A Graded Monad for the Local Freshness Semantics of Nominal Automata with Name Allocation

Bachelor's Thesis

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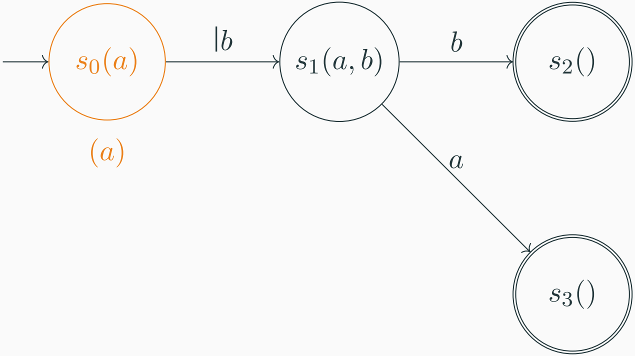
June 17, 2025

Motivation

We can use automata with name binding to generate **data languages** over an infinite alphabet $\mathbb{A}$.



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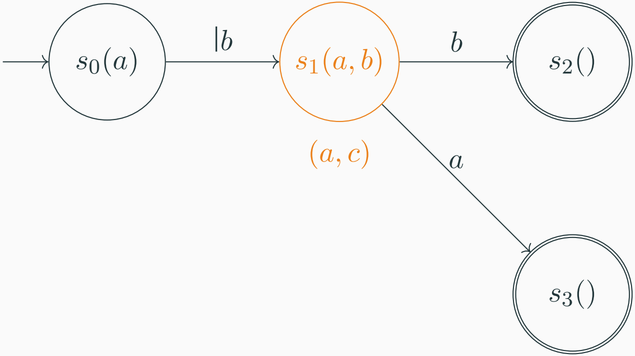

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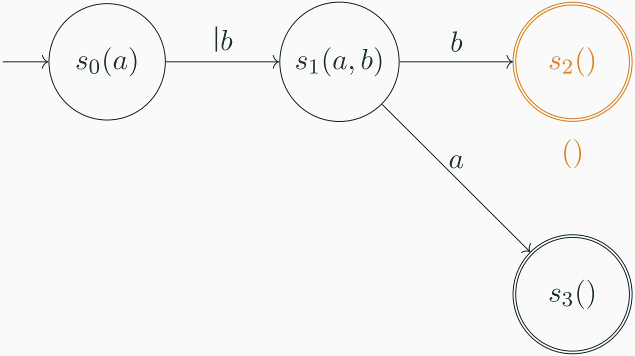

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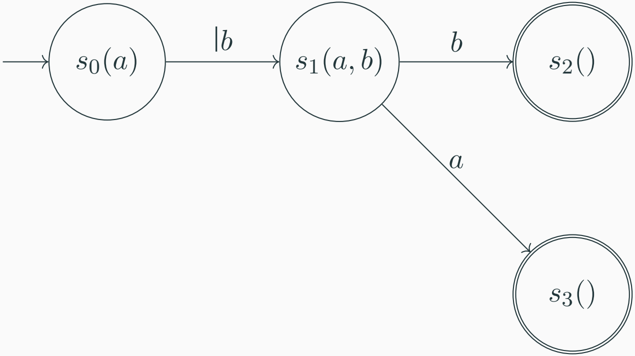

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Motivation

We want to describe the trace semantics of such automata in a more general setting using coalgebra.

Goal: Describe the semantics of nominal automata with name allocation as a **graded semantics**, which provide a generic framework for describing trace semantics coalgebraically.

Outline

Preliminaries – Nominal Sets [1]

Intuitively, nominal sets are sets where the elements *depend on* a finite set of atoms, their **support**. We write $\mathsf{supp}(x)$ for the smallest support of x.

Every nominal set X comes equipped with a permutation action $(\cdot) : \mathsf{Perm}(\mathbb{A}) \times \to X$ to allow renaming of atoms.

