

Towards Generalizing Probability Monads

Nico Wittrock

Oberseminar at Chair for Theoretical Computer Science,
University of Erlangen-Nuremberg,

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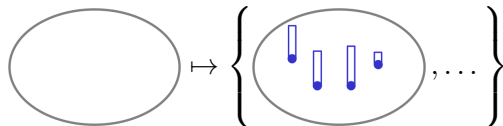
Discrete Monads

- ▶ power-set monad

$$\begin{aligned}\mathcal{P} : \text{Set} &\rightarrow \text{Set} \\ X &\mapsto \{S \subseteq X\}\end{aligned}$$

- ▶ distribution monad

$$\begin{aligned}\mathcal{D} : \text{Set} &\rightarrow \text{Set} \\ X &\mapsto \{\text{finite distributions on } X\}\end{aligned}$$

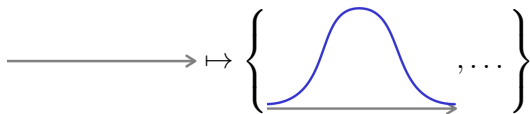


Continuous Monads

- ▶ Giry monad

$$\mathcal{G} : \text{Meas} \rightarrow \text{Meas}$$

$$(X, \Sigma) \mapsto (\{\text{probability-measures on } X\}, \Sigma_{\text{init}})$$



- ▶ variations thereof:

$$\mathcal{G}|_{\text{Pol}} : \text{Pol} \rightarrow \text{Pol}$$

on polish spaces

$$\text{Radon monad } \mathcal{R} : \text{CHaus} \rightarrow \text{CHaus}$$

on cpct. Hausdorff sp.

Kleisli Categories

consider monad $\mathcal{T} : D \rightarrow D$

Kleisli cat.

$$\text{Kl}_{\mathcal{T}} \begin{cases} \text{objects: } \text{Obj}_{\text{Kl}_{\mathcal{T}}} := \text{Obj}_D \\ \text{morphisms: } \text{Hom}_{\text{Kl}_{\mathcal{T}}}(A, B) := \text{Hom}_D(A, \mathcal{T}B) \end{cases}$$

Kleisli Categories

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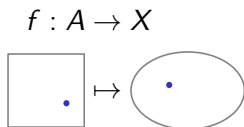
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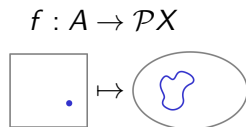
Kleisli Categories

Example (Power-Set Monad)

► maps in Set:



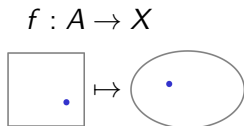
► maps in $\text{Kl}_{\mathcal{P}}$:



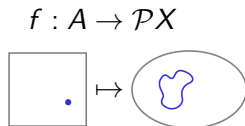
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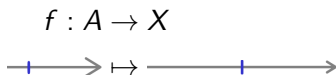


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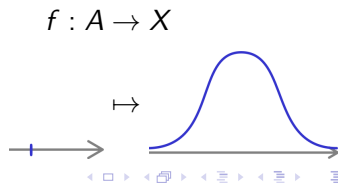


Example (Giry Monad)

► maps in Meas:



► maps in $\text{Kl}_{\mathcal{G}}$:



Symmetric Monoidal Kleisli Categories

We consider monad $\mathcal{T} : D \rightarrow D$.

- ▶ D is cartesian monoidal
- ▶ \mathcal{T} is *commutative*, i. e. , comes with “zipper map”

$$\nabla_{X,Y} : \mathcal{T}X \times \mathcal{T}Y \rightarrow \mathcal{T}(X \times Y)$$

$\text{Kl}_{\mathcal{T}}$ is symmetric monoidal via \otimes

$$\left\{ \begin{array}{l} \text{on objects: } X \otimes Y := X \times Y \\ \text{on morphisms } f : A \rightarrow \mathcal{T}X, \\ \quad \quad \quad g : B \rightarrow \mathcal{T}Y \\ f \otimes g : X \times Y \xrightarrow{f \times g} \mathcal{T}X \times \mathcal{T}Y \xrightarrow{\nabla_{X,Y}} \mathcal{T}(X \times Y) \end{array} \right.$$

