Towards Generalizing Probability Monads

Nico Wittrock

Oberseminar at Chair for Theoretical Computer Science,
University of Erlangen-Nuremberg,

27 August 2024

Table of contents

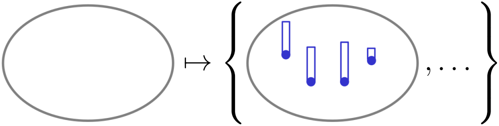
Discrete Monads

- power-set monad

$$\begin{aligned} \mathcal{P} : \mathsf{Set} &\to \mathsf{Set} \\ X &\mapsto \{S \subseteq X\} \end{aligned}$$

- distribution monad

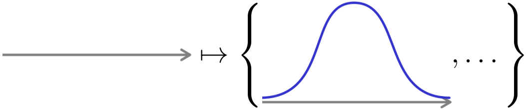
$$\begin{aligned} \mathcal{D} : \mathsf{Set} &\to \mathsf{Set} \\ X &\mapsto \{\text{finite distributions on } X\} \end{aligned}$$



Continuous Monads

- Giry monad

$$\mathcal{G} : \mathsf{Meas} \to \mathsf{Meas}$$
$$(X, \Sigma) \mapsto (\{\text{probability-measures on } X\}, \Sigma_{\text{init}})$$



- variations thereof:

$$\mathcal{G}|_{\mathsf{Pol}} : \mathsf{Pol} \to \mathsf{Pol} \qquad \text{on polish spaces}$$
$$\text{Radon monad } \mathcal{R} : \mathsf{CHaus} \to \mathsf{CHaus} \qquad \text{on cpct. Hausdorff sp.}$$

Kleisli Categories

consider monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$

Kleisli cat.

$$\mathsf{Kl}_{\mathcal{T}} \begin{cases} \text{objects: } \mathsf{Obj}_{\mathsf{Kl}_{\mathcal{T}}} := \mathsf{Obj}_{\mathsf{D}} \\ \text{morphisms: } \mathsf{Hom}_{\mathsf{Kl}_{\mathcal{T}}}(A, B) := \mathsf{Hom}_{\mathsf{D}}(A, \mathcal{T}B) \end{cases}$$

Kleisli Categories

consider monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$

Kleisli cat.

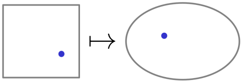
$$\mathcal{K}\ell : \mathsf{D} \to \mathsf{Kl}_{\mathcal{T}} \begin{cases} \text{objects: } \mathsf{Obj}_{\mathsf{Kl}_{\mathcal{T}}} := \mathsf{Obj}_{\mathsf{D}} \\ \text{morphisms: } \mathsf{Hom}_{\mathsf{Kl}_{\mathcal{T}}}(A, B) := \mathsf{Hom}_{\mathsf{D}}(A, \mathcal{T}B) \end{cases}$$

Kleisli Categories

Example (Power-Set Monad)

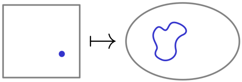
- maps in Set:

$$f : A \to X$$



- maps in $\mathsf{Kl}_{\mathcal{P}}$:

$$f : A \to \mathcal{P}X$$

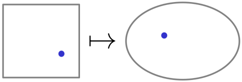


Kleisli Categories

Example (Power-Set Monad)

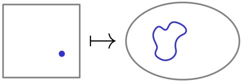
- maps in Set:

$$f : A \to X$$



- maps in $\mathsf{Kl}_{\mathcal{P}}$:

$$f : A \to \mathcal{P}X$$



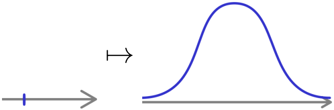
Example (Giry Monad)

- maps in Meas:

$$f : A \to X$$

- maps in $\mathsf{Kl}_{\mathcal{G}}$:

$$f : A \to X$$



Symmetric Monoidal Kleisli Categories

We consider monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$.