Example (Nominal Sets)

- The set of lambda terms: $\mathsf{supp}(\lambda x.\ x\ y) = \{x, y\}$
- The set of lambda terms modulo alpha-equivalence: $\mathsf{supp}([\lambda x.\ x\ y]_\alpha) = \{y\}$

Definition (Category of Nominal Sets)

Nominal sets form a category Nom:

- Objects: **Nominal sets**
- Morphisms: **Equivariant functions** $f : X \to Y$ with $f(\pi \cdot x) = \pi \cdot f(x)$

Equivariant functions *preserve supports*.

Definition (Name Abstraction Functor)

We can define alpha equivalence through a functor $[\mathbb{A}](-)$: Nom $\to$ Nom, where $[\mathbb{A}]X$ contains **name abstractions** $\langle a \rangle x$.

equivalence class of (a, x) in $(\mathbb{A} \times X)/\sim_\alpha$

Definition (Bar Strings)

- **Extended Bar Alphabet:** $\bar{\mathbb{A}} = \mathbb{A} \cup \{|a : a \in \mathbb{A}\}$
- **Bar Strings:** Words over $\bar{\mathbb{A}}$
- **Alpha-Equivalence on Bar Strings:** Equivalence $\equiv_\alpha$ generated by

$$w|av \equiv_\alpha w|bu \quad \text{if } \langle a \rangle v = \langle b \rangle u \text{ in } [\mathbb{A}]\bar{\mathbb{A}}^\star.$$

Example

$$a|aab \equiv_\alpha a|ccb \not\equiv_\alpha a|bbb$$

Preliminaries – Nominal Automata with Name Allocation [2]

Definition (RNNA [2])

A **regular nondeterministic nominal automaton (RNNA)** is a tuple $(A, \to, s, F)$ consisting of

- an orbit-finite set Q of **states**,
- an equivariant subset $\to \subseteq Q \times \bar{\mathbb{A}} \times Q$ called the **transition relation**,
- an **initial state** $s \in Q$,
- an equivariant subset $F \subseteq Q$ of **final states**,

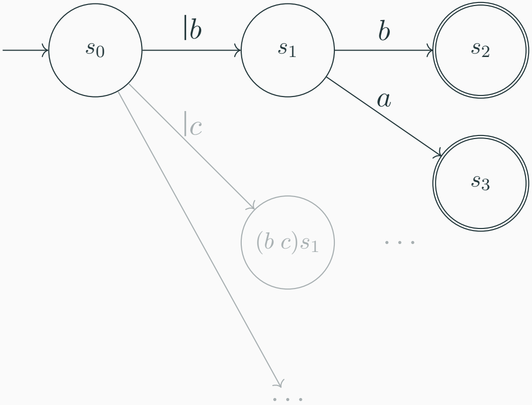
such that

if $s \xrightarrow{|a} q$ and $\langle a \rangle q = \langle a' \rangle q'$, then $s \xrightarrow{|a'} q'$

1. The relation $\to$ is α-invariant.
2. The relation $\to$ is finitely branching up to α-equivalence.

$\{(a, q) : s \xrightarrow{a} q\}$ and $\{\langle a \rangle q : s \xrightarrow{|a} q\}$ are finite for $s \in Q$

Preliminaries – Nominal Automata with Name Allocation [2]

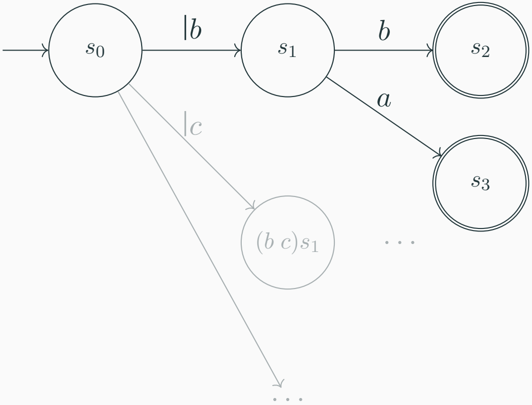


Literal Language: $L_0(s_0) = \{|bb, |ba, |cc, |ca, |dd, |da, \ldots\}$

$\rightsquigarrow$ Bar Language: $L_\alpha(s_0) = \{[|bb]_\alpha, [|ba]_\alpha\}$

