Dual Bisimilarity Games

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The Setting

Given a coalgebra $\langle X, \gamma \rangle$, are two states in X bisimilar?

States are bisimilar iff there exists a bisimulation...

What is a bisimulation?

Huh?

Bisimulation

A bisimulation is an eq.rel R, s.t.

$2 \times X^A$

$$\begin{tikzcd} x \arrow[d, "a"'] \arrow[r, phantom, "R"] & y \arrow[d, "a"] \\ x' \arrow[r, phantom, "R"] & y' \end{tikzcd}$$

&

x final $\Leftrightarrow$ y final

$A \to PX$

$$\begin{tikzcd} x \arrow[d, "a"'] \arrow[r, phantom, "R"] & y \arrow[d, "a"] \\ S \arrow[r, phantom, "\tilde{R}"] & T \end{tikzcd}$$

„Just lift R along functor"

$\forall\, x' \in S.$
$\exists\, y' \in T.\ x' R y'$
and $\forall\, y' \in T$
$\exists\, x' \in S.\ x' R y'$

The Standard Game

We play on an LTS $(A \to PX)$

Positions are pairs $(x, y) \in X \times X$

I win infinite games!

Players are Spoiler and Duplicator

$x \neq y$!

$x = y$!

A

1. Spoiler chooses an a-successor of either x or y
2. Duplicator must copy said move, repeat

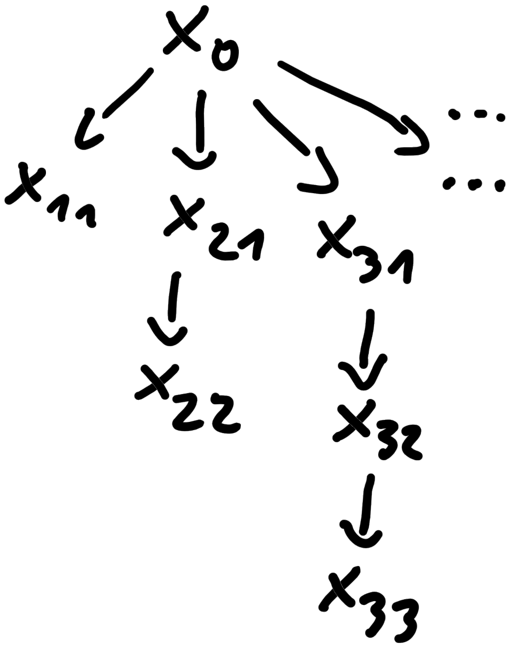
Whoever cannot move loses!

