

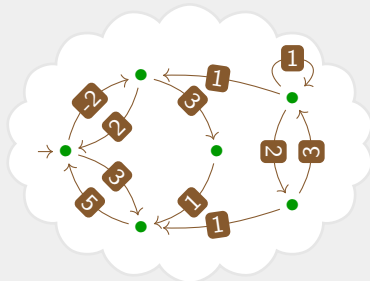
Reachability vs Bisimilarity Minimization

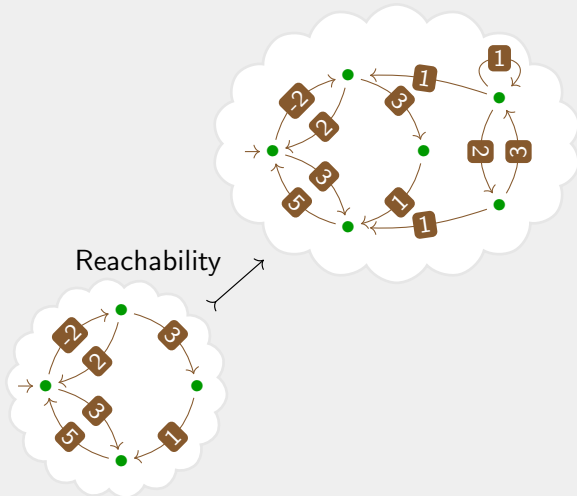
Thorsten Wißmann

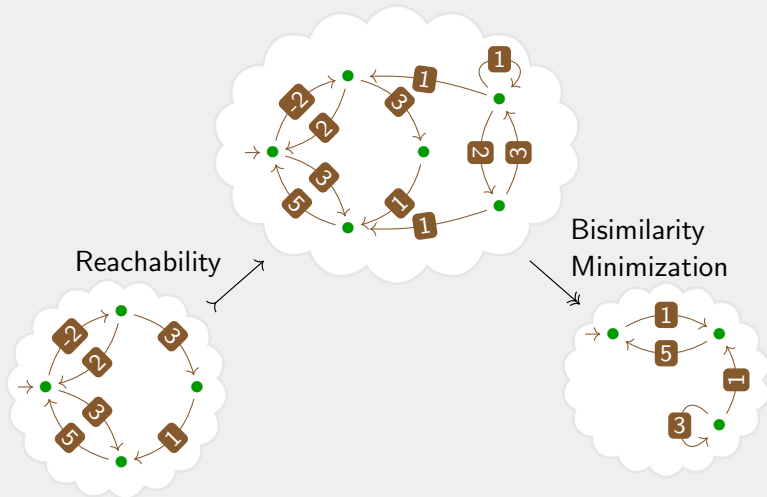
Friedrich-Alexander-Universität Erlangen-Nürnberg

