

Interpolation in Modal and Fuzzy Modal Logic

A Sequent Calculus-Based Approach

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Roadmap

- 1 Motivation
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- Craig Interpolation shows that if one formula entails another, then the entailment can be witnessed using only the vocabulary common to both formulas.
- It gives a precise way to measure which symbols are actually responsible for a consequence.
- Uniform interpolation strengthens this by providing systematic variable elimination, similar in spirit to quantifier elimination.
- Interpolants can be used as approximations of formulas after forgetting selected propositional variables.
- However, Interpolation is often proved as an existence result.
- In classical modal logic, there are already proof-theoretic approaches to Craig and uniform interpolation.
- The perspective here is explicitly computational: the sequent calculi should not only prove that interpolants exist, but show how to construct them.
- We first develop this idea for K , and then adapt it to non-expansive fuzzy modal logic.

Constructive viewpoint

Craig interpolation is treated by extracting an interpolant from a labelled sequent proof.

Uniform interpolation

Uniform interpolation is handled differently: we modify the calculus so that each derivation step computes $\Gamma \mid C_q(\Gamma)$, where $C_q(\Gamma)$ is the strongest q -free object making $\Gamma, C_q(\Gamma)$ provable.

The contribution is to make these constructions explicit, especially in the signed-sequent setting of non-expansive fuzzy modal logic.

- Language:

$$\varphi ::= p \mid \perp \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi$$

- Kripke models:

$$\mathfrak{M} = (W \in \text{Set}, R \subseteq W \times W, V : \text{At} \rightarrow \mathcal{P}(W))$$

- Truth clause for \Box :

$$\mathfrak{M}, w \Vdash \Box\varphi \iff \forall v. wRv \Rightarrow (\mathfrak{M}, v \Vdash \varphi).$$

One-Sided Sequent Calculus for K

- Sequents are finite (multi)sets of formulas Γ and are read disjunctively as

$$\bigvee_{\varphi \in \Gamma} \varphi.$$

- Sequent calculus rules for sequents with formulas in NNF (to be read from the bottom-up):

$$(\text{Ax}) \frac{}{\Gamma, p, \neg p} \quad (\perp) \frac{}{\Gamma, \neg \perp} \quad (\wedge) \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \quad (\vee) \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}$$

$$(\Box_K) \frac{\Gamma, \varphi}{\Diamond \Gamma, \Box \varphi, \Delta} \quad (\Delta \text{ propositionally saturated and contains no other } \Diamond \text{ formulas})$$

- Optional weakening rule:

$$(\text{W}) \frac{\Gamma}{\Gamma, \varphi}$$

Definition (Craig Interpolation Property)

A logic L has CIP if whenever

$$L \vdash \varphi \rightarrow \psi,$$

there is a formula θ such that

$$L \vdash \varphi \rightarrow \theta \quad \text{and} \quad L \vdash \theta \rightarrow \psi,$$

with

$$\text{Var}(\theta) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi).$$

Definition (Uniform Interpolation Property)

A logic L has UIP if for every formula φ and every variable p , there exist formulas $\exists p.\varphi$ and $\forall p.\varphi$ not containing p such that they are respectively the strongest p -free consequence and weakest p -free antecedent of φ .

$$\varphi \vdash \chi \iff \exists p.\varphi \vdash \chi \quad (p \notin \text{Var}(\chi)).$$

Craig Interpolation for K

- Possible proof methods:
 - proof-theoretic, via sequent calculi;
 - semantic, via bisimulations or model amalgamation;
 - algebraic, via modal algebras.

Theorem

Modal logic K has Craig interpolation.

- Idea: For $\varphi \vdash \psi$, we construct a proof tree for $\neg\varphi_A, \psi_C$, and construct the interpolant along it.
- The subscripts allow us to retrace whether a formula is part of the antecedent or the consequence.
- In more detail we define a function I , mapping a sequent with antecedent and consequence labels to an interpolant of the negated conjunction over all antecedent formulas and the disjunction of the consequence formulas.

