# Trees in Coalgebra from Generalized Reachability



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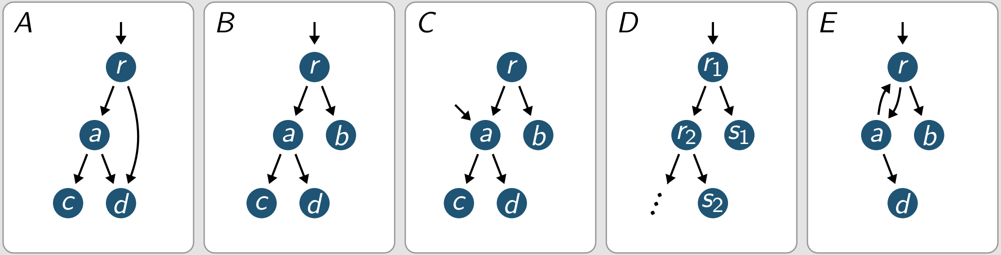
E.g. for the Set-functor $FX = 1 + X^2$, which of the following is a tree?



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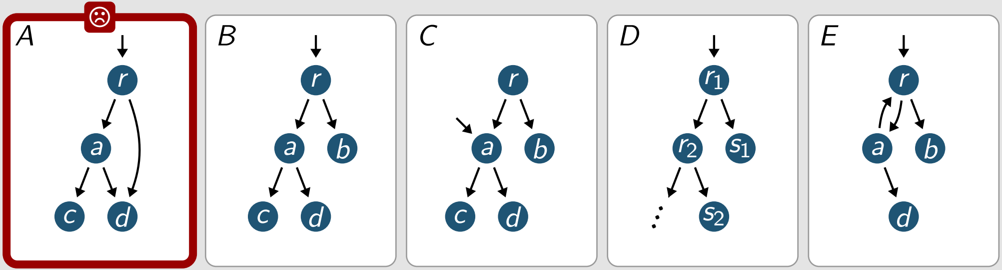

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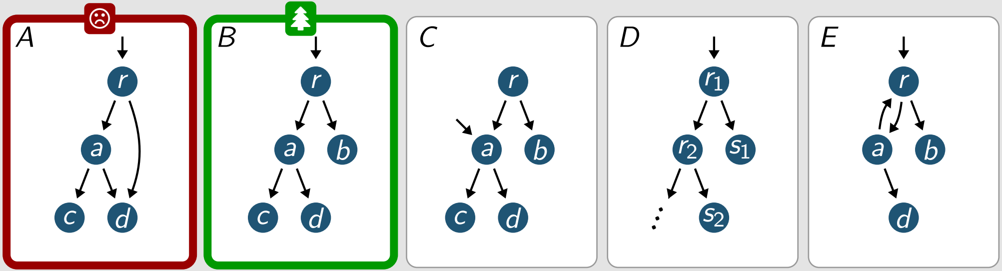

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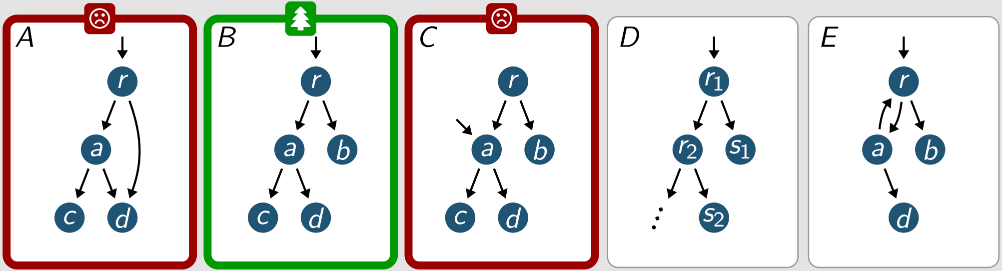

