Names and Substitution

Theme and Variations

Fabian Lenke,

joint work with Henning and Stefan

Oberseminar 13.05.2025

[This talk and the material it contains was created without aid of LLMs]

What are variables?

$$(x\cdot y)^{-1} = y^{-1}\cdot x^{-1}$$

$$\downarrow$$

$$(abc)^{-1} = c^{-1}b^{-1}a^{-1}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$S \to aAa$

$A \to$

$$\int_a^b xxy\,dy = \int_a^b xxy\,dy \overset{[y/x]}{\neq} \int_a^b yyy\,dy$$

$$(\lambda x.xy)(\lambda a.aa) \to_\beta (\lambda a.aa)y \to_\beta yy$$

What are variables?

$$(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$$

$$\downarrow$$

$$(abc)^{-1} = c^{-1} b^{-1} a^{-1}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$S \to aAa$

$A \to$

$$\int_a^b xxy\,dy = \int_a^b xxy\,dy \overset{[y/x]}{\neq} \int_a^b yyy\,dy$$

$$(\lambda x. xy)(\lambda a. aa) \to_\beta (\lambda a. aa) y \to_\beta yy$$

- Variables are "placeholders"

↳ variable name does $\wedge$ not matter ⇝ renaming should be allowed

(almost: avoid capture)

What are variables?

$$(x\cdot y)^{-1} = y^{-1}\cdot x^{-1}$$

$$\downarrow$$

$$(abc)^{-1} = c^{-1}b^{-1}a^{-1}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$S \to aAa$

$A \to$

$$\int_a^b xxy\,dy = \int_a^b xxy\,dy \neq {}^{[y/x]}\!\int_a^b yyy\,dy$$

$$(\lambda x.\,xy)(\lambda a.\,aa) \to_\beta (\lambda a.\,aa)y \to_\beta yy$$

- Variables are "placeholders"

↳ variable name does not matter ⇒ renaming should be allowed

(almost: avoid capture)

- Algebra: substitute variables by complex terms

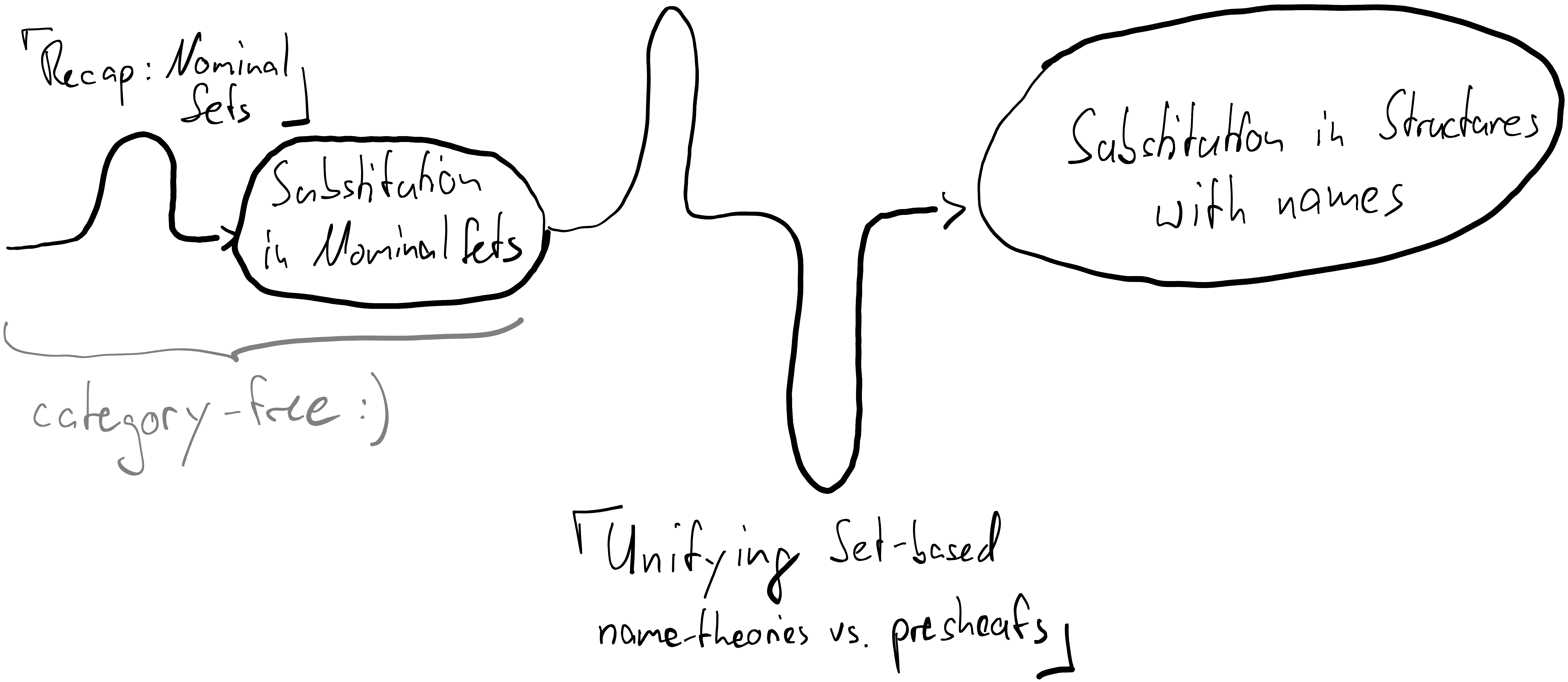
↳ abstraction + substitution ≙ operational semantics of FP languages

↳ again, be careful to avoid variable capture

Mathmatical Models of Variables and Substitution

Outline:

⌜Presheaf calculus⌟



⌜Recap: Nominal Sets⌟

Substitution in Nominal Sets

category-free :)

⌜Unifying Set-based name-theories vs. presheafs⌟

Substitution in Structures with names

Nominal Sets: Fix "infinite" $\mathbb{A}$ of "atoms"/"variables"/"names"/"urelements"

⌜ "Definition": Set X such that

↳ every $x \in X$ "**depends**" on finitely many names

↳ closed under swapping $\begin{pmatrix} a \mapsto b \\ b \mapsto a \end{pmatrix}$ of names. ⌟

Nominal Sets: Fix "infinite" $\mathbb{A}$ of "atoms"/"variables"/"names"/"urelements"

"Definition": Set X such that

↳ every $x \in X$ "**depends**" on finitely many names

↳ closed under swapping $\begin{pmatrix} a \mapsto b \\ b \mapsto a \end{pmatrix}$ of names.

Examples:

✓	✗
$\mathbb{A}^{*2} = \{(a,b) \mid a \neq b\}$	$\{a, b, c\}$ ← supported set
λ-terms, e.g. $\lambda(a \in \mathbb{A}).ab$	$\mathbb{A}^{\mathbb{N}}$ ← Perm$\mathbb{A}$-set

Nominal Sets: Fix "infinite" $\mathbb{A}$ of "atoms"/"variables"/"names"/"urelements"

"**Definition**": Set X such that

↳ every $x \in X$ "**depends**" on finitely many names

↳ closed under swapping $\begin{pmatrix} a \mapsto b \\ b \mapsto a \end{pmatrix}$ of names.

Examples:

✓

$\mathbb{A}^{*2} = \{(a,b) \mid a \neq b\}$

λ-terms, e.g. $\lambda(a \in \mathbb{A}).\, ab$

✗

$\{a, b, c\}$ ← supported set

$\mathbb{A}^{\mathbb{N}}$ ← Perm$\mathbb{A}$-set

Intuition: terms over finitary syntax + permutation symmetries

↳ e.g. $\dfrac{\{f(a,b,c) \mid \overset{\times}{a} \overset{b}{\neq} \overset{\times}{c}\}}{f(a,b,c) = f(c,b,a)} \quad \dot{\cup} \quad \{a+b \mid a \neq b\}$

Nominal Sets:

- **Morphisms**: can forget names and compare for (in-)equality:

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A} + 1$$

✓ $(a,b) \mapsto \begin{cases} 1 & a = b \\ b & \text{else} \end{cases}$

✗: $(a,b) \mapsto \begin{cases} b & a = c \\ d & \text{else} \end{cases}$

Nominal Sets:

- Morphisms: can forget names and compare for (in-)equality:

$$\mathbb{A}\times\mathbb{A} \to \mathbb{A}+1$$

✓ $(a,b) \mapsto \begin{cases} 1 & a=b \\ b & \text{else} \end{cases}$

✗: $(a,b) \mapsto \begin{cases} b & a=c \\ d & \text{else} \end{cases}$

↳ Category Nom with rich structure

function space Y^X

powerset construction

fresh/separated product: $X * Y$

"independent" elements (no shared names)

abstraction $[\mathbb{A}]X$

for fresh b: $\lambda a.x = \lambda b.x$

$\mathbb{A} * [\mathbb{A}]X \to X$ "capture-avoiding evaluation"

$(a, \lambda b.t) \mapsto (a\,b)\cdot t$

Nominal Sets:

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A} + 1$$

✓ $(a,b) \mapsto \begin{cases} 1 & a=b \\ b & \text{else} \end{cases}$

✗: $(a,b) \mapsto \begin{cases} b & a=c \\ d & \text{else} \end{cases}$

- **Morphisms**: can forget names and compare for (in-)equality:

↳ Category Nom with rich structure

function space Y^X

powerset construction

abstraction $[\mathbb{A}]X$

for fresh b: $\lambda a.x = \lambda b.x$

fresh/separated product: $X * Y$

"independent" elements (no shared names)

$\mathbb{A} * [\mathbb{A}]X \to X$ "capture-avoiding evaluation"

$(a, \lambda b.t) \mapsto (a\,b) \cdot t$

↳ We have syntax over variables, so what about "algebra-like" term substitution as in

$$\mathcal{P}_f \mathbb{A} \times (\mathbb{A} \to \mathcal{P}_f \mathbb{A}) \to \mathcal{P}_f \mathcal{P}_f \mathbb{A} \to \mathcal{P}_f \mathbb{A}$$

$$(A,\ \lambda a.\, B_a) \mapsto \{B_a \mid a \in A\} \mapsto \bigcup_{a \in A} B_a$$

Nominal Sets: Substitution Structure.

