Quantalic(?!) Relational Connectors

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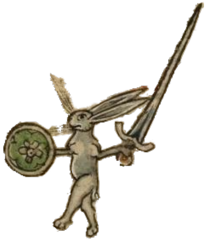
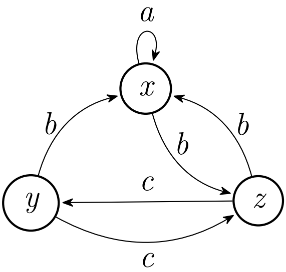
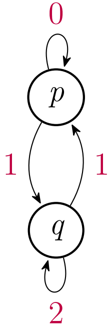


Systems of Different Type





$$FX = \mathcal{P}\left(\mathcal{A} \times X\right)$$

$$GX = \mathcal{P}\left(\mathbb{N} \times X\right)$$

Relations

Notation

We write $r\colon X \nrightarrow Y$ for a relation $r \subseteq X \times Y$ or $r\colon X \times Y \to 2$

Relations $s\colon Y \nrightarrow Z$ and $t\colon X \nrightarrow Y$ *combose*:

$$(s \cdot t)(x, z) \iff \exists y \in Y.\, s(y, z) \wedge t(x, y)$$

The *graph* of a function $f\colon X \to Y$ is the relation $\mathrm{gr}(f)\colon X \nrightarrow Y$ given by

$$\mathrm{gr}(f)(x, y) = \begin{cases} \top & f(x) = y \\ \bot & \text{else} \end{cases}$$

The *identity relation* $\Delta_X\colon X \nrightarrow X$ is given by $\Delta_X = \mathrm{gr}(id_X)$

Couniversal Factorization

A relation $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ can be factorized in a couniversal manner:

$$
\begin{tikzcd}
X \arrow[rr, "r", "|" marking] \arrow[dr, "t_r"', "|" marking] \arrow[ddr, bend right, "t"', "|" marking] & & Y \\
& C_r \arrow[ur, "s_r"', "|" marking] & \\
& Z \arrow[u, dashed, "\exists! h"'] \arrow[uur, bend right, "s"', "|" marking] &
\end{tikzcd}
$$

$$
\begin{aligned}
C_r &= \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y) \mid A \times B \subseteq r\} \\
s_r &= \{((A, B), y) \mid y \in B\} \\
t_r &= \{(x, (A, B)) \mid x \in A\}
\end{aligned}
$$

Relational Connectors*

Definition

Let $F, G\colon \mathsf{Set} \to \mathsf{Set}$. A relational connector $L\colon F \to G$ maps a relation $r\colon X \rightarrow\!\!\!\!\!+ Y$ to

$$Lr\colon FX \rightarrow\!\!\!\!\!+ GY$$

such that for $r'\colon X \rightarrow\!\!\!\!\!+ Y$, $f\colon X' \to X$, $g\colon Y' \to Y$

$$r \subseteq r' \implies Lr \subseteq Lr' \qquad \text{(monotonicity)}$$
$$L(g^\circ \circ r \circ f) = (Gg)^\circ \circ Lr \circ Ff \qquad \text{(naturality)}$$

Constructions

Definition

For $K\colon F \to G$, $L\colon G \to H$ the *composite* $L \cdot K\colon F \to H$ is

$$\begin{aligned}(L \cdot K)r &= \bigvee_{r = s \cdot t} Ls \cdot Kt \\ &= Ls_r \cdot Kt_r\end{aligned}$$

couniversal factorization

Remark

Associativity of composition relies crucially on join preservation of relational composition:

$$\left(\bigvee_{i \in I} r_i\right) \cdot s = \bigvee_{i \in I} r_i \cdot s$$

Constructions

Definition

The *identity relational connector* $\mathrm{Id}_F\colon F \to F$ is given on $r\colon X \nrightarrow Y$ by

$$\mathrm{Id}_F(r)(a,b) \iff \forall G\colon \mathsf{Set} \to \mathsf{Set}, L\colon G \to F, s\colon Z \nrightarrow X, c \in GZ. \\ Ls(c,a) \implies L(r \cdot s)(c,b)$$

Theorem

As expected, for $L\colon F \to G$ *we have*

$$L = \mathrm{Id}_G \cdot L = L \cdot \mathrm{Id}_F$$

Quantitative Systems

- Two-valued Bisimulation often not appropriate for quantitative systems
- Relational Connectors are in parts specific to the two-valued setting

---

Generalize!