- D is cartesian monoidal
- $\mathcal{T}$ is *commutative*, i. e. , comes with "zipper map"

$$\nabla_{X,Y} : \mathcal{T}X \times \mathcal{T}Y \to \mathcal{T}(X \times Y)$$

$\mathsf{Kl}_{\mathcal{T}}$ is symmetric monoidal via $\otimes$

$$\begin{cases} \text{on objects: } X \otimes Y := X \times Y \\ \text{on morphisms } f : A \to \mathcal{T}X, \\ \qquad\qquad\qquad g : B \to \mathcal{T}Y \\ f \otimes g : X \times Y \xrightarrow{f \times g} \mathcal{T}X \times \mathcal{T}Y \xrightarrow{\nabla_{X,Y}} \mathcal{T}(X \times Y) \end{cases}$$

Symmetric Monoidal Kleisli Categories

We consider monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$.

- D is cartesian monoidal
- $\mathcal{T}$ is *commutative*, i. e. , comes with "zipper map"

$$\nabla_{X,Y} : \mathcal{T}X \times \mathcal{T}Y \to \mathcal{T}(X \times Y)$$

$\mathcal{K}\ell : \mathsf{D} \to \mathsf{Kl}_{\mathcal{T}}$ preserves tensor products

Symmetric Monoidal Kleisli Categories

Example (Power-Set Monad)

$\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$

$$\nabla_{X,Y} : \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X \times Y)$$
$$(S, T) \mapsto S \times T \subseteq X \times Y$$

Symmetric Monoidal Kleisli Categories

Example (Power-Set Monad)

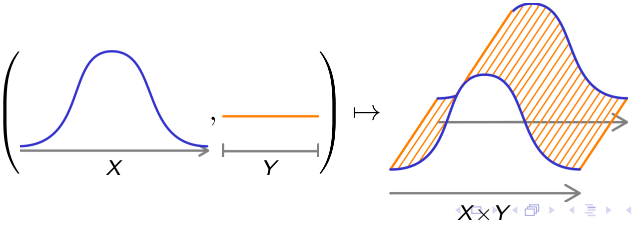
$\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$

$$\nabla_{X,Y} : \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X \times Y)$$
$$(S, T) \mapsto S \times T \subseteq X \times Y$$

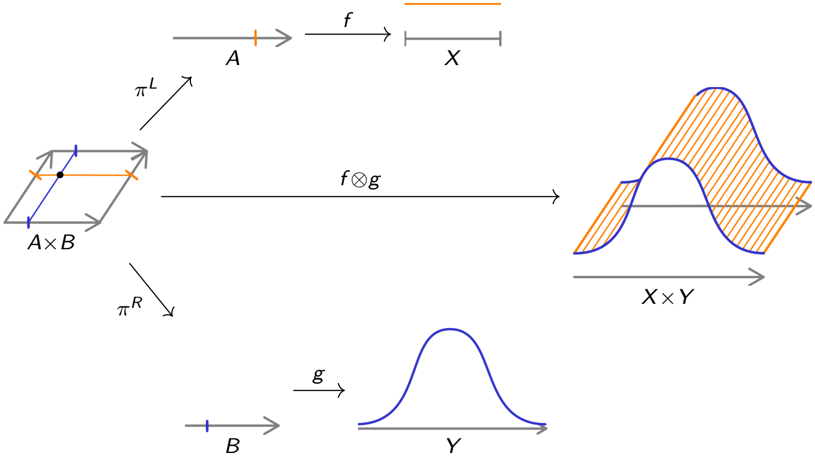
Example (Giry monad)