Symmetric Monoidal Kleisli Categories

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$$\nabla_{X,Y} : \mathcal{T}X \times \mathcal{T}Y \rightarrow \mathcal{T}(X \times Y)$$

$Kl : D \rightarrow Kl_{\mathcal{T}}$ preserves tensor products

Symmetric Monoidal Kleisli Categories

Example (Power-Set Monad)

$\mathcal{P} : \text{Set} \rightarrow \text{Set}$

$$\nabla_{X,Y} : \mathcal{P}X \times \mathcal{P}Y \rightarrow \mathcal{P}(X \times Y)$$

$$(S, T) \mapsto S \times T \subseteq X \times Y$$

Symmetric Monoidal Kleisli Categories

Example (Power-Set Monad)

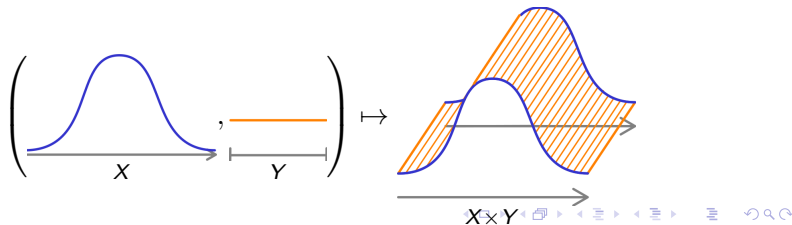
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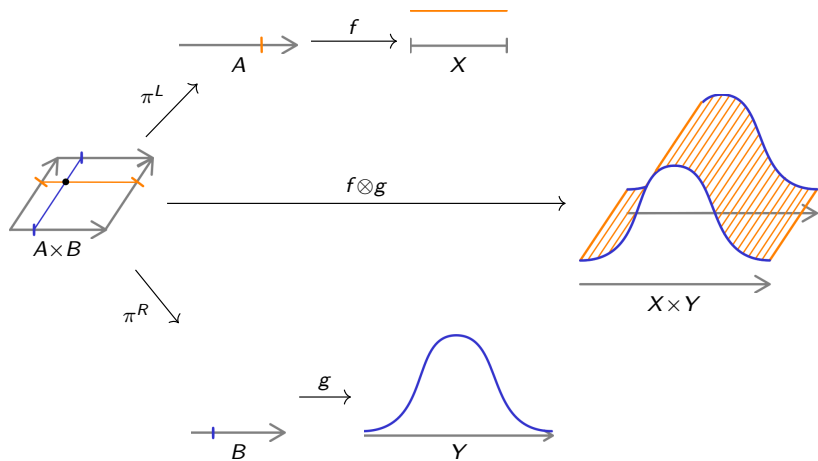
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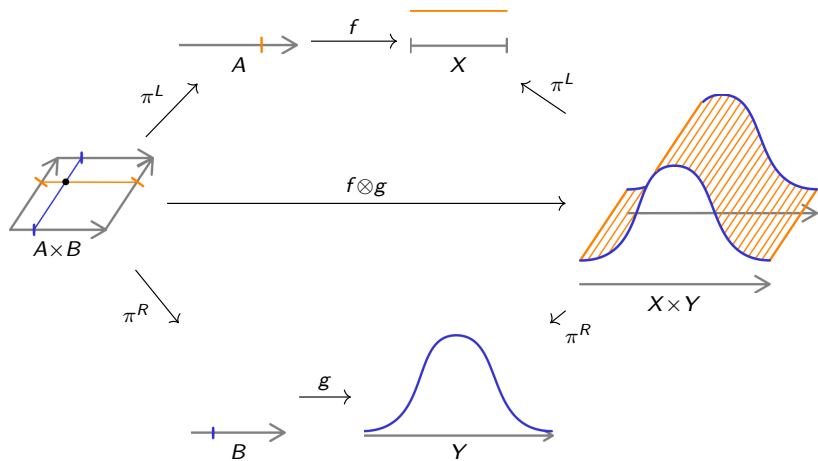
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Symmetric Monoidal Kleisli Categories: Giry



