Modal Logic   Summer semester 2025

Exercise Sheet 3

Solutions due: 30.06.2025

Exercise 1 Incompleteness — 7 points

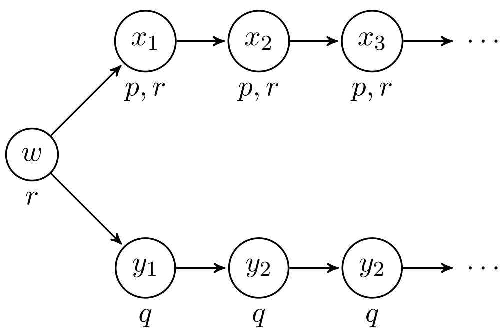
Complete the proof that **KvB** is a normal modal logic by showing that it is closed under uniform substitution. *Hint:* Show by induction on formulae that the extension of every formula in a model over $\mathfrak{F}_\omega$ is either finite and does not contain ω, or cofinite and does contain ω. The actual proof of closure under uniform substitution will then require a substitution lemma as discussed in the lecture; it will otherwise be analogous to the standard proof of soundness of uniform substitution for satisfaction over frames.

Exercise 2 Smallest and Largest Filtrations — 5 points

Show that the smallest and the largest filtration as defined in the lecture are indeed filtrations.

Exercise 3 Filtration: Examples — 8 points

1. Consider the model $\mathfrak{M} = (W, R, V)$



given by

$$\begin{aligned} W &= \{w\} \cup \{x_i \mid i > 0\} \cup \{y_i \mid i > 0\} \\ R &= \{(w, x_1), (w, y_1)\} \cup \{(x_i, x_{i+1}) \mid i > 0\} \cup \{(y_i, y_{i+1}) \mid i > 0\} \\ V(p) &= \{x_i \mid i > 0\} \\ V(q) &= \{y_i \mid i > 0\} \\ V(r) &= \{w\} \cup \{x_i \mid i > 0\} \end{aligned}$$

Let $\Gamma = \{\Diamond p, \Diamond q, \Diamond r, p, q, r\}$. Compute $\sim_\Gamma$ and use this relation to find

- the smallest filtration of $\mathfrak{M}$ through Γ, and
- the largest filtration of $\mathfrak{M}$ through Γ.

2. Prove that for all models $\mathfrak{M}$ based on a symmetric frame and all sets Γ of formulae,

(a) the largest filtration of $\mathfrak{M}$ through Γ does *not* necessarily have a symmetric relation.

(b) the smallest filtration of $\mathfrak{M}$ through Γ has a symmetric relation.

To prove Item (a), it suffices to find a counterexample. Use Item (b) to show that satisfiability and validity are decidable in **B** are decidable.