$\rightsquigarrow$ Local Freshness Semantics: $D(L_\alpha(s_0)) = \{aa, bb, ba, cc, ca, dd, da, \ldots\}$

Preliminaries – Nominal Automata with Name Allocation [2]



poststate

Literal Language Pretraces: $\hat{L}_0(s_0) = \{|bbs_2, |bas_3, |ccs_2, |cas_3, \dots\}$

$\rightsquigarrow$ Bar Language Pretraces: $\hat{L}_\alpha(s_0) = \{[|bbs_2]_{\hat{\alpha}}, [|bas_3]_{\hat{\alpha}}\}$

$\rightsquigarrow$ Local Freshness Semantics: $\hat{D}(\hat{L}_\alpha(s_0)) = \{aas_2, bbs_2, bas_3, ccs_2, cas_3, \dots\}$

Going forward, we will assume that every state is final.

Thus, we can consider an RNNA as an orbit-finite coalgebra $\gamma : X \to H(X)$ for the functor H : Nom $\to$ Nom with

$$H(X) = \mathcal{P}_{\mathsf{f}}(\mathbb{A} \times X) \times \mathcal{P}_{\mathsf{f}}([\mathbb{A}]X).$$

free transitions   bound transitions

Definition (Graded Monad)

A **graded monad** on a category $\mathbf{C}$ is a tuple $((M_n)_{n\in\mathbb{N}_0}, \eta, (\mu^{nk})_{n,k\in\mathbb{N}_0})$ containing

- for every $n \in \mathbb{N}_0$, an endofunctor $M_n : \mathbf{C} \to \mathbf{C}$,
- a **unit** transformation $\eta : \mathsf{Id} \to M_0$,
- for every $n, k \in \mathbb{N}_0$, a **multiplication** transformation $\mu^{nk} : M_n M_k \to M_{n+k}$,

satisfying

1. the **unit law**: $\forall n \in \mathbb{N}_0.\ \mu^{0,n} \cdot \eta M_n = \mathsf{id}_{M_n} = \mu^{n,0} \cdot M_n\eta$,
2. the **associative law**: $\forall n, k, m \in \mathbb{N}_0.\ \mu^{n,k+m} \cdot M_n\mu^{k,m} = \mu^{n+k,m} \cdot \mu^{n,k} M_m$.

Definition (Graded Semantics)

A **graded semantics** for G-coalgebras consists of

- a graded monad $((M_n), \eta, (\mu^{nk}))$,
- a natural transformation $\alpha : G \to M_1$.

Given a G-coalgebra $\gamma : X \to G(X)$, the α-**pretrace sequence** is then given by

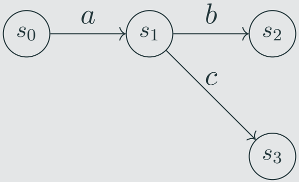
$$\gamma^{(0)} = (X \xrightarrow{\eta_X} M_0(X)),$$
$$\gamma^{(n+1)} = (X \xrightarrow{\alpha_X \circ \gamma} M_1(X) \xrightarrow{M_1(\gamma^{(n)})} M_1(M_n(X)) \xrightarrow{\mu_X^{1n}} M_{n+1}(X)),$$

and the α-**trace sequence** is defined as $(M_n(!) \circ \gamma^{(n)} : X \to M_n(1))_{n \in \mathbb{N}_0}$.

Example (Graded Semantics [4])

We consider labelled transition systems as coalgebras for $G(X) = \mathcal{P}(\mathbb{A} \times X)$.