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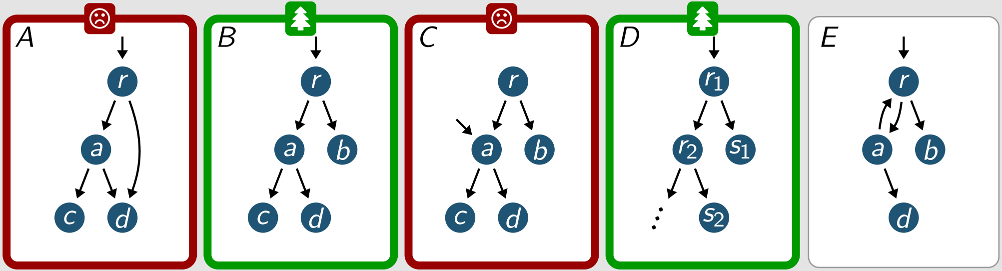

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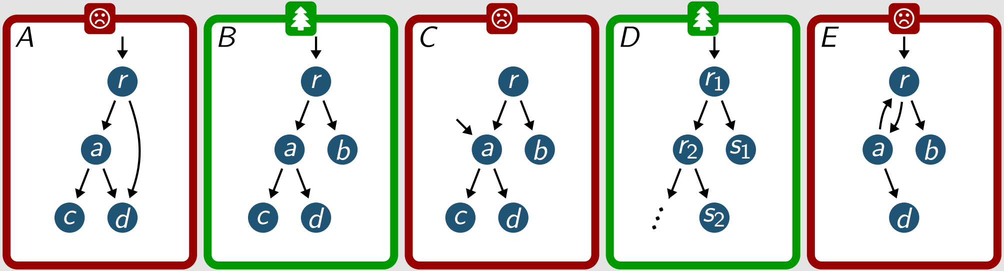
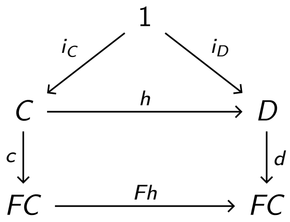


## Definition: (Pointed) Coalgebras

$$\mathbf{C} = \left( I \xrightarrow{i_C} C \xrightarrow{c} FC \right)$$

$I := 1$ in this talk

## Definition: $\mathrm{Coalg}_1(F)$

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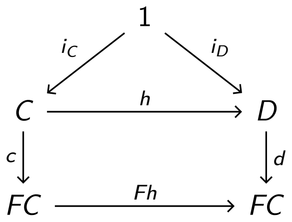


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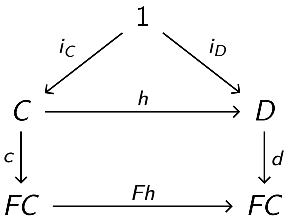


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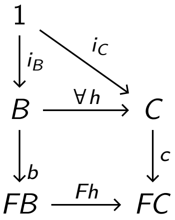


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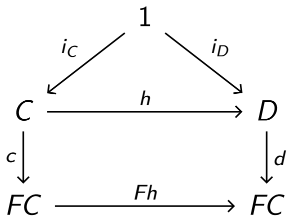


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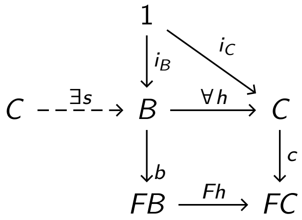


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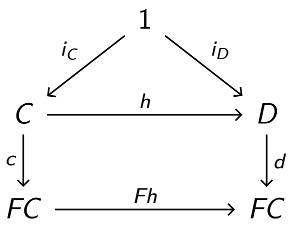


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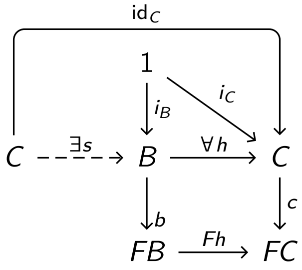