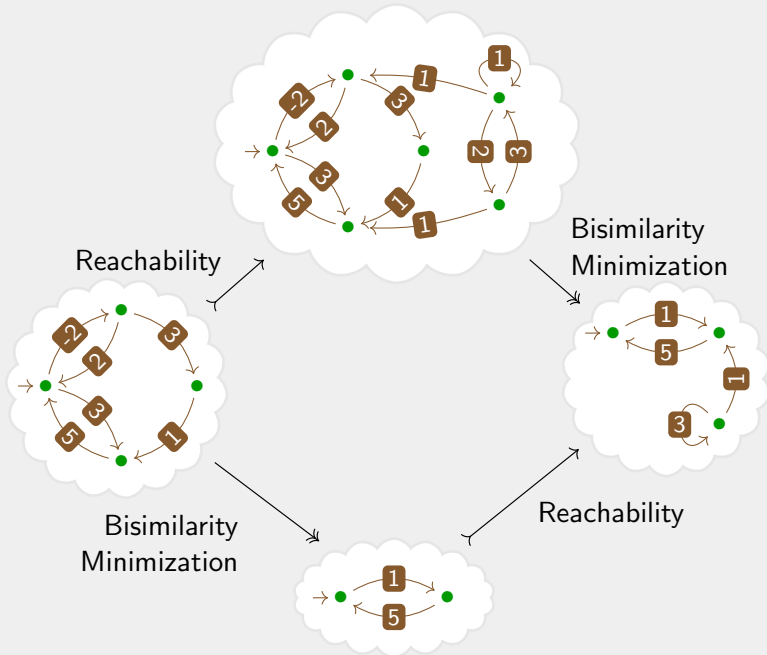
Informatik 8, Oberseminar

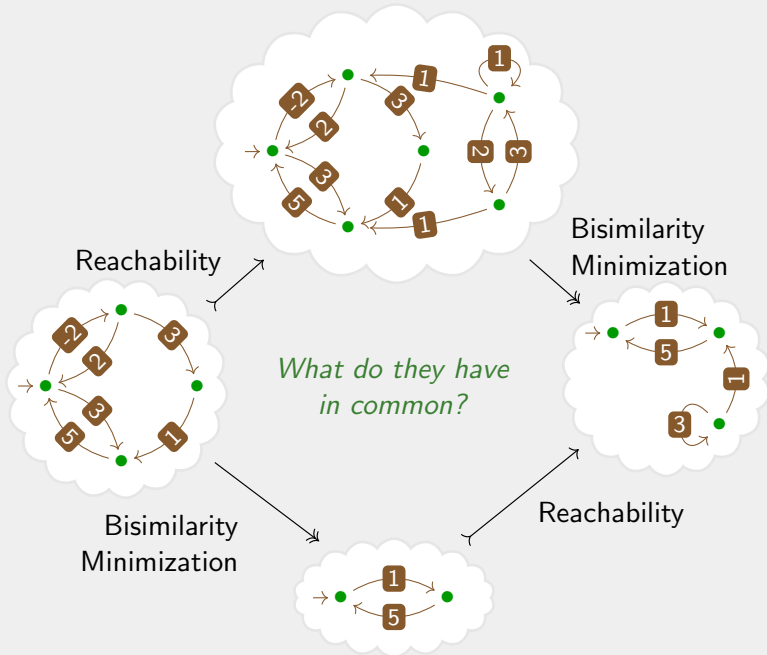
June 23, 2020











Category $\text{Coalg}(F)$

For a functor $F: \mathcal{C} \rightarrow \mathcal{C}$:

$(F-)$ coalgebra $C \xrightarrow{c} FC$

Coalgebra homomorphisms:

$h: (C, c) \rightarrow (D, d)$
in $\text{Coalg}(F)$

$:\Leftrightarrow$

$$\begin{array}{ccc} C & \xrightarrow{c} & FC \\ h \downarrow & & \downarrow Fh \\ D & \xrightarrow{d} & FD \end{array}$$

States

Successor map

Successor type

Category $\text{Coalg}(F)$

For a functor $F: \mathcal{C} \rightarrow \mathcal{C}$: $(F\text{-})\text{coalgebra}$ $C \xrightarrow{c} FC$

Coalgebra homomorphisms:

$$h: (C, c) \longrightarrow (D, d) \quad \text{in } \text{Coalg}(F) \quad \iff \quad \begin{array}{ccc} C & \xrightarrow{c} & FC \\ h \downarrow & & \downarrow Fh \\ D & \xrightarrow{d} & FD \end{array}$$

States Successor map Successor type

Examples

From now on: $\mathcal{C} = \text{Set}$. Coalgebras for functors $F: \text{Set} \rightarrow \text{Set}$:

- $FX = 2 \times X^A$ deterministic automata (without initial state)
- $FX = \mathcal{P}X$ transition systems
- $FX = \mathbb{R}^{(X)}$ weighted systems (negative weights!)

Coalgebraic Task

Given a coalgebra

$$\begin{array}{ccc} C & \xrightarrow{c} & FC \\ h \downarrow & & \downarrow Fh \\ D & \xrightarrow{d} & FD \end{array}$$

find the simple quotient

no proper
quotient



Gumm '03

Coalgebraic Task

Given a coalgebra

$$\begin{array}{ccc}
 C & \xrightarrow{c} & FC \\
 h \downarrow & & \downarrow Fh \\
 D & \xrightarrow{d} & FD
 \end{array}$$

find the simple quotient

no proper
quotient



Gumm '03

Examples

This task for specific F -coalgebras:

