Monad-Based Programming SS 2023

Assignment 6

Deadline for solutions: 20.07.2023

Exercise 1 Kleisi Triples as Monads (6 Points)

Complete the proof from the lecture that Kleisli triples bijectively correspond to monads (T, η, μ). Recall that you need to

(a) define a Kleisli triple from a monad, given as a triple (T, η, μ) and verify the axioms of Kleisli triples;

(b) define a monad in the form (T, η, μ) from a Kleisli triple and verify the axioms of monads;

(c) show that the passage $(T, \eta, (-)^\star) \to (T, \eta, \mu) \to (T, \eta, (-)^\star)$ yields an identity;

(d) show that the passage $(T, \eta, \mu) \to (T, \eta, (-)^\star) \to (T, \eta, \mu)$ yields an identity.

Exercise 2 (Non-)Commutative Monads (6 Points)

A strong monad T is commutative if

$$
\begin{array}{ccccc}
TA \times TB & \xrightarrow{\tau_{A,B}} & T(TA \times B) & \xrightarrow{T\hat{\tau}_{TA,B}} & TT(A \times B) \\
\downarrow{\scriptstyle \hat{\tau}_{A,B}} & & & & \\
T(A \times TB) & & & & \downarrow{\scriptstyle \mu_{A \times B}} \\
\downarrow{\scriptstyle T\tau_{A,TB}} & & & & \\
TT(A \times B) & & \xrightarrow{\qquad\mu_{A \times B}\qquad} & & T(A \times B)
\end{array}
$$

where $\hat{\tau}_{A,B} \colon TA \times B \to T(A \times B)$ is the following dual of $\tau_{A,B}$:

$$TA \times B \xrightarrow{\langle \mathsf{snd}, \mathsf{fst} \rangle} B \times TA \xrightarrow{\tau_{B,TA}} T(B \times A) \xrightarrow{T\langle \mathsf{snd}, \mathsf{fst} \rangle} T(A \times B).$$

(a) Consider the exception monad $TX = X + E$ over the category of sets and functions. For which E it is commutative? Justify your answer with a formal proof.

(b) Consider the lifting monad $TX = X_\perp$ over the category of complete partial orders and continuous functions. Is it commutative? Justify your answer with a formal proof.

(c) Prove that the reader monad $TX = X^S$ over the category of sets and functions is commutative for every S.

Exercise 3 Monads on Posets (8 Points)

A *closure operator* T over a poset (=partially ordered set), say $\mathcal{C}$, satisfies properties:

$$X \leqslant TX \qquad \text{(extensiveness)}$$
$$X \leqslant Y \quad \text{implies} \quad TX \leqslant TY \qquad \text{(monotonicity)}$$
$$TTX = TX \qquad \text{(idempotence)}$$

For example, if $\mathcal{C}$ is the standard partial order on real numbers, then the operator that rounds up a real number to the closest integer is a closure operator.

Recall from the lecture that we can view $\mathcal{C}$ as a category: $|\mathcal{C}|$ is the set of elements, $\mathcal{C}(X, Y) = \{*\}$ if $X \leqslant Y$ and $\mathcal{C}(X, Y) = \{\,\}$ otherwise.

(a) Prove that T is a monad on $\mathcal{C}$ iff T is a closure operator.

(b) Prove that if $\mathcal{C}$ is a *total order*, i.e. for any two objects X and Y either $X \leqslant Y$ or $Y \leqslant X$, then every monad T on $\mathcal{C}$ is strong. **Hint:** You need to explain first, what binary products and what terminal objects in $\mathcal{C}$ are.

(c) Construct an example of a monad on a poset category that is not strong.

Hint: You can consider $\mathcal{C}$ to be the poset of geometric shapes on the plane, ordered by inclusion, i.e. $X \leqslant Y$ if Y (as a set of points) contains X. As the closure operator, consider *convex closure*, i.e. the operator that sends every shape X to the smallest convex shape TX that contains X – see Fig. Exercise 3 for an example.

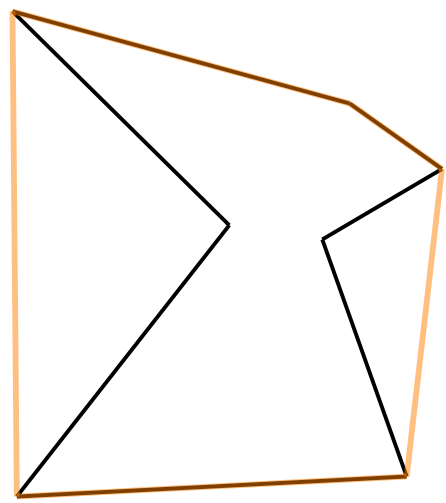

Figure 1: Example of convex closure: black lines – original shape, orange lines – the induced closure.