Modal Logic   Summer semester 2025

Exercise Sheet 2

Solutions due: 11.06.2025

Exercise 1 The K Axiom as a Proof Rule

Let Λ be a modal logic.

1. Show that Λ is normal iff Λ is closed under proof rule

$$(RK)\ \frac{\phi_1 \wedge \dots \phi_n \to \phi_0}{\Box_i \phi_1 \wedge \dots \Box_i \phi_n \to \Box_i \phi_0}\ (n \geq 0, i \in I).$$

2. Conclude that every normal modal logic is closed under replacement of equivalents, i.e. under the rule

$$(RPE)\ \frac{\phi \leftrightarrow \psi}{\Box_i \phi \leftrightarrow \Box_i \psi}.$$

Exercise 2 Hilbert-style Proofs

(Blackburn et al., Ex. 1.6.1 and 1.6.3.)

1. Give proofs of the formulae

$$(\Box p \wedge \Diamond q) \to \Diamond (p \wedge q)$$
$$\Diamond (p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$$

   in $\mathbb{K}$.

2. Recall that **S5** is defined by the axioms $\Box p \to p$, $\Box p \to \Box\Box p$, $p \to \Box\Diamond p$. Give a proof of the formula

$$\Diamond\Box p \to \Box p$$

   in **S5**.

Exercise 3 Provability in K

(Blackburn et al., Ex. 1.6.2.)

Given a modal formula ϕ, write ϕ^- for the propositional formula obtained by erasing all modalities from ϕ (preserving the bracketing, of course). Give an explicit definition of ϕ^- by structural recursion on formulae. Show by induction on derivations that whenever $\phi \in \mathbf{K}$, then ϕ^- is a propositional tautology. Conclude that **K** is consistent.

Exercise 4 Canonical models

We consider the normal modal logic **KF** that is generated by the axiom

$$\text{(F)} \quad \Box p \leftrightarrow \Diamond p.$$

Find a suitable class of frames $S_{\mathbf{KF}}$ for which **KF** is sound and complete; in detail:

1. Show that **KF** is sound with respect to your candidate $S_{\mathbf{KF}}$, that is, *prove* that indeed $S_{\mathbf{KF}} \models \text{(F)}$.
2. Use the canonical model construction to show that **KF** is complete with respect to your candidate $S_{\mathbf{KF}}$; that is, show that the canonical **KF**-model is in your class $S_{\mathbf{KF}}$.