$\mathcal{G} : \mathsf{Meas} \to \mathsf{Meas}$

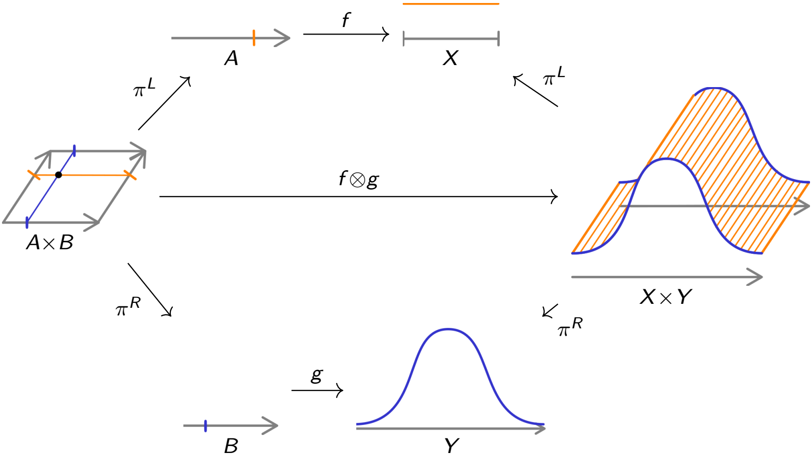
$$\nabla_{X,Y} : \mathcal{G}X \times \mathcal{G}Y \to \mathcal{G}(X \times Y)$$



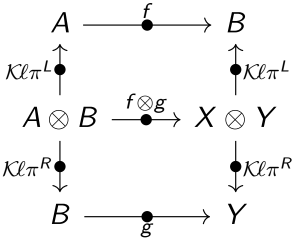
Symmetric Monoidal Kleisli Categories: Giry



Symmetric Monoidal Kleisli Categories: Giry



Probability Monads



Probability Monads

Definition

A *probability monad* is a monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$ such that

- D is cartesian monoidal
- $\mathcal{T}$ is commutative monad
- $\mathcal{K}\ell : \mathsf{D} \to \mathsf{Kl}_{\mathcal{T}}$ *preserves projections*:

in $\mathsf{Kl}_{\mathcal{T}}$ commutes

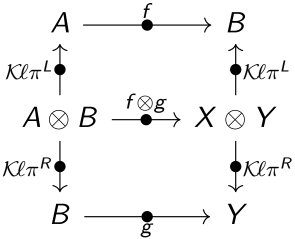
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \mathcal{K}\ell\pi^L} & & \uparrow{\scriptstyle \mathcal{K}\ell\pi^L} \\ A \otimes B & \xrightarrow{f \otimes g} & X \otimes Y \\ \downarrow{\scriptstyle \mathcal{K}\ell\pi^R} & & \downarrow{\scriptstyle \mathcal{K}\ell\pi^R} \\ B & \xrightarrow{g} & Y \end{array}$$

Probability Monads

Definition

A *probability monad* is a monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$ such that

- D is cartesian monoidal
- $\mathcal{T}$ is commutative monad
- $\mathcal{K}\ell : \mathsf{D} \to \mathsf{Kl}_{\mathcal{T}}$ *preserves projections*:



in $\mathsf{Kl}_{\mathcal{T}}$ commutes

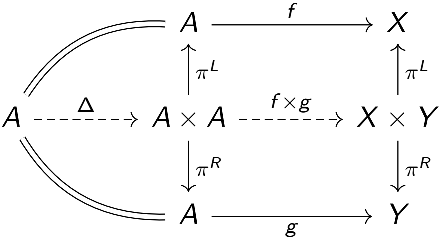
Example

Giry monad $\mathcal{G}$, Radon monad $\mathcal{R}$, distribution monad $\mathcal{D}$.
Power-set monad $\mathcal{P}$ is *no* probability monad.

Weak Products

Consider probability monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$ with $\mathcal{K}\ell : \mathsf{D} \to \mathsf{Kl}_{\mathcal{T}}$.

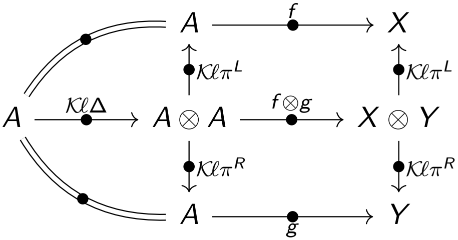
- D has products:



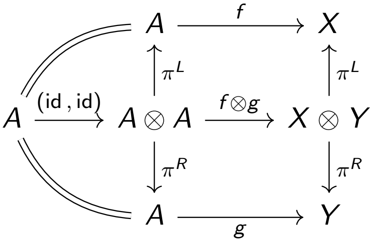
Weak Products

Consider probability monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$ with $\mathcal{K}\ell : \mathsf{D} \to \mathsf{Kl}_{\mathcal{T}}$.