Interpolant Extraction Rules

Initial and weakening cases

$$\begin{array}{l} \overline{\Gamma, (\neg\perp)_A} \mapsto \perp \\ \overline{\Gamma, (\neg\perp)_C} \mapsto \top \\ \frac{\Gamma}{\Gamma, \varphi_x} \mapsto I(\Gamma) \end{array}$$

Atomic axioms

$$\begin{array}{l} \overline{\Gamma, p_A, (\neg p)_A} \mapsto \perp \\ \overline{\Gamma, p_C, (\neg p)_C} \mapsto \top \\ \overline{\Gamma, p_A, (\neg p)_C} \mapsto \neg p \\ \overline{\Gamma, p_C, (\neg p)_A} \mapsto p \end{array}$$

Propositional rules

$$\frac{\Gamma, \varphi_x \quad \Gamma, \psi_x}{\Gamma, (\varphi \wedge \psi)_x} \mapsto I(\Gamma, \varphi_x) \vee I(\Gamma, \psi_x)$$
$$\frac{\Gamma, \varphi_x, \psi_x}{\Gamma, (\varphi \vee \psi)_x} \mapsto I(\Gamma, \varphi_x, \psi_x)$$

Modal rule

$$\frac{\Gamma, \varphi_A}{\diamond \Gamma, (\Box \varphi)_A, \Delta} \mapsto \diamond I(\Gamma, \varphi_A)$$
$$\frac{\Gamma, \varphi_C}{\diamond \Gamma, (\Box \varphi)_C, \Delta} \mapsto \Box I(\Gamma, \varphi_C)$$

- Uniform interpolation is stronger than Craig interpolation.
- In modal logic K, uniform interpolants can be obtained via fixpoint-style or automata-theoretic constructions.

Theorem

Modal logic K has uniform interpolation.

A Sequent Calculus for Post-Interpolants in \mathcal{K}

- Fix a variable q . We write

$$\Gamma \mid C_q(\Gamma)$$

to mean that $C_q(\Gamma)$ is the strongest q -free formula such that

$$\Gamma, C_q(\Gamma)$$

is provable.

- The post-interpolant $\exists q., \varphi$ of φ is then obtained as

$$C_q(\neg\varphi).$$

- One obviously obtains $\forall q. \varphi$ as $\neg\exists q., \neg\varphi$.

Initial & Propositional Rules

- Initial Rules:

$$\begin{array}{ccc} (\text{Ax}) \frac{}{\Gamma, p, \neg p \mid \perp} & (\perp) \frac{}{\Gamma, \neg \perp \mid \perp} & (\text{At}) \frac{}{\Gamma \mid \bigwedge_{\substack{\varphi \in \Gamma \\ \varphi \notin \{q, \neg q\}}} \neg \varphi} \end{array}$$

where in (At), Γ contains only atoms, \perp , and their negations.

- Propositional Rules:

$$\begin{array}{cc} (\wedge) \frac{\Gamma, \varphi \mid C_q(\Gamma, \varphi) \quad \Gamma, \psi \mid C_q(\Gamma, \psi)}{\Gamma, \varphi \wedge \psi \mid C_q(\Gamma, \varphi) \vee C_q(\Gamma, \psi)} & (\vee) \frac{\Gamma, \varphi, \psi \mid C_q(\Gamma, \varphi, \psi)}{\Gamma, \varphi \vee \psi \mid C_q(\Gamma, \varphi, \psi)} \end{array}$$

- Modal Rule:

$$(\Box_K) \frac{\Gamma, \varphi_1 \mid C_q(\Gamma, \varphi_1) \quad \cdots \quad \Gamma, \varphi_n \mid C_q(\Gamma, \varphi_n)}{\Gamma \mid C_q(\Gamma)}$$

$$\diamond\Gamma, \Box\varphi_1, \dots, \Box\varphi_n, \Delta$$

$$\mid \bigwedge_{1 \leq i \leq n} \diamond C_q(\Gamma, \varphi_i) \wedge \Box C_q(\Gamma) \wedge \bigwedge_{\substack{\chi \in \Delta \\ \chi \notin \{q, \neg q\}}} \neg \chi$$

Here Δ is propositionally saturated and contains no other formulas of the form $\diamond\chi$ or $\Box\chi$.

- Syntax:

$$\varphi, \psi ::= 0 \mid p \mid \varphi \ominus c \mid \neg\varphi \mid \varphi \sqcap \psi \mid \diamond\varphi$$

where p ranges over propositional atoms and $c \in [0, 1]$.