- $FX = 2 \times X^A \rightsquigarrow$ quotient by: language equivalence
- $FX = \mathcal{P}X \rightsquigarrow$ quotient by: (strong) bisimilarity
- $FX = \mathbb{R}^{(X)} \rightsquigarrow$ quotient by: weighted bisimilarity

Category $\text{Coalg}_I(F)$ for $I \in \text{Set}$

- I -pointed F -coalgebra $(C, c, i_C) \quad I \xrightarrow{i_C} C \xrightarrow{c} FC$
- Homomorphism:

$$\begin{array}{ccc}
 (C, c, i_C) & & I \xrightarrow{i_C} C \xrightarrow{c} FC \\
 \downarrow h & \text{in } \text{Coalg}_I(F) & \swarrow i_D \quad \downarrow h \quad \downarrow Fh \\
 (D, d, i_D) & \iff & D \xrightarrow{d} FD \quad \text{in } \mathcal{C}
 \end{array}$$

Category $\text{Coalg}_I(F)$ for $I \in \text{Set}$

- I -pointed F -coalgebra $(C, c, i_C) \quad I \xrightarrow{i_C} C \xrightarrow{c} FC$
- Homomorphism:

$$\begin{array}{ccc}
 (C, c, i_C) & & I \xrightarrow{i_C} C \xrightarrow{c} FC \\
 \downarrow h & \text{in } \text{Coalg}_I(F) & \swarrow \text{id} \quad \downarrow h \quad \downarrow Fh \\
 (D, d, i_D) & \iff & D \xrightarrow{d} FD
 \end{array}$$

Examples

Coalgebras for Set-functors $F: \text{Set} \rightarrow \text{Set}$ and $I = 1 = \{*\}$:

- $FX = 2 \times X^A$ deterministic automata incl. initial state
- $FX = \mathcal{P}X$ pointed graphs / transition systems
- $FX = \mathbb{R}^{(X)}$ pointed weighted systems


Coalgebraic Task

Given a pointed coalgebra $I \xrightarrow{i_C} C \xrightarrow{c} FC$

find the reachable subcoalgebra $I \xrightarrow{i_D} D \xrightarrow{d} FD$

$$\begin{array}{ccccc}
 I & \xrightarrow{i_C} & C & \xrightarrow{c} & FC \\
 & \searrow i_D & \uparrow h & & \uparrow Fh \\
 & & D & \xrightarrow{d} & FD
 \end{array}$$

no proper
pointed subcoalgebra



Adámek, Milius, Moss, Sousa '13

Coalgebraic Task

Given a pointed coalgebra $I \xrightarrow{i_C} C \xrightarrow{c} FC$

find the reachable subcoalgebra $I \xrightarrow{i_D} D \xrightarrow{d} FD$

$$\begin{array}{ccccc}
 I & \xrightarrow{i_C} & C & \xrightarrow{c} & FC \\
 & \searrow i_D & \uparrow h & & \uparrow Fh \\
 & & D & \xrightarrow{d} & FD
 \end{array}$$

no proper
pointed subcoalgebra

Adámek, Milius, Moss, Sousa '13

Examples

This task for specific F -coalgebras:

- $FX = 2 \times X^A \rightsquigarrow$ restrict to: reachable states
- $FX = \mathcal{P}X \rightsquigarrow$ restrict to: reachable states
- $FX = \mathbb{R}^{(X)} \rightsquigarrow$ restrict to: reachable states

Definition: \mathcal{M} -Minimality

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

object C is (\mathcal{M}) -minimal $:\Leftrightarrow$ every $h: C' \twoheadrightarrow C$ is an isomorphism

in \mathcal{M}

Definition: \mathcal{M} -Minimality

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

object C is (\mathcal{M}) -minimal $:\Leftrightarrow$ every $h: C' \twoheadrightarrow C$ is an isomorphism

\swarrow in \mathcal{M}



$\mathcal{K} = \text{Coalg}_I(F)$

(C, c, i_C) is $(\mathcal{M}$ -carried-)minimal

\Leftrightarrow

(C, c, i_C) is reachable

Definition: \mathcal{M} -Minimality

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

object C is (\mathcal{M}) -minimal $:\Leftrightarrow$ every $h: C' \twoheadrightarrow C$ is an isomorphism