Symmetric Monoidal Kleisli Categories: Giry



Probability Monads

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow \mathcal{K}l\pi^L & & \uparrow \mathcal{K}l\pi^L \\ A \otimes B & \xrightarrow{f \otimes g} & X \otimes Y \\ \downarrow \mathcal{K}l\pi^R & & \downarrow \mathcal{K}l\pi^R \\ B & \xrightarrow{g} & Y \end{array}$$

Probability Monads

Definition

A *probability monad* is a monad $\mathcal{T} : \mathcal{D} \rightarrow \mathcal{D}$ such that

- ▶ \mathcal{D} is cartesian monoidal
- ▶ \mathcal{T} is commutative monad
- ▶ $\mathcal{Kl} : \mathcal{D} \rightarrow \mathcal{Kl}_{\mathcal{T}}$ preserves projections:
in $\mathcal{Kl}_{\mathcal{T}}$ commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow \mathcal{Kl}\pi^L & & \uparrow \mathcal{Kl}\pi^L \\ A \otimes B & \xrightarrow{f \otimes g} & X \otimes Y \\ \downarrow \mathcal{Kl}\pi^R & & \downarrow \mathcal{Kl}\pi^R \\ B & \xrightarrow{g} & Y \end{array}$$

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Example

Giry monad \mathcal{G} , Radon monad \mathcal{R} , distribution monad \mathcal{D} .

Power-set monad \mathcal{P} is *no* probability monad.

Weak Products

Consider probability monad $\mathcal{T} : \mathcal{D} \rightarrow \mathcal{D}$ with $\mathcal{Kl} : \mathcal{D} \rightarrow \mathcal{Kl}_{\mathcal{T}}$.

- ▶ \mathcal{D} has products:

$$\begin{array}{ccccc} & & A & \xrightarrow{f} & X \\ & & \uparrow \pi^L & & \uparrow \pi^L \\ A & \xrightarrow{\Delta} & A \times A & \xrightarrow{f \times g} & X \times Y \\ & & \downarrow \pi^R & & \downarrow \pi^R \\ & & A & \xrightarrow{g} & Y \end{array}$$

Weak Products

Consider probability monad $\mathcal{T} : \mathcal{D} \rightarrow \mathcal{D}$ with $\mathcal{Kl} : \mathcal{D} \rightarrow \mathcal{Kl}_{\mathcal{T}}$.

- ▶ $\mathcal{Kl}_{\mathcal{T}}$ has *weak* products:

The diagram illustrates the weak product property in the Kleisli monad. It consists of the following components:

- Top row:** $A \xrightarrow{f} X$
- Bottom row:** $A \xrightarrow{g} Y$
- Middle row:** $A \otimes A \xrightarrow{f \otimes g} X \otimes Y$
- Left vertical arrows:** $A \xrightarrow{\mathcal{Kl}\Delta} A \otimes A$ (pointing right) and $A \otimes A \xrightarrow{\mathcal{Kl}\pi^L} A$ (pointing up) and $A \otimes A \xrightarrow{\mathcal{Kl}\pi^R} A$ (pointing down).
- Right vertical arrows:** $X \otimes Y \xrightarrow{\mathcal{Kl}\pi^L} X$ (pointing up) and $X \otimes Y \xrightarrow{\mathcal{Kl}\pi^R} Y$ (pointing down).
- Curved arrows:** Two curved arrows originate from the leftmost A and terminate at the top and bottom A nodes, representing the multiplication μ of the monad.