- Fuzzy Models:

$$\mathfrak{M} = (W \in \text{Set}, R : W \times W \rightarrow [0, 1], V : \text{At} \times W \rightarrow [0, 1])$$

- Sequents Γ are finite (multi)sets of literals, where literals are of the form:

$$\varphi \bowtie c, \quad \bowtie \in \{\geq, >, \leq, <\}.$$

- We will also use $\triangleleft \in \{\leq, <\}, \triangleleft \in \{\geq, >\}$.
- Semantics of \diamond :

$$\llbracket \diamond\phi \rrbracket_{\mathfrak{M}}(w) = c \iff \sup_{w' \in W} \min(R(w, w'), \llbracket \phi \rrbracket_{\mathfrak{M}}(w')) = c$$

One-Sided Sequent Calculus for Non-expansive Fuzzy Modal Logic

$$(W) \frac{\Gamma}{\Gamma, \varphi \boxtimes c} \quad (Ax_p) \frac{}{\Gamma, p \triangleleft c, p \triangleright d} \quad (\forall e \in [0, 1]: e \triangleleft c \text{ or } e \triangleright d)$$

$$(Ax_0) \frac{}{\Gamma, 0 \boxtimes c} \quad (0 \boxtimes c) \quad (Ax_{p0}) \frac{}{\Gamma, p \triangleright c} \quad (0 \triangleright c) \quad (Ax_{p1}) \frac{}{\Gamma, p \triangleleft c} \quad (1 \triangleleft c)$$

$$(\neg) \frac{\Gamma, \varphi \boxtimes^\circ (1 - c)}{\Gamma, \neg \varphi \boxtimes c} \quad (\Pi \triangleright) \frac{\Gamma, \varphi \triangleright c \quad \Gamma, \psi \triangleright c}{\Gamma, \varphi \Pi \psi \triangleright c} \quad (\Pi \triangleleft) \frac{\Gamma, \varphi \triangleleft c, \psi \triangleleft c}{\Gamma, \varphi \Pi \psi \triangleleft c}$$

$$(\Theta \triangleright) \frac{\Gamma, \varphi \triangleright (d + c)}{\Gamma, \varphi \Theta c \triangleright d} \quad (\text{not } 0 \triangleright d) \quad (\Theta \triangleleft) \frac{\Gamma, \varphi \triangleleft (d + c)}{\Gamma, \varphi \Theta c \triangleleft d} \quad (0 \triangleleft d)$$

$$(\Diamond) \frac{\varphi \triangleleft c, \{\varphi_i \triangleright_i c_i \mid \forall e \in [0, 1], \text{ not } e \triangleleft c: e \triangleright_i c_i\}}{\Gamma, \Diamond \varphi \triangleleft c, \{\varphi_i \triangleright_i c_i\}}$$

(where Γ is propositionally saturated and contains no literals of the form $\Diamond \psi \triangleright d$)

Definition (Craig Interpolation)

The logic has *Craig interpolation* if, for all literals $\varphi \boxtimes_1 c$ and $\psi \boxtimes_2 d$, whenever $\varphi \boxtimes_1 c \models \psi \boxtimes_2 d$, there exists an interpolating literal $I(\varphi, \psi)$, such that

$$\text{Var}(I(\varphi, \psi)) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi),$$

and

$$\varphi \boxtimes_1 c \models I(\varphi, \psi) \quad \text{and} \quad I(\varphi, \psi) \models \psi \boxtimes_2 d.$$

Definition (Uniform Interpolation)

The logic has *uniform interpolation* if, for every literal $\varphi \boxtimes c$ and every propositional variable q , there exist q -free literals

$$\exists q(\varphi \boxtimes c) \quad \text{and} \quad \forall q(\varphi \boxtimes c)$$

such that

$$\varphi \boxtimes c \models \exists q(\varphi \boxtimes c) \quad \text{and} \quad \forall q(\varphi \boxtimes c) \models \varphi \boxtimes c.$$

Moreover, for every q -free literal χ ,

$$\varphi \boxtimes c \models \chi \iff \exists q(\varphi \boxtimes c) \models \chi,$$

and

$$\chi \models \varphi \boxtimes c \iff \chi \models \forall q(\varphi \boxtimes c).$$

Craig Interpolation: Reduction Step

- Using negation if necessary, it suffices to consider entailments of the form

$$\varphi \triangleleft_1 c \models \psi \triangleleft_2 d.$$

- Define the two fixed literals

$$S := 0 \triangleleft_1 0.5, \quad U := \neg 0 \triangleleft_1 0.5.$$

- We build a proof of the sequent

$$\varphi_A \overline{\triangleleft_1}^\circ c, \psi_C \triangleleft_2 d.$$

- We then once again use this proof tree to build an interpolant.