Def: $$X \Diamond Y = \{ (x \in X, \gamma : \mathrm{supp}\, x \to Y) \mid \forall a,b : \underbrace{\gamma a \# \gamma b}_{\text{no shared names}} \} / \underbrace{(\pi \cdot x, \gamma) \sim (x, \gamma\pi)} \text{ for } \pi : \mathbb{A} \cong \mathbb{A}$$

$$\begin{tikzcd}
\mathrm{supp}\, x \arrow[r, "\pi"] \arrow[dr, "\gamma\pi"'] & \mathrm{supp}(\pi \cdot x) \arrow[d, "\gamma"] \\
 & Y
\end{tikzcd}$$

↳ equivalence classes $x[\gamma]$

↳ action: $\pi \cdot x[\gamma] = x[\lambda a.\ \pi \cdot \gamma(a)]$

Nominal Sets: Substitution Structure.

Def: $X \lozenge Y = \{ (x \in X, \gamma : \mathrm{supp}\, x \to Y) \mid \forall_{a,b} : \underbrace{\gamma a \# \gamma b}_{\text{no shared names}} \} / \underbrace{(\pi \cdot x, \gamma) \sim (x, \gamma \pi)}$ for $\pi : \mathbb{A} \cong \mathbb{A}$

$$\begin{tikzcd}
\mathrm{supp}\, x \arrow[r, "\pi"] \arrow[dr, "\gamma\pi"'] & \mathrm{supp}(\pi \cdot x) \arrow[d, "\gamma"] \\
& Y
\end{tikzcd}$$

↳ equivalence classes $x[\gamma]$

↳ action: $\pi \cdot x[\gamma] = x[\lambda a.\ \pi \cdot \gamma(a)]$

Theorem:

- $X \lozenge Y$ is a nominal set and $\mathrm{supp}\, x[\gamma] = \mathrm{supp}\, \gamma = \bigcup_{a \in \mathrm{supp}\, x} \gamma a$
- $\lozenge$ is monoidal: $X \lozenge (Y \lozenge Z) \cong (X \lozenge Y) \lozenge Z$, $\mathbb{A} \lozenge Y \cong Y \cong Y \lozenge \mathbb{A}$, (note: $X \lozenge Y \neq Y \lozenge X$ almost always)
- $(X * X') \lozenge Y \cong (X \lozenge Y) * (X' \lozenge Y)$, $(X + X') \lozenge Y \cong X \lozenge Y + X' \lozenge Y$

Nominal Sets: Substitution Structure.

Def: $X \diamond Y = \Big\{ (x \in X, \gamma : \operatorname{supp} x \to Y) \;\Big|\; \forall a,b : \underbrace{\gamma a \# \gamma b}_{\text{no shared names}} \Big\} \Big/ \underbrace{(\pi \cdot x, \gamma) \sim (x, \gamma\pi)}$ for $\pi : \mathbb{A} \cong \mathbb{A}$

$$\begin{tikzcd}
\operatorname{supp} x \arrow[r, "\pi"] \arrow[dr, "\gamma\pi"'] & \operatorname{supp}(\pi \cdot x) \arrow[d, "\gamma"] \\
 & Y
\end{tikzcd}$$

↳ equivalence classes $x[\gamma]$

↳ action: $\pi \cdot x[\gamma] = x[\lambda a.\, \pi \cdot \gamma(a)]$

Theorem:

- $X \diamond Y$ is a nominal set and $\operatorname{supp} x[\gamma] = \operatorname{supp} \gamma = \bigcup_{a \in \operatorname{supp} x} \gamma a$
- $\diamond$ is monoidal: $X \diamond (Y \diamond Z) \cong (X \diamond Y) \diamond Z$, $\mathbb{A} \diamond Y \cong Y \cong Y \diamond \mathbb{A}$, (note: $X \diamond Y \neq Y \diamond X$ almost always)
- $(X * X') \diamond Y \cong (X \diamond Y) * (X' \diamond Y)$, $(X + X') \diamond Y \cong X \diamond Y + X' \diamond Y$

⌜ Note: there also exists an "unfresh version", but it is not monoidal!

$$X \hat{\diamond} Y = \{ (x \in X, \gamma : \operatorname{supp} x \to Y) \} /_{\sim}$$ ⌟

Nominal Sets: Substitution Structure.

Def: $$X \diamond Y = \{ (x \in X, \gamma : \mathrm{supp}\, x \to Y) \mid \forall a,b : \underbrace{\gamma a \# \gamma b}_{\text{no shared names}} \} / (\pi \cdot x, \gamma) \sim (x, \gamma \pi) \quad \text{for } \pi : \mathbb{A} \cong \mathbb{A}$$

Def: $$Y \multimap\!\!\diamond\, Z = \{ f : Y^{*\mathbb{A}} \to Z \mid f \text{ equivariant} + \text{almost constant} \}$$

$\forall a,b : \gamma a \# \gamma b$

$$\underbrace{\text{И}a : f(\ldots, \gamma_{a_i}, \gamma_a, \gamma_{a_j}) = f(\ldots, \gamma_{a_i}, \gamma'_a, \gamma_{a_i})}_{}$$

f uses only finitely many components:

$\iff$ factors through a finite projection:

$$\exists B \subseteq_f \mathbb{A}: \quad \begin{tikzcd} Y^{*\mathbb{A}} \arrow[rr, "f"] \arrow[dr] & & Z \\ & Y^{*B} \arrow[ur] & \end{tikzcd}$$

$$\left(\text{for } \pi : \mathbb{A} \cong \mathbb{A} : f(\pi \cdot \gamma_a)_{a \in \mathbb{A}} = \pi \cdot f(\gamma_a)_{a \in \mathbb{A}} \right)$$

Nominal Sets: Substitution Structure.

Def: $X \diamond Y = \{ (x \in X, \gamma : \operatorname{supp} x \to Y) \mid \forall a,b : \underbrace{\gamma a \# \gamma b}_{\text{no shared names}} \} / (\pi \cdot x, \gamma) \sim (x, \gamma\pi)$ for $\pi : \mathbb{A} \cong \mathbb{A}$

Def: $Y \mathrel{-\!\!\diamond} Z = \{ f : Y^{*\mathbb{A}} \to Z \mid f$ equivariant + almost constant$\}$

$\forall a,b : \gamma a \# \gamma b$

$\left(\text{for } \pi : \mathbb{A} \cong \mathbb{A} : \; f(\pi \cdot \gamma_a)_{a \in \mathbb{A}} = \pi \cdot f(\gamma_a)_{a \in \mathbb{A}} \right)$

$\underbrace{\text{И} a : \; f(\ldots, \gamma_{a_i}, \gamma_a, \gamma_{a_j}) = f(\ldots, \gamma_{a_i}, \gamma'_a, \gamma_{a_i})}_{}$

f uses only finitely many components:

$\Leftrightarrow$ factors through a finite projection:

$\exists B \subseteq_f \mathbb{A}$:

$$\begin{tikzcd}
Y^{*\mathbb{A}} \arrow[rr, "f"] \arrow[dr] & & Z \\
& Y^{*B} \arrow[ur] &
\end{tikzcd}$$

Theorem:

- $Y \mathrel{-\!\!\diamond} Z$ is nominal for $\pi \cdot f = \lambda(\gamma_a)_a . f(\gamma_{\pi a})_a \Rightarrow \operatorname{supp} f = \{ a \mid f \text{ not constant in } a \}$
- $\diamond$ is right closed: $\dfrac{X \diamond Y \to Z}{X \to (Y \mathrel{-\!\!\diamond} Z)}$ in Nom.