The graded semantics with $M_n(X) = \mathcal{P}(\mathbb{A}^n \times X)$ and $\alpha = \mathsf{id}$ describes the trace-semantics of LTS.



$$
\begin{aligned}
\gamma^{(2)}(s_0) &= \mu^{1,1}(\{(a, \gamma^{(1)}(s_1)\}) \\
&= \mu^{1,1}(\{(a, \mu^{1,0}(\{(b, \gamma^{(0)}(s_2)), (c, \gamma^{(0)}(s_3))\}))\}) \\
&= \mu^{1,1}(\{(a, \mu^{1,0}(\{(b, \{s_2\}), (c, \{s_3\})\}))\}) \\
&= \mu^{1,1}(\{(a, \{(b, s_2), (c, s_3)\}))\}) \\
&= \{(ab, s_2), (ac, s_3)\}
\end{aligned}
$$

trace poststate

$$(M_n(!) \circ \gamma^{(2)})(s_0) \cong \{ab, ac\}$$

Preliminaries – Graded Semantics [3]

algebraic theory where every operation has a depth and axioms have uniform depth

Fact (Graded Theories on Set)

On Set, every graded monad corresponds to a ***graded theory****.*

The graded theory (Σ, E, d) *induces a graded monad* $((M_n), \eta, (\mu^{nk}))$*, where*

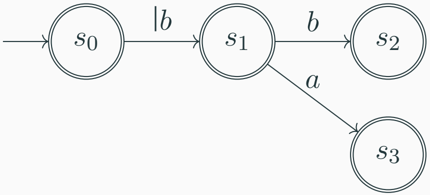
- $M_n(X)$ *are depth-n terms over* X *modulo derivable equality,*
- η *converts variables into terms,*
- μ^{nk} *collapses layered terms, "removing" the inner equivalence classes.*

This motivates a similar construction on Nom.

Outline

Goal

Given an RNNA



we want to describe the pretraces as terms:

$$|b(bs_2 + as_3) \quad \rightsquigarrow \quad |bbs_2 + |bas_3.$$

We will first define these terms and theories over them in a more general setting.

Terms

pure operations free operations bound operations

Definition (Terms)

Given a nominal set X and a graded signature $\Sigma = (\Sigma_0, \Sigma_{\mathsf{f}}, \Sigma_{\mathsf{b}})$, define a **nominal Σ-term** as

$$t ::= x \mid f(t_1, \dots, t_p) \mid a.g(t_1, \dots, t_q) \mid \nu a.h(t_1, \dots, t_r),$$

ranging over all $x \in X$, $a \in \mathbb{A}$, $f/p \in \Sigma_0$, $g/q \in \Sigma_{\mathsf{f}}$, and $h/r \in \Sigma_{\mathsf{b}}$.

Example (Pretrace Terms)

To describe pretraces of RNNA, we will use the signature Σ with $\Sigma_0 = \{+/2, \bot/0\}$, $\Sigma_{\mathsf{f}} = \{\mathsf{pre}/1\}$, and $\Sigma_{\mathsf{b}} = \{\mathsf{abs}/1\}$, where we will write $+$ in infix notation and

$$at := a.\mathsf{pre}(t) \quad \text{and} \quad |at := \nu a.\mathsf{abs}(t).$$

Terms

Definition (Uniform Depth)

A term t over X has **(uniform) depth** $n \in \mathbb{N}_0$ iff

- $t = x \in X$ and $n = 0$.
- $t = f(t_1, \ldots, t_p)$, where $t_1, \ldots, t_p$ have uniform depth n', and $n = n' + d(f)$.
- Similarly for $t = a.f(t_1, \ldots, t_p)$ and $t = \nu a.f(t_1, \ldots, t_p)$.

Example (Pretrace Terms)

For pretraces of RNNA, $d(+) = d(\bot) = 0$ and $d(\mathsf{pre}) = d(\mathsf{abs}) = 1$.

The depth of a term is exactly the length of the pretraces. For example, the term $|b(bs_2 + as_3)$ has uniform depth 2.

If c is a constant, then it has uniform depth $n \in \mathbb{N}_0$ for all $n \geq d(c)$.

We refer to the set of Σ-terms over X with uniform depth n as $\mathsf{Term}_{\Sigma,n}(X)$.