\nearrow in \mathcal{M}

$\mathcal{K} = \text{Coalg}_I(F)$

(C, c, i_C) is $(\mathcal{M}$ -carried-)minimal

\Leftrightarrow

(C, c, i_C) is reachable

$\mathcal{K} = \text{Coalg}(F)^{\text{op}}$

(C, c) is $(\mathcal{E}$ -carried-)minimal

\Leftrightarrow

(C, c) is simple

Definition: $(\mathcal{E}, \mathcal{M})$ -factorization system in \mathcal{K}

Classes of Morphisms: $\mathcal{M} \rightarrow \mathcal{E}$

each closed under composition with isomorphisms

Definition: $(\mathcal{E}, \mathcal{M})$ -factorization system in \mathcal{K}

Classes of Morphisms: $\mathcal{M} \rightarrow \mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f: A \rightarrow B$

$$\begin{array}{c} \xrightarrow{\quad f \quad} \\ \downarrow \quad \quad \quad \downarrow \\ A \xrightarrow{e} \text{Im}(f) \xrightarrow{m} B \end{array}$$

Definition: $(\mathcal{E}, \mathcal{M})$ -factorization system in \mathcal{K}

Classes of Morphisms: $\mathcal{M} \rightarrow \mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f: A \rightarrow B$
- Diagonal fill-in:

$$\begin{array}{ccc}
 & f & \\
 \lrcorner & & \searrow \\
 A & \xrightarrow{e} \text{Im}(f) \xrightarrow{m} & B
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{e} & B \\
 f \downarrow & \exists! d \swarrow & \downarrow g \\
 C & \xrightarrow{m} & D
 \end{array}$$

Definition: $(\mathcal{E}, \mathcal{M})$ -factorization system in \mathcal{K}

Classes of Morphisms: $\xrightarrow{\quad} \mathcal{M} \quad \twoheadrightarrow \mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f: A \rightarrow B$
- Diagonal fill-in:

$$\begin{array}{ccc}
 & \xrightarrow{\quad f \quad} & \\
 \lrcorner & & \searrow \\
 A & \xrightarrow{e} \twoheadrightarrow \text{Im}(f) \xrightarrow{m} & B
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{e} \twoheadrightarrow & B \\
 f \downarrow & \exists! d \swarrow \text{dashed} & \downarrow g \\
 C & \xrightarrow{m} & D
 \end{array}$$

Example:

$\mathcal{K} = \text{Set}$ $\mathcal{E} = \text{surjective maps}$ $\mathcal{M} = \text{injective maps}$

Definition: $(\mathcal{E}, \mathcal{M})$ -factorization system in \mathcal{K}

Classes of Morphisms: $\xrightarrow{\mathcal{M}}$ $\twoheadrightarrow \mathcal{E}$

each closed under composition with isomorphisms

- Factorization of every $f: A \rightarrow B$
- Diagonal fill-in:

$$\begin{array}{ccc}
 & \xrightarrow{f} & \\
 \lrcorner & & \searrow \\
 A & \xrightarrow{e} \twoheadrightarrow \text{Im}(f) & \xrightarrow{m} B
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{e} \twoheadrightarrow & B \\
 f \downarrow & \exists! d \swarrow \text{dashed} & \downarrow g \\
 C & \xrightarrow{m} & D
 \end{array}$$

Example:

$\mathcal{K} = \text{Set}$ $\mathcal{E} = \text{surjective maps}$ $\mathcal{M} = \text{injective maps}$

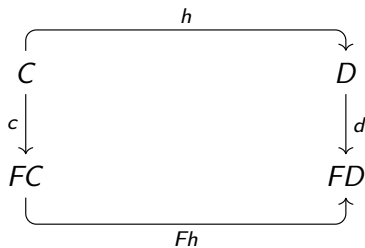
Example:

$(\mathcal{E}, \mathcal{M})$ -factorization system in \mathcal{K} \iff $(\mathcal{M}, \mathcal{E})$ -factorization system in \mathcal{K}^{op}

$(\mathcal{E}\text{-carried}, \mathcal{M}\text{-carried})\text{-factorization system in } \text{Coalg}(F)?$

Assume: F preserves \mathcal{M}

Factorization of a homomorphism $h: (C, c) \rightarrow (D, d)$



[Standard]

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$?

