Modal Logic   Summer semester 2025

Exercise Sheet 4

Solutions due: 14.07.2025

Exercise 1 Complexity of $\mathbf{S5_n}$ 8 points

Recall that **S5** is the normal modal logic that is generated by $\{(T), (4), (B)\}$ and recall that **S5** is sound and complete w.r.t. the class of frames $(\mathcal{W}, \mathcal{R})$ where $\mathcal{R}$ is an equivalence relation. We have seen that $SAT(\mathbf{S5})$ (the satisfiability problem of **S5**) is NP-complete.

This is not true for $\mathbf{S5_n}$, the logic of frames that have n equivalence relations $\mathcal{R}_1, \dots, \mathcal{R}_n$, when $n \geq 2$. These logics are generated by n copies of the axioms (T), (4) and (B), where the axioms

$$\begin{array}{ll} (\mathrm{T}_i) & p \to \Diamond_i p \\ (4_i) & \Diamond_i \Diamond_i p \to \Diamond_i p \\ (\mathrm{B}_i) & p \to \Box_i \Diamond_i p \end{array}$$

axiomatize the relation $\mathcal{R}_i$ to be an equivalence relation.

We note without proof that the satisfiability problem of the logic **T** (the logic of reflexive frames, axiomatized as the least normal modal logic containing the (T) axiom $\Box p \to p$) is PSpace-hard (a proof can be found in the lecture notes for 'Description logics and formal ontologies'). Show that satisfiability in $\mathbf{S5_2}$ is PSpace-hard by giving a satisfiability-preserving translation of **T** into $\mathbf{S5_2}$)

Hint: The translation really is only satisfiability-preserving, not equivalent. You may need to construct new models from given ones; in particular, the models you construct may need to have additional states.

Exercise 2 Hoare Logic 7 points

We can encode *Hoare logic* into PDL: A *Hoare triple* $\{\phi\}\ \alpha\ \{\psi\}$ states that *if program α is executed in a state satisfying ϕ, then every poststate reached after termination of α satisfies ψ.* (Note the universal quantification over poststates, owed to the fact that α is nondeterministic.) We can view $\{\phi\}\alpha\{\psi\}$ as an abbreviation for the PDL formula $\phi \to [\alpha]\psi$. Recall that we can encode the *while* loop construct into PDL-style programs as *while* ϕ *do* $\alpha = (\phi?; \alpha)^*(\neg\phi?)$.

1. For $\beta =$ *while* ϕ *do* α, derive the unfolding law

$$[\beta]\psi \leftrightarrow \big((\neg\phi \to \psi) \wedge (\phi \to [\alpha][\beta]\psi)\big)$$

2. Derive the standard Hoare rule for while,

$$\frac{\{\psi \wedge \phi\}\ \alpha\ \{\psi\}}{\{\psi\}\ \textit{while}\ \phi\ \textit{do}\ \alpha\ \{\psi \wedge \neg\phi\}}$$

Exercise 3 Induction as a Rule 5 points

Show that the PDL induction axiom is interderivable with the induction rule

$$\frac{\phi \to [\alpha]\phi}{\phi \to [\alpha^*]\phi}$$

– that is, give deriviations of one from the other in both directions, using the remaining rules and axioms of the system. Derivability of the above rule means that its conclusion is derivable from the premiss.