

# A coincidence of partial and total correctness: Intrinsically Correct Sorting with a slice of Cubical Agda

Cass Alexandru<sup>1</sup>   Vikraman Choudhury<sup>2</sup>   Jurriaan Rot<sup>3</sup>  
Niels van der Weide<sup>3</sup>

<sup>1</sup>RPTU Kaiserslautern-Landau

<sup>2</sup>University of Bologna & INRIA OLAS

<sup>3</sup>Radboud University Nijmegen

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- “Sorting with Bialgebras and Distributive Laws” (Hinze et al. 2012)

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- Contribution: intrinsic verification of business logics & setting in which correctness of the dual algorithms follows
- Key idea: Index data by the multiset of their elements

# Outline

- 1 Sorting as an Index-Preserving Map
- 2 Recap of “Sorting with Bialgebras and Distributive Laws”
  - Base Functors
  - Bialgebraic Semantics
- 3 Correct Sorting using Distributive Laws
  - Base Functors for Element-Indexed (Ordered) Lists
  - The FMSet Index as a Termination Measure
- 4 Conclusion & Future Work

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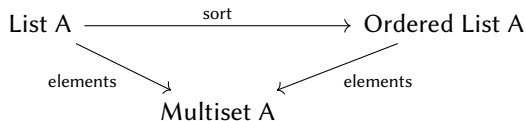
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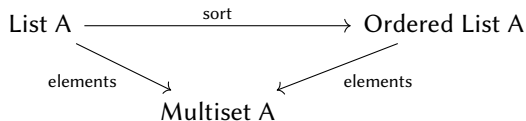
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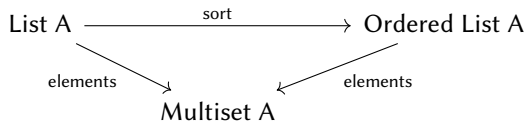
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$$\{g : \text{FMSet } A\} \rightarrow \text{EList } g \rightarrow \text{OEList } g$$

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# Base Functors of Recursive Datatypes

- Recursive datatypes have a shape given by a **base functor**  $F$
- E.g. Natural numbers:  $(1 + -)$ . Lists of element type  $A$ :  $(1 + A \times -)$ .
- Recursive datatype is given by fixpoint of composition of base functor  $F$  with itself
- Least fixpoint  $(\mu F)$ : Inductive datatype. Greatest  $(\nu F)$ : coinductive – not necessarily well founded

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- Algebraically: `fold alg` where  $\text{alg} : F (\text{Rec } G) \rightarrow \text{Rec } G$
- Coalgebraically: `unfold coalg` where  $\text{coalg} : \text{Rec } F \rightarrow G (\text{Rec } F)$

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- A way that gives us both...

# Insertion- / Bubble Sort

(Hinze et al. 2012)

```
data L (r : Type) : Type where
  []      : L r
  _::_    : A → r → L r
-- aliasing
O = L
pattern _≤::_ x xs = x :: xs
```

```
swap : ∀ {x} → L (O x) → O (L x)
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swap [] = []
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swap (a :: []) = a ≤:: []
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```
swap (a :: (b ≤:: r)) with a ≤?≥ b
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...| inl a ≤ b = a ≤:: (b :: r)
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insertSort = fold ( unfold (swap ∘ L1 out))
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# The Finite Multiset Quotient Inductive Type

(Choudhury and Fiore 2023)

```
data FMSet (A : Type ℓ) : Type ℓ where
  []      : FMSet A
  _::__   : (x : A) → (xs : FMSet A) → FMSet A
  comm    : ∀ {x y xs} → x :: y :: xs ≡ y :: x :: xs
  trunc   : isSet (FMSet A)
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1 :: 2 :: 3 :: [] ≡⟨ cong (1 ::_) comm ⟩ 1 :: 3 :: 2 :: []
  
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# The Finite Multiset Quotient Inductive Type

(Choudhury and Fiore 2023)

**data** FMSet ( $A : \text{Type } \ell$ ) :  $\text{Type } \ell$  **where**

$[]$  : FMSet  $A$

$\_::\_$  :  $(x : A) \rightarrow (xs : \text{FMSet } A) \rightarrow \text{FMSet } A$

**comm** :  $\forall \{x \ y \ xs\} \rightarrow x :: y :: xs \equiv y :: x :: xs$

**trunc** :  $\text{isSet } (\text{FMSet } A)$

$1 :: 2 :: 3 :: [] \equiv \langle \text{cong } (1 :: \_) \text{ comm} \rangle 1 :: 3 :: 2 :: []$

**pattern**  $[] \mathcal{M} = []$

**pattern**  $\_::\mathcal{M}\_ \ x \ xs = x :: xs$



# Base Functors for (Ordered) Element-Indexed Lists

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data L (r : Type)                : Type                where
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data L (r : Type) : Type where
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data L (r : FMSet A → Type) : FMSet A → Type where
  [] : L r []M
  _::_ : ∀ {g} → (x : A) → (r g) → L r (x ::M g)

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```
data O (r : FMSet A → Type) : FMSet A → Type where
  [] : O r []M
  _≤::_ : ∀ {g} (x : A) → (r g) → All (x ≤_) g → O r (x ::M g)
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# The FMSet Index as a Termination Measure