?

But to what domain?

Complete Semirings

Definition

A *complete semiring* is a tuple $(R, \sum_I, \cdot, 0, 1)$ such that

- $(R, \sum_I, 0)$ is a complete monoid
  - $(R, +, 0)$ is a commutative monoid
  - $\sum_I$ agrees with $+$ on finite I
  - for all $j \neq j'$, $I_j \cap I_{j'} = \varnothing$

$$\sum_{j \in J} \sum_{i \in I_j} m_i = \sum_{i \in \bigcup_{j \in J} I_j} m_i$$

- $(R, \cdot, 1)$ is a monoid
- $\cdot$ distributes over $\sum_I$:

$$\sum_{i \in I} (a \cdot a_i) = a \cdot (\sum_{i \in I} a_i) \qquad \sum_{i \in I} (a_i \cdot a) = (\sum_{i \in I} a_i) \cdot a$$

R-Valued Relations

Definition

A R-valued relation $r\colon X \nrightarrow Y$ is a map $X \times Y \to R$.
R-valued relations $s\colon Y \nrightarrow Z$ and $t\colon X \nrightarrow Y$ *combose*:

$$(s \cdot t)(x, z) = \sum_{y \in Y} s(y, z) \cdot r(x, y)$$

Failure of Join Preservation

Definition

Composition of R-valued relations preserves joins if

$$\left(\bigvee_{i\in I} r_i\right)\cdot s = \bigvee_{i\in I} r_i \cdot s$$

for all $s\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $(r_i\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z)_{i\in I}$

Counterexample

Let $R = \mathbb{N}\cup\{\infty\}$, $I = Y = \mathbb{N}$ and $r_i(y,z) = \begin{cases} 1 & y = i \\ 0 & \text{else}\end{cases}$

$$\left(\bigvee_{i\in I} r_i\right)\cdot \Delta_{\mathbb{N}} = \sum_{y\in\mathbb{N}} 1 = \infty \neq \bigvee_{i\in I} r_i\cdot\Delta_{\mathbb{N}} = \bigvee_{i\in\mathbb{N}} 1 = 1$$

Quantales

Definition

A (commutative and unital) quantale is a tuple $(\mathcal{V}, \otimes, k)$ where

- $\mathcal{V}$ is a complete lattice
- $(\mathcal{V}, \otimes, k)$ is a commutative monoid
- $\otimes$ preserves joins

Example

- the extended positive reals with addition $[0,\infty]_{+}^{\leq^{\mathrm{op}}}$
- the extended positive reals with maximum $[0,\infty]_{\max}^{\leq^{\mathrm{op}}}$
- the unit interval with truncated addition $[0,1]_{\oplus}^{\leq^{\mathrm{op}}}$
- the booleans with conjunction

Quantales

Since a map $a \otimes - : \mathcal{V} \to \mathcal{V}$ preserves joins, it has a right adjoint $\hom(a, -) : \mathcal{V} \to \mathcal{V}$:

$$a \otimes b \leq c \iff b \leq \hom(a, c)$$

Example

- truncated subtraction for $[0, \infty]_{+}^{\leq^{\mathrm{op}}}$
- minimum for $[0, \infty]_{\mathrm{max}}^{\leq^{\mathrm{op}}}$
- truncated subtraction for $[0, 1]_{\oplus}^{\leq^{\mathrm{op}}}$
- implication for booleans