Nominal Sets: λ-Calculus: operational model:

λ-terms: initial algebra for $\underline{\Lambda} X = A + X * X + [A]X$

affine: use var at most once,
(*not:* $\lambda a.aa$)

Nominal Sets: λ-Calculus: operational model:

λ-terms: initial algebra for $\Lambda X = \mathbb{A} + X * X + [\mathbb{A}]X$

affine: use var at most once,
(not: $\lambda a.aa$)

Substitution $\hat{=}$ β-reduction

$$\underbrace{\mu\Lambda \diamond \mu\Lambda \longrightarrow \mu\Lambda}_{\text{substitution by structural recursion}}$$

base case: $\mathbb{A} \diamond \mu\Lambda \cong \mu\Lambda$

$$[\mathbb{A}](\mu\Lambda) * \mu\Lambda \longrightarrow \mu\Lambda$$

$$\lambda a.s \, , \; t \longmapsto s[{}^{t}/_{a}]$$

Nominal Sets: λ-Calculus: operational model:

λ-terms: initial algebra for $\Lambda X = \mathbb{A} + X * X + [\mathbb{A}]X$

affine: use var at most once,
(not: $\lambda a.aa$)

Substitution ≙ β-reduction

$$\underbrace{\mu\Lambda \diamond \mu\Lambda \longrightarrow \mu\Lambda}_{\text{substitution by structural recursion}}$$

base case: $\mathbb{A} \diamond \mu\Lambda \cong \mu\Lambda$

$$[\mathbb{A}](\mu\Lambda) * \mu\Lambda \longrightarrow \mu\Lambda$$

$$\lambda a.s\ ,\quad t \longmapsto s[t/a]$$

more theory: [📖 Fiore, Turi, Plotkin LICS'99]
(and many more)

New here: Set-based models! (Instead of Presheaf-based)

Nominal Sets: Theme and Variations

$\ulcorner$ $\underline{\mathrm{Nom}}$ [📖 Gabbay Pitts '02] $\lrcorner$

Nominal Sets: Theme and Variations

⌜ **Nom** [📖 Gabbay / Pitts '02] ⌟

⌜ **Ren** [📖 Gabbay / Hofmann '08]

- Ob: Renaming sets
  ↳ finite supports + closure under $\mathbb{A} \to \mathbb{A}$ renaming
- Mor: equivariant

⌜ rediscovered every n years:
Staton '08, Gabbay '09, ...
Popescu '22, Pitts '23 ⌟

Nominal Sets: Theme and Variations

SuppSet [📖 Wißmann '23]

- Ob: no renamings
  ↳ just supports: $X \xrightarrow{s} \mathcal{P}_f\mathbb{A}$
- Mor: $s(fx) \subseteq sx$

Nom [📖 Gabbay/Pitts '02]

Ren [📖 Gabbay/Hofmann '08]

- Ob: Renaming Sets
  ↳ finite supports +
  closure under $\mathbb{A} \to \mathbb{A}$ renaming
- Mor: equivariant

rediscovered every n years:
Staton '08, Gabbay '09, ...
Popescu '22, Pitts '23

Nominal Sets: Theme and Variations

SuppSet [📖 Wißmann '23]

- Ob: no renamings
  - ↳ just supports: $X \xrightarrow{s} \mathcal{P}_f \mathbb{A}$
- Mor: $s(fx) \subseteq sx$

Nom [📖 Gabbay / Pitts '02]

$\mathrm{Nom}_=$ "linear Nom" (here)

- Ob: Nom
- Mor: equiv. + supp-pres
  - $\mathrm{supp}(fx) = \mathrm{supp}\, x$
  - no: $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$

⇝ different limits

RelRen (here)

- Ob: "Relevant" renaming sets
  - $\mathrm{supp}(g \cdot x) = g \cdot \mathrm{supp}\, x$
  - no: $\mathbb{A} * \mathbb{A}$
  - no: $\mathbb{A} * \mathbb{A} + 1$
- Mor: equivariant

Ren [📖 Gabbay / Hofmann '08]

- Ob: Renaming sets
  - ↳ finite supports + closure under $\mathbb{A} \to \mathbb{A}$ renaming
- Mor: equivariant

rediscovered every n years:
Staton '08, Gabbay '09, ...
Popescu '22, Pitts '23

Nominal Sets: Theme and Variations

SuppSet [Wißmann '23]

- Ob: no renamings
  - ↳ just supports: $X \xrightarrow{s} \mathcal{P}_f\mathbb{A}$
- Mor: $s(fx) \subseteq sx$

SuppSet$_=$ (here)

- Ob: SuppSets
- Mor: supp-pres

Nom [Gabbay/Pitts '02]

Nom$_=$ "linear Nom" (here)

- Ob: Nom
- Mor: equiv. + supp-pres

$\mathrm{supp}(fx) = \mathrm{supp}\,x$

no: $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$

⇝ different limits

RelRen$_=$ (here)

- Ob: RelRen
- Mor: equiv + supp-pres

RelRen (here)

- Ob: "Relevant" renaming sets
  - $\mathrm{supp}(g \cdot x) = g \cdot \mathrm{supp}\,x$
  - no: $\mathbb{A} * \mathbb{A}$
  - no: $\mathbb{A} * \mathbb{A} + 1$
- Mor: equivariant

Ren [Gabbay/Hofmann '08]

- Ob: Renaming sets
  - ↳ finite supports + closure under $\mathbb{A} \to \mathbb{A}$ renaming
- Mor: equivariant

rediscovered every n years:
Staton '08, Gabbay '09, …
Popescu '22, Pitts '23

Nominal Sets: Theme and Variations

SuppSet [Wißmann '23]

- Ob: no renamings
  ↳ just supports: $X \xrightarrow{s} \mathcal{P}_f \mathbb{A}$
- Mor: $s(fx) \subseteq sx$

SuppSet$_=$ (here)

- Ob: SuppSets
- Mor: supp-pres

For which does substitution exist?

Nom [Gabbay/Pitts '02]

Nom$_=$ "linear Nom" (here)

- Ob: Nom
- Mor: equiv. + supp-pres

$\mathrm{supp}(fx) = \mathrm{supp}\, x$

no: $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$

⇝ different limits

RelRen$_=$ (here)

- Ob: RelRen
- Mor: equiv + supp-pres

RelRen (here)

- Ob: "Relevant" renaming sets
  $\mathrm{supp}(g \cdot x) = g \cdot \mathrm{supp}\, x$
  no: $\mathbb{A} * \mathbb{A}$
  no: $\mathbb{A} * \mathbb{A} + 1$
- Mor: equivariant

Ren [Gabbay/Hofmann '08]

- Ob: Renaming sets
  ↳ finite supports + closure under $\mathbb{A} \to \mathbb{A}$ renaming
- Mor: equivariant

rediscovered every n years:
Staton '08, Gabbay '09, ...
Popescu '22, Pitts '23

Substitution originally: on Presheafs $\in \mathsf{Set}^{\mathbb{F}}$ [📖 Fiore, Turi, Plotkin LICS'99]

Idea: Instead of $X \xrightarrow{supp} \mathcal{P}_f\mathbb{A}$ (what names does X use?)

Give $\mathcal{P}_f\mathbb{A}_{sub} \xrightarrow{F} \mathsf{Set}$ with $FA \mathrel{\hat{=}} \{x \in X \mid x \text{ contains only names from } A \subseteq_f \mathbb{A}\}$

Substitution originally: on Presheafs $\in Set^{\mathbb{F}}$ [📖 Fiore, Turi, Plotkin LICS'99]

Idea: Instead of $X \xrightarrow{supp} \mathcal{P}_f \mathbb{A}$ (what names does X use?)

Give $\mathcal{P}_f \mathbb{A}_{sub} \xrightarrow{F} Set$ with $FA \hat{=} \{ x \in X \mid x \text{ contains only names from } A \subseteq_f \mathbb{A} \}$

"OG" substitution

(operational model for λ-calc)

versions + generalizations

↳ Tanaka '01: linear λ-calc

↳ Power 03: generalization

↳ Fiore 2001-2007

⇝ Relation to nominal approach?

We need to talk about presheafs...

⌜This is the preparation for people interested in formulas⌟

⌜Def: $\mathcal{C}$ category (small). $\mathrm{PSh}\,\mathcal{C} = \mathrm{Set}^{\mathcal{C}}$ is the category of (covariant) presheaves⌟

We need to talk about presheafs...

⌜This is the preparation for people interested in formulas⌟

⌜Def: $\mathcal{C}$ category (small). $\mathrm{PSh}\,\mathcal{C} = \mathrm{Set}^{\mathcal{C}}$ is the category of (covariant) presheaves⌟

Intuitions

- "generalized sets"
- generalized/typed monoid actions
  $\mathcal{C} = \mathcal{M} \rightsquigarrow \mathrm{PSh}\,\mathcal{C} \cong \mathcal{M}\text{-Set}$

We need to talk about presheafs...

⌜This is the preparation for people interested in formulas⌟

⌜Def: $\mathcal{C}$ category (small). $\mathrm{PSh}\,\mathcal{C} = \mathrm{Set}^{\mathcal{C}}$ is the category of (covariant) presheaves⌟

Intuitions

- "generalized sets"
- generalized/typed monoid actions
  $\mathcal{C} = M \rightsquigarrow \mathrm{PSh}\,\mathcal{C} \cong M\text{-Set}$

Rich Structure

- (even if $\mathcal{C}$ does not have any)
  (free cocompletion of $\mathcal{C}$)
- inherits structure from $\mathcal{C}$
  (e.g. Day convolution)

We need to talk about presheafs...