Derivations

Definition (Graded Theories and Derivable Equality)

- **Equations:** pairs of terms $t, u \in \mathsf{Term}_{\Sigma,n}(X)$ written as $X \vdash_m t = u$.
- **Graded Theory** $T = (\Sigma, E)$: graded signature and class E of **axiom** equations.
- **Derivable Equality:** Congruence with additional rules

$$(\mathsf{ax}_{r=s})\frac{X \vdash_l \pi\sigma(x) = \sigma(\pi x) \qquad \forall \pi \in \mathsf{Perm}(\mathbb{A}), x \in Y}{X \vdash_{m+l} (\tau r)\sigma = (\tau s)\sigma}$$

$$(\mathsf{perm}_f)\frac{a\#u_i \qquad X \vdash_m t_i = (a\ b)u_i \qquad \forall i \in \{1, \ldots, p\}}{X \vdash_{m+d(f)} \nu a.f(t_1, \ldots, t_p) = \nu b.f(u_1, \ldots, u_p)} \quad (a \neq b)$$

ranging over $Y \vdash_m r = s \in E$, $\tau \in \mathsf{Perm}(\mathbb{A})$, $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$, $f/p \in \Sigma_{\mathsf{b}}$.

Definition (Nominal Algebra)

A **nominal** (Σ, n)**-algebra** A consists of

- a family $(A_i)_{0 \leq i \leq n}$ of nominal sets, called **carriers**,
- for $f/p \in \Sigma_0$, a family $(f_{A,m} : A_m^p \to A_{m+d(f)})$ of equivariant functions,
- for $f/p \in \Sigma_\mathsf{f}$, a family $(f_{A,m} : \mathbb{A} \times A_m^p \to A_{m+d(f)})$ of equivariant functions,
- for $f/p \in \Sigma_\mathsf{b}$, a family $(f_{A,m} : [\mathbb{A}]A_m^p \to A_{m+d(f)})$ of equivariant functions.

A **morphism between** (Σ, n)**-algebras** A, B is a family $(h_i)_{0 \leq i \leq n}$ of equivariant functions $h_i : A_i \to B_i$ such that, for $f/p \in \Sigma_0$ and $x_1, \dots, x_p \in A_m$,

$$h_{m+d(f)}(f_{A,m}(x_1, \dots, x_p)) = f_{B,m}(h_m(x_1), \dots, h_m(x_p)),$$

and similarly for Σ_f and Σ_b.

Semantics

Definition (Evaluation)

Given a nominal (Σ, n)-algebra A and an **environment** $\iota : X \to A_k$, the **evaluation map** $[\![\cdot]\!]^\iota_m$: $\mathsf{Term}_{\Sigma,m}(X) \to A_{k+m}$ of depth-m terms is defined as

$$
\begin{aligned}
[\![x]\!]^\iota_0 &= \iota(x), \\
[\![f(t_1, \ldots, t_p)]\!]^\iota_{m+d(f)} &= f_{A,k+m}([\![t_1]\!]^\iota_m, \ldots, [\![t_p]\!]^\iota_m), \\
[\![a.f(t_1, \ldots, t_p)]\!]^\iota_{m+d(f)} &= f_{A,k+m}(a, [\![t_1]\!]^\iota_m, \ldots, [\![t_p]\!]^\iota_m), \\
[\![\nu a.f(t_1, \ldots, t_p)]\!]^\iota_{m+d(f)} &= f_{A,k+m}(\langle a \rangle([\![t_1]\!]^\iota_m, \ldots, [\![t_p]\!]^\iota_m)).
\end{aligned}
$$

Definition (Nominal Model)

A (T, n)**-model** is a (Σ, n)-algebra such that, for every axiom $X \vdash_m t = u$ and every environment ι, $[\![t]\!]^\iota_m = [\![u]\!]^\iota_m$.

Theorem (Soundness)

Let A be a (T, n)-model and $t, u \in \mathsf{Term}_{\Sigma,m}(X)$ with $m \leq n$.

If $X \vdash_m t = u$ is derivable, then $[\![t]\!]^\iota_m = [\![u]\!]^\iota_m$ for every environment ι.