Assume: F preserves \mathcal{M}

Factorization of a homomorphism $h: (C, c) \rightarrow (D, d)$

$$\begin{array}{ccccc}
 & & \xrightarrow{h} & & \\
 & \swarrow & & \searrow & \\
 C & \xrightarrow{e} & \text{Im}(h) & \xrightarrow{m} & D \\
 \downarrow c & & & & \downarrow d \\
 FC & & & & FD \\
 & \swarrow & & \searrow & \\
 & & \xrightarrow{Fh} & &
 \end{array}$$

[Standard]

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$?

Assume: F preserves \mathcal{M}

Factorization of a homomorphism $h: (C, c) \rightarrow (D, d)$

$$\begin{array}{ccccc}
 & & \xrightarrow{h} & & \\
 & \swarrow & & \searrow & \\
 C & \xrightarrow{e} & \text{Im}(h) & \xrightarrow{m} & D \\
 \downarrow c & & & & \downarrow d \\
 FC & \xrightarrow{Fe} & F\text{Im}(h) & \xrightarrow{Fm} & FD \\
 & \swarrow & & \searrow & \\
 & & \xrightarrow{Fh} & &
 \end{array}$$

[Standard]

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$?

Assume: F preserves \mathcal{M}

Factorization of a homomorphism $h: (C, c) \rightarrow (D, d)$

$$\begin{array}{ccccc}
 & & h & & \\
 & \overbrace{\hspace{10em}} & & \searrow & \\
 C & \xrightarrow{e} & \text{Im}(h) & \xrightarrow{m} & D \\
 \downarrow c & & \downarrow \exists! u & & \downarrow d \\
 FC & \xrightarrow{Fe} & F\text{Im}(h) & \xrightarrow{Fm} & FD \\
 & \underbrace{\hspace{10em}} & & \nearrow & \\
 & & Fh & &
 \end{array}$$

[Standard]

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$?

Assume: F preserves \mathcal{M}

Factorization of a homomorphism $h: (C, c) \rightarrow (D, d)$

$$\begin{array}{ccccc}
 & & \xrightarrow{h} & & \\
 & \swarrow & & \searrow & \\
 C & \xrightarrow{e} & \text{Im}(h) & \xrightarrow{m} & D \\
 \downarrow c & & \downarrow \exists! u & & \downarrow d \\
 FC & \xrightarrow{Fe} & F\text{Im}(h) & \xrightarrow{Fm} & FD \\
 & \swarrow & & \searrow & \\
 & & \xrightarrow{Fh} & &
 \end{array}$$

[Standard]

\implies Not enough for a factorization system in $\text{Coalg}(F)$

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$!

Proposition: Diagonal fill-in in $\text{Coalg}(F)$, if F preserves \mathcal{M}

$$\begin{array}{ccc} (A, a) & \xrightarrow{e} \twoheadrightarrow & (B, b) \\ f \downarrow & \swarrow \exists! u \text{ } \dashrightarrow & \downarrow g \\ (C, c) & \xrightarrow{m} \rightarrow & (D, d) \end{array}$$

[New]

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$!

Proposition: Diagonal fill-in in $\text{Coalg}(F)$, if F preserves \mathcal{M}

$$\begin{array}{ccc}
 (A, a) & \xrightarrow{e} \twoheadrightarrow & (B, b) \\
 f \downarrow & \swarrow \exists! u \dashrightarrow & \downarrow g \\
 (C, c) & \xrightarrow{m} \rightarrow & (D, d)
 \end{array}$$

[New]

Proof. Obtain u as the diagonal in \mathcal{C} . It is a coalgebra morphism!

$(\mathcal{E}$ -carried, \mathcal{M} -carried)-factorization system in $\text{Coalg}(F)$!