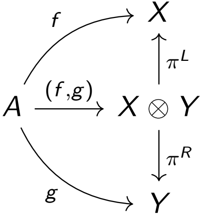
- $\mathsf{Kl}_{\mathcal{T}}$ has *weak* products:



Markov Categories

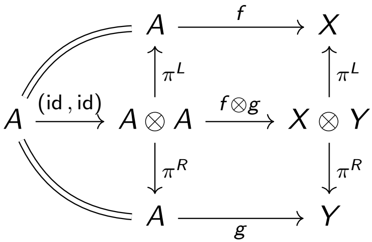


(1)

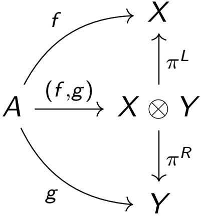


(2)

Markov Categories



(1)



(2)

Definition

A *Markov Category* is cat. with weak products and chosen morphisms (f, g) as in (2), such that

- they canonically induce a symmetric monoidal structure
- diagrams (1), (2) coincide: $(f \otimes g) \circ (\mathrm{id}, \mathrm{id}) = (f, g)$

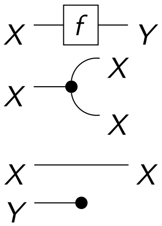
Markov Categories

- Symmetric monoidal cat.s have graphical notation

$$f : X \to Y \quad \equiv$$

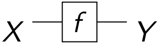
$$(\mathrm{id}, \mathrm{id}) : X \to X \otimes X \quad \equiv$$

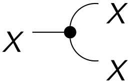
$$\pi^L : X \otimes Y \to X \quad \equiv$$

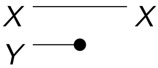


Markov Categories

- Symmetric monoidal cat.s have graphical notation

$f : X \to Y \quad \equiv$ 

$(\mathrm{id}, \mathrm{id}) : X \to X \otimes X \quad \equiv$ 

$\pi^L : X \otimes Y \to X \quad \equiv$ 

- leads to usual definition of Markov cat.s
  - more explicit [Fri20]
  - generalization: CD-cat.s (Cho and Jacobs [CJ19])

Markov categories

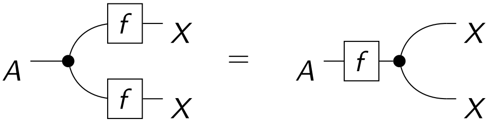
Example

Kleisli cat.s of probability monads.
In particular: cartesian monoidal cat.s.

Deterministic Morphisms

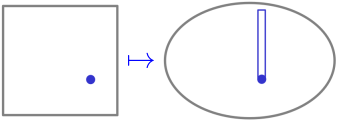
Definition

Morphism $f : A \to X$ in Markov cat. is *deterministic*, if



Example

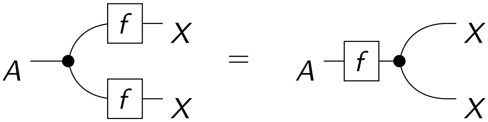
1. In Kleisli cat. $\mathrm{Kl}_{\mathcal{D}}$ of distribution monad: deterministic morphisms are of the form



Deterministic Morphisms

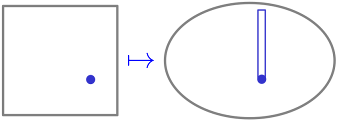
Definition

Morphism $f : A \to X$ in Markov cat. is *deterministic*, if



Example

1. In Kleisli cat. $\mathrm{Kl}_{\mathcal{D}}$ of distribution monad: deterministic morphisms are of the form



2. In Kleisli cat. $\mathrm{Kl}_{\mathcal{T}}$: all $\mathcal{K}\ell(f)$ are deterministic

Representable Markov Categories

Consider Markov cat. C.

Lemma

Deterministic morphisms form subcategory $\mathsf{C}_{\mathsf{det}} \subseteq \mathsf{C}$.