⌜This is the preparation for people interested in formulas⌟

⌜Def: $\mathcal{C}$ category (small). $\mathrm{PSh}\,\mathcal{C} = \mathrm{Set}^{\mathcal{C}}$ is the category of (covariant) presheaves⌟

Intuitions

- "generalized sets"
- generalized/typed monoid actions
  $\mathcal{C} = M \rightsquigarrow \mathrm{PSh}\,\mathcal{C} \cong M\text{-Set}$

"Roof" for different fields

- Set theory/topology
  (Grothendieck toposes: left exact refl. subcat)
- Algebra
  (Lfp-categories: refl. subcat w. filtered colimits)

Rich Structure

- (even if $\mathcal{C}$ does not have any)
  (free cocompletion of $\mathcal{C}$)
- inherits structure from $\mathcal{C}$
  (e.g. Day convolution)

We need to talk about presheafs...

⌜This is the preparation for people interested in formulas⌟

⌜Def: $\mathcal{C}$ category (small). $\mathrm{PSh}\,\mathcal{C} = \mathrm{Set}^{\mathcal{C}}$ is the category of (covariant) presheaves⌟

Intuitions

- "generalized sets"
- generalized/typed monoid actions
  $\mathcal{C} = M \rightsquigarrow \mathrm{PSh}\,\mathcal{C} \cong M\text{-}\mathrm{Set}$

"Roof" for different fields

- Set theory/topology
  (Grothendieck toposes: left exact refl. subcat)
- Algebra
  (lfp-categories: refl. subcat w. filtered colimits)

Rich Structure

- (even if $\mathcal{C}$ does not have any)
  (free cocompletion of $\mathcal{C}$)
- inherits structure from $\mathcal{C}$
  (e.g. Day convolution)

Notation: type inference

- For $F \in \mathrm{PSh}\,\mathcal{C}$, $f \in \mathcal{C}(c,d)$, $x \in Fc$: ⌜$f.x := F(f)(x)$⌟
  ↳ also if $F: \mathcal{C}^{op} \to \mathrm{Set}$: $g \in \mathcal{C}(d,c)$, $x \in Fc$: $x.g = F(g)(x)$
  ↳ $\alpha: F \Rightarrow G$ natural, $x \in FA$: $\alpha(x) := \alpha_A(x)$

↳ $y := \mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$

Some more category theory:

⌜ Co/Powers: Let $S \in \mathrm{Set}$

(generalizing "S copies" and "S-indexed family")

$S \cdot c \hat{=} \coprod_S c$ / $S \pitchfork c \hat{=} c^S$ (iso if S co/complete)

formally: adjoints to $yc = \mathcal{C}(c,-)$, $y^{(-)}c = \mathcal{C}(-,c)$ ⌟

$\rightsquigarrow$ generalize co/limits

Some more category theory:

Co/Powers: Let $S \in Set$

("generalizing "S copies" and "S-indexed family")

$S \cdot c \hat{=} \coprod_S c$ / $S \pitchfork c \hat{=} c^S$ (iso if S co/complete)

formally: adjoints to $yc = \mathcal{C}(c,-)$, $y^{(-)}c = \mathcal{C}(-,c)$

→ generalize co/limits

Co/Ends: Let $H : \mathcal{C}^{op} \times \mathcal{C} \to Set$

(generalizing co/limits)

↳ recall $\lim(F : \mathcal{C} \to Set)$ = compatible F-families $\{(x_c \in Fc)_{c \in \mathcal{C}} \mid f \cdot x_c = x_d \text{ for } f : c \to d\}$

$\operatorname{colim}(F : \mathcal{C} \to Set)$ = F-sum modulo F-orbits $\left(\coprod_c Fc\right) / _{f \cdot x = x}$

Some more category theory:

⌈ **Co/Powers:** Let $S \in Set$

(generalizing "S copies" and "S-indexed family")

$S \cdot c \mathrel{\hat{=}} \coprod_S c$ / $S \pitchfork c \mathrel{\hat{=}} c^S$ (iso if S co/complete)

formally: adjoints to $y c = \mathcal{C}(c,-)$, $y^{(-)}c = \mathcal{C}(-,c)$ ⌋

→ generalize co/limits ↗

⌈ **Co/Ends:** Let $H : \mathcal{C}^{op} \times \mathcal{C} \to Set$

(generalizing co/limits)

↳ recall $\lim(F : \mathcal{C} \to Set)$ = compatible F-families $\{(x_c \in Fc)_{c \in \mathcal{C}} \mid f \cdot x_c = x_d \text{ for } f : c \to d\}$

$\operatorname{colim}(F : \mathcal{C} \to Set)$ = F-sum modulo F-orbits $\left(\coprod_c Fc\right)/_{f \cdot x = x}$

- End: $\int_c H(c,c)$: two-sided compatible H-families $\{(x_c \in H(c,c))_{c \in \mathcal{C}} \mid \overset{H(c,d)}{\overset{\ni \qquad \ni}{f \cdot x_c = x_d \cdot f}} ,\ f : c \to d\}$
- Coend: $\int^c H(c,c)$: H-diagonal modulo H-relations $\left(\coprod_c H(c,c)\right) \Big/_{\underbrace{r \cdot f}_{\overset{\uparrow}{H(d,d)}} = f \cdot \underset{\overset{\uparrow}{H(c,c)}}{r}}$ $\begin{matrix} f : c \to d \\ r \in H(d,c) \end{matrix}$

Some more category theory:

⌜Presheaf calculus looks complicated, but is easy; only one Theorem⌟ (and it's a lemma)

$\boxed{\text{Yoneda Lemma:}}$ For $F \in PSh\,\mathcal{C}$, $c \in \mathcal{C}$:

$$\int_d \mathrm{Set}(yc(d), Fd) \overset{(1)}{\cong} \mathrm{Nat}(yc, F) \overset{(2)}{\cong} Fc \overset{(3)}{\cong} \int^b yb(c) \times Fb$$

⌜categorical version of the fact that (M, e) is the initial M-automaton⌟

Some more category theory:

(and it's a lemma)

⌈Presheaf calculus looks complicated, but is easy; only one Theorem⌋

$\boxed{\text{Yoneda Lemma:}}$ For $F \in PSh\,\mathcal{C}$, $c \in \mathcal{C}$:

$$\int_d \mathrm{Set}(yc(d), Fd) \overset{(1)}{\cong} \mathrm{Nat}(yc, F) \overset{(2)}{\cong} Fc \overset{(3)}{\cong} \int^b yb(c) \times Fb$$

⌈categorical version of the fact that (M,e) is the initial M-automaton⌋

Proof: (2) reorganizing of the functor structure: $\forall c\, \forall d\, \forall f : c \to d\, \forall x \in Fc : f \cdot x \in Fd$

$$\hookrightarrow \forall c.\ \forall x \in Fc : \overbrace{[\forall d.\ \forall f \in yc(d).\ f \cdot x \in Fd]}^{\alpha_x : yc \Rightarrow F}$$

(1) (more generally $\mathrm{Nat}(H,K) \cong \int_c \mathcal{D}(Hc, Kc)$ for $H, K : \mathcal{C} \to \mathcal{D}$)

(3) evaluation: $\mathcal{C}(b,c) \times Fb \to Fc$ and its properties

$$f\ ,\ x \mapsto f \cdot x$$

Some more category theory: Extending ....

Functors to Presheaves:

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{E} \\
\mathrm{PSh}\,\mathcal{C} \arrow[ur, dotted, "?"'] &
\end{tikzcd}$$

canonical solutions?

approximating "from the left" ($\mathcal{E}$ cocomplete)

$$\mathrm{Lan}\,F : \mathrm{PSh}\,\mathcal{C} \to \mathcal{E}$$

$$X \mapsto \int^{c} Fc \cdot Xc$$

↳ Lan F restricts to $\mathcal{C}^{op}$: $\mathrm{Lan}F(yc) \cong Fc$

Yoneda (3)

Some more category theory: Extending ....

Functors to Presheaves:

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{E} \\
\mathrm{PSh}\,\mathcal{C} \arrow[ur, dotted, "?"'] &
\end{tikzcd}$$

canonical solutions?

approximating "from the left" ($\mathcal{E}$ cocomplete)

$$\mathrm{Lan}\,F : \mathrm{PSh}\,\mathcal{C} \to \mathcal{E}$$
$$X \mapsto \int^{c} Fc \cdot Xc$$

↳ Lan F restricts to $\mathcal{C}^{op}$: $\mathrm{Lan}F(yc) \cong Fc$

Yoneda (3)

Presheaves themselves:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "K"'] & \mathrm{Set} \\
\mathcal{D} \arrow[ur, dotted, "?"'] &
\end{tikzcd}$$

approximating ? "from the left": $\hat{F} d = \int^{c} \mathcal{D}(Kc, d) \cdot Gc = \dfrac{\{[f : Kc \to d, x \in Gc]\}}{[f \cdot Kg, x] = [f, g \cdot x]}$

... universal properties etc. included

Some more category theory: Day convolution

Let M be a monoid

↳ $\mathcal{P}M$ monoid s.t.