Rules involving 0

$$\frac{}{\Gamma, (0 \boxtimes c)_A} \mapsto U$$

$$\frac{}{\Gamma, (0 \boxtimes c)_C} \mapsto S$$

Weakening

$$\frac{\Gamma}{\Gamma, \varphi \boxtimes c} \mapsto I(\Gamma)$$

Contradictory literals

$$\overline{\Gamma, (p \triangleleft c)_A, (p \triangleright d)_A} \mapsto U$$

$$\overline{\Gamma, (p \triangleleft c)_C, (p \triangleright d)_C} \mapsto S$$

$$\overline{\Gamma, (p \boxtimes c)_A, (p \boxtimes' d)_C} \mapsto p \overline{\boxtimes}^\circ c$$

$$\overline{\Gamma, (p \boxtimes c)_C, (p \boxtimes' d)_A} \mapsto p \overline{\boxtimes'}^\circ d$$

Craig Interpolant Assignment: Propositional Rules

Rules preserving the interpolant

$$\frac{\Gamma, (\varphi \boxtimes^\circ (1 - c))_x}{\Gamma, (\neg\varphi \boxtimes c)_x} \mapsto I(\Gamma, (\varphi \boxtimes^\circ (1 - c))_x)$$

$$\frac{\Gamma, (\varphi \triangleleft c)_x, (\psi \triangleleft c)_x}{\Gamma, (\varphi \sqcap \psi \triangleleft c)_x} \mapsto I(\Gamma, (\varphi \triangleleft c)_x, (\psi \triangleleft c)_x)$$

$$\frac{\Gamma, (\varphi \triangleright d + c)_x}{\Gamma, (\varphi \ominus c \triangleright d)_x} \mapsto I(\Gamma, (\varphi \triangleright d + c)_x)$$

$$\frac{\Gamma, (\varphi \triangleleft d + c)_x}{\Gamma, (\varphi \ominus c \triangleleft d)_x} \mapsto I(\Gamma, (\varphi \triangleleft d + c)_x)$$

Branching meet rule

$$\frac{\Gamma, (\varphi \triangleright c)_x \quad \Gamma, (\psi \triangleright c)_x}{\Gamma, (\varphi \sqcap \psi \triangleright c)_x} \mapsto Q(I(\Gamma, (\varphi \triangleright c)_x)) \sqcap Q(I(\Gamma, (\psi \triangleright c)_x)) \triangleleft_1 0.5$$

Diamond rule

$$\frac{\Sigma}{\Gamma, (\Diamond\varphi \triangleleft c)_A, \{(\varphi_i \triangleright_i c_i)_x\}} \mapsto \Diamond Q(I(\Sigma)) \triangleleft_1 0.5$$

$$\frac{\Sigma}{\Gamma, (\Diamond\varphi \triangleleft c)_C, \{(\varphi_i \triangleright_i c_i)_x\}} \mapsto \Diamond \neg Q(I(\Sigma)) \triangleleft_1^{\circ} 0.5$$

Here Σ denotes the premise sequent generated by the modal rule.

Idea

As in the case of K , we introduce a modified sequent calculus whose rules compute, together with a sequent Γ , a strongest q -free literal $C_q(\Gamma)$ such that $\Gamma, C_q(\Gamma)$ is provable.

- Thus a derivation has the form $\Gamma \mid C_q(\Gamma)$.
- If we want the post-interpolant of a literal $\varphi \boxtimes c$ we construct $C_q(\varphi \boxtimes^\circ c)$.
- We can once again reduce to the case $\varphi \triangleleft c$.

- Axioms:

$$(Ax_p) \frac{}{\Gamma, p \triangleleft c, p \triangleright d \mid 1 \triangleleft 0.5} \quad (\forall e \in [0, 1]: e \triangleleft c \text{ or } e \triangleright d)$$

$$(Ax_0) \frac{}{\Gamma, 0 \bowtie c \mid 1 \triangleleft 0.5} \quad (0 \bowtie c)$$

$$(Ax_{p0}) \frac{}{\Gamma, p \triangleright c \mid 1 \triangleleft 0.5} \quad (0 \triangleright c) \quad (Ax_{p1}) \frac{}{\Gamma, p \triangleleft c \mid 1 \triangleleft 0.5} \quad (1 \triangleleft c)$$

- Atoms introduction:

$$(At) \frac{}{\Gamma \mid \bigsqcup_{\substack{p \bowtie c \in \Gamma \\ p \neq q}} Q(p \bowtie^\circ c) \triangleleft 0.5}$$