L-coalgebras are well founded

**pattern**  $\_ :: \_ ^ \_ \times \_ \text{xs } g = \_ :: \_ \{g = g\} \times \_ \text{xs}$

**unfoldL** :  $\{r : \text{FMSet } A \rightarrow \text{Type}\} \rightarrow$   
 $(\forall \{g_r\} \rightarrow (r \text{ } g_r) \rightarrow \text{L } r \text{ } g_r) \rightarrow (\forall \{g\} \rightarrow (r \text{ } g) \rightarrow \text{EList } g)$

**unfoldL** *grow*  $\{\_ \}$  *seed with grow seed*

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- Index-preservation forces index of **seed** and **grow seed** to coincide

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$\text{unfoldL } \text{grow } \{\_ \}$        $\text{seed with } \text{grow seed}$   
 $\text{unfoldL } \text{grow } \{[\mathcal{M}]\}$        $\_ \mid [\_]$        $=$   
 $\text{unfoldL } \text{grow } \{x :: \mathcal{M} \ g'\}$        $\_ \mid x :: \text{seed}' ^ g'$        $=$

- Index-preservation forces index of **seed** and **grow seed** to coincide
- **with**-abstraction: Pattern matching refines earlier arguments, propagates information about the indexee back to the index

# The FMSet Index as a Termination Measure

L-coalgebras are well founded

pattern  $\_ :: \_ \wedge \_ \times xs \ g = \_ :: \_ \{g = g\} \times xs$

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$\text{unfoldL } \text{grow } \{[\_]\mathcal{M}\}$        $\_ \mid [\_] = [\_]$

$\text{unfoldL } \text{grow } \{x :: \mathcal{M} \ g'\}$        $\_ \mid x :: \text{seed}' \wedge g' =$   
 $x :: \text{unfoldL } \text{grow } \{g'\} \text{ seed}'$

- Index-preservation forces index of **seed** and **grow seed** to coincide
- **with**-abstraction: Pattern matching refines earlier arguments, propagates information about the indexee back to the index
- Index of recursive argument is smaller



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- Syntactic termination checking based on dot-patterns of HITs inconsistent (Pitts 2020)

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- Syntactic termination checking based on dot-patterns of HITs inconsistent (Pitts 2020)
- Define a family of maps indexed by **FMSet A** by well founded induction on the length of the index
- **length** defined by eliminating from **FMSet A** as the free commutative monoid to  $(\mathbb{N}, +)$  by  $\lambda a \rightarrow 1$

# swap, Revisited

```

swap : {r : FMSet A → Type} {g : FMSet A} →
  L (O r) g → O (L r) g
swap [] = []
swap (a :: []) = (a ≤:: []) []-A
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...| inl a≤b = (a ≤:: (b :: r)) $ a≤b ≤::# a≤#r
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  subst (O (L _)) comm

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- `transpX-O (λ n → ...) i0 ...` (Cavallo and Harper 2019)

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- Evaluation of `subst ...`?
- `transpX-O (λ n → ...) i0 ...` (Cavallo and Harper 2019)
- Discard index with `toList : {g : FMSet A} → OEIList g → List A`

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- Intrinsically correct algorithms from correct distr. law
- For verified quick/treesort & heapsort following (Hinze et al. 2012), semantics via slice category → see paper

# Future Work

- Conditions under which coalgebras are recursive in an indexed/fibered setting

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- More algorithms to verify with a distributive law as business logic

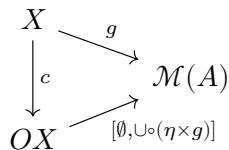
# Well-foundedness of $O$ -Coalgebras

$$c : (X, g) \rightarrow O(X, g)$$

$$c^n : X \rightarrow 1 + X$$

$$c^0(x) := \text{inr}(x)$$

$$c^{n+1}(x) := \begin{cases} \text{inl}(\star) & c^n(x) = \text{inl}(\star) \\ c(y) & c^n(x) = \text{inr}(y) \end{cases}$$



- $c$  well-founded:  $\forall x \in X. \exists n. c^n(x) = \text{inl}(\star)$
- Idea: Use  $g$  as a ranking function into well-order  $(\mathcal{M}(A), \subset)$
- Case  $c(x) = \text{inr}(a, r)$ :  $g(x) = \eta(a) \cup g(r) \Rightarrow g(r) \subset g(x)$   $\square$

# Slice Topos over a Monoid object

$$L_{\text{elt}}(X, g):$$

$$1 + A \times X$$

$$\downarrow Lg$$

$$1 + A \times \mathcal{M}(A)$$

$$\downarrow [\emptyset, \cup \circ (\eta \times \text{id})]$$

$$\mathcal{M}(A)$$

$$\begin{array}{ccccc}
 1 & \oplus & A & \otimes & X \\
 \searrow \emptyset & & \downarrow \eta & \swarrow g & \\
 & & \mathcal{M}(A)^2 & & \\
 & \swarrow \cup & & & \\
 & \mathcal{M}(A) & & & 
 \end{array}$$

$$\hat{1}, \hat{A} : \mathbf{Set}/\mathcal{M}(A)$$

$$\hat{1} := (1, \emptyset)$$

$$\hat{A} := (A, \eta)$$

$$L_{\text{elt}}(g) \simeq \hat{1} \oplus \hat{A} \otimes g$$

$$\oplus : (\mathbf{Set}/\mathcal{M}(A))^2 \rightarrow \mathbf{Set}/\mathcal{M}(A)$$

$$g \oplus h := [g, h]$$

$$\otimes : (\mathbf{Set}/\mathcal{M}(A))^2 \rightarrow \mathbf{Set}/\mathcal{M}(A)$$

$$g \otimes h := \cup \circ (g \times h)$$