Markov Categories

$$\begin{array}{ccc} & A & \xrightarrow{f} X \\ & \uparrow \pi^L & \uparrow \pi^L \\ A & \xrightarrow{(\text{id}, \text{id})} A \otimes A & \xrightarrow{f \otimes g} X \otimes Y \\ & \downarrow \pi^R & \downarrow \pi^R \\ & A & \xrightarrow{g} Y \end{array}$$

Diagram (1) illustrates a Markov category structure. It shows a commutative square with an additional map from the left object A to the top-left object A . The top-left object is A , the top-right is X , the bottom-left is $A \otimes A$, and the bottom-right is $X \otimes Y$. The map from A to X is f . The map from $A \otimes A$ to $X \otimes Y$ is $f \otimes g$. The map from A to Y is g . The map from A to $A \otimes A$ is (id, id) . The map from $A \otimes A$ to A is π^R . The map from $X \otimes Y$ to X is π^L . The map from $X \otimes Y$ to Y is π^R . There are two curved arrows from the left A to the top A and the bottom A .

(1)

$$\begin{array}{ccc} & & X \\ & \nearrow f & \uparrow \pi^L \\ A & \xrightarrow{(f, g)} X \otimes Y & \\ & \searrow g & \downarrow \pi^R \\ & & Y \end{array}$$

Diagram (2) illustrates a Markov category structure. It shows a commutative square with an additional map from the left object A to the top-right object $X \otimes Y$. The top-left object is A , the top-right is X , the bottom-left is $X \otimes Y$, and the bottom-right is Y . The map from A to X is f . The map from $X \otimes Y$ to X is π^L . The map from $X \otimes Y$ to Y is π^R . The map from A to $X \otimes Y$ is (f, g) . The map from A to Y is g .

(2)

Markov Categories

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \uparrow \pi^L & & \uparrow \pi^L \\ A \otimes A & \xrightarrow{f \otimes g} & X \otimes Y \\ \downarrow \pi^R & & \downarrow \pi^R \\ A & \xrightarrow{g} & Y \end{array} \quad (1)$$
$$\begin{array}{ccc} & & X \\ & \nearrow f & \uparrow \pi^L \\ A & \xrightarrow{(f,g)} & X \otimes Y \\ & \searrow g & \downarrow \pi^R \\ & & Y \end{array} \quad (2)$$

Definition

A *Markov Category* is cat. with weak products and chosen morphisms (f, g) as in (2), such that

- ▶ they canonically induce a symmetric monoidal structure
- ▶ diagrams (1), (2) coincide: $(f \otimes g) \circ (\text{id}, \text{id}) = (f, g)$

Markov Categories

- ▶ Symmetric monoidal cat.s have graphical notation

$$\begin{aligned} f : X \rightarrow Y &\equiv \begin{array}{c} X \text{ --- } \boxed{f} \text{ --- } Y \end{array} \\ (\text{id}, \text{id}) : X \rightarrow X \otimes X &\equiv \begin{array}{c} X \text{ --- } \bullet \begin{array}{l} \text{--- } X \\ \text{--- } X \end{array} \end{array} \\ \pi^L : X \otimes Y \rightarrow X &\equiv \begin{array}{c} X \text{ --- } X \\ Y \text{ --- } \bullet \end{array} \end{aligned}$$

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- ▶ leads to usual definition of Markov cat.s
 - ▶ more explicit [Fri20]
 - ▶ generalization: CD-cat.s (Cho and Jacobs [CJ19])

Markov categories

Example

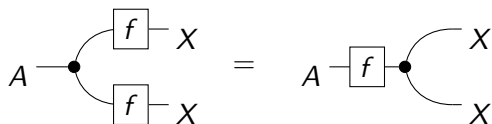
Kleisli cat.s of probability monads.

In particular: cartesian monoidal cat.s.

Deterministic Morphisms

Definition

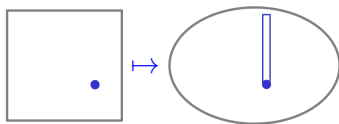
Morphism $f : A \rightarrow X$ in Markov cat. is *deterministic*, if



The diagram shows an equality between two morphisms. On the left, a morphism from A to X is represented by a dot with two outgoing arcs, each leading to a box labeled f and then to X . On the right, a morphism from A to X is represented by a box labeled f followed by a dot with two outgoing arcs, each leading to X .