Proposition: Diagonal fill-in in $\text{Coalg}(F)$, if F preserves \mathcal{M}

$$\begin{array}{ccc}
 (A, a) & \xrightarrow{e} \twoheadrightarrow & (B, b) \\
 f \downarrow & \swarrow \exists! u \text{ ---} & \downarrow g \\
 (C, c) & \xrightarrow{m} & (D, d)
 \end{array}$$

[New]

Proof. Obtain u as the diagonal in \mathcal{C} . It is a coalgebra morphism!

$$\begin{array}{ccccc}
 A & \xrightarrow{e} \twoheadrightarrow & B & & A & \xrightarrow{e} \twoheadrightarrow & B \\
 a \downarrow & \searrow f & \swarrow u & \downarrow g & \searrow b & \downarrow b & \\
 FA & \xrightarrow{f \text{ hom.}} & C & \xrightarrow{m} & D & \xrightarrow{g \text{ hom.}} & FB \\
 Ff \downarrow & \swarrow c & \downarrow d & \swarrow Fg & & & \\
 FC & \xrightarrow{Fm} & FD & & FC & \xrightarrow{Fm} & Fm \\
 & & & & & & \\
 & & & & FA & \xrightarrow{Fe} & FB & \xrightarrow{\text{trivial}} & FB \\
 & & & & Ff \downarrow & \swarrow Fu & \searrow Fg & \downarrow Fg & \\
 & & & & FC & \xrightarrow{Fm} & Fm & &
 \end{array}$$

Uniqueness of the diagonal fill-in $\Rightarrow c \cdot u = Fu \cdot b$ □

Proposition:

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

object C is \mathcal{M} -minimal \iff every $h: C' \rightarrow C$ is in \mathcal{E}

Proposition:

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

object C is \mathcal{M} -minimal \iff every $h: C' \rightarrow C$ is in \mathcal{E}



$\mathcal{K} = \text{Coalg}_I(F)$

(C, c, i_C) is reachable



every $h: (C', c', i_{C'}) \rightarrow (C, c, i_C)$
is surjective

Proposition:

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

object C is \mathcal{M} -minimal \iff every $h: C' \rightarrow C$ is in \mathcal{E}



$\mathcal{K} = \text{Coalg}_l(F)$

(C, c, i_C) is reachable

\iff

every $h: (C', c', i_{C'}) \rightarrow (C, c, i_C)$
is surjective

$\mathcal{K} = \text{Coalg}(F)^{\text{op}}$

(C, c) is simple

\iff

every $h: (C, c) \rightarrow (C', c')$
is injective

Definition:

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

$$\mathcal{M}\text{-minimization of } C = \begin{array}{l} m: D \twoheadrightarrow C \\ \text{where } D \text{ is } \mathcal{M}\text{-minimal} \end{array}$$

Definition:

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

$$\mathcal{M}\text{-minimization of } C = \begin{array}{l} m: D \twoheadrightarrow C \\ \text{where } D \text{ is } \mathcal{M}\text{-minimal} \end{array}$$



$$\mathcal{K} = \text{Coalg}_I(F)$$

\mathcal{M} -minimization of (C, c, i_C)

=

reachable subcoalgebra of
 (C, c, i_C)

Definition:

Category \mathcal{K} with an $(\mathcal{E}, \mathcal{M})$ -factorization System:

$$\mathcal{M}\text{-minimization of } C = \begin{array}{l} m: D \twoheadrightarrow C \\ \text{where } D \text{ is } \mathcal{M}\text{-minimal} \end{array}$$



$$\mathcal{K} = \text{Coalg}_I(F)$$

$$\begin{array}{l} \mathcal{M}\text{-minimization of } (C, c, i_C) \\ = \\ \text{reachable subcoalgebra of} \\ (C, c, i_C) \end{array}$$

$$\mathcal{K} = \text{Coalg}(F)^{\text{op}}$$

$$\begin{array}{l} \mathcal{E}\text{-minimization of } (C, c) \\ = \\ \text{simple quotient of } (C, c) \end{array}$$

Proposition: \mathcal{M} -minimizations ...

- are unique (up to iso) if \mathcal{K} has pullbacks of \mathcal{M} -morphisms
- exist if \mathcal{K} has wide pullbacks of \mathcal{M} -morphisms, \mathcal{M} is a class of monomorphisms, and \mathcal{K} is \mathcal{M} -wellpowered

Proposition: \mathcal{M} -minimizations ...

- are unique (up to iso) if \mathcal{K} has pullbacks of \mathcal{M} -morphisms
- exist if \mathcal{K} has wide pullbacks of \mathcal{M} -morphisms, \mathcal{M} is a class of monomorphisms, and \mathcal{K} is \mathcal{M} -wellpowered



$\mathcal{K} = \text{Coalg}_I(F)$

$F: \text{Set} \rightarrow \text{Set}$

reachable subcoalgebras ...