Representable Markov Categories

Consider Markov cat. C.

Lemma

Deterministic morphisms form subcategory $\mathsf{C}_{\mathsf{det}} \subseteq \mathsf{C}$.

Definition

Kleisli cat. $\mathsf{Kl}_{\mathcal{T}}$ of probability monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$ is *representable*, if

$$(\mathsf{Kl}_{\mathcal{T}})_{\mathsf{det}} \cong \mathsf{D}$$

Representable Markov Categories

Consider Markov cat. C.

Lemma

Deterministic morphisms form subcategory $\mathsf{C}_{\mathsf{det}} \subseteq \mathsf{C}$.

Definition

Kleisli cat. $\mathsf{Kl}_{\mathcal{T}}$ of probability monad $\mathcal{T} : \mathsf{D} \to \mathsf{D}$ is *representable*, if

$$(\mathsf{Kl}_{\mathcal{T}})_{\mathsf{det}} \cong \mathsf{D}$$

Example

1. Kleisli cat. $\mathsf{Kl}_{\mathcal{D}}$ of distribution monad is representable.
2. Kleisli cat. $\mathsf{Kl}_{\mathcal{G}}$ of Giry monad is *not*.

Polish Spaces

$\mathsf{Kl}_{\mathcal{G}}$ of Giry monad is *not* representable.

Polish Spaces

$\mathsf{Kl}_{\mathcal{G}}$ of Giry monad is *not* representable.
But: restriction to Polish spaces $\mathsf{Pol} \subseteq \mathsf{Meas}$

$$\mathcal{B} := \mathcal{G}|_{\mathsf{Pol}} : \mathsf{Pol} \to \mathsf{Pol}$$

has representable Kleisli cat. $\mathsf{Kl}_{\mathcal{B}} =: \mathsf{BorelStoch}$:

Polish Spaces

$\mathsf{Kl}_{\mathcal{G}}$ of Giry monad is *not* representable.
But: restriction to Polish spaces $\mathsf{Pol} \subseteq \mathsf{Meas}$

$$\mathcal{B} := \mathcal{G}|_{\mathsf{Pol}} : \mathsf{Pol} \to \mathsf{Pol}$$

has representable Kleisli cat. $\mathsf{Kl}_{\mathcal{B}} =: \mathsf{BorelStoch}$:

$$\begin{array}{ccc} \mathsf{Pol}(A, \mathcal{B}X) & \overset{\text{def}}{=\!=\!=} & \mathsf{Kl}_{\mathcal{B}}(A, X) \\ \wr\| & & \| \\ \mathsf{BorelStoch}_{\mathsf{det}}(A, \mathcal{B}X) & & \mathsf{BorelStoch}(A, X) \end{array}$$

$$\mathrm{id}_{\mathcal{B}X} \longmapsto [\mathsf{samp}_X : \mathcal{B}X \to X]$$

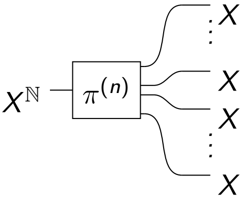
Countable Products

Lemma

BorelStoch *has inverse limits*

$$X^{\mathbb{N}} := \varprojlim_{n \in \mathbb{N}} X^{\otimes n}$$

with projections $\pi^{(n)} : X^{\mathbb{N}} \to X^{\otimes n}$, *depicted as*

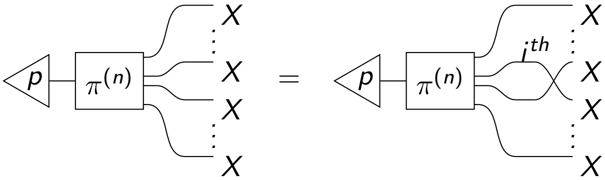


De Finetti's Theorem

Theorem

For a morphism $p : I \to X^{\mathbb{N}}$ *in* BorelStoch, *it is equivalent:*

1. *p is invariant under finite permutations:*
   for all $n \in \mathbb{N}$ *and* $i < n$