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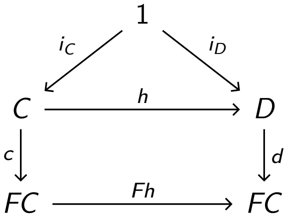


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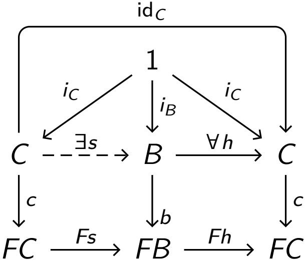


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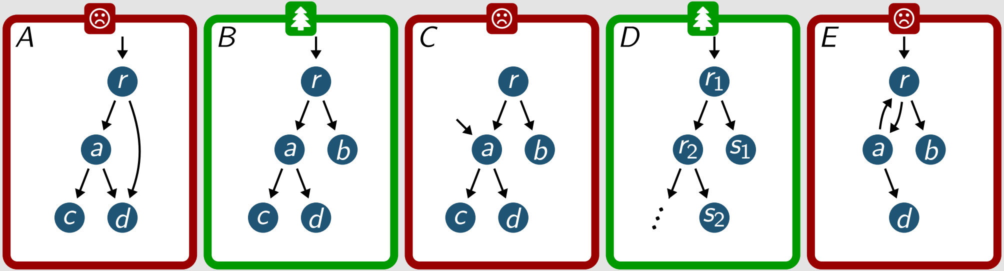


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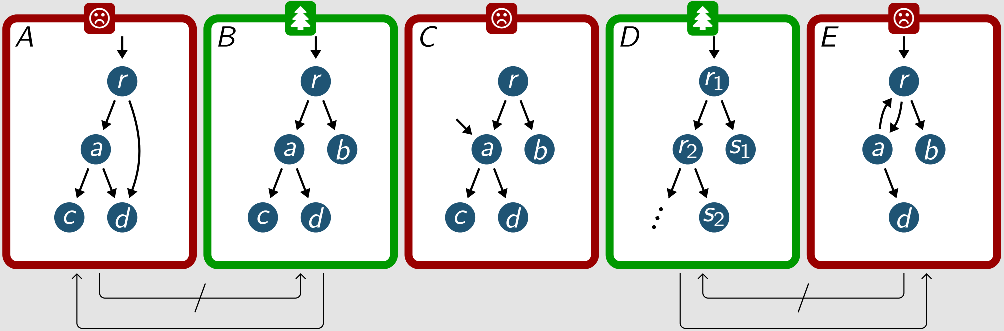


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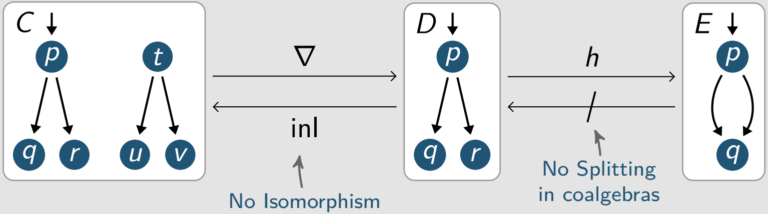


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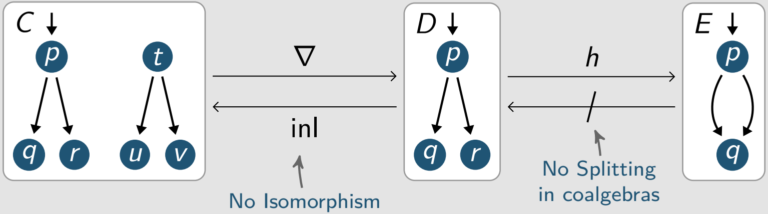


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A morphism $h\colon \mathbf{T} \to \mathbf{C}$, where $\mathbf{T}$ is a tree.