- $M \to \mathcal{P}M$ monoid hom
- $\mathcal{P}M \times \mathcal{P}M \xrightarrow{\bullet} \mathcal{P}M$ $\mathcal{P}$-bilinear

$$\left(\bigcup_i A_i\right) \bullet \left(\bigcup_j B_j\right) = \bigcup_{ij} A_i \cdot B_j$$

$\Rightarrow$ determines $\bullet$ uniquely

$A \cdot B = \{ab \mid a \in A, b \in B\}$

Some more category theory: Day convolution

Let M be a monoid

↳ $\mathcal{P}M$ monoid s.t.

- $\cdot$ $M \to \mathcal{P}M$ monoid hom
- $\cdot$ $\mathcal{P}M \times \mathcal{P}M \xrightarrow{\bullet} \mathcal{P}M$ $\mathcal{P}$-bilinear

$$\left(\bigcup_i A_i\right) \bullet \left(\bigcup_j B_j\right) = \bigcup_{ij} A_i \cdot B_j$$

⇒ determines $\bullet$ uniquely

$$A \cdot B = \{ab \mid a \in A, b \in B\}$$

↳ $\mathcal{D}X$ monoid s.t.

- $\cdot$ $M \to \mathcal{D}M$ monoid hom
- $\cdot$ $\mathcal{D}M \times \mathcal{D}M \xrightarrow{\bullet} \mathcal{D}M$ biaffine

$$\left(\sum_i r_i m_i\right) \bullet \left(\sum_j s_j n_j\right) = \sum_{ij} r_i s_j \, m_i n_j$$

⇒ determines $\bullet$ uniquely

↳ $\mathbb{R}^{(X)}$ monoid

...

Some more category theory: Day convolution

Let $\mathcal{M}$ be a monoid

↳ $\mathcal{P}\mathcal{M}$ monoid s.t.

- $\mathcal{M} \to \mathcal{P}\mathcal{M}$ monoid hom
- $\mathcal{P}\mathcal{M} \times \mathcal{P}\mathcal{M} \xrightarrow{\cdot} \mathcal{P}\mathcal{M}$ $\mathcal{P}$-bilinear

$$\left(\bigcup_i A_i\right) \cdot \left(\bigcup_j B_j\right) = \bigcup_{ij} A_i \cdot B_j$$

⇒ determines $\cdot$ uniquely

$$A \cdot B = \{ab \mid a \in A, b \in B\}$$

↳ $\mathcal{D}X$ monoid s.t.

- $\mathcal{M} \to \mathcal{D}\mathcal{M}$ monoid hom
- $\mathcal{D}\mathcal{M} \times \mathcal{D}\mathcal{M} \xrightarrow{\cdot} \mathcal{D}\mathcal{M}$ biaffine

$$\left(\sum_i r_i m_i\right) \cdot \left(\sum_j s_j n_j\right) = \sum_{ij} r_i s_j \, m_i n_j$$

⇒ determines $\cdot$ uniquely

↳ $\mathbb{R}^{(X)}$ monoid

…

Categorical version: Let $(\mathcal{C}, \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}, e \in \mathcal{C})$ be monoidal: $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$, $e \otimes c \cong c \cong c \otimes e$

⇝ $\mathrm{PSh}\,\mathcal{C}$ monoidal via $(\mathrm{PSh}\,\mathcal{C} \times \mathrm{PSh}\,\mathcal{C} \xrightarrow{*} \mathrm{PSh}\,\mathcal{C}, \; ye \in \mathrm{PSh}\,\mathcal{C})$

- $y: \mathcal{C}^{op} \to \mathrm{PSh}\,\mathcal{C}$ "homomorphism" (strong monoidal)
- $*$ is "$\mathrm{PSh}\,\mathcal{C}$-bimorphism" (cocont. in both args)

determines

$$F * G = \int^{c,d \in \mathcal{C}} (Fc \times Gd) \cdot y(c \otimes d)$$

Presheaf Presentations:

well-known: Nom $\simeq$ Schanuel

- $\mathcal{C} = \mathbb{I}$: finite sets (or $\mathcal{P}_f \mathbb{A}$)
  with injective maps (renamings)

- $\mathrm{Sh}\,\mathbb{I} \subseteq \mathrm{PSh}\,\mathbb{I}$: intersection-preserving presheaves

  ↳ every "element" is contained in a smallest $A \in \mathbb{I}$ w.r.t. inclusions $\rightsquigarrow$ support

Presheaf Presentations:

well-known: Nom $\simeq$ Schanuel

- $\mathcal{C} = \mathbb{I}$: finite sets (or $\mathcal{P}_f \mathbb{A}$) with injective maps (renamings)
- $\mathrm{Sh}\,\mathbb{I} \subseteq \mathrm{PSh}\,\mathbb{I}$: intersection-preserving presheaves
  - ↳ every "element" is contained in a smallest $A \in \mathbb{I}$ w.r.t. inclusions ⇝ support

$$\mathrm{Nom} \simeq \mathrm{Sh}\,\mathbb{I} \subseteq \mathrm{PSh}\,\mathbb{I}$$

$$X \xmapsto{I_*} I_*X : \mathbb{I} \to \mathrm{Set}$$

$$A \mapsto \{x \mid \mathrm{supp}\, x \subseteq A\}$$

$$\begin{tikzcd}
\mathrm{Nom} & & \mathrm{PSh}\,\mathbb{I} \arrow[ll, "I^*"'] \\
& \mathbb{I}^{op} \arrow[ul, "\mathbb{A}^{*(-)}"] \arrow[ur, "Y"'] &
\end{tikzcd}$$

Presheaf Presentations:

well-known: Nom $\simeq$ Schanuel

- $\mathcal{C} = \mathbb{I}$: finite sets (or $\mathcal{P}_f \mathbb{A}$) with injective maps (renamings)

- $\mathrm{Sh}\,\mathbb{I} \subseteq \mathrm{PSh}\,\mathbb{I}$: intersection-preserving presheaves

  ↳ every "element" is contained in a smallest $A \in \mathbb{I}$ w.r.t. inclusions ⇝ support

$$\mathrm{Nom} \simeq \mathrm{Sh}\,\mathbb{I} \subseteq \mathrm{PSh}\,\mathbb{I}$$

$$X \xmapsto{I_*} I_* X : \mathbb{I} \to \mathrm{Set}$$

$$A \mapsto \{x \mid \mathrm{supp}\, x \subseteq A\}$$

$$\begin{tikzcd}
\mathrm{Nom} & & \mathrm{PSh}\,\mathbb{I} \arrow[ll, "I^*"'] \\
& \mathbb{I}^{op} \arrow[ul, "\mathbb{A}^{*(-)}"] \arrow[ur, "y"'] &
\end{tikzcd}$$

My opinion: Nom people know this but they don't use it

↳ So far: "dictionary" is folklore + incomplete!

Shame: Concepts exist only in one of the two representations

(some papers work out stuff that is trivial on the other side)

A little Dictionary for Nom

Nominal Sets

Presheaves $\mathbb{I} \to Set$

A little Dictionary for Nom

Nominal Sets

- Disc : Set → Nom

  $X \mapsto X,\ \pi \cdot x = x$

Presheaves $\mathbb{I} \to$ Set

Discretes

Δ : Set → PSh$\mathbb{I}$

$X \mapsto (\lambda X.\, A)$ constant

A little Dictionary for Nom

Nominal Sets

- $\mathrm{Disc} : \mathrm{Set} \to \mathrm{Nom}$

  $X \mapsto X$, $\pi \cdot x = x$

- $$\begin{tikzcd} \mathrm{Set} \arrow[r, "\top"] & \mathrm{Nom} \arrow[l, bend right, "\mathrm{equiv}"'] \arrow[l, bend left, "\mathrm{orb}", "\top"'] \end{tikzcd}$$

Discretes

(u. adjoints)

Presheaves $\mathbb{I} \to \mathrm{Set}$

$\Delta : \mathrm{Set} \to \mathrm{PSh}\,\mathbb{I}$

$X \mapsto (\lambda X. A)$ constant

$$\begin{tikzcd} \mathrm{Set} \arrow[r, "\top"] & \mathrm{PSh}\,\mathbb{I} \arrow[l, bend right, "\mathrm{lim}"'] \arrow[l, bend left, "\mathrm{colim}", "\top"'] \end{tikzcd}$$

A little Dictionary for Nom

Nominal Sets

- Disc : Set → Nom

  $X \mapsto X, \ \pi \cdot x = x$

- $$\begin{tikzcd} \mathrm{Set} \arrow[r, ""] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"] & \mathrm{Nom} \arrow[l, bend right=50, "\mathrm{equiv}"'] \arrow[l, bend left=50, "\mathrm{orb}"] \end{tikzcd}$$

Discretes

(w. adjoints)

Presheaves $\mathbb{I} \to \mathrm{Set}$

$\Delta : \mathrm{Set} \to \mathrm{PSh}\,\mathbb{I}$

$X \mapsto (\lambda X. A)$ constant

$$\begin{tikzcd} \mathrm{Set} \arrow[r, ""] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"] & \mathrm{PSh}\,\mathbb{I} \arrow[l, bend right=50, "\lim"'] \arrow[l, bend left=50, "\mathrm{colim}"] \end{tikzcd}$$