Quantale-Valued Relations

Definition

A $\mathcal{V}$-valued relation $r\colon X \nrightarrow Y$ is a map $X \times Y \to V$.
The *graph* of a function $f\colon X \to Y$ is the relation

$$\mathrm{gr}(f)(x, y) = \begin{cases} k & f(x) = y \\ \perp & \text{else} \end{cases}$$

Definition

Given $\mathcal{V}$-valued relations $s\colon Y \nrightarrow Z$, $t\colon X \nrightarrow Y$ their *composition* is given by

$$(s \cdot t)(x, z) = \bigvee_{y \in Y} s(y, z) \otimes t(x, y)$$

Quantale-Valued Relations

Remark

$\mathcal{V}$-valued relations form a complete lattice with the pointwise order.

Theorem

$\mathcal{V}$-valued relational composition preserves joins

Extension[†]

Definition

The *extension* $s \mathrel{\bullet\!\!-} r$ of a a $\mathcal{V}$-relation $s\colon X \mathrel{\mapstochar\rightarrow} Y$ along a $\mathcal{V}$-relation $r\colon X \mathrel{\mapstochar\rightarrow} Z$ is defined by

$$t \cdot r \leq s \iff t \leq s \mathrel{\bullet\!\!-} r$$

$$\begin{tikzcd}
X \arrow[r, "s", "\shortmid"{marking}] \arrow[d, "r"', "\shortmid"{marking}] \arrow[dr, phantom, "\geq"] & Y \\
Z \arrow[ur, "s \bullet\!- r"', "\shortmid"{marking}] \arrow[ur, bend right=70, "t"', "\shortmid"{marking}] \arrow[ur, phantom, shift right=4ex, "\geq"] &
\end{tikzcd}$$

$$(s \mathrel{\bullet\!\!-} r)(z, y) = \bigwedge_{x \in X} \hom(r(x, z), s(x, y))$$

[†] Goncharov et al. 2024.

Quantale-Valued Relational Connectors

Definition

Let $F, G\colon \mathsf{Set} \to \mathsf{Set}$. A relational connector $L\colon F \to G$ maps a relation $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ to

$$Lr\colon FX \relbar\joinrel\mapstochar\joinrel\rightarrow GY$$

such that for $r'\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$,$f\colon X' \to X$, $g\colon Y' \to Y$

$$r \subseteq r' \implies Lr \subseteq Lr' \qquad \text{(monotonicity)}$$
$$L(g^\circ \circ r \circ f) = (Gg)^\circ \circ Lr \circ Ff \qquad \text{(naturality)}$$

(Bi)simulations

Definition

Let $(C, \gamma\colon C \to FC)$, $(D, \delta\colon D \to GD)$ be coalgebras. A relation $r\colon C \nrightarrow D$ is an *L-simulation* for a relational connector $L\colon F \to G$ if

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\mid\wedge"] & D \\ FC \arrow[r, "Lr"'] & GD \arrow[u, "\delta^\circ"'] \end{tikzcd}$$

POMDPs

Definition

A *partially observable Markov decision process* is a coalgebra for the functor

$$FX = (\mathcal{D}(X \times \Omega) \times \mathbb{R})^{\mathcal{A}}$$

set of observations

set of actions

Definition

An *policy* for an agent interacting with a POMDP is a coalgebra for the functor

$$GY = \mathcal{A} \times Y^{\Omega}$$

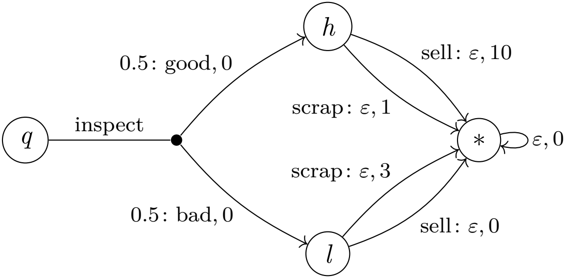
POMDPs

We model lower bounds on the optimal reward for a policy in a POMDP as a $[0,\infty]_+$-valued relational connector $L\colon F \to G$:

$$L(r)(\langle d, b\rangle, (\ell, g)) = b(\ell) + \mathbb{E}_{(x,\omega)\sim d(\ell)} r(x, g(\omega))$$

An L-bisimulation then serves as a witness for a lower bound on the optimal reward.