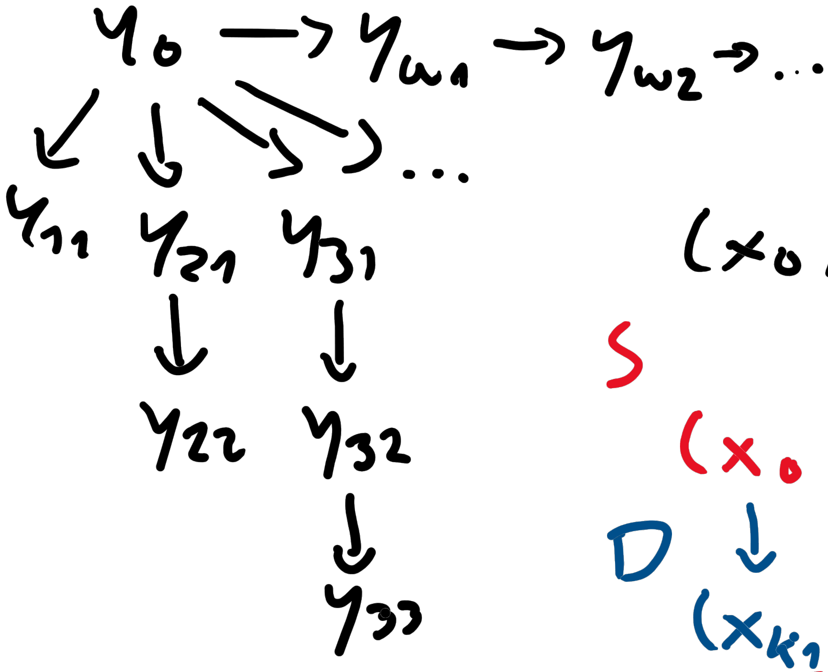
The Standard Examples

$$
\begin{tikzcd}
 & x_0 \arrow[d, "a"'] & \\
 & x_1 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
x_2 & & x_3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y_0 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_1 \arrow[d, "b"'] & & y_2 \arrow[d, "c"] \\
y_3 & & y_4
\end{tikzcd}
$$

$$
\begin{array}{ll}
 & (x_0, y_0) \\
S & \qquad\quad \downarrow a \\
 & (x_0, y_1) \\
D & a \downarrow \\
 & (x_1, y_1) \\
S & c \downarrow \\
 & (x_3, y_1) \\
D & \text{¯\\_(ツ)\_/¯}
\end{array}
$$

The Standard Examples





$$(x_0, y_0)$$

S $\downarrow$

$$(x_0, y_{\omega_1})$$

D $\downarrow$

$$(x_{k_1}, y_{\omega_1})$$

S wins in k turns

The Quest for Generelization

Graded Semantics

Codensity Liftings

induce

games

(Ford et al.)

(Forster et al.)

(Komorida et al.)

„They seem almost dual to each other"

– Jonas Forster

(also everyone else I've talked to)

Games via Graded Semantics*

Position	Pl	Move
$(x,y) \in X \times X$	D	„Equations“ $R \subseteq X \times X$ s.t. $\gamma(x) =_R \gamma(y)$
$R \subseteq X \times X$	S	$(x',y') \in R$

$\boxed{FX = \mathcal{P}(A \times X)}$

local bisimulation

believable for one step

Spoiler doesn't agree with this equation

*simplified

Games via Graded Semantics*

For Trace Equivalence!

Position	Pl	Move
$(x, y) \in PX \times PX$	D	„Equations" $R \subseteq PX \times PX$ s.t. $\gamma^{\#}(x) =_R \gamma^{\#}(y)$
$R \subseteq X \times X$	S	$(x', y') \in R$

$\boxed{FX = P(A \times X)}$

Just play on the determinization

via

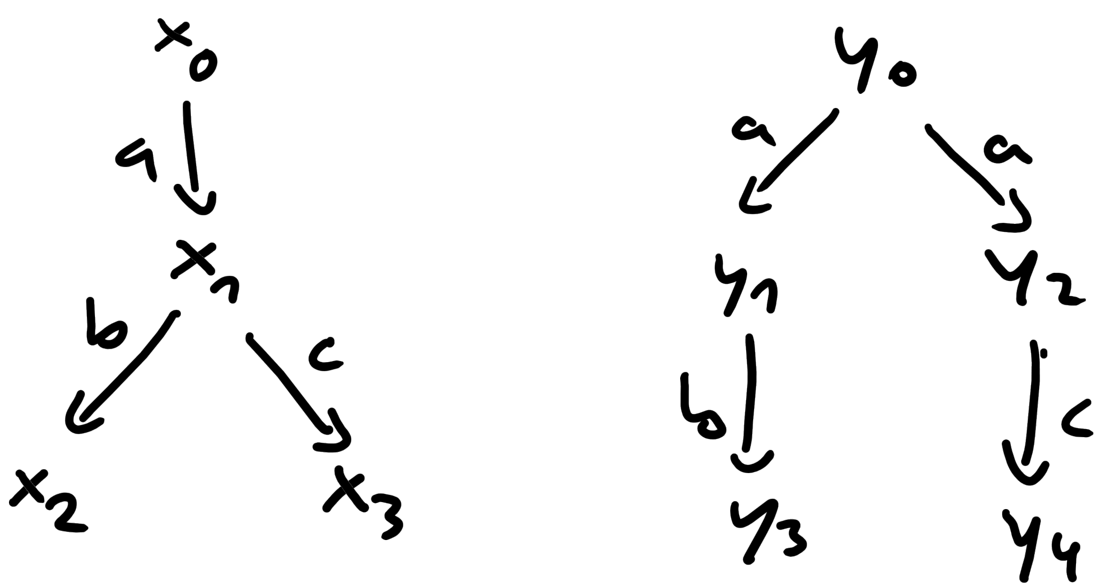
abstract nonsense!