A little Dictionary for Nom

Nominal Sets

- Disc : Set → Nom
  $X \mapsto X$, $\pi \cdot x = x$
- $$\begin{tikzcd} \mathrm{Set} \arrow[r] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"'] & \mathrm{Nom} \arrow[l, bend right, "\mathrm{equiv}"'] \arrow[l, bend left, "\mathrm{orb}"] \end{tikzcd}$$
- orbit finite: $X = \bigsqcup_{i=1}^{n} \{\pi \cdot x_i \mid \pi : \mathbb{A} \cong \mathbb{A}\}$

Presheaves $\mathbb{I} \to \mathrm{Set}$

Discretes

$\Delta : \mathrm{Set} \to \mathrm{PSh}\,\mathbb{I}$
$X \mapsto (\lambda X. A)$ constant

(u. adjoints)

$$\begin{tikzcd} \mathrm{Set} \arrow[r] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"'] & \mathrm{PSh}\,\mathbb{I} \arrow[l, bend right, "\lim"'] \arrow[l, bend left, "\mathrm{colim}"] \end{tikzcd}$$

Finite

Superfinitary: $\exists x_i \in FA_i , i \leq n : \forall y : \exists f, i : y = f \cdot x$

A little Dictionary for Nom

Nominal Sets		Presheaves $\mathbb{I} \to Set$
Disc: $Set \to Nom$ $X \mapsto X, \; \pi \cdot x = x$	**Discretes**	$\Delta : Set \to PSh\,\mathbb{I}$ $X \mapsto (\lambda X. A)$ constant
$\begin{tikzcd} Set \arrow[r, ""] & Nom \arrow[l, bend right, "equiv"'] \arrow[l, bend left, "orb"] \end{tikzcd}$ $\top$ $\top$	(u. adjoints)	$\begin{tikzcd} Set \arrow[r, ""] & PSh\,\mathbb{I} \arrow[l, bend right, "lim"'] \arrow[l, bend left, "colim"] \end{tikzcd}$ $\top$ $\top$
orbit finite: $X = \coprod_{i=1}^{n} \{\pi \cdot x_i \mid \pi : \mathbb{A} \cong \mathbb{A}]$	**Finite**	Superfinitary: $\exists x_i \in FA_i, \; i \le n : \forall y : \exists f, i : y = f \cdot x$
strongly transitive: $\mathbb{A}^{*n}$	**Representables**	$yA \in PSh\,\mathbb{I}$

A little Dictionary for Nom

Nominal Sets

Presheaves $\mathbb{I} \to Set$

- Disc: $Set \to Nom$
  $X \mapsto X, \ \pi \cdot x = x$

Discretes

$\Delta : Set \to PSh\,\mathbb{I}$
$X \mapsto (\lambda X.\, A)$ constant

-
$$\begin{tikzcd} Set \arrow[r] & Nom \arrow[l, bend right=50, "equiv"'] \arrow[l, bend left=50, "orb"] \arrow[l, phantom, shift right=3, "\top"] \arrow[l, phantom, shift left=3, "\top"] \end{tikzcd}$$

(u. adjoints)

$$\begin{tikzcd} Set \arrow[r] & PSh\,\mathbb{I} \arrow[l, bend right=50, "lim"'] \arrow[l, bend left=50, "colim"] \arrow[l, phantom, shift right=3, "\top"] \arrow[l, phantom, shift left=3, "\top"] \end{tikzcd}$$

- orbit finite: $X = \coprod_{i=1}^{n} \{\pi \cdot x_i \mid \pi : \mathbb{A} \cong \mathbb{A}]$

Finite

Superfinitary: $\exists x_i \in FA_i, i \leq n : \forall y : \exists f, i : y = f \cdot x$

- strong, transitive: $\mathbb{A}^{*n}$

Representables

$yA \in PSh\,\mathbb{I}$

- $[\mathbb{A}]X$

Abstraction

$F(1+-)$ context extension

- $[Y]Z$

Generalized Abstraction

? (Conjecture)

A little Dictionary for Nom

Nominal Sets		Presheaves $\mathbb{I} \to \mathrm{Set}$
• Disc: $\mathrm{Set} \to \mathrm{Nom}$ $X \mapsto X,\ \pi \cdot x = x$	Discretes	$\Delta : \mathrm{Set} \to \mathrm{PSh}\,\mathbb{I}$ $X \mapsto (\lambda X.\ A)$ constant
• $\begin{tikzcd} \mathrm{Set} \arrow[r, "\top", "\top"'] & \mathrm{Nom} \arrow[l, bend right, "equiv"'] \arrow[l, bend left, "orb"] \end{tikzcd}$	(w. adjoints)	$\begin{tikzcd} \mathrm{Set} \arrow[r, "\top", "\top"'] & \mathrm{PSh}\,\mathbb{I} \arrow[l, bend right, "\lim"'] \arrow[l, bend left, "\mathrm{colim}"] \end{tikzcd}$
• orbit finite: $X = \coprod_{i=1}^{n} \{\pi \cdot x_i \mid \pi : \mathbb{A} \cong \mathbb{A}]$	Finite	Superfinitary: $\exists x_i \in F A_i,\ i \leq n : \forall y : \exists f, i : y = f \cdot x$
• strong, transitive: $\mathbb{A}^{*n}$	Representables	$y A \in \mathrm{PSh}\,\mathbb{I}$
• $[\mathbb{A}]X$	Abstraction	$F(1+-)$ context extension
• $[Y]Z$	Generalized Abstraction	?
• fresh product $X * Y$	Convolution	Day Convolution $F * G$
• fresh uniform substitution	x-Convolution	"Dirichlet"/"Boardman-Vogt" tensor product

A little Dictionary for Nom

Nominal Sets		Presheaves $\mathbb{I} \to Set$
• Disc: $Set \to Nom$ $X \mapsto X, \ \pi \cdot x = x$	Discretes	$\Delta : Set \to PSh\,\mathbb{I}$ $X \mapsto (\lambda X.\, A)$ constant
• $\begin{tikzcd} Set \arrow[r, ""{name=M}] & Nom \arrow[l, bend right, "equiv"'] \arrow[l, bend left, "orb"] \end{tikzcd}$ with $\top$, $\top$	(w. adjoints)	$\begin{tikzcd} Set \arrow[r] & PSh\,\mathbb{I} \arrow[l, bend right, "lim"'] \arrow[l, bend left, "colim"] \end{tikzcd}$ with $\top$, $\top$
• orbit finite: $X = \bigsqcup_{i=1}^{n} \{\pi \cdot x_i \mid \pi : \mathbb{A} \cong \mathbb{A}\}$	Finite	Superfinitary: $\exists x_i \in FA_i,\ i \leq n : \forall y : \exists f, i : y = f \cdot x$
• strong, transitive: $\mathbb{A}^{*n}$	Representables	$yA \in PSh\,\mathbb{I}$
• $[\mathbb{A}]X$	Abstraction	$F(1+-)$ context extension
• $[Y]Z$	Generalized Abstraction	?
• fresh product $X * Y$	Convolution	Day Convolution $F * G$
• fresh uniform substitution	x-Convolution	"Dirichlet"/"Boardman-Vogt" tensor product
• Substitution product	Substitution	Substitution product
	⋮	

An Atlas of the Nominal World:

All index categories $\mathbb{C}$ have $\mathrm{Ob}\,\mathbb{C} = \mathcal{P}_f \mathbb{A}$ and

$\mathbb{C}$	Id	$\subseteq$	$\mathbb{B}$	$\mathbb{I}$	$\mathbb{S}$	$\mathbb{F}$
$\mathrm{Mor}\,\mathbb{C}$	identities	inclusions	bijections	injections	surjections	all
$\mathrm{Sh}\,\mathbb{C}$	$\mathrm{SuppSet}_=$	$\mathrm{SuppSet}$	$\mathrm{Nom}_=$	Nom	$\mathrm{RelRen}_=$	Ren

← Not a sheaf topos

An Atlas of the Nominal World:

All index categories $\mathbb{C}$ have $\mathrm{Ob}\,\mathbb{C} = \mathcal{P}_f\mathbb{A}$ and

$\mathbb{C}$	Id	$\subseteq$	$\mathbb{B}$	$\mathbb{I}$	$\mathbb{S}$	$\mathbb{F}$
$\mathrm{Mor}\,\mathbb{C}$	identities	inclusions	bijections	injections	surjections	all
$\mathrm{Sh}\,\mathbb{C}$	$\mathrm{SuppSet}_=$	$\mathrm{SuppSet}$	$\mathrm{Nom}_=$	Nom	$\mathrm{RelRen}_=$	Ren

← Not a sheaf topos

If $\mathbb{C} \overset{\iota}{\hookrightarrow} \mathbb{D}$ monadic adjunction

$\mathrm{PSh}(\iota) : \mathrm{PSh}\,\mathbb{C} \dashv \mathrm{PSh}\,\mathbb{D} : \iota^*$

↳ adj. restricts to sheaves (not monadic)

↳ $\mathrm{PSh}(\iota)$ preserves Day convolution

↳ cocontinuous right adj ι^*

$$\begin{tikzcd}
 & & \mathbb{S} \arrow[ddr, tail] & \\
 & \mathbb{B} \arrow[ur, tail] \arrow[dr, tail] & & \\
\mathrm{Id} \arrow[ur, tail] \arrow[dr, tail] & & \mathbb{I} \arrow[r, tail] & \mathbb{F} \\
 & \subseteq \arrow[ur, tail] \arrow[urr, tail] & &
\end{tikzcd}$$