Example

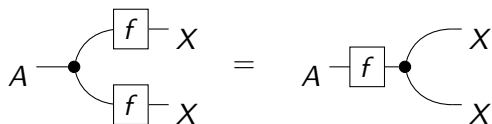
1. In Kleisli cat. $\text{Kl}_{\mathcal{D}}$ of distribution monad: deterministic morphisms are of the form



Deterministic Morphisms

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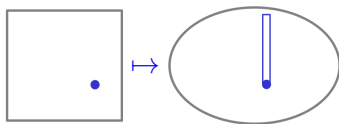
Morphism $f : A \rightarrow X$ in Markov cat. is *deterministic*, if



The diagram shows an equality between two morphisms. On the left, a morphism from A to X is represented by a dot on the left with two curved lines branching out to the right, each ending in a box labeled f and then X . On the right, the same morphism is represented by a box labeled f on the left, followed by a dot, and then two curved lines branching out to the right, each ending in X .

Example

1. In Kleisli cat. $\text{Kl}_{\mathcal{D}}$ of distribution monad: deterministic morphisms are of the form



2. In Kleisli cat. $\text{Kl}_{\mathcal{T}}$: all $\mathcal{Kl}(f)$ are deterministic

Representable Markov Categories

Consider Markov cat. \mathcal{C} .

Lemma

Deterministic morphisms form subcategory $\mathcal{C}_{\text{det}} \subseteq \mathcal{C}$.

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Kleisli cat. $\text{Kl}_{\mathcal{T}}$ of probability monad $\mathcal{T} : D \rightarrow D$ is *representable*, if

$$(\text{Kl}_{\mathcal{T}})_{\text{det}} \cong D$$

Representable Markov Categories

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$$(\text{Kl}_{\mathcal{T}})_{\text{det}} \cong D$$

Example

1. Kleisli cat. $\text{Kl}_{\mathcal{D}}$ of distribution monad is representable.
2. Kleisli cat. $\text{Kl}_{\mathcal{G}}$ of Giry monad is *not*.

Polish Spaces

Kl_G of Giry monad is *not* representable.

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$\text{Kl}_{\mathcal{G}}$ of Giry monad is *not* representable.

But: restriction to Polish spaces $\text{Pol} \subseteq \text{Meas}$

$$\mathcal{B} := \mathcal{G}|_{\text{Pol}} : \text{Pol} \rightarrow \text{Pol}$$

has representable Kleisli cat. $\text{Kl}_{\mathcal{B}} =: \text{BorelStoch}$:

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has representable Kleisli cat. $\text{Kl}_{\mathcal{B}} =: \text{BorelStoch}$:

$$\begin{array}{ccc} \text{Pol}(A, \mathcal{B}X) & \xlongequal{\text{def}} & \text{Kl}_{\mathcal{B}}(A, X) \\ \wr \parallel & & \parallel \\ \text{BorelStoch}_{\text{det}}(A, \mathcal{B}X) & & \text{BorelStoch}(A, X) \end{array}$$

$$\text{id}_{\mathcal{B}X} \longmapsto [\text{samp}_X : \mathcal{B}X \rightarrow X]$$

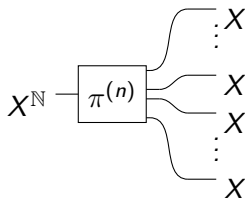
Countable Products

Lemma

BorelStoch *has inverse limits*

$$X^{\mathbb{N}} := \varprojlim_{n \in \mathbb{N}} X^{\otimes n}$$

with projections $\pi^{(n)} : X^{\mathbb{N}} \rightarrow X^{\otimes n}$, depicted as