$x \sim_{tr} y$ iff D wins $(\{x\}, \{y\})$

finite!

*simplified

Playing the Game



$$x_0 \stackrel{?}{=} y_0$$

$$\{a(x_1)\} \stackrel{?}{=} \{a(y_1), a(y_2)\}$$

D: $\{x_1 = y_1,\ x_1 = y_2\}$

S: $\downarrow$

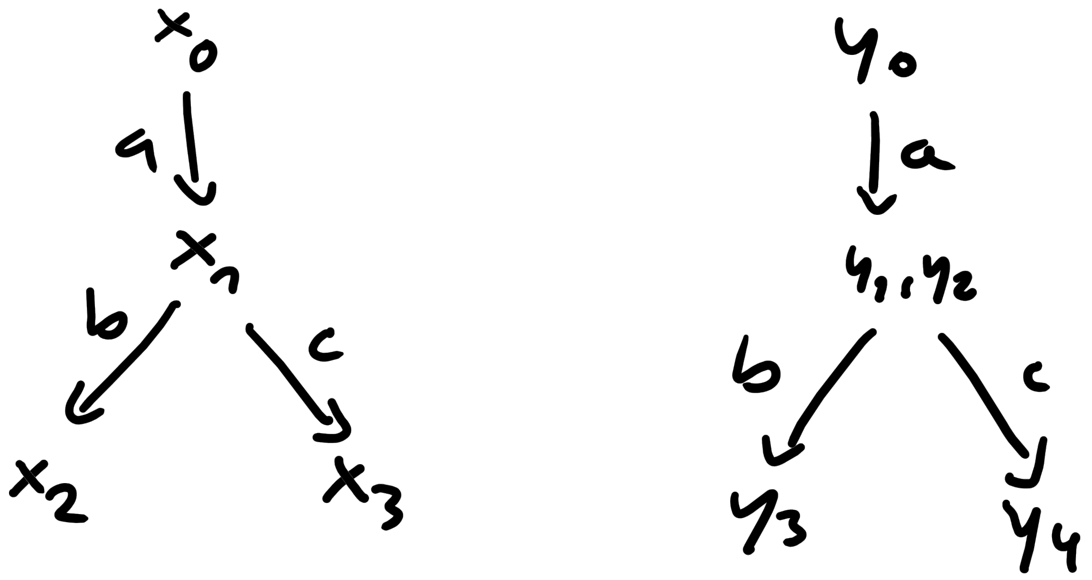
$$x_1 \stackrel{?}{=} y_1$$

$$\{b(x_2), c(x_3)\} \stackrel{?}{=} \{b(y_3)\}$$

D ¯_(ツ)_/¯

Playing the Game

For Trace Equivalence!



$$\{x_0\} \overset{?}{=} \{y_0\}$$

$$\gamma^\# \downarrow \qquad \qquad \downarrow \gamma^\#$$

$$\{a(\{x_1\})\} \overset{?}{=} \{a(\{y_1, y_2\})\}$$

D: $\{\{x_1\} = \{y_1, y_2\}\}$

S: $\downarrow$

...