An Atlas of the Nominal World:

All index categories $\mathbb{C}$ have $\mathrm{Ob}\,\mathbb{C} = \mathcal{P}_f \mathbb{A}$ and

$\mathbb{C}$	Id	$\subseteq$	$\mathbb{B}$	$\mathbb{I}$	$\mathbb{S}$	$\mathbb{F}$
$\mathrm{Mor}\,\mathbb{C}$	identities	inclusions	bijections	injections	surjections	all
$\mathrm{Sh}\,\mathbb{C}$	$\mathrm{SuppSet}_=$	$\mathrm{SuppSet}$	$\mathrm{Nom}_=$	Nom	$\mathrm{RelRen}_=$	Ren

↖ Not a sheaf topos

If $\mathbb{C} \overset{\iota}{\hookrightarrow} \mathbb{D}$ monadic adjunction

$\mathrm{PSh}(\iota) : \mathrm{PSh}\,\mathbb{C} \dashv \mathrm{PSh}\,\mathbb{D} : \iota^*$

↳ adj. restricts to sheaves (not monadic)

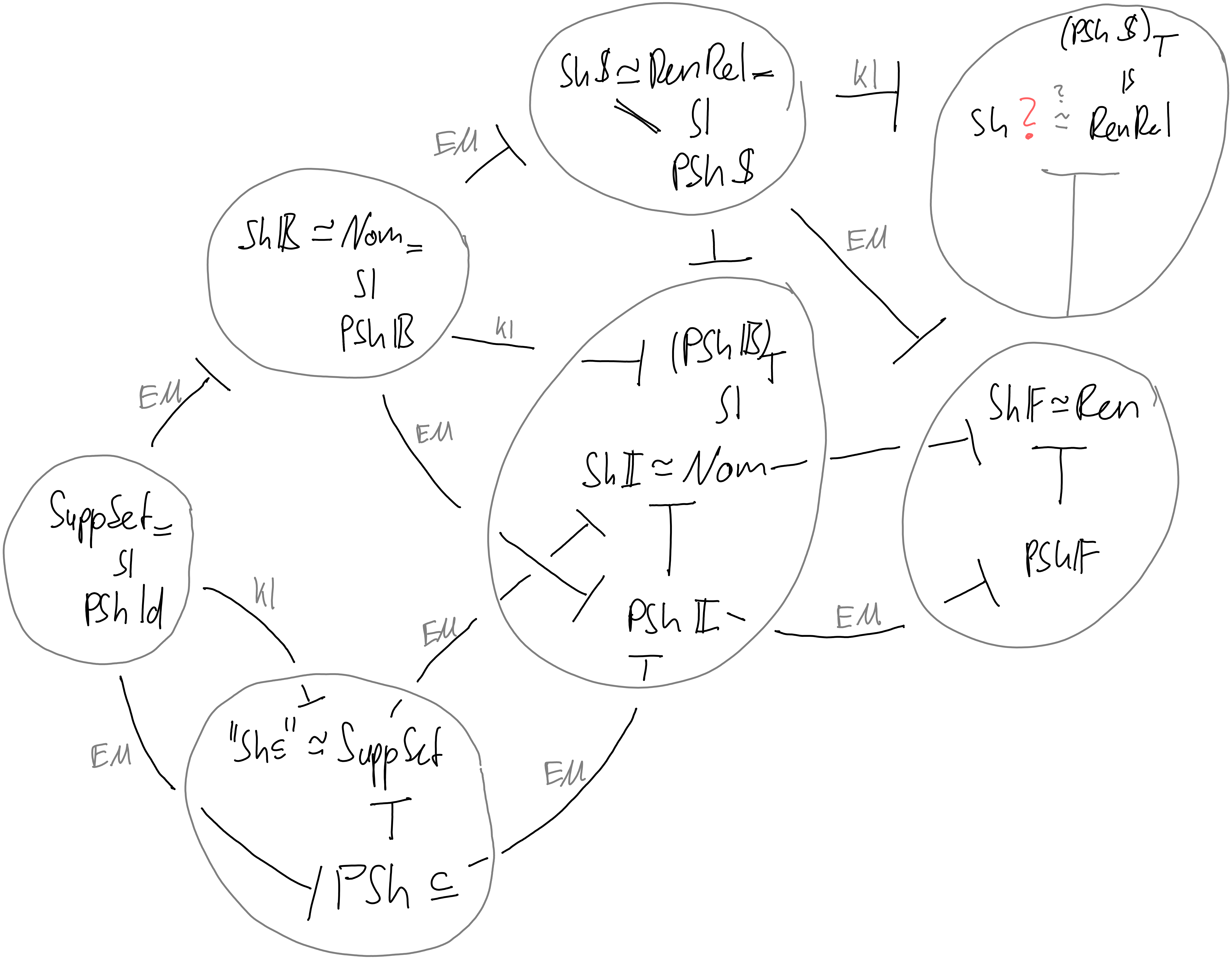
↳ $\mathrm{PSh}(\iota)$ preserves Day convolution

↳ cocontinuous right adj ι^*

$$\begin{tikzcd}
 & & \mathbb{S} \arrow[rdd, tail] & \\
 & \mathbb{B} \arrow[ru, tail] \arrow[rd, tail] & & \\
\mathrm{Id} \arrow[ru, tail] \arrow[rd, tail] & & \mathbb{I} \arrow[r, tail] & \mathbb{F} \\
 & \subseteq \arrow[ru, tail] \arrow[rru, tail] & &
\end{tikzcd}$$

[Fiore 2001]

- $\mathrm{Nom}_= \dashv \mathrm{Nom}$
- $\mathrm{SuppSet}_= \dashv \mathrm{SuppSet}$ are Kleisli
- $\mathrm{RelRen}_= \dashv \mathrm{RelRen}$


Sh𝕊 ≃ RenRel ≃
Sl
PSh𝕊
Kl
(PSh𝕊)_T
≅
Sh ? ≃ RenRel
?
EM
Sh𝔹 ≃ Nom ≃
Sl
PSh𝔹
Kl
EM
(PSh𝔹)_T
Sl
Sh𝕀 ≃ Nom ≃
PSh𝕀 ≃
EM
Sh𝔽 ≃ Ren
PSh𝔽
EM
SuppSet ≃
Sl
PSh Id
Kl
EM
EM
"Sh≤" ≃ SuppSet
PSh ≤
EM
EM

Goal: Bring everything together!

Q: When does substitution in $\mathrm{PSh}\,\mathbb{C}$ / $\mathrm{Sh}\,\mathbb{C}$ exist

Q: What is the relation between substitution in $\mathrm{PSh}\,\mathbb{C}$ and $\mathrm{PSh}\,\mathbb{D}$?

For now: only subcategories of $\mathbb{F}$.

Reindexing Categories:

Recall: family $\underline{X} = (X_b)_{b \in B}$ over B

"pulled back" along $f : A \to B$:

to A-family $f^*\underline{X} = (X_{fa})_{a \in A}$

Reindexing Categories:

Recall: family $\underline{X} = (X_b)_{b \in B}$ over B

"pulled back" along $f: A \to B$:

to A-family $f^*\underline{X} = (X_{fa})_{a \in A}$

↳
$$\begin{tikzcd}
p_f : \prod \underline{X} = \prod_b X_b \arrow[r] \arrow[dr, "\pi_{fa}"'] & \prod_a X_{fa} = \prod f^*\underline{X} \arrow[d, "\pi_a"] \\
 & X_{fa}
\end{tikzcd}$$

Reindexing Categories:

Recall: family $\underline{X} = (X_b)_{b\in B}$ over B

"pulled back" along $f: A \to B$:

to A-family $f^*\underline{X} = (X_{fa})_{a\in A}$

↳ $p_f:$

$$\begin{tikzcd}
\prod \underline{X} = \prod_b X_b \arrow[r] \arrow[dr, "\pi_{fa}"'] & \prod_a X_{fa} = \prod f^*\underline{X} \arrow[d, "\pi_a"'] \\
 & X_{fa}
\end{tikzcd}$$

↳ $r_f:$

$$\begin{tikzcd}
\coprod f^*\underline{X} = \coprod_a X_{fa} \arrow[r] & \coprod_b X_b = \coprod \underline{X} \\
X \arrow[u, "\iota_{\substack{a\\fa}}"'] \arrow[ur, "\iota_{fa}"']
\end{tikzcd}$$

Reindexing Categories:

Recall: family $\underline{X} = (X_b)_{b \in B}$ over B

"pulled back" along $f : A \to B$:

to A-family $f^* \underline{X} = (X_{fa})_{a \in A}$

$\hookrightarrow$ $p_f : \prod \underline{X} = \prod_b X_b \longrightarrow \prod_a X_{fa} = \prod f^* \underline{X}$

$$\begin{tikzcd} \prod_b X_b \arrow[r] \arrow[dr, "\pi_{fa}"'] & \prod_a X_{fa} \arrow[d, "\pi_a"] \\ & X_{fa} \end{tikzcd}$$

$\hookrightarrow$ $r_f : \coprod f^* \underline{X} = \coprod_a X_{fa} \longrightarrow \coprod_b X_b = \coprod \underline{X}$

$$\begin{tikzcd} \coprod_a X_{fa} \arrow[r] & \coprod_b X_b \\ X \arrow[u, "\iota_{fa}"] \arrow[ur, "\iota_{fa}"'] & \end{tikzcd}$$

Def: A reindexing category $\mathbb{C}$ is a subcategory $\mathbb{C} \rightarrowtail \mathbb{F}$ s.t.