Free Models

Fix a graded theory $T = (\Sigma, E)$ and a depth $n \leq \omega$.

We will refer to derivable equality as the binary relation $\sim$ between terms.

Since derivable equality is an equivalence, we can partition $\mathsf{Term}_{\Sigma,m}(X)$ into equivalence classes $[t]_m \in \mathsf{Term}_{\Sigma,m}(X)/\sim$.

Definition (Free Model)

The (T, n)-algebra $F(X)$ is defined as follows:

$$(F(X))_i = \mathsf{Term}_{\Sigma,i}(X)/\sim,$$

$$f_{F(X),m}([t_1]_m, \dots, [t_p]_m) = [f(t_1, \dots, t_p)]_{m+d(f)} \qquad \text{for } f/p \in \Sigma_0,$$

$$f_{F(X),m}(a, [t_1]_m, \dots, [t_p]_m) = [a.f(t_1, \dots, t_p)]_{m+d(f)} \qquad \text{for } f/p \in \Sigma_{\mathsf{f}},$$

$$f_{F(X),m}(\langle a\rangle([t_1]_m, \dots, [t_p]_m)) = [\nu a.f(t_1, \dots, t_p)]_{m+d(f)} \qquad \text{for } f/p \in \Sigma_{\mathsf{b}}.$$

Free Models

The class of (T, n)-models forms a category $\mathsf{Alg}(T, n)$.

Definition (Forgetful Functor)

We define the forgetful functor $G : \mathsf{Alg}(T, n) \to \mathsf{Nom}$ with

$$G(A) = A_0$$
$$G(h) = h_0.$$

Theorem

The definition of $F(X)$ yields a left adjoint functor F to the forgetful functor G.

Free Models

Corollary

Every graded theory $T = (\Sigma, E)$ *induces a graded monad* $((M_n), \eta, (\mu^{nk}))$ *with*

$$M_n : \mathsf{Nom} \to \mathsf{Nom},$$
$$M_n(X) = \mathsf{Term}_{\Sigma,n}(X)/{\sim},$$
$$M_n(f)([t]_n) = [t\sigma_f]_n,$$

$$\eta_X : X \to M_0(X),$$
$$\eta_X(x) = [x]_0,$$

$$\mu^{nk}_X : M_n(M_k(X)) \to M_{n+k}(X),$$
$$\mu^{nk}_X([t]_n) = [\![t]\!]^{\mathsf{id}}_n,$$

with the substitution $\sigma_f = (X \xrightarrow{f} Y \hookrightarrow \mathsf{Term}_{\Sigma,0}(Y))$.

Outline

A Graded Theory for Local Freshness Semantics

We can now define a graded theory over pretrace terms with the axioms

$$
\begin{aligned}
X &\vdash_0 x + y = y + x, \\
X &\vdash_0 (x + y) + z = x + (y + z), \\
X &\vdash_0 x + x = x, \\
X &\vdash_0 x + \bot = x, \\
X &\vdash_1 a(x + y) = ax + ay, \\
X &\vdash_1 |a(x + y) = |ax + |ay, \\
X &\vdash_1 a\bot = \bot, \\
X &\vdash_1 |a\bot = \bot, \\
X &\vdash_1 |ax = |ax + ax,
\end{aligned}
$$

ranging over all $a \in \mathbb{A}$, all nominal sets X, and all elements $x, y, z \in X$.

A Model For Local Freshness Semantics

The (Σ, n)-model $F'(X)$ is defined as follows:

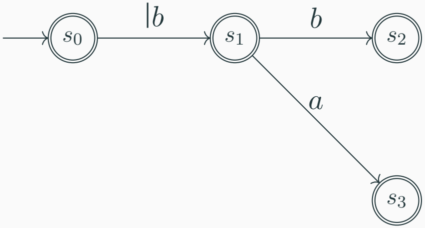
$$\begin{aligned}
(F'(X))_m &= \mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times X), \\
\mathsf{pre}_{F'(X),m}(a, L) &= \{aw : w \in L\}, \\
\mathsf{abs}_{F'(X),m}(\langle a \rangle L) &= \{bv : \langle a \rangle L = \langle a' \rangle L', w' \in L', \langle a' \rangle w' = \langle b \rangle v\}, \\
+_{F'(X),m}(L_1, L_2) &= L_1 \cup L_2, \\
\bot_{F'(X),m} &= \emptyset.
\end{aligned}$$