- are unique (up to iso) if F preserves intersections
- exist if F preserves infinite intersections

Adámek, Milius, Moss, Sousa '13

Proposition: \mathcal{M} -minimizations ...

- are unique (up to iso) if \mathcal{K} has pullbacks of \mathcal{M} -morphisms
- exist if \mathcal{K} has wide pullbacks of \mathcal{M} -morphisms, \mathcal{M} is a class of monomorphisms, and \mathcal{K} is \mathcal{M} -wellpowered



$$\mathcal{K} = \text{Coalg}_I(F) \quad F: \text{Set}^{\circlearrowleft}$$

reachable subcoalgebras ...

- are unique (up to iso) if F preserves intersections
- exist if F preserves infinite intersections

Adámek, Milius, Moss, Sousa '13

$$\mathcal{K} = \text{Coalg}(F)^{\text{op}} \quad F: \text{Set}^{\circlearrowleft}$$

simple quotients ...

- are unique (up to iso)
- exist

Gumm '08

Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Proposition:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then

- \mathcal{M} -minimal objects form a coreflective subcategory $\mathcal{K}_{\min} \hookrightarrow \mathcal{K}$
- \mathcal{M} -minimization is the coreflector $\mathcal{K} \rightarrow \mathcal{K}_{\min}$ (\Rightarrow functorial)

Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Proposition:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then

- \mathcal{M} -minimal objects form a coreflective subcategory $\mathcal{K}_{\min} \hookrightarrow \mathcal{K}$
- \mathcal{M} -minimization is the coreflector $\mathcal{K} \rightarrow \mathcal{K}_{\min}$ (\Rightarrow functorial)



$$\mathcal{K} = \text{Coalg}_I(F)$$

$$F: \text{Set} \rightarrow \text{Set}$$

If F preserves inverse images,

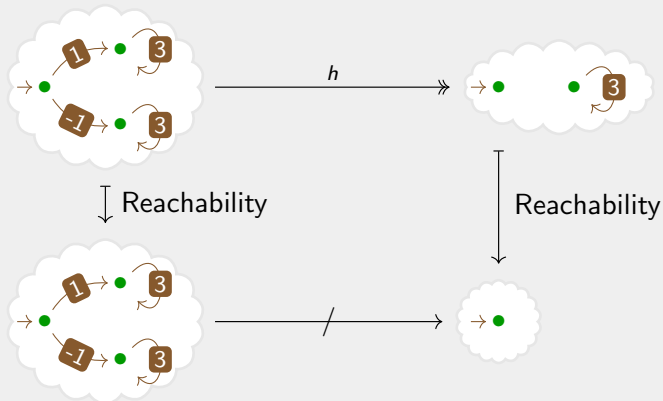
- reachable coalgebras form a coreflective subcategory $\text{Coalg}_I^{\text{reach}}(F) \hookrightarrow \text{Coalg}_I(F)$
- reachability is functorial

Inverse image preservation

- $FX = 2 \times X^A$ preserves inverse images
- $FX = \mathcal{P}X$ preserves inverse images
- $FX = \mathbb{R}^{(X)}$ does not preserve inverse images, because the monoid $(\mathbb{R}, +, 0)$ is not positive.

Inverse image preservation

- $FX = 2 \times X^A$ preserves inverse images
- $FX = \mathcal{P}X$ preserves inverse images
- $FX = \mathbb{R}^{(X)}$ does not preserve inverse images, because the monoid $(\mathbb{R}, +, 0)$ is not positive.



Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Proposition:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then

- \mathcal{M} -minimal objects form a coreflective subcategory $\mathcal{K}_{\min} \hookrightarrow \mathcal{K}$
- \mathcal{M} -minimization is the coreflector $\mathcal{K} \rightarrow \mathcal{K}_{\min}$ (\Rightarrow functorial)



$$\mathcal{K} = \text{Coalg}_I(F)$$

$$F: \text{Set} \rightarrow \text{Set}$$

If F preserves inverse images,

- reachable coalgebras form a coreflective subcategory $\text{Coalg}_I^{\text{reach}}(F) \hookrightarrow \text{Coalg}_I(F)$
- reachability is functorial

Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Proposition:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then

- \mathcal{M} -minimal objects form a coreflective subcategory $\mathcal{K}_{\min} \hookrightarrow \mathcal{K}$
- \mathcal{M} -minimization is the coreflector $\mathcal{K} \rightarrow \mathcal{K}_{\min}$ (\Rightarrow functorial)



$$\mathcal{K} = \text{Coalg}_I(F) \quad F: \text{Set}^{\downarrow}$$

If F preserves inverse images,

- reachable coalgebras form a coreflective subcategory $\text{Coalg}_I^{\text{reach}}(F) \hookrightarrow \text{Coalg}_I(F)$
- reachability is functorial

$$\mathcal{K} = \text{Coalg}(F)^{\text{op}} \quad F: \text{Set}^{\downarrow}$$

- Simple coalgebras form a reflective subcategory $\text{Coalg}^{\text{simple}}(F) \hookrightarrow \text{Coalg}(F)$
- Finding the simple quotient is functorial

Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Corollary:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then
 \mathcal{M} -minimal objects are closed under \mathcal{E} -morphisms

Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Corollary:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then \mathcal{M} -minimal objects are closed under \mathcal{E} -morphisms



$\mathcal{K} = \text{Coalg}_I(F)$

$F: \text{Set}^{\circlearrowleft}$

If F preserves inverse images,
then reachable coalgebras are
closed under quotients

Assume: \mathcal{M} is a class of monos and all \mathcal{M} -minimizations exist

Corollary:

If pullbacks along \mathcal{M} morphisms exist in \mathcal{K} , then
 \mathcal{M} -minimal objects are closed under \mathcal{E} -morphisms



$\mathcal{K} = \text{Coalg}_I(F)$ $F: \text{Set}^{\circlearrowleft}$

If F preserves inverse images,
 then reachable coalgebras are
 closed under quotients

$\mathcal{K} = \text{Coalg}(F)^{\text{op}}$ $F: \text{Set}^{\circlearrowleft}$

Simple coalgebras are closed
 under subcoalgebras

Assume:

- all \mathcal{M} -minimizations in \mathcal{K} and all \mathcal{E} -minimizations in \mathcal{K}^{op} exist
- \mathcal{K} has pullbacks along \mathcal{M}
- \mathcal{K} has pushouts along \mathcal{E}

Proposition

For $C \in \mathcal{K}$, the following yield the same:

- \mathcal{E} -minimization (in \mathcal{K}^{op}) of the \mathcal{M} -minimization of C (in \mathcal{K})
- \mathcal{M} -minimization (in \mathcal{K}) of the \mathcal{E} -minimization of C (in \mathcal{K}^{op})

Assume:

- all \mathcal{M} -minimizations in \mathcal{K} and all \mathcal{E} -minimizations in \mathcal{K}^{op} exist
- \mathcal{K} has pullbacks along \mathcal{M}
- \mathcal{K} has pushouts along \mathcal{E}

Proposition

For $C \in \mathcal{K}$, the following yield the same:

- \mathcal{E} -minimization (in \mathcal{K}^{op}) of the \mathcal{M} -minimization of C (in \mathcal{K})
- \mathcal{M} -minimization (in \mathcal{K}) of the \mathcal{E} -minimization of C (in \mathcal{K}^{op})

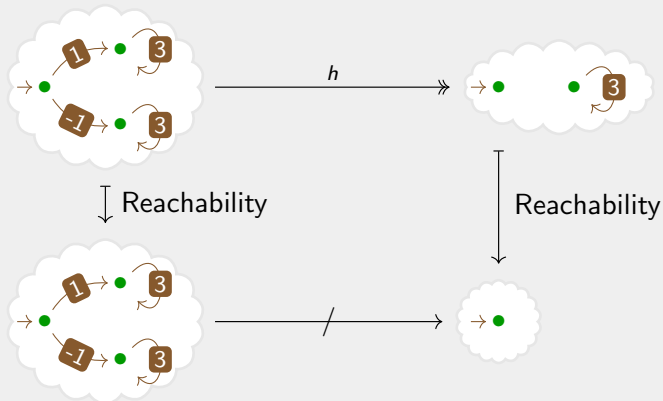