- $\mathbb{C}$ is $+$-monoidal and $1 \in \mathbb{C}$
  + closed under isos ?
- $\mathbb{C}$ has reindexing:
  if $B \in \mathbb{C}$, $\underline{B} = (B_b)_b \in \mathbb{C}^B$, $f \in \mathbb{C}(A, B)$
  then $\left(r_f : \coprod f^* \underline{B} \to \coprod \underline{B}\right) \in \mathbb{C}$

Reindexing Categories:

Recall: family $\underline{X} = (X_b)_{b\in B}$ over B

"pulled back" along $f: A \to B$:

to A-family $f^*\underline{X} = (X_{fa})_{a\in A}$

$$\begin{tikzcd} p_f: \prod \underline{X} = \prod_b X_b \arrow[r] \arrow[dr, "\pi_{fa}"'] & \prod_a X_{fa} = \prod f^*\underline{X} \arrow[d, "\pi_a"] \\ & X_{fa} \end{tikzcd}$$

$$\begin{tikzcd} r_f: \coprod f^*\underline{X} = \coprod_a X_{fa} \arrow[r] & \coprod_b X_b = \coprod \underline{X} \\ X \arrow[u, "\iota_{fa}"] \arrow[ur, "\iota_{fa}"'] & \end{tikzcd}$$

Def: A reindexing category $\mathbb{C}$ is a subcategory $\mathbb{C} \rightarrowtail \mathbb{F}$ s.t.

- $\mathbb{C}$ is $+$-monoidal and $1 \in \mathbb{C}$
  + closed under isos ?
- $\mathbb{C}$ has reindexing:
  if $B \in \mathbb{C}$, $\underline{B} = (B_b)_b \in \mathbb{C}^B$, $f \in \mathbb{C}(A, B)$
  then $(r_f: \coprod f^*\underline{B} \to \coprod \underline{B}) \in \mathbb{C}$

Observation: For $+$-ideals ($f+g \in \mathbb{C}$ iff $f, g \in \mathbb{C}$) $\mathbb{C} \subseteq \mathbb{F}$:

$\mathbb{C}$ reindexing $\Leftrightarrow$ $\mathbb{C}$ is $\times$-monoidal

Day Power:

⌜**Def:** For $X \in \mathrm{PSh}\,\mathbb{C}$ and $A \in \mathbb{C}$. Define $X^{*A} := \int^{A_a \in \mathbb{C},\, a \in A} \prod_a X A_a \cdot y\left(\coprod_a A_a\right)$ ⌟

Day Power:

⌜Def: For $X \in \mathrm{PSh}\,\mathbb{C}$ and $A \in \mathbb{C}$. Define $X^{*A} := \int^{A_a \in \mathbb{C},\, a \in A} \prod_a X A_a \cdot y\left(\coprod_a A_a\right)$ ⌟

⌜Theorem: If $\mathbb{C}$ is a reindexing category then

- $(-)^{*(-)} : \mathbb{C}^{op} \times \mathrm{PSh}\,\mathbb{C} \to \mathrm{PSh}\,\mathbb{C}$ functorial
- $X^{*-} : (\mathbb{C}^{op}, +, 0) \to (\mathrm{PSh}\,\mathbb{C}, *, y0)$ is strict monoidal
- $(-)^{*(-)} : (\mathbb{C}^{op}, \times, 1) \times \mathrm{PSh}\,\mathbb{C} \to \mathrm{PSh}\,\mathbb{C}$ is a graded monad on $\mathrm{PSh}\,\mathbb{C}$
- for $\iota : \mathbb{C} \rightarrowtail \mathbb{D}$ we have $\mathrm{PSh}(\iota)(X^{*A}) \cong (\mathrm{PSh}(\iota) X)^{*\iota A}$

$\mathrm{PSh}(\iota) = \mathrm{Lan}_{y\iota} : \mathrm{PSh}\,\mathbb{C} \to \mathrm{PSh}\,\mathbb{D}$

⌟

Substitution Product:

Def: $X \diamond Y = \int^A XA \cdot Y^{*A}$, $Y \mathbin{-\!\!\diamond} Z = \mathrm{Nat}(Y^{*-}, Z)$

$\uparrow$ replace by $\int^A XA \cdot A \triangleright X$ + $\uparrow$ $\mathrm{Nat}(- \triangleright Y, Z)$ for $\mathbb{C}^{op} \times \mathrm{PSh}\,\mathbb{C} \xrightarrow{\triangleright} \mathrm{PSh}\,\mathbb{C}$ satisfying ?

Substitution Product:

Def: $X \diamond Y = \int^{A} XA \cdot Y^{*A}$, $Y \multimap Z = \mathrm{Nat}(Y^{*-}, Z)$

$\uparrow$ replace by $\int^{A} XA \cdot A \triangleright X$ + $\uparrow$ $\mathrm{Nat}(- \triangleright Y, Z)$ for $\mathbb{C}^{op} \times \mathrm{PSh}\,\mathbb{C} \xrightarrow{\triangleright} \mathrm{PSh}\,\mathbb{C}$ satisfying ?

Theorem:

(1) $(-) \diamond Y$ preserves Day convolution

(2) $\diamond$ is monoidal with unit $y1$

(3) $\diamond$ is right-closed with internal hom $\multimap$

(4) $\diamond$ is preserved by the canonical functors $\mathrm{PSh}\,\mathbb{C} \to \mathrm{PSh}\,\mathbb{D}$ for $\iota : \mathbb{C} \to \mathbb{D}$

(5) If J is a subcanonical topology on $\mathbb{C}$ then $\diamond$ restricts to $\mathrm{Sh}\,\mathbb{C}$

$\diamond$ restricts to the nominal structures use Sau (not: SuppSet!)

(6) $\mathrm{Monoid}(\mathrm{PSh}\,\mathbb{C}, \diamond, y1) \simeq \mathrm{Monad}^{*\text{-pres}, \text{cocont}}(\mathrm{PSh}\,\mathbb{C})$

Substitution Product: Instances

Instances:

$\mathbb{C}$	Id	$\mathbb{B}$	$\mathbb{I}$	$\mathbb{S}$	$\mathbb{F}$
Mor $\mathbb{C}$	identities	bijections	injections	surjections	all
PSh $\mathbb{C}$	linear+strict	linear	affine	relevant+strict	classical
Sh $\mathbb{C}$	Supp$_=$	**Nom**$_=$	**Nom**	**RelRen**$_=$	**Ren**

No Instance: SuppSet, **RelRen** (I think it does, but not worked out yet)

Index categories beyond $\mathbb{C} \rightarrowtail \mathbb{F}$:

↳ e.g. "probabilistic nominal Sets":

$\mathbb{C} \subseteq$ Enhanced Measurable Spaces

[📖 Li et. al. LICS'24]

↳ different symmetries on $\mathbb{A}$
(order, typed, ...)

Future Work

Future Work

Index categories beyond $\mathbb{C} \rightarrowtail \mathbb{F}$:

- e.g. "probabilistic nominal Sets":
  - $\mathbb{C} \subseteq$ Enhanced Measurable Spaces
  - [📖 Li et. al. LICS'24]
- different symmetries on $\mathbb{A}$ (order, typed, …)

Complete the dictionary

- Generalized abstraction for presheaves?
- Interpret Dirichlet product in $\mathbf{Nom}$

Index categories beyond $\mathbb{C} \twoheadrightarrow \mathbb{F}$:

↳ e.g. "probabilistic nominal Sets":

$\mathbb{C} \subseteq$ Enhanced Measurable Spaces

[📖 Li et.al. LICS'24]

↳ different symmetries on $\mathbb{A}$
(order, typed, ...)

Future Work

Relation to other generalizations
(by Power/Fiore):

take two 2-monads on CAT +

one for $\mathbb{C}$ one for PSh/Sh

pseudo-distr. law

substitution structures are an instance!

(I think it's instructive nonetheless)

Complete the dictionary

↳ Generalized abstraction for presheaves?

↳ Interpret Dirichlet product in Nom

Future Work

Index categories beyond $\mathbb{C} \rightarrowtail \mathbb{F}$:

- e.g. "probabilistic nominal Sets":
  - $\mathbb{C} \subseteq$ Enhanced Measurable Spaces
  - [Li et. al. LICS'24]
- different symmetries on $\mathbb{A}$ (order, typed, ...)

Relation to other generalizations (by Power/Fiore):

take two 2-monads on CAT +

one for $\mathbb{C}$ one for PSh/Sh

pseudo-distr. law

substitution structures are an instance!

(I think its instructive nonetheless)

Your ~~Ad~~ ideas could be here!

Complete the dictionary

- Generalized abstraction for presheaves?
- Interpret Dirichlet product in Nom