- Rules preserving $Q_q(\Gamma)$:

$$(\neg) \frac{\Gamma' = \{\Gamma, \varphi \boxtimes^\circ (1 - c)\} \mid C_q(\Gamma')}{\Gamma, \neg \varphi \boxtimes c \mid C_q(\Gamma')} \quad (\sqcap \triangleleft) \frac{\Gamma' = \{\Gamma, \varphi \triangleleft c, \psi \triangleleft c\} \mid C_q(\Gamma')}{\Gamma, \varphi \sqcap \psi \triangleleft c \mid C_q(\Gamma')}$$

$$(\ominus) \frac{\Gamma' = \{\Gamma, \varphi \triangleright (d + c)\} \mid C_q(\Gamma')}{\Gamma, \varphi \ominus c \triangleright d \mid C_q(\Gamma')} \quad (\text{not } 0 \triangleright d)$$

$$(\ominus \triangleleft) \frac{\Gamma' = \{\Gamma, \varphi \triangleleft (d + c)\} \mid C_q(\Gamma')}{\Gamma, \varphi \ominus c \triangleleft d \mid C_q(\Gamma')} \quad (0 \triangleleft d)$$

- Branching meet rule:

$$(\sqcap \triangleright) \frac{\Gamma, \varphi \triangleright c \mid C_q(\Gamma, \varphi \triangleright c) \quad \Gamma, \psi \triangleright c \mid C_q(\Gamma, \psi \triangleright c)}{\Gamma, \varphi \sqcap \psi \triangleright c \mid Q(C_q(\Gamma, \varphi \triangleright c)) \sqcap Q(C_q(\Gamma, \psi \triangleright c)) \triangleleft 0.5}$$

- For the modal rule and $\{\Diamond\varphi_i \triangleleft c_i \mid 1 \leq i \leq n\}, \{\Diamond\psi_i \triangleright d_i \mid 1 \leq i \leq m\}$ (where w.l.o.g. we assume the terms are ordered, i.e. the most restrictive bounds have the highest index), let:

$$\Gamma_i = \varphi_i \triangleleft c_i, \{\psi \triangleright d_j \mid \forall e \in [0, 1], \text{ not } e \triangleleft c_i: e \triangleright d_j\}$$

$$\Gamma'_i = \{\psi \triangleright d_j \mid 1 \leq j \leq i\}$$

- The modal rule then is:

$$(\Diamond) \frac{\begin{array}{ccc} \Gamma_1 \mid C_q(\Gamma_1) & \dots & \Gamma_n \mid C_q(\Gamma_n) \\ \Gamma'_1 \mid C_q(\Gamma'_1) & \dots & \Gamma'_m \mid C_q(\Gamma'_m) \end{array}}{\Gamma, \{\Diamond\varphi_i \triangleleft c_i \mid 1 \leq i \leq n\}, \{\Diamond\psi_i \triangleright d_i \mid 1 \leq i \leq m\}} \\
 \mid \bigsqcup_{1 \leq i \leq n} Q(\Diamond Q_{\triangleright c_i/2}(C_q(\Gamma_i)) \triangleright c_i/2) \sqcup \\
 \bigsqcup_{1 \leq i \leq m} Q(\Diamond Q_{\triangleleft (1+d_i)/2}(C_q(\Gamma'_i)) \triangleleft (1 + d_i)/2) \sqcup \bigsqcup_{\substack{p \boxtimes c \in \Gamma \\ p \neq q}} Q(p \boxtimes^\circ c) \triangleleft 0.5$$

(where Γ is propositionally saturated and contains no literals of the form $\Diamond\psi \boxtimes e$)

- We began with interpolation in classical modal logic K , using one-sided sequents as the basic proof system.
- For Craig interpolation, the main construction is proof-transformational: labelled derivations are converted into interpolants.
- For uniform interpolation, the construction is different: a modified calculus computes expressions

$$\Gamma \mid C_q(\Gamma),$$

where $C_q(\Gamma)$ is the strongest q -free object making $\Gamma, C_q(\Gamma)$ provable.

- The same two proof-theoretic patterns are then adapted to non-expansive fuzzy modal logic using signed literals.

Main message

Craig and uniform interpolation are both constructive and deeply rooted in sequent calculi, but in different ways.

Craig interpolation : extract an interpolant from a labelled proof,

Uniform interpolation : compute $C_q(\Gamma)$ inside a modified calculus.