Modal Logic Summer semester 2025

Exercise Sheet 5

Solutions due: 21.07.2025

Exercise 1 PDL axioms **8 points**

(a) Show that the axioms of regular PDL (without '?') are sound over regular PDL frames.

(b) Let $\mathcal{I} = \{a, a^*\}$. Show that the axioms

$$[a^*]p \leftrightarrow (p \wedge [a][a^*]p) \tag{1}$$
$$[a^*](p \rightarrow [a]p) \rightarrow p \rightarrow [a^*]p \tag{2}$$

define the class of frames where $R_{a^*} = (R_a)^*$.

Exercise 2 Fischer-Ladner Closure **7 points**

Show that the Fischer-Ladner closure $\mathsf{FL}(\phi)$ of a formula ϕ is finite and of linear size in ϕ. Proceed according to the following outline:

We can rewrite the definition of $\mathsf{FL}(\phi)$ as being the least set Γ of formulae containing ϕ and closed under the rules

1. if $\neg\psi \in \Gamma$, then $\psi \in \Gamma$;
2. if $\chi \wedge \psi \in \Gamma$, then $\chi, \psi \in \Gamma$;
3. if $[a]\psi \in \Gamma$, then $\psi \in \Gamma$
4. if $[\alpha \cup \beta]\psi \in \Gamma$, then $[\alpha]\psi, [\beta]\psi, \psi \in \Gamma$
5. if $[\alpha; \beta]\psi \in \Gamma$, then $[\alpha][\beta]\psi, [\beta]\psi \in \Gamma$;
6. if $[\alpha^*]\psi \in \Gamma$, then $\psi, [\alpha][\alpha^*]\psi \in \Gamma$.

(a) Prove this, by showing that the respective sets of closure conditions are equivalent.

Next, we define a variant $\mathsf{FL}'(\phi)$ of the FL closure where some formulae ψ are replaced with fresh propositional letters q_ψ: $\mathsf{FL}'(\phi)$ is the least set Γ of formulae containing ϕ and closed under the rules

1. if $\neg\psi \in \Gamma$, then $\psi \in \Gamma$;
2. if $\chi \wedge \psi \in \Gamma$, then $\chi, \psi \in \Gamma$;
3. if $[a]\psi \in \Gamma$, then $\psi \in \Gamma$
4. if $[\alpha \cup \beta]\psi \in \Gamma$, then $[\alpha]q_\psi, [\beta]q_\psi, \psi \in \Gamma$
5. if $[\alpha; \beta]\psi \in \Gamma$, then $[\alpha]q_{[\beta]\psi}, [\beta]\psi \in \Gamma$;

6. if $[\alpha^*]\psi \in \Gamma$, then $\psi, [\alpha]q_{[\alpha^*]\psi} \in \Gamma$.

We have an obvious function e on formulae that fully expands these abbreviations (e.g. $e([a]q_{[b]q_{[c]p}}) = [a][b][c]p$).

(b) Give an explicit recursive definition of e (arguing why it terminates).

(c) Show by induction on the generation of $\mathsf{FL}'(\phi)$ that whenever q_ψ occurs in a formula in $\mathsf{FL}'(\phi)$, then $\psi \in \mathsf{FL}'(\phi)$.

(d) Show by induction on the generation of $\mathsf{FL}'(\phi)$ that $e(\psi) \in \mathsf{FL}(\phi)$ for every $\psi \in \mathsf{FL}'(\phi)$.

(e) Show by induction on the generation of $\mathsf{FL}(\phi)$ that for every $\psi \in \mathsf{FL}(\phi)$, there is some $\psi' \in \mathsf{FL}'(\phi)$ such that $e(\psi') = \psi$ (in other words, e defines a surjective map $\mathsf{FL}'(\phi) \to \mathsf{FL}(\phi)$). This part requires some care; in particular, you will need to use (c).

Now we define a size measure δ on formulae that discounts the fresh letters; specifically, $\delta(\psi)$ is the number of occurrences of propositional connectives $\bot, \neg, \wedge$, program constructs $;, \cup, *$, atomic programs, and propositional letters other than the q_χ in ψ. The rules effectively all have the form 'if $\psi \in \Gamma$, then $\chi_1, \dots, \chi_n \in \Gamma$'.

(f) Show that in all rules,

$$\delta(\{\psi\}) = 1 + \delta(\chi_1) + \cdots + \delta(\chi_n).$$

Note (no proof necessary) that each formula ψ is matched by exactly one rule of the above form, and then

$$\mathsf{FL}'(\psi) = \{\psi\} \cup \mathsf{FL}'(\chi_1) \cup \cdots \cup \mathsf{FL}'(\chi_n).$$

(g) Using the above, show by induction on $\delta(\psi)$ that

$$|\mathsf{FL}'(\psi)| \leq \delta(\psi)$$

for all ψ, and conclude the claim from this fact.