$$\{x_2\} \overset{?}{=} \{y_3\}$$

$$\gamma^\# \downarrow \qquad \qquad \downarrow \gamma^\#$$

$$\emptyset \overset{?}{=} \emptyset$$

D: $\emptyset$

¯_(ツ)_/¯ S can't choose $\swarrow$

Try both versions with this one

$$
\begin{tikzcd}
& x_0 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
x_{11} & x_{21} \arrow[d] & x_{31} \arrow[d] & \cdots \\
& x_{22} & x_{32} \arrow[d] & \\
& & x_{33} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& y_0 \arrow[r] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & y_{\omega 1} \arrow[r] & y_{\omega 2} \arrow[r] & \cdots \\
y_{11} & y_{21} \arrow[d] & y_{31} \arrow[d] & \cdots & \\
& y_{22} & y_{32} \arrow[d] & & \\
& & y_{33} & &
\end{tikzcd}
$$

Games via Codensity Liftings*

Position	Pl	Move
$(x,y) \in X \times X$	S	„Predicate" $k : X \to 2$ s.t. $\exists a.\ \top \in Fk \circ \gamma(x)(a)$!! $\top \notin Fk \circ \gamma(y)(a)$
$k : X \to 2$	D	(x',y') s.t. $k(x') \neq k(y')$

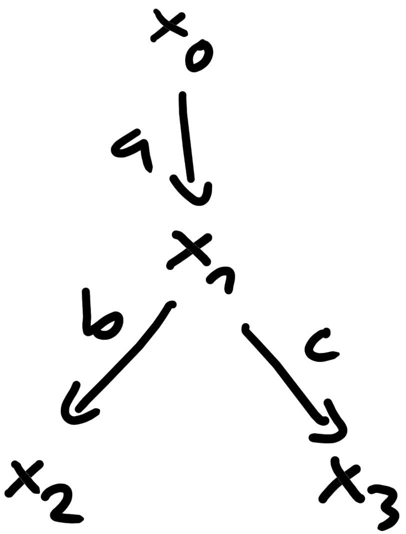
$\boxed{FX = P(A \times X)}$

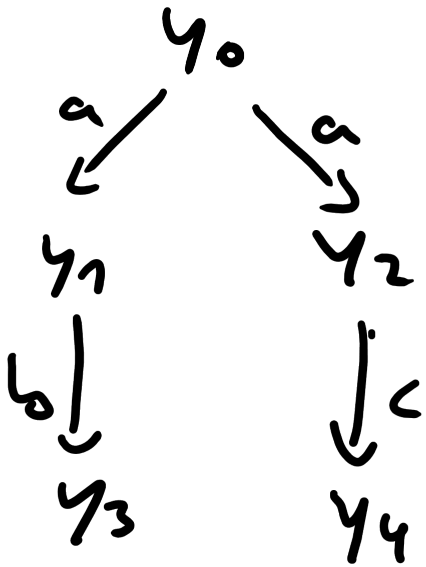
Spoiler plays a set of distinguishing behaviours

Duplicator thinks this isn't respecting bisimilarity

*simplified

Playing the Game





$x_0 \overset{?}{=} y_0$

$\{a(x_1)\}$ $\{a(y_1), a(y_2)\}$

S: $k = \{x_1\}$

D: (x_1, y_1)

$\{b(x_2), c(x_3)\}$ $\{b(y_3)\}$

S: $k = X$

D: ¯_(ツ)_/¯

Try with this one

$$\begin{tikzcd}
 & x_0 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \cdots \\
x_{11} & x_{21} \arrow[d] & x_{31} \arrow[d] & \cdots \\
 & x_{22} & x_{32} \arrow[d] & \\
 & & x_{33} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y_0 \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[r] & y_{\omega 1} \arrow[r] & y_{\omega 2} \arrow[r] & \cdots \\
y_{11} & y_{21} \arrow[d] & y_{31} \arrow[d] & \cdots & \\
 & y_{22} & y_{32} \arrow[d] & & \\
 & & y_{33} & &
\end{tikzcd}$$

The Seeming Duality

Graded Semantics

(x, y)

D plays
Equalities to show
an Equality

S picks a pair R
$(x', y') \in$

Codensity Liftings

S plays
a Predicate to show
an Inequality

k D picks a ...