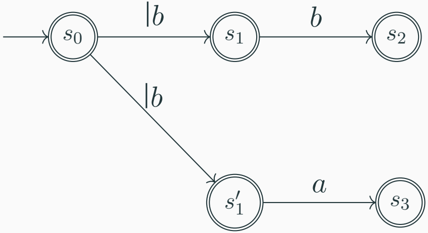
We also define the *interpretation* as the morphism $\Phi : F(X) \to F'(X)$ with

$$\Phi_m([t]_m) = [\![t]\!]_m^{\eta'_X},$$

with the equivariant environment $\eta'_X : X \to (F'(X))_0, x \mapsto \{x\}$.

$$X \vdash_1 \, |a(x+y) = |ax + |ay$$





$$|b(bs_2 + as_3)$$

$$\mathsf{abs}_{F'(X),1}(\langle b\rangle\{bs_2, as_3\})$$

$$|bbs_2 + |bas_3$$

$$\mathsf{abs}_{F'(X),1}(\langle b\rangle\{bs_2\}) \cup \mathsf{abs}_{F'(X),1}(\langle b\rangle\{as_3\})$$

For this case, we require $\mathsf{abs}_{F'(X),m}$ to be monotone, in particular:

$$\mathsf{abs}_{F'(X),1}(\langle b\rangle\{bs_2, as_3\}) \quad \supseteq \quad \mathsf{abs}_{F'(X),1}(\langle b\rangle\{bs_2\})$$

If we assume $s_2 \neq s_3$, $a \neq b$, and $a\#s_2$, then $\langle b\rangle\{bs_2\} = \langle a\rangle\{(a\ b)bs_2)\}$.

$$\implies \quad (a\ b)bs_2 \in \mathsf{abs}_{F'(X),1}(\langle b\rangle\{bs_2\})$$

However, on the left-hand side, we have $\langle b\rangle\{bs_2, as_3\} \neq \langle a\rangle(a\ b)\{bs_2, as_3\}$.

We can instead pick a $c \in \mathbb{A} \setminus \mathsf{supp}(\{bs_2\}, \{bs_2, as_3\})$.

$$\implies \quad bs_2 = (a\ c)bs_2 \quad \text{and} \quad \langle b\rangle\{bs_2, as_3\} = \langle c\rangle\{(b\ c)(a\ c)bs_2, (b\ c)as_3\}.$$

Since a, b, c are pairwise distinct, $(a\ c)(a\ b) = (b\ c)(a\ c)$.

$$\implies \quad \langle c\rangle(b\ c)(a\ c)bs_2 = \langle c\rangle(a\ c)(a\ b)bs_2 = \langle a\rangle(a\ b)bs_2.$$

Interpretation under Local Freshness Semantics

We can identify the set of single pretraces $\bar{\mathbb{A}}^m \times X$ as a fragment of Σ-terms.

Theorem

If $k \in \mathbb{N}_0$ and $w_1, \ldots, w_k \in \bar{\mathbb{A}}^m \times X$ are pretraces, then

$$\Phi_m\left(\left[\sum_{i=1}^k w_i\right]_m\right) = \hat{D}\left(\{[w_i]_{\hat{\alpha}} : i \in \{1, \ldots, k\}\}\right).$$

Theorem

Let $t \in \mathsf{Term}_{\Sigma,m}(X)$ be a term and $w \in \bar{\mathbb{A}}^m \times X$ a pretrace.

If $\hat{D}_m(\{[w]_{\hat{\alpha}}\}) \subseteq \Phi_m([t]_m)$, then $[t]_m = [t+w]_m$.

