Towards Reframing Probabilistic Monads

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Thread One: Collaboration

Joint Work with

- Laura Gonzales Bravo (PhD Student in Matrid),
- Paolo Perrone (Research Associate in Oxford),
- Tomáš Gonda (Postdoc in Innsbruck)

Categorical Probability

We use Markov categories [Fri20]. Recent research ...

- unifies and generalizes different notions of probability (discrete, continuous, quantum, possibility, ...)
- abstract, graphical characterizations for conditionals, independence, almost sure equality, Bayesian inversion, ...
- generalizations of theorems (de Finetti, zero-one-laws, strong law of large numbers)

Markov Categories

symmetric monoidal cats (SMC)

$\cup|$

SMC with projections

$\cup|$

SMC with weak products

$\cup|$

Markov cats

$\cup|$

cartesian monoidal cats

Markov Categories: Examples

Example (Probability Matrices)

- objects: $\mathbb{N}_{>0}$
- morphisms: probability matrices
  composition: matrix multiplication

Example (Giry's Approach to Probability [Gir82])

- objects: measurable spaces
- morphisms: measurable maps $X \to \{\text{prob. measures on } Y\}$
  composition: Chapman–Kolmogorov equation

Distribution Monads

Lemma

1. *Unital semiring* $(R, +, 0, \cdot, 1)$ *induces* Set-*monad* D_R.
2. $\cdot$ *commutative* $\Rightarrow$ *Kleisli cat.* Kl_{D_R} *is Markov cat.*

Example (Markov Categories from Distribution Monads)

1. $R = \mathbb{R}_{\geq 0}$: D_R is distribution monad:

$$\mathrm{D}_R X = \{\text{finitely supported distribution on } X\}$$

2. $R = \{0, 1\}$: D_R is a power-set monad:

$$\mathrm{D}_R X \cong \{S \subseteq X \text{ finite, non-empty}\}$$

Notions of Imprecise Probability

Imprecise distribution on set X could be ...

- a subset of $\mathrm{D}X$
- a *convex* subset of $\mathrm{D}X$
- pairs of lower and upper estimates $(l_x, u_x)_{x \in X}$

In particular: probability + non-determinism
Application:

- Philosophy (ignorance, incomplete knowledge)
- Engineering (efficient software, unreliable sensors)

Lower and Upper Probability

Goal: Set-monad T with

- $\mathsf{T}X \subseteq ([0,1]\times[0,1])^X$
- $(0,1)_{x\in X} \in \mathsf{T}X$(for $|X| > 1$)
- $\mathsf{D} \subseteq \mathsf{T}$ is submonad:

$$\begin{aligned} \iota_X : \mathsf{D}X &\to \mathsf{T}X \\ p &\mapsto (p_x, p_x)_{x\in X} \end{aligned}$$

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Fuzzy Monads

Example (Fuzzy Powerset Monad)

$$\begin{aligned} F : \mathsf{Set} &\to \mathsf{Set} \\ X &\mapsto \{\text{functions } X \to [0,1]\} \end{aligned}$$

Lemma ([Man76])

Completely distributive lattice L induces fuzzy power set monad F_L.

Reframe Fuzzy Functors

Notation: CompDistLat is cat. of complete distributive lattices
CABA $\cong$ $\mathsf{Set}^{\mathsf{op}}$ is cat. of complete atomic boolean algebras

Lemma

Functor F_L factorizes

$$\mathsf{Set} \xrightarrow{2^-} \mathsf{CABA}^{op} \subseteq \mathsf{CompDistLat}^{op} \xrightarrow{L^-} \mathsf{CompDistLat} \to \mathsf{Set}.$$

Effect Algebras

Definition

An *effect algebra* is set E with

- constants $0, 1 \in E$
- involution $\neg : E \to E$
- commutative, associative, partial addition

$$\oplus : E \times E \supseteq \perp \to E$$

s.th.

$$a \perp 1 \Leftrightarrow a = 0 \qquad a \oplus b = 1 \Leftrightarrow b = \neg a$$

An *effect algebra morphism* is a function preserving $1, \perp, \oplus$.

Examples

$[0, 1]$, 2^X, Boolean algebras,

Reframe Finite Distribution Functors

Notation: FinSet is cat. of finite sets
FinBool $\cong$ FinSet$^{\mathsf{op}}$ is cat. of finite boolean algebras

Lemma

Functor D factorizes

$$\mathsf{FinSet} \xrightarrow{2^{-}} \mathsf{FinBool}^{op} \subseteq \mathsf{EffAlg}^{op} \xrightarrow{[0,1]^{-}} \mathsf{EffAlg} \rightarrow \mathsf{Set}.$$

Summary

$$\mathsf{D}_{\mathbb{R}_{\geq 0}} : \mathsf{FinSet} \xrightarrow{2^{-}} \mathsf{EffAlg}^{\mathsf{op}} \xrightarrow{[0,1]^{-}} \mathsf{EffAlg} \to \mathsf{Set}$$

$$\mathsf{F}_{[0,1]} : \mathsf{Set} \xrightarrow{2^{-}} \mathsf{CompDistLat}^{\mathsf{op}} \xrightarrow{[0,1]^{-}} \mathsf{CompDistLat} \to \mathsf{Set}$$

? role of dualizing objects ?

? recover *monad* ?

? recover (more types of) imprecise probability ?

? recover (notions of) quantum probability ?

Thank You.

[Fri20] Tobias Fritz. "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics". In: *Advances in Mathematics* 370 (2020), p. 107239.

[Gir82] Michèle Giry. "A categorical approach to probability theory". In: *Categorical Aspects of Topology and Analysis*. Ed. by B. Banaschewski. Berlin, Heidelberg: Springer Berlin Heidelberg, 1982, pp. 68–85.

[Man76] Ernest G Manes. "Algebraic Theories". In: *Graduate Texts in Mathematics* (1976).