The Seeming Duality

Graded Semantics

(x, y)

D plays
Equalities to show
an Equality

S picks a pair $\in R$
(x', y')

Codensity Liftings

S plays
~~a Predicate to show~~
~~an Inequality~~

~~k~~

~~D picks a ...~~

The Seeming Duality

Graded Semantics

(x, y)

D plays

Equalities to show an Equality

S picks a pair $(x', y') \in R$

Codensity Liftings

S plays

$k : X \to Z$ are special cases

Equalities to show an Inequality

$R \not\ni$ D picks a pair (x', y')

The Dual Games

Position	Pl	Move
$(x,y) \in X \times X$		Eq. rel. $R \subseteq X \times X$ s.t.
	S	$\gamma(x) \neq_R \gamma(y)$
	D	$\gamma(x) =_R \gamma(y)$
$R \subseteq X \times X$	D	$(x',y') \notin R$
	S	$(x',y') \in R$

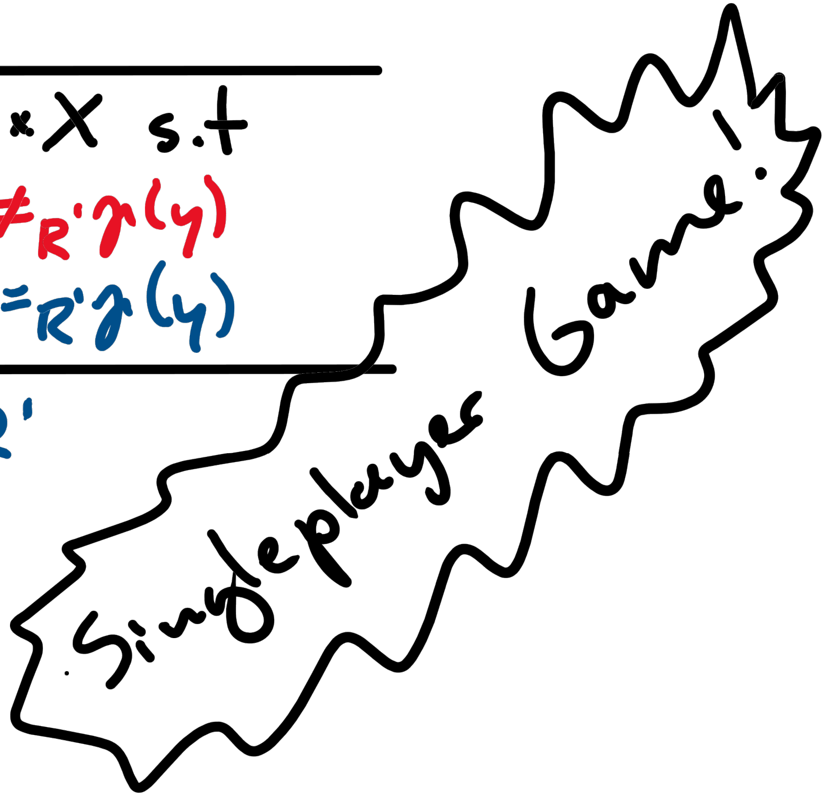
You can alternate non-deterministically!

The Dual Games

Position	Pl	Move
$R \subseteq X \times X$		Eq. rel. $R' \subseteq X \times X$ s.t
	S	$\exists (x,y) \in R$ $\quad \gamma(x) \neq_{R'} \gamma(y)$
	D	$\gamma(x) =_{R'} \gamma(y)$
$R \subseteq X \times X$	D	$R \subseteq X \times X \setminus R'$
	S	$R \subseteq R'$

The Duality Games

Position	Pl	Move
$R \subseteq X \times X$		Eq. rel. $R' \subseteq X \times X$ s.t
	S	$\forall (x,y) \in R$ $\gamma(x) \neq_{R'} \gamma(y)$
	D	$\gamma(x) =_{R'} \gamma(y)$
$R \subseteq X \times X$	D	$R \subseteq X \times X \setminus R'$
	S	$R \subseteq R'$



TO-DOs

Abstraction?

Relation to Apartness?

Conversion of Winning Strategies?