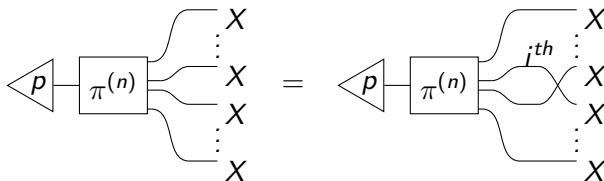


De Finetti's Theorem

Theorem

For a morphism $p : I \rightarrow X^{\mathbb{N}}$ in BorelStoch, it is equivalent:

1. p is invariant under finite permutations:
for all $n \in \mathbb{N}$ and $i < n$

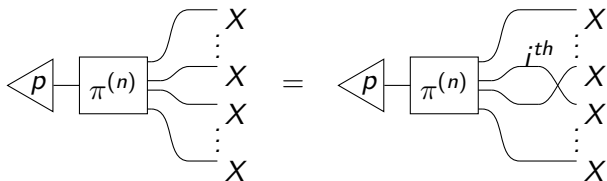


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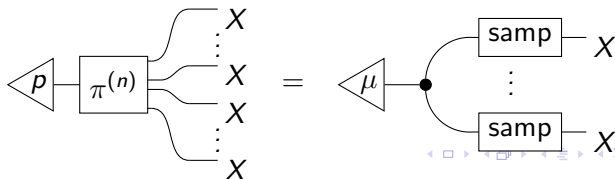
Theorem

For a morphism $p : I \rightarrow X^{\mathbb{N}}$ in BorelStoch, it is equivalent:

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2. there is a (unique) morphism $\mu : I \rightarrow \mathcal{B}X$ such that



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What's that?

Towards Imprecise Probability Monads

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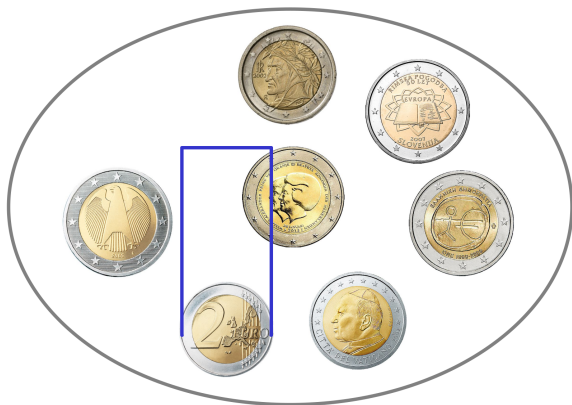
What's that?

Towards Non-Commutative Probability Monads

Flip a 2€ Coin



Flip a 2€ Coin



Probability theory is too “fine-grained”.

What is Imprecise Probability?

Many different uncertainty frameworks available:

What is Imprecise Probability?

Many different uncertainty frameworks available:

- ▶ Dempster-Schafer belief functions
- ▶ super-additive measures
- ▶ non-additive measures
- ▶ ...

Do they have probability monads?

Towards Imprecise Probability Monads

$$\begin{array}{ccc} \alpha \mapsto (f \mapsto \int f d\alpha) & & \\ \mathcal{R} & \begin{array}{c} \xrightarrow{\hspace{1.5cm}} \\ \left[[-, \mathbb{R}_{\geq 0}], \mathbb{R}_{\geq 0} \right]_{\text{lin}} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \longrightarrow \text{Cont} \\ (A \mapsto \omega(1_A)) \leftarrow \omega & & \end{array}$$

$$\begin{array}{ccc} \alpha \mapsto (f \mapsto \int f d\alpha) & & \\ \text{Gr} & \begin{array}{c} \xrightarrow{\hspace{1.5cm}} \\ \text{Rd} \\ \xleftarrow{\hspace{1.5cm}} \end{array} & \longrightarrow \text{Cont} \\ (A \mapsto \omega(1_A)) \leftarrow \omega & & \end{array}$$