Corollary

The function $\Phi_m : \mathsf{Term}_{\Sigma,m}(X)/{\sim} \to \mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times X)$ is injective for every $m \in \mathbb{N}_0$.

Outline

A Graded Semantics for RNNAs

Recall that we can view RNNAs as orbit-finite coalgebras for the functor $H : \mathsf{Nom} \to \mathsf{Nom}$ given by

$$H(X) = \mathcal{P}_{\mathsf{f}}(\mathbb{A} \times X) \times \mathcal{P}_{\mathsf{f}}([\mathbb{A}]X).$$

We define a graded semantics for RNNA using $\alpha : H \to M_1$ with

$$\alpha_X(S_{\mathsf{f}}, S_{\mathsf{b}}) = \left[\sum_{(a,x) \in S_{\mathsf{f}}} ax + \sum_{(a,x) \in u_X[S_{\mathsf{b}}]} |ax \right]_1,$$

where $u_X : [\mathbb{A}]X \to \mathbb{A} \times X$ is any splitting of $[\mathbb{A}]X$.

Capturing Pretraces

Theorem

Let $s \in Q$ be a state in an RNNA defined by $\gamma : Q \to H(Q)$. Then

$$\gamma^{(n)}(s) = \left[\sum_{wq \in v_n\left[\hat{L}_\alpha^{(n)}(s)\right]} wq \right]_n ,$$

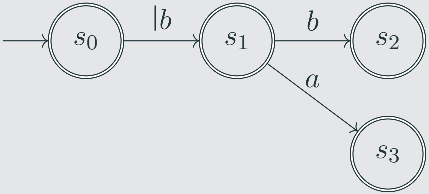
where

$$\hat{L}_\alpha^{(n)}(s) = \{w \in \hat{L}_\alpha(s) : |w| = n\},$$

and $v_n : (\bar{\mathbb{A}}^n \times Q)/\hat{\equiv}_\alpha \to \bar{\mathbb{A}}^n \times Q$ is any splitting.

Capturing Pretraces

Example



$$
\begin{aligned}
\gamma^{(2)}(s_0) &= \mu_X^{1,1}([|b(\gamma^{(1)}(s_1))]_1) \\
&= \mu_X^{1,1}([|b(\mu_X^{1,0}([b(\gamma^{(0)}(s_2)) + a(\gamma^{(0)}(s_3))]_1)]_1) \\
&= \mu_X^{1,1}([|b(\mu_X^{1,0}([b[s_2]_0 + a[s_3]_0]_1)]_1) \\
&= \mu_X^{1,1}([|b[bs_2 + as_3]_1) \\
&= [|b(bs_2 + as_3)]_2 = [|bbs_2 + |bas_3]_2
\end{aligned}
$$

Capturing Traces

Theorem

Let $s \in Q$ be a state in an RNNA defined by $\gamma : Q \to H(Q)$ and $n \in \mathbb{N}_0$. Then

$$\Phi_n(M_n(!)(\gamma^{(n)}(s))) \cong \{w \in D(L_\alpha(s)) : |w| = n\}.$$

Corollary

Let $q \in Q$, $s \in S$ be states in RNNAs defined by $\gamma : Q \to H(Q)$ and $\delta : S \to H(S)$.

The states q and s have the same α-trace sequence iff $D(L_\alpha(q)) = D(L_\alpha(s))$.

Outline

Conclusion and Future Work

We have:

- introduced a concept of graded theories over Nom inducing graded monads.
- defined a graded theory capturing local freshness semantics of pretraces.
- defined graded semantics capturing the local freshness semantics of RNNAs.

Future Work:

- It remains to be shown that the induced graded monad is depth-1 if all operations and axioms in the theory are at most depth-1.
- It may be possible to replace the infinitary $(\mathsf{ax}_{r=s})$ rule with a finitary one.
- Turn $F'(X)$ into a functor (possibly using $\mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times \mathsf{Frs}(X))$, not $\mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times X)$).
- Give an alternative description of the graded monad based on $F'(X)$.
- Extend the graded semantics to work on ufs sets of transitions.

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