Behavioural Conformances based on Lax Couplings

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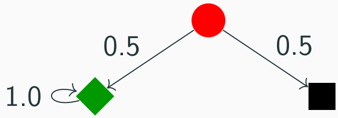
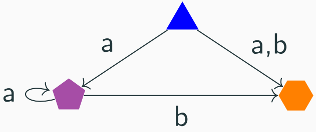
-  and  are not bisimilar
-  and  have behavioural distance 0.01

Coalgebra



$\alpha\colon X \to 1 + \mathcal{D}X$

Markov chain with deadlock



$\alpha\colon X \to \mathcal{P}(\mathcal{A} \times X)$

Labelled transition system

$\alpha\colon X \to FX$, where $F\colon \mathsf{Set} \to \mathsf{Set}$ is a functor

Functor Liftings and Lax Extensions

Functor lifting

Given a pseudometric d on X, construct one on FX.

$$(F\colon \mathsf{Set} \to \mathsf{Set}) \quad \rightsquigarrow \quad (\overline{F}\colon \mathsf{PMet} \to \mathsf{PMet})$$

Behavioural distance = fixpoint of $d \mapsto \overline{F}(d) \cdot (\alpha \times \alpha)$

Lax extension

Given $r\colon X \times Y \to [0, \infty]$, construct $Lr\colon FX \times FY \to [0, \infty]$

L-simulation: r such that $r \leq Lr \cdot (\alpha \times \beta)$

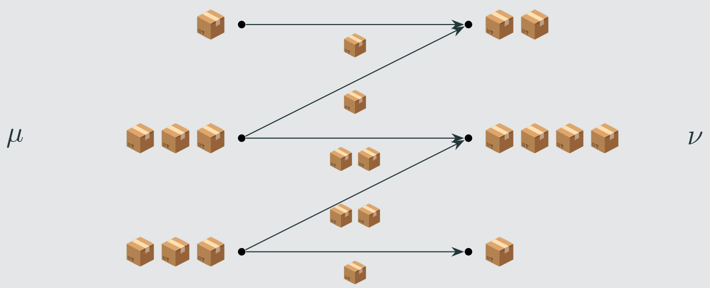
The Wasserstein Metric

Functor lifting/lax extension for $\mathcal{D}$

$$W(d)(\mu,\nu) = \inf\{\mathbb{E}_\rho(d) \mid \rho \text{ is a coupling of } \mu \text{ and } \nu\}$$

ρ is a coupling of μ and ν $\iff$ $\forall x.\, \mu(x) = \sum_y \rho(x,y)$ and $\forall y.\, \nu(y) = \sum_x \rho(x,y)$

Optimal transport



The Coalgebraic Wasserstein Metric

Generalized couplings

ρ is a coupling of μ and $\nu \iff \forall x.\, \mu(x) = \sum_y \rho(x,y)$ and $\forall y.\, \nu(y) = \sum_x \rho(x,y)$

The Coalgebraic Wasserstein Metric

Generalized couplings

ρ is a coupling of μ and $\nu \iff \mathcal{D}\pi_1(\rho) = \mu$ and $\mathcal{D}\pi_2(\rho) = \nu$

The Coalgebraic Wasserstein Metric

Generalized couplings

t is a coupling of t_1 and $t_2 \iff F\pi_1(t) = t_1$ and $F\pi_2(t) = t_2$

The Coalgebraic Wasserstein Metric

Generalized couplings

t is a coupling of t_1 and t_2 $\iff$ $F\pi_1(t) = t_1$ and $F\pi_2(t) = t_2$

Predicate liftings

$$f\colon X \to [0,\infty] \quad \rightsquigarrow \quad \mathbb{E}_{(-)}(f)\colon \mathcal{D}X \to [0,\infty]$$

The Coalgebraic Wasserstein Metric

Generalized couplings

t is a coupling of t_1 and t_2 $\iff$ $F\pi_1(t) = t_1$ and $F\pi_2(t) = t_2$

Predicate liftings

$$f\colon X \to [0,\infty] \quad \rightsquigarrow \quad \lambda_X(f)\colon FX \to [0,\infty]$$