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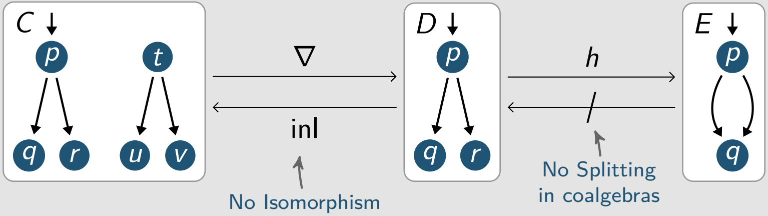


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### Definition: Tree unravelling of a pointed coalgebra $\mathbf{C}$

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### Proposition

- There is at most one tree unravelling of $\mathbf{C}$, if $\mathrm{Coalg}_1(F)$ has weak pullbacks.
- Every tree is reachable
- $\mathbf{T}$ is a tree $\iff$ every $h\colon \mathbf{R} \to \mathbf{T}$ with $\mathbf{R}$ reachable is an isomorphism

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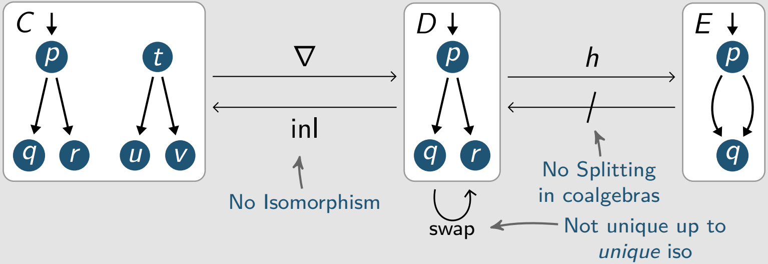


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Def.: $\mathbf{C} \in \mathrm{Coalg}_1(F)$ is reachable

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Adámek, Milius, Moss, Sousa '13

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**Existing Generlization to $(\mathcal{E},\mathcal{M})$**

Requires $\mathcal{M} \subseteq$ Mono

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Today:
Generalization to arbitrary
$(\mathcal{E},\mathcal{M})$-factorization systems

## Definition: $(\mathcal{E},\mathcal{M})$-factorization system on a category $\mathcal{C}$

$$A \overset{e}{\twoheadrightarrow} \mathcal{I}m(f) \overset{m}{\rightarrowtail} B, \qquad f = m \cdot e$$

+ closure of $\mathcal{E}$,$\mathcal{M}$ + diagonal lift

## Examples

- $\mathcal{E}$ = Epi, $\mathcal{M}$ = Mono in Set
- $\mathcal{E}$ = Iso, $\mathcal{M}$ = Mor in every category
- If $(\mathcal{E},\mathcal{M})$ in $\mathcal{C}$, $F\colon \mathcal{C} \to \mathcal{C}$, $F[\mathcal{M}] \subseteq \mathcal{M} \Longrightarrow (\mathcal{E},\mathcal{M})$ in $\mathsf{Coalg}_1(F)$ Wißmann '22

Assumption: $(\mathcal{E},\mathcal{M})$-factorization system on $\mathcal{C}$ and $F[\mathcal{M}] \subseteq \mathcal{M}$

## Definition: Generalized Reachability

- $\mathcal{M}$-subcoalgebra of $\mathbf{C}$: $m\colon \mathbf{B} \rightarrowtail \mathbf{C}$ in $\mathcal{M}$ in $\mathrm{Coalg}_1(F)$
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$\Downarrow$ Instance

## For $\mathcal{M} \subseteq$ Mono

Earlier concepts of reachability

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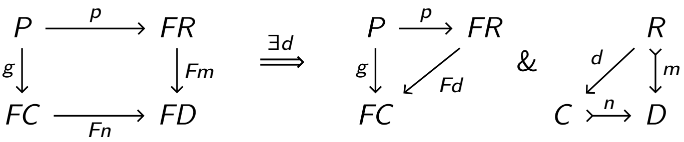
### How to construct the $\mathcal{M}$-reachable subcoalgebra of a coalgebra?

