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Weak completeness of PDL, cont'd

$A \overset{!}{S_\alpha} B \iff \hat{A} \wedge \langle\alpha\rangle\hat{B}$ consistent

Regular frame: R_α defined recursively

$$R_a = S_a \quad (a \text{ atomic})$$

$$R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$$

$$R_{\alpha;\beta} = R_\alpha ; R_\beta$$

$$R_{\alpha^*} = (R_\alpha)^*$$

Plan: Show $S_\alpha \subseteq R_\alpha$

Show existence lemma for S_α, inherit it for R_α

Truth lemma: New treatment of R_{α^*}

<u>Existence lemma</u>: Let $\langle\alpha\rangle\varphi \in^{(\neg)} FL(\varphi)$, $\langle\alpha\rangle\varphi \in A$. Then there exists B s.t. $\varphi \in B$ and $A \, S_\alpha \, B$.

<u>Proof</u>: Let $FL(\varphi) = \{\sigma_1, \ldots, \sigma_n\}$

Define $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n$ s.t. $\hat{A} \wedge \langle\alpha\rangle\hat{B}_i$ consistent.

$(B_0 = \{\varphi\})$

Induction base: $\hat{A} \wedge \underbrace{\langle\alpha\rangle\varphi}_{\in A}$ consistent

$B_i \rightsquigarrow B_{i+1}$:

Note: $(\hat{B}_i \wedge \sigma_{i+1}) \vee (\hat{B}_i \wedge \neg\sigma_{i+1}) \equiv \hat{B}_i$

So $\quad \langle\alpha\rangle \hat{B}_i \equiv \langle\alpha\rangle\big((\hat{B}_i \wedge \sigma_{i+1}) \vee (\hat{B}_i \wedge \neg\sigma_{i+1})\big)$

$\equiv \langle\alpha\rangle(\hat{B}_i \wedge \sigma_{i+1}) \vee \langle\alpha\rangle(\hat{B}_i \wedge \neg\sigma_{i+1})$

Here (by IH) $\hat{A} \wedge \langle\alpha\rangle \hat{B}_i$ consistent

[N. B. $\varphi \wedge (\psi \vee \chi)$ consistent $\Rightarrow$

$\varphi \wedge \psi$ consistent or $\varphi \wedge \chi$ consistent.

For otherwise: $\varphi \to \neg\psi$, $\varphi \to \neg\chi$ derivable

$\Rightarrow$ $\varphi \to (\neg\psi \wedge \neg\chi)$ derivable

$\Rightarrow$ $\varphi \wedge (\psi \vee \chi)$ inconsistent]

$\Rightarrow$ one of $\hat{A} \wedge \langle\alpha\rangle(\hat{B}_i \wedge \sigma_{i+1})$, $\hat{A} \wedge \langle\alpha\rangle(\hat{B}_i \wedge \neg\sigma_{i+1})$

consistent, add that $\sigma_{i+1} / \neg\sigma_{i+1}$ to B_i to get B_{i+1}

B_n maximally consistent (i.e. atom),

$A \, S_\alpha \, B_n$, $\psi \in B_n$ $\quad \square$

Lemma: $S_\alpha \subseteq R_\alpha$

Proof: Induction on α.

($\alpha =$) a: By definition

$\alpha ; \beta$: Done by IH once

$S_{\alpha;\beta} \subseteq S_\alpha ; S_\beta$

So let $\hat{A} \wedge \langle\alpha;\beta\rangle \hat{B}$ consistent

$\Rightarrow_{\text{Axioms}}$ $\hat{A} \wedge \langle\alpha\rangle\langle\beta\rangle \hat{B}$ "

Need C s.t. $\hat{A} \wedge \langle\alpha\rangle \hat{C}$, $\hat{C} \wedge \langle\beta\rangle \hat{B}$ consistent

Put $\mathcal{C} = \{ C \mid \hat{C} \wedge \langle\beta\rangle \hat{B} \}$

$\gamma = \bigvee_{C \in \mathcal{C}} \hat{C}$

Suffices: $\hat{A} \wedge \langle\alpha\rangle\gamma$ consistent

Suppose not; then $\vdash \hat{A} \to [\alpha]\neg\gamma$

$\Rightarrow$ $\hat{A} \wedge \langle\alpha\rangle(\neg\gamma \wedge \langle\beta\rangle\hat{B})$ consistent

$\Rightarrow$ $\neg\gamma \wedge \langle\beta\rangle\hat{B}$ consistent $\lightning$

$\alpha \cup \beta$: Done by IH once

$$S_{\alpha\cup\beta} \subseteq S_\alpha \cup S_\beta$$

So let $\hat{A} \wedge \langle\alpha\cup\beta\rangle\hat{B}$ consistent

$\underset{\text{Axioms}}{\Rightarrow}$ $\hat{A} \wedge (\langle\alpha\rangle\hat{B} \vee \langle\beta\rangle\hat{B})$ consistent

$\Rightarrow$ wlog $\hat{A} \wedge \langle\alpha\rangle\hat{B}$ consistent

$\Rightarrow$ $A \, S_\alpha \, B$ ✓

α^*: Done by IH once

$$S_{\alpha^*} \subseteq (S_\alpha)^*$$

So let $A \, S_{\alpha^*} \, B$

$$\mathcal{D} = \{ D \mid A (S_\alpha)^* D \}$$

$$\delta = \bigvee_{D \in \mathcal{D}} \hat{D}$$

Show $\delta \to [\alpha^*]\delta$ $\quad (*)$

Then: Let $\hat{A} \wedge \langle\alpha^*\rangle\hat{B}$ consistent

$\underset{\vdash \hat{A} \to \delta, (*)}{\Rightarrow}$ $\hat{A} \wedge \langle\alpha^*\rangle(\delta \wedge \hat{B})$ consistent

$\Rightarrow$ $\delta \wedge \hat{B}$ consistent $\Rightarrow B \in \mathcal{D}$, done.

$(*)$ by Segeberg: suffices $\vdash [\alpha^*](\delta \to [\alpha]\delta)$,
suffices $\vdash \delta \to [\alpha]\delta$

Suppose not $\Rightarrow$ $\delta \wedge \langle\alpha\rangle\neg\delta$ consistent

$$\neg\delta \equiv \bigvee_{D\notin\mathcal{D}} \hat{D}$$

$\Rightarrow$ ex. $D\in\mathcal{D}$, $E\notin\mathcal{D}$ s.t. $\hat{D} \wedge \langle\alpha\rangle\hat{E}$ consistent

$\Rightarrow$ $D \; S_\alpha \; E$ $\lightning$

Details: right-to-left:

$E\notin\mathcal{D} \Rightarrow \forall D\in\mathcal{D}\; \underbrace{\exists\psi\in FL(\varphi).\ \psi\in E \not\Leftrightarrow \psi\in D}_{\hat{E}\rightarrow\neg\hat{D}}$

thus, $\hat{E}\rightarrow\neg\delta$

thus, rhs $\rightarrow\neg\delta$

left-to-right: Show $\vdash \bigvee_{A\in At} \hat{A}$

Suppose not, then ex. B s.t.

$\forall\psi\in FL(\varphi).\ \psi\in B \vee \neg\psi\in B \;\wedge$

B consistent, $B\notin At$ $\lightning$