The Coalgebraic Wasserstein Metric

Generalized couplings

t is a coupling of t_1 and t_2 $\iff$ $F\pi_1(t) = t_1$ and $F\pi_2(t) = t_2$

Predicate liftings

$$f \colon X \to [0,\infty] \quad \rightsquigarrow \quad \lambda_X(f) \colon FX \to [0,\infty]$$

Coalgebraic Wasserstein [Hofmann 07, Baldan et al 14]

$$W_\lambda^{=}(d)(t_1,t_2) = \inf\{\lambda_{X\times Y}(t)(d) \mid t \text{ is a coupling of } t_1 \text{ and } t_2\}$$

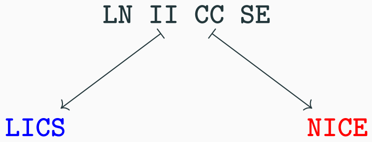
Example: Hamming Distance

$$\mathrm{List}X = \text{ set of finite sequences over } X$$

$$\lambda_X(f)((x_1, \ldots, x_n)) = f(x_1) + \cdots + f(x_n)$$

$$\Delta_{01}(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise} \end{cases}$$

$$\mathrm{Ham} := W_\lambda^{=}(\Delta_{01}) = \text{ Hamming distance on List}X$$



$$\mathrm{Ham}(\texttt{LICS}, \texttt{NICE}) = 2$$

Failure of Couplings

Couplings need not exist in general:

Finite measures

- Meas $X = \{\mu \colon X \to [0, \infty] \mid \sum_x \mu(x) < \infty\}$
- μ and ν have couplings $\iff \sum_x \mu(x) = \sum_x \nu(x)$

Lists

$\ell_1, \ell_2 \in$ List$X$ have a coupling $\iff$ ℓ_1 and ℓ_2 have the same length

Labelled transition systems

Couplings only exist if the labels match exactly

$$t_1 \text{ and } t_2 \text{ have no coupling} \implies W_\lambda^{=}(d)(t_1, t_2) = \infty$$

Example: Unbalanced Optimal Transport

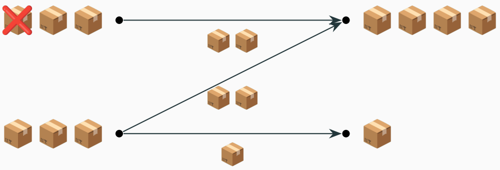
Total variation distance for Meas

$$d_{\mathsf{TV}}(\mu, \nu) = \sup_{U \subseteq X} \max(\nu(U) - \mu(U), 0)$$

Intuition: produce missing goods and/or scrap excess goods

Distance

$d_{\mathsf{UOT}}(\mu, \nu) = \inf\{d_{\mathsf{TV}}(\mu, \mu') + W_{\mathbb{E}}^{=}(\mu', \nu') + d_{\mathsf{TV}}(\nu', \nu) \mid \mu', \nu' \text{ finite measures}\}$



Example: Levenshtein Distance

Insertion/deletion distance

$$d_{\pm}(s,t) = \begin{cases} ||t| - |s||, & \text{if one of } s \text{ or } t \text{ is subsequence of the other} \\ \infty, & \text{otherwise} \end{cases}$$

Levenshtein distance

$$\mathsf{Lev}(s,t) = \inf\{d_{\pm}(s,s') + \mathsf{Ham}(s',t') + d_{\pm}(t',t) \mid s',t' \in \mathsf{List}X\}$$

$$\texttt{WORLD} \underset{1}{\text{———}} \texttt{WORD} \underset{1}{\text{———}} \texttt{BORD} \underset{2}{\text{———}} \texttt{ABOARD}$$

$$\mathsf{Lev}(\texttt{WORLD},\texttt{ABOARD}) = d_{\pm}(\texttt{WORLD},\texttt{WORD}) + \mathsf{Ham}(\texttt{WORD},\texttt{BORD}) + d_{\pm}(\texttt{BORD},\texttt{ABOARD}) = 4$$