Get rid of assumption ~~$\mathcal{M} \subseteq$ Mono~~ of Wißmann, Milius, Katsumata, Dubut '19

## Definition for a functor $F\colon \mathcal{C} \to \mathcal{D}$

$p\colon P \to FR$ is $F$-precise if for all $m, n \in \mathcal{M}$



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$$
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P & \xrightarrow{\;p\;} & FR \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Fm} \\
FC & \xrightarrow[\;Fn\;]{} & FD
\end{array}
\quad\overset{\exists d}{\Longrightarrow}\quad
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{\scriptstyle g}\downarrow & \swarrow{\scriptstyle Fd} & \\
FC & &
\end{array}
\quad \& \quad
\begin{array}{ccc}
 & & R \\
 & \swarrow{\scriptstyle d} & \downarrow{\scriptstyle m} \\
C & \xrightarrow{\;n\;} & D
\end{array}
$$

⇓

## For $\mathcal{M} \subseteq$ Mono

Least bound (also called *base*) for the functor $F$ Blok '12

## Intuition: $p\colon P \to FR$ is precise

iff every $y \in R$ is mentioned *at least once* in the definition of $p$

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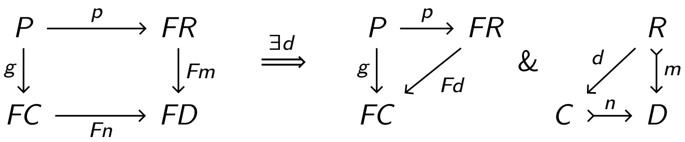


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Wißmann, Dubut, Katsumata, Hasuo '19

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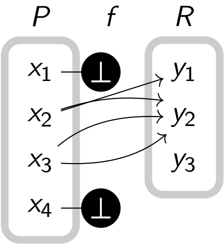
## Example for $FX = \{\bot\} + X \times X$ and $\mathcal{M} = \mathrm{Mor}$




$$f\colon P \to FR$$
$$x_1 \mapsto \bot$$
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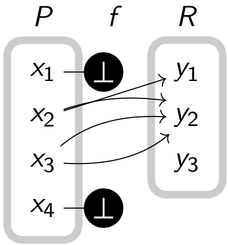


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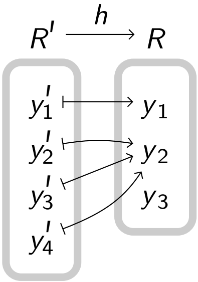

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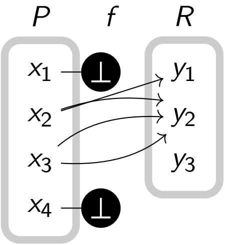


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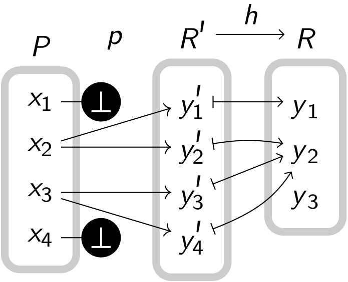
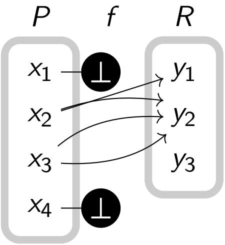


## Example for $FX = \{\bot\} + X \times X$ and $\mathcal{M} = \mathrm{Mor}$

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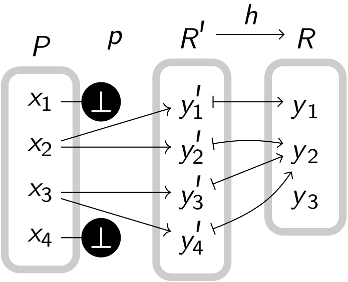


## Definition: $F$ admits precise factorizations for $\mathcal{M}$

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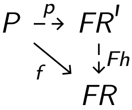