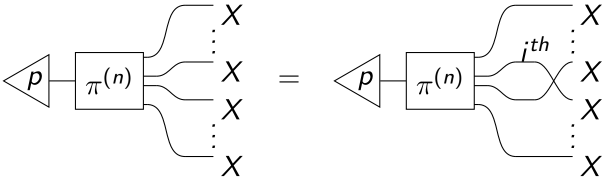


De Finetti's Theorem

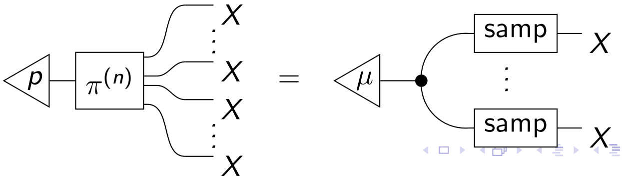
Theorem

For a morphism $p : I \to X^{\mathbb{N}}$ in BorelStoch, *it is equivalent:*

1. *p is invariant under finite permutations:*
   for all $n \in \mathbb{N}$ and $i < n$

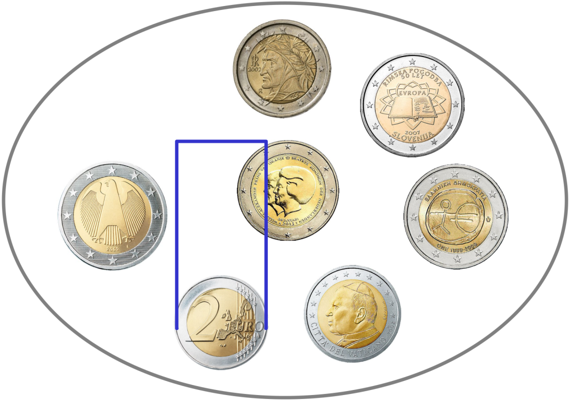


2. *there is a (unique) morphism $\mu : I \to \mathcal{B}X$ such that*



Flip a 2€ Coin

Flip a 2€ Coin



Probability theory is too “fine-grained”.

What is Imprecise Probability?

Many different uncertainty frameworks available:

What is Imprecise Probability?

Many different uncertainty frameworks available:

- Dempster-Schafer belief functions
- super-additive measures
- non-additive measures
- ...

Do they have probability monads?

Towards Imprecise Probability Monads

$$\mathcal{R} \underset{(A \mapsto \omega(1_A)) \leftarrow\!\shortmid \omega}{\overset{\alpha \mapsto (f \mapsto \int f d\alpha)}{\rightleftarrows}} [[-, \mathbb{R}_{\geq 0}], \mathbb{R}_{\geq 0}]_{\text{lin}} \hookrightarrow \textbf{Cont}$$

$$\textbf{Gr} \underset{(A \mapsto \omega(1_A)) \leftarrow\!\shortmid \omega}{\overset{\alpha \mapsto (f \mapsto \oint f d\alpha)}{\rightleftarrows}} \textbf{Rd} \hookrightarrow \textbf{Cont}$$

Towards Super-Additive Measures

$$\mathbf{Gr} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Rd} \hookrightarrow \mathbf{Cont}$$

$$\mathbf{Rd} = [[-,\mathbb{R}_{\geq 0}],\mathbb{R}_{\geq 0}]_{??} \qquad \mathbf{Cont} = [[-,\mathbb{R}_{\geq 0}],\mathbb{R}_{\geq 0}]$$

super-additive measures on X:

$$\alpha : PX \to [0,1]$$
$$\alpha(X) = 1$$
$$\alpha(\emptyset) = 0$$
$$\alpha(A \cup B) \geq \alpha(A) + \alpha(B) - \alpha(A \cap B)$$

?? restrict $\omega : [X,\mathbb{R}_{\geq 0}] \to \mathbb{R}_{\geq 0}$

$$\omega(1) \overset{!}{=} 1$$
$$\omega(\lambda f) \overset{!}{=} \lambda\omega(f)$$
$$f \leq g \overset{!}{\Rightarrow} \omega(f) \leq \omega(g)$$
$$\vdots$$

An Italian Flag

Further Work

- Further restrictions on **Rd** s.th.
  - **Rd** is monad
  - **Gr** are Dempster-Shafer belief functions / ...

