Exercise 1  Kleisi Triples as Monads  (6 Points)

Complete the proof from the lecture that Kleisli triples bijectively correspond to monads \((T, \eta, \mu)\). Recall that you need to

(a) define a Kleisli triple from a monad, given as a triple \((T, \eta, \mu)\) and verify the axioms of Kleisli triples;

(b) define a monad in the form \((T, \eta, \mu)\) from a Kleisli triple and verify the axioms of monads;

(c) show that the passage \((T, \eta, (-)^*) \to (T, \eta, \mu) \to (T, \eta, (-)^*)\) yields an identity;

(d) show that the passage \((T, \eta, \mu) \to (T, \eta, (-)^*) \to (T, \eta, \mu)\) yields an identity.

Exercise 2  (Non-)Commutative Monads  (6 Points)

A strong monad \(T\) is commutative if

\[
\begin{align*}
TA \times TB & \xrightarrow{\tau_{A,B}} T(TA \times B) \xrightarrow{T\tau_{TA,B}} TT(A \times B) \\
\hat{T}_{A,B} & \downarrow \hspace{2cm} \mu_{A \times B} \\
T(A \times TB) & \downarrow \hspace{2cm} \mu_{A \times B} \\
TT(A \times B) & \xrightarrow{\mu_{A \times B}} T(A \times B)
\end{align*}
\]

where \(\hat{T}_{A,B} : TA \times B \to T(A \times B)\) is the following dual of \(\tau_{A,B}\):

\[
\begin{align*}
TA \times B & \xrightarrow{(\text{snd}, \text{fst})} B \times TA \xrightarrow{\tau_{B,T\mu}} T(B \times A) \xrightarrow{T(\text{snd}, \text{fst})} T(A \times B).
\end{align*}
\]

(a) Consider the exception monad \(TX = X + E\) over the category of sets and functions. For which \(E\) it is commutative? Justify your answer with a formal proof.

(b) Consider the lifting monad \(TX = X_\perp\) over the category of complete partial orders and continuous functions. Is it commutative? Justify your answer with a formal proof.

(c) Prove that the reader monad \(TX = X^S\) over the category of sets and functions is commutative for every \(S\).
Exercise 3  Monads on Posets  (8 Points)

A closure operator $T$ over a poset (=partially ordered set), say $C$, satisfies properties:

\[
\begin{align*}
X \leq TX & \quad \text{(extensiveness)} \\
X \leq Y & \implies TX \leq TY \quad \text{(monotonicity)} \\
TTX = TX & \quad \text{(idempotence)}
\end{align*}
\]

For example, if $C$ is the standard partial order on real numbers, then the operator that rounds up a real number to the closest integer is a closure operator.

Recall from the lecture that we can view $C$ as a category: $|C|$ is the set of elements, $C(X,Y) = \{\ast\}$ if $X \leq Y$ and $C(X,Y) = \{\}$ otherwise.

(a) Prove that $T$ is a monad on $C$ iff $T$ is a closure operator.

(b) Prove that if $C$ is a total order, i.e. for any two objects $X$ and $Y$ either $X \leq Y$ or $Y \leq X$, then every monad $T$ on $C$ is strong.  
\textbf{Hint:} You need to explain first, what binary products and what terminal objects in $C$ are.

(c) Construct an example of a monad on a poset category that is not strong.  
\textbf{Hint:} You can consider $C$ to be the poset of geometric shapes on the plane, ordered by inclusion, i.e. $X \leq Y$ if $Y$ (as a set of points) contains $X$. As the closure operator, consider convex closure, i.e. the operator that sends every shape $X$ to the smallest convex shape $TX$ that contains $X$ – see Fig. Exercise 3 for an example.

Figure 1: Example of convex closure: black lines – original shape, orange lines – the induced closure.