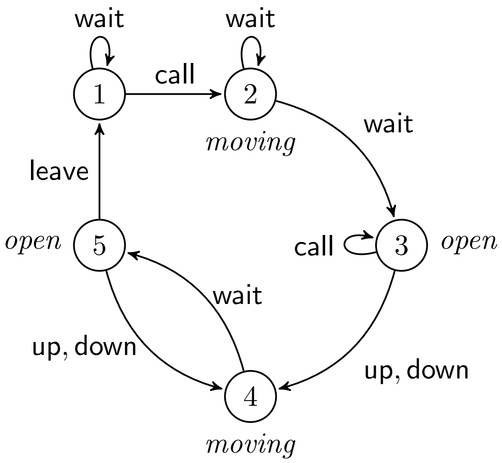
Modal Logic Summer semester 2025

Exercise Sheet 1

Solutions due: 26.05.2025

Exercise 1 Satisfaction **6 Points**

Let $\mathcal{A} = \{moving, open\}$ and $I = \{\mathsf{call}, \mathsf{wait}, \mathsf{up}, \mathsf{down}, \mathsf{leave}\}$. Consider the following model of an elevator controller.



Formally, the model is defined as $\mathfrak{M} = (W, (R_{\mathsf{call}}, R_{\mathsf{wait}}, \dots), V)$ where

$$\begin{aligned} W &= \{1,2,3,4,5\} & R_{\mathsf{call}} &= \{(1,2),(3,3)\} \\ R_{\mathsf{leave}} &= \{(5,1)\} & R_{\mathsf{wait}} &= \{(1,1),(2,2),(2,3),(4,5)\} \\ R_{\mathsf{up}} &= \{(3,4),(5,4)\} & R_{\mathsf{down}} &= \{(3,4),(5,4)\} \\ V(open) &= \{3,5\} & V(moving) &= \{2,4\}. \end{aligned}$$

Of course, *open* is meant to indicate that the elevator door is open, and *moving* says that the elevator is in motion. Transitions correspond to user interactions (call – the user presses the call button; wait – the user is waiting; $\mathsf{up}, \mathsf{down}$ – the user presses the up / down button in the – somewhat simplistic – elevator ; leave – the user leaves the elevator).

1. For three pairs of states $i, j \in W$, find modal formulae ψ that distinguish i from j, that is, $\mathfrak{M}, i \models \psi$ but $\mathfrak{M}, j \not\models \psi$. In each case, explain briefly why the respective formula does or does not hold. Are there distinct states in the model that cannot be distinguished by any modal formula? If yes, explain for your example why the two states cannot be distinguished.

2. Check whether the following hold:

   (a) $\mathfrak{M}, 1 \models open \lor moving$

   (b) $\mathfrak{M}, 1 \models \Diamond_{\mathsf{call}} \Box_{\mathsf{wait}}\, open$

   (c) $\mathfrak{M}, 4 \models \Box_{\mathsf{wait}}(\Diamond_{\mathsf{leave}} \top \land \Box_{\mathsf{call}}\, open)$

(d) $\mathfrak{M}, 2 \models \Box_{\mathsf{wait}}\Diamond_{\mathsf{down}}\mathit{moving} \to \Diamond_{\mathsf{call}}\mathit{open}$

In each case, explain briefly why the formula does or does not hold.

3. Formalize the following statements as modal formulae:

   (a) "If the doors are open, then one can go up or down." (What does the natural language expression 'or' really mean here?)

   (b) "The elevator is moving, and in the next step, the elevator will be open (no matter what happens)."

   For both formulae, determine in which states of the model they hold (no explanation necessary this time). Is one of the formulae globally satisfied in the model?

Exercise 2 Satisfiability, Validity 5 Points

1. Show that the following formulae are not valid in the class of all frames:

$$\Diamond\top \qquad\qquad \Diamond p \to \Box p \qquad\qquad \Diamond\Box p \to \Box\Diamond p$$

   For each of these formulae, find and describe a class of frames where the formula is valid. Choose the respective classes as large as possible. Prove these claims.

2. Let S, S_R, S_S, and S_T be the classes of all frames, all reflexive frames, all symmetric frames, and all transitive frames, respectively.

   (a) Find a formula ψ_1 such that $\models_{S_T} \psi_1$ and $\not\models_{S_R} \psi_1$.

   (b) Find a satisfiable formula ψ_2 that is unsatisfiable over S_R.

   (c) Find a formula ψ_3 such that $\models_{S_S} \psi_3$ and $\not\models_{S} \psi_3$.

   In each case, prove your claims.

Exercise 3 Standard Translation 3 Points

Translate the following modal formulae into FOL, i.e.. compute $ST_x(\psi_i)$ for $i = 1, \ldots, 4$:

$$\psi_1 := \Box p \to p \qquad\qquad \psi_2 := \Diamond\Diamond p \to \Diamond p$$
$$\psi_3 := \Diamond\Diamond p \land \Box(q \to \Diamond p) \qquad\qquad \psi_4 := \Box_3(q \to \Diamond_5 p) \land \neg\Diamond_4\Diamond_1 p, \text{ wobei } I = \mathbb{N}$$

Use only two different variable names x, y.

Exercise 4 Bisimulation Games 6 Points

Show that bisimulation games characterize bisimilarity, that is, for all models $\mathfrak{M}$, $\mathfrak{M}'$ and all states w in $\mathfrak{M}$, w' in $\mathfrak{M}'$,

$$\mathfrak{M}, w \simeq \mathfrak{M}', w' \quad\Leftrightarrow\quad \exists \text{ wins the position } (w, w') \text{ in } \mathfrak{G}(\mathfrak{M}, \mathfrak{M}')$$

where $\mathfrak{G}(\mathfrak{M}, \mathfrak{M}')$ is the bisimulation game between $\mathfrak{M}$ and $\mathfrak{M}'$.