Functor Liftings Revisited

Setup

- Instead of $F\colon \mathsf{Set} \to \mathsf{Set}$, consider $F\colon \mathsf{Set} \to \mathsf{PMet}$
- The pseudometric d_{FX} handles failure of couplings

List functor

$FX = (\mathsf{List}X, d_{\pm})$

Finite measure functor

$FX = (\mathsf{Meas}X, d_{\mathsf{TV}})$

The Distributorial Wasserstein Metric

Distributorial Wasserstein

$$W_\lambda(d)(t_1, t_2) = \inf\{d_{FX}(t_1, t_1') + W_\lambda^{=}(d)(t_1', t_2') + d_{FY}(t_2', t_2) \mid t_1' \in FX, t_2' \in FY\}$$

The Distributorial Wasserstein Metric

Distributorial Wasserstein

$$W_\lambda(d) = d_{FY} \cdot W_\lambda^{=}(d) \cdot d_{FX}$$

The Distributorial Wasserstein Metric

Distributorial Wasserstein

$$W_\lambda(d)(t_1, t_2) = \inf\{d_{FX}(t_1, F\pi_1(t)) + \lambda_{X\times Y}(d)(t) + d_{FY}(F\pi_2(t), t_2) \mid t \in F(X \times Y)\}$$

The Distributorial Wasserstein Metric

Distributorial Wasserstein

$$W_\lambda(d)(t_1, t_2) = \inf\{\epsilon + \lambda_{X \times Y}(d)(t) \mid t \text{ is an } \epsilon\text{-coupling of } t_1 \text{ and } t_2\}$$

$$t \text{ is an } \epsilon\text{-coupling of } t_1 \text{ and } t_2 \iff d_{FX}(t_1, F\pi_1(t)) + d_{FY}(F\pi_2(t), t_2) \leq \epsilon$$

The Distributorial Wasserstein Metric

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$$W_\lambda(d)(t_1, t_2) = \inf\{\epsilon + \lambda_{X \times Y}(d)(t) \mid t \text{ is an } \epsilon\text{-coupling of } t_1 \text{ and } t_2\}$$

$$t \text{ is an } \epsilon\text{-coupling of } t_1 \text{ and } t_2 \iff d_{FX}(t_1, F\pi_1(t)) + d_{FY}(F\pi_2(t), t_2) \leq \epsilon$$

"Distributorial"?

$W_\lambda(d)$ is compatible with d_{FX} and d_{FY}:

$$d_{FY} \cdot W_\lambda(d) = W_\lambda(d) \cdot d_{FX} = W_\lambda(d)$$

Viewing d_{FX}, d_{FY} as categories (à la Lawvere), $W_\lambda(d)$ is a distributor/profunctor

Conditions on F and λ

Preservation of exact squares

$$\begin{array}{ccc} P & \xrightarrow{v} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} FP & \xrightarrow{Fv} & FY \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle Fg} \\ FX & \xrightarrow[Ff]{} & FZ \end{array}$$

$$g^\circ \cdot f \geq v \cdot u^\circ \quad \Longrightarrow \quad (Fg)^\circ \cdot d_{FZ} \cdot Ff \geq d_{FY} \cdot Fv \cdot (Fu)^\circ \cdot d_{FX}$$

Conditions on λ

Monotonicity, preservation of 0, $\mathcal{V}$-subadditivity

Simulations

Also works in the two-valued setting or for a quantale $\mathcal{V}$:

$[0, \infty]$	$\{\bot, \top\}$	$\mathcal{V}$
pseudometric	preorder	$\mathcal{V}$-category
behavioural distance	behavioural preorder	behavioural conformance

Examples in the two-valued setting:

- Egli-Milner simulation (forward and backward)
- Ready and complete simulation (forward and backward)
- Refinement for modal transition systems

Conclusions

Results

- New method of defining behavioural preorders and metrics
- Abstraction over type of system and type of conformance
- Conditions under which this construction works

Future work

- Combine distances in other ways than with $+$
- Different preorders/distances on either side
- Non-pseudometric distances, e.g. divergences