c$.
- Filter of positive elements $\{\varepsilon \mid \varepsilon \gg 0\}$.

Key properties

$$0 = \bigwedge \{\varepsilon \mid \varepsilon \gg 0\} \qquad \text{and} \qquad \varepsilon \gg 0 \implies \exists \delta \gg 0.\ \delta \oplus \delta \leq \varepsilon$$

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Main Examples

$$2 = \{0, 1\} \qquad [0, 1] \qquad \{[a, b] \mid 0 \leq a \leq b \leq 1\}$$

Coalgebras

$$\gamma\colon X \to TX \qquad (T \text{ endofunctor on Set})$$

Some choices of T:

- LTS with edge labels in A: $TX = \mathcal{P}(A \times X)$
- Markov chains with deadlocks: $TX = 1 + \mathcal{D}X$
- Metric transition systems with state labels in (S, d_S): $TX = S \times \mathcal{P}X$

Quantitative Coalgebraic Modal Logic

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Example

Probabilistic modal logic: $\mathbb{E}\, X(f)(\mu) =$ expected value of f under μ

Behavioural Distance via Relation Lifting

Kantorovich Lifting

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Behavioural distance as least fixed point

$$d^K = K_\Lambda(d^K) \circ (\gamma \times \gamma)$$

$$d^K(x, x) = 0 \qquad \text{and} \qquad d^K(x, z) \leq d^K(x, y) \oplus d^K(y, z)$$

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$d^K(x, y) = d^K(y, x)$ if Λ closed under duals

Quantitative Hennessy-Milner Theorem

Theorem (Quantitative Hennessy-Milner theorem)

Let Λ be finite and $\mathcal{V}$ totally bounded and continuous from below.

If T is finitary, then

$$d^K(a, b) = \bigvee \{ [\![\varphi]\!](a) \ominus [\![\varphi]\!](b) \mid \varphi \text{ a modal formula} \}.$$

Quantitative van Benthem Theorem

Quantitative Coalgebraic *Predicate* Logic

$$\varphi, \psi ::= c \mid x = y \mid \varphi \oplus c \mid \varphi \ominus c \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \exists x.\varphi \mid \forall x.\varphi \mid x\lambda\lceil y \colon \varphi \rceil$$

Theorem (Quantitative van Benthem theorem)

Let Λ be finite and $\mathcal{V}$ totally bounded. Let $\varphi \in$ QCPL be non-expansive wrt. d^K.

For every $\varepsilon \gg 0$ there exists a modal formula ψ such that for all γ, x:

$$[\![\varphi]\!]_\gamma(x) \ominus [\![\psi]\!]_\gamma(x) \leq \varepsilon \qquad \text{and} \qquad [\![\psi]\!]_\gamma(x) \ominus [\![\varphi]\!]_\gamma(x) \leq \varepsilon$$

Instantiations

Existing instances of Hennessy-Milner and van Benthem theorems we cover:

- Classical modal logic with $\mathcal{V} = 2$ and $TX = \mathcal{P}(A \times X)$
- Probabilistic modal logic with $\mathcal{V} = [0,1]$ and $TX = 1 + \mathcal{D}X$
- Two-valued ($\mathcal{V} = 2$) coalgebraic modal logic [Schröder/Pattinson/Litak 2017]

Instantiations

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New instances include:

- Metric modal logic with $TX = S \times \mathcal{P}X$ and modalities based on d_S.
- For $\mathcal{V} = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$: convex-nondeterministic metric modal logic.
- Simulation-based versions of all the above.