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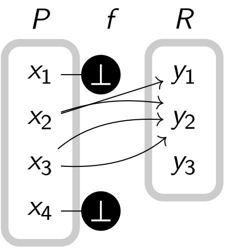


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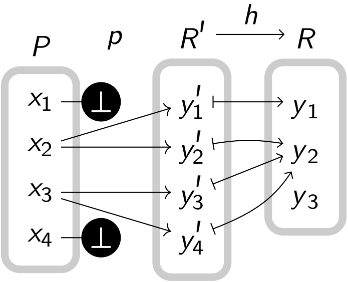


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Polynomials, Right-Adjoints, Analytic Functors

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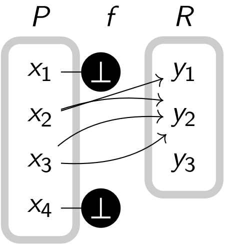


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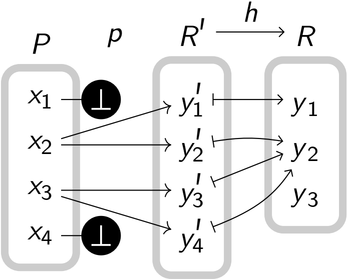


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## For $\mathcal{M} \subseteq \mathrm{Mono}$

$\iff$ $F$ preserves infinite $\mathcal{M}$-intersections

## Examples for $\mathcal{M} = \mathrm{Mor}$

Polynomials, Right-Adjoints, Analytic Functors

Assumption $F$ admits precise factorization w.r.t. $\mathcal{M}$

## Construction for $\mathbf{C} = (C, c, i_C) \in \mathsf{Coalg}_1(F)$

Define $m_k\colon T_k \rightarrowtail C$ = “states $k$ steps away from $i_C$”:

$(\mathcal{E},\mathcal{M})$-factorization:

$$I \overset{i_T}{\twoheadrightarrow} \underset{=\mathcal{I}m(i_C)}{T_0} \overset{m_0}{\rightarrowtail} C, \qquad i_C = m_0 \cdot i_T$$

$F$-precise factorization:

$$\begin{array}{ccc} T_k & \xrightarrow{t_k} & FT_{k+1} \\ {\scriptstyle m_k}\downarrow & & \downarrow{\scriptstyle Fm_{k+1}} \\ C & \xrightarrow{c} & FC \end{array}$$

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$\mathcal{I}m$ = Union of reachable states

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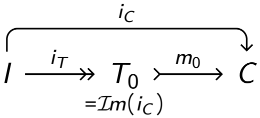


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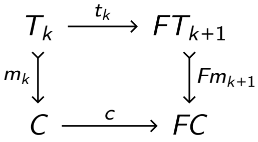


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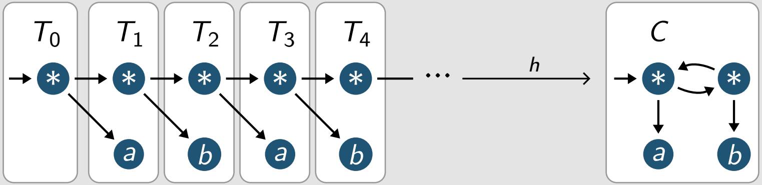
## Instance for $\mathcal{M} = \mathrm{Mor}$, $\mathcal{E} = \mathrm{Iso}$:

$\mathcal{I}m = \coprod_k T_k$ is the tree-unravelling of $\mathbf{C}$

For $FX = \{a\} + \{b\} + X \times X$, $F\colon \mathsf{Set} \to \mathsf{Set}$:



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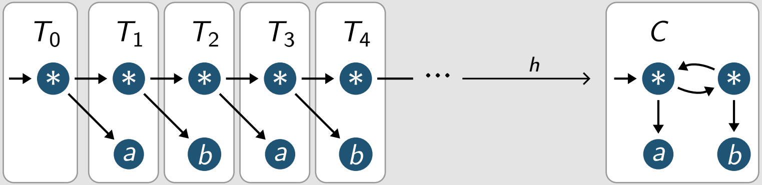


### Adamek & Porst 2004 for polynomial functors $F\colon \mathsf{Set} \to \mathsf{Set}$

Every element $e \in \nu F$ in the final (point-free) $F$-coalgebra induces a tree coalgebra.