Towards Super-Additive Measures

$$\begin{array}{ccc} \mathbf{Gr} & \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} & \mathbf{Rd} & \xrightarrow{\quad} & \mathbf{Cont} \\ & & \Downarrow & & \Downarrow \\ & & [[-, \mathbb{R}_{\geq 0}], \mathbb{R}_{\geq 0}]?? & & [[-, \mathbb{R}_{\geq 0}], \mathbb{R}_{\geq 0}] \end{array}$$

super-additive measures on X : ?? restrict $\omega : [X, \mathbb{R}_{\geq 0}] \rightarrow \mathbb{R}_{\geq 0}$

$$\begin{array}{ll} \alpha : PX \rightarrow [0, 1] & \omega(1) \stackrel{!}{=} 1 \\ \alpha(X) = 1 & \omega(\lambda f) \stackrel{!}{=} \lambda \omega(f) \\ \alpha(\emptyset) = 0 & f \leq g \stackrel{!}{\Rightarrow} \omega(f) \leq \omega(g) \\ \alpha(A \cup B) \geq \alpha(A) + \alpha(B) & \vdots \\ \quad - \alpha(A \cap B) & \end{array}$$

An Italian Flag



Further Work

- ▶ Further restrictions on **Rd** s.th.
 - ▶ **Rd** is monad
 - ▶ **Gr** are Dempster-Shafer belief functions / ...

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- ▶ What does that mean for its Kleisli cat.?
 - ▶ It is *not* monoidal.

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 - ▶ But *premonoidal*?

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- ▶ Further restrictions on **Rd** s.th.
 - ▶ **Rd** is monad
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- ▶ Is **Gr** a commutative monad? (Probably not)
- ▶ What does that mean for its Kleisli cat.?
 - ▶ It is *not* monoidal.
 - ▶ But *premonoidal*? A *monoidal effectful cat.*?
- ▶ Results from Markov cat.s?
- ▶ Relation to (other) Markov cat.s of imprecise probability
(So far: "just" special cases)

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What is Non-Commutative Probability?

- Probabilistic Gelfand duality [FJ15, Eq. (1.1)]:

$$\begin{array}{ccc} \begin{array}{c} \mathcal{R} \\ \curvearrowright \\ \mathbf{CH} \end{array} & \xrightarrow[\text{Gelfand}]{\cong} & (\mathbf{CCstar}_{\text{MIU}})^{\text{op}} \\ \begin{array}{c} \uparrow \\ \downarrow \\ \mathbf{Kl}(\mathcal{R}) \end{array} & \xrightarrow[\text{new}]{\cong} & (\mathbf{CCstar}_{\text{PU}})^{\text{op}} \end{array}$$

What is Non-Commutative Probability?

- ▶ Probabilistic Gelfand duality [FJ15, Eq. (1.1)]:

$$\begin{array}{ccc} \overset{\mathcal{R}}{\text{CH}} & \xrightarrow[\text{Gelfand}]{\cong} & (\text{CCstar}_{\text{MIU}})^{\text{op}} \\ \uparrow \scriptstyle{-1} & & \downarrow \\ \text{Kl}(\mathcal{R}) & \xrightarrow[\text{new}]{\cong} & (\text{CCstar}_{\text{PU}})^{\text{op}} \end{array}$$

- ▶ (Subcategories of) C^* -algebras!

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- ▶ Generalizations to
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- ▶ Do they come from “quantum probability monads”?
 - probably!

References

- [CJ19] Kenta Cho and Bart Jacobs. “Disintegration and Bayesian inversion via string diagrams”. In: *Mathematical Structures in Computer Science* 29.7 (Mar. 2019), pp. 938–971.
- [FJ15] Robert WJ Furber and Bart PF Jacobs. “From Kleisli categories to commutative C^* -algebras: probabilistic Gelfand duality”. In: *Logical methods in computer science* 11 (2015).
- [FL24] Tobias Fritz and Antonio Lorenzin. *Involutive Markov categories and the quantum de Finetti theorem*. 2024.
- [Fri20] Tobias Fritz. “A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics”. In: *Advances in Mathematics* 370 (2020), p. 107239.
- [Par20] Arthur J Parzygnat. “Inverses, disintegrations, and Bayesian inversion in quantum Markov categories”. In: *arXiv preprint arXiv:2001.08375* (2020).

Thank You!

Questions?