Example



$$Y = \{0, g, b, *\}$$

$$\beta(0) = \left(\text{inspect}, \omega \mapsto \begin{cases} g & \omega = \text{good} \\ b & \omega = \text{bad} \\ * & \omega = \varepsilon \end{cases}\right) \qquad r(x, y) = \begin{cases} 6.5 & x = q, y = 0 \\ 10 & x = h, y = g \\ 3 & x = l, y = b \\ 0 & \text{else} \end{cases}$$

$$\beta(g) = (\text{sell}, \omega \mapsto *)$$

$$\beta(b) = (\text{scrap}, \omega \mapsto *)$$

$$\beta(*) = (\text{scrap}, \omega \mapsto *)$$

Composition of Relational Connectors

Definition

For $K\colon F \to G$, $L\colon G \to H$ the *composite* $L \cdot K\colon F \to H$ is given on $r\colon X \relbar\joinrel\mapsto Y$:

$$(L \cdot K)r = \bigvee_{s \cdot t \leq r} Ls \cdot Kt$$

Theorem

Relational connectors are closed under composition.

Couniversal Subfactorization

A relation $r\colon X \to\!\!\!\!\!\!+\;\, Y$ can be factorized in a couniversal manner:

$$
\begin{tikzcd}
X \arrow[rr, "r", "|" marking] \arrow[dr, "t_r"', "|" marking] \arrow[ddr, bend right, "t"', "|" marking] & & Y \\
& C_r \arrow[ur, "s_r"', "|" marking] & \\
& Z \arrow[u, dashed, "\exists! h"'] \arrow[uur, bend right, "s"', "|" marking] &
\end{tikzcd}
$$

$$
\begin{aligned}
C_r &= \left\{ (A, B) \in \mathcal{V}^X \times \mathcal{V}^Y \,\middle|\, \forall x, y.\ A(x) \otimes B(y) \le r(x, y) \right\} \\
s_r &= ((A, B), y) \mapsto B(y) \\
t_r &= (x, (A, B)) \mapsto A(x) \\
h &= z \mapsto (t(-, z), s(z, -))
\end{aligned}
$$

Composition

Theorem

Let $K\colon F \to G$, $L\colon G \to H$ *and* $r\colon X \nrightarrow Z$, *then*

$$(L \cdot K)r = Ls_r \cdot Kt_r$$

Lemma

Composition of relational connectors is associative, i.e. for $M\colon H \to D$, $K\colon G \to H$, $L\colon F \to G$:

$$(M \cdot K) \cdot L = M \cdot (K \cdot L)$$

Identity Connector

Definition

The *identity relational connector* $\mathrm{Id}_F \colon F \to F$ is given on $r \colon X \mathrel{\ooalign{$\rightarrow$\cr\hidewidth$+$\hidewidth}} Y$ by

$$\mathrm{Id}_F(r) = \bigwedge_{\substack{G \colon \mathsf{Set} \to \mathsf{Set} \\ L \colon G \to F \\ s \colon Z \mathrel{\ooalign{$\rightarrow$\cr\hidewidth$+$\hidewidth}} X}} L(r \cdot s) \bullet\!\!- Ls$$

Conjecture

The connector defined above is an identity for composition:

$$L = \mathrm{Id}_G \cdot L = L \cdot \mathrm{Id}_F$$

Further Work

- Generalize the FoSSaCS paper to quantales
- Find examples