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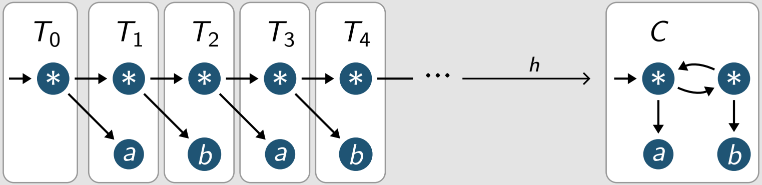


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⇑ Instance

### Our Construction for final coalgebras in Set:

For every element $e \in \nu F$, consider the pointed coalgebra

$$1 \xrightarrow{e} \nu F \xrightarrow{f} F(\nu F)$$

Its tree unravelling = Adamek & Porst's tree coalgebra.

## Partial automata $FX = O \times (A \rightharpoonup X)$

- Trees = For every state $q$, there is precisely one $w \in A^*$ with $q_0 \xrightarrow{w} q$.
- Tree unravelling of an automaton = Words for which the automaton is defined.

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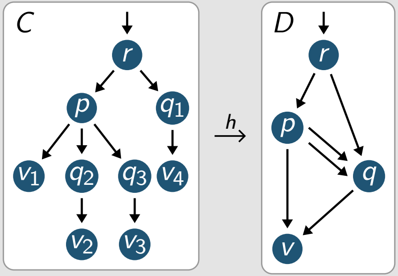

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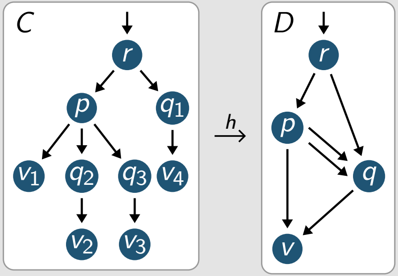
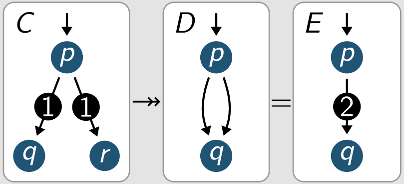


# Powerset

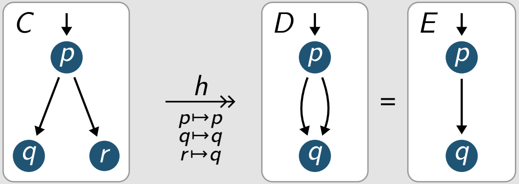
## No precise maps for powerset $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$

$$p\colon P \to \mathcal{P}R \text{ is } \mathcal{P}\text{-precise} \iff R = \emptyset$$

$$\Downarrow$$

## $\mathcal{P}$-Coalgebras

☹ Only one tree: the $\mathcal{P}$-coalgebra of one state and no transitions.




$$d(p) = \{q, q\} \quad d(q) = \emptyset$$
$$e(p) = \{q\} \quad e(q) = \emptyset$$

# Conclusions

- Being a tree = instance of reachability
  - $\Longrightarrow$ Leveraging reachability to arbitrary $(\mathcal{E},\mathcal{M})$-factorizations
- Universal property of tree unravellings
  - $\Longrightarrow$ Unique up to iso $\neq$ Unique up to *unique* iso
- Construction & existence of the tree unravelling and $\mathcal{M}$-reachability
  - $\Longrightarrow$ Subsumes earlier explicit definitions for polynomial functors by Adamek & Porst
  - $\Longrightarrow$ Also works for all analytic Set-functors ...
  - $\Longrightarrow$ ... e.g. the finite multiset functor

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