Further Work

- Further restrictions on **Rd** s.th.
  - **Rd** is monad
  - **Gr** are Dempster-Shafer belief functions / ...
- Is **Gr** a commutative monad? (Probably not)
- What does that mean for its Kleisli cat.?
  - It is *not* monoidal.

Further Work

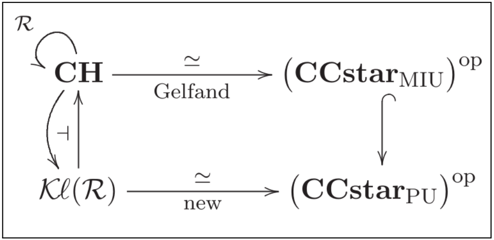
- Further restrictions on **Rd** s.th.
  - **Rd** is monad
  - **Gr** are Dempster-Shafer belief functions / . . .
- Is **Gr** a commutative monad? (Probably not)
- What does that mean for its Kleisli cat.?
  - It is *not* monoidal.
  - But *pre*monoidal?

Further Work

- Further restrictions on **Rd** s.th.
  - **Rd** is monad
  - **Gr** are Dempster-Shafer belief functions / ...
- Is **Gr** a commutative monad? (Probably not)
- What does that mean for its Kleisli cat.?
  - It is *not* monoidal.
  - But *pre*monoidal? A *monoidal effectful cat.*?
- Results from Markov cat.s?
- Relation to (other) Markov cat.s of imprecise probability
  (So far: "just" special cases)

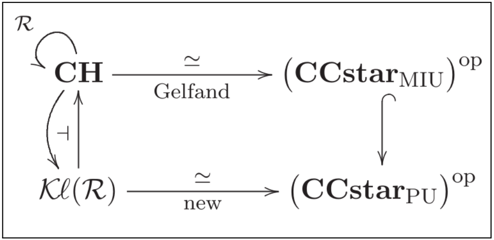
What is Non-Commutative Probability?

- Probabilistic Gelfand duality [FJ15, Eq. (1.1)]:



What is Non-Commutative Probability?

- Probabilistic Gelfand duality [FJ15, Eq. (1.1)]:



- (Subcategories of) C^*-algebras!

Towards Non-Commutative Probability Monads

- Are C^*-algebras dual Markov cat.?

Towards Non-Commutative Probability Monads

- Are C^*-algebras dual Markov cat.?
  – NO!

Towards Non-Commutative Probability Monads

- Are C^*-algebras dual Markov cat.?
  – NO!
- Generalizations to
  - *involutive Markov cat.s* in [FL24]
  - *quantum Markov cat.s* in [Par20]
- Do they come from "quantum probability monads"?

Towards Non-Commutative Probability Monads

- Are C^*-algebras dual Markov cat.?
  – NO!
- Generalizations to
  - *involutive Markov cat.s* in [FL24]
  - *quantum Markov cat.s* in [Par20]
- Do they come from "quantum probability monads"?
  – probably!

References

[CJ19] Kenta Cho and Bart Jacobs. "Disintegration and Bayesian inversion via string diagrams". In: *Mathematical Structures in Computer Science* 29.7 (Mar. 2019), pp. 938–971.

[FJ15] Robert WJ Furber and Bart PF Jacobs. "From Kleisli categories to commutative C*-algebras: probabilistic Gelfand duality". In: *Logical methods in computer science* 11 (2015).

[FL24] Tobias Fritz and Antonio Lorenzin. *Involutive Markov categories and the quantum de Finetti theorem*. 2024.

[Fri20] Tobias Fritz. "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics". In: *Advances in Mathematics* 370 (2020), p. 107239.

[Par20] Arthur J Parzygnat. "Inverses, disintegrations, and Bayesian inversion in quantum Markov categories". In: *arXiv preprint arXiv:2001.08375* (2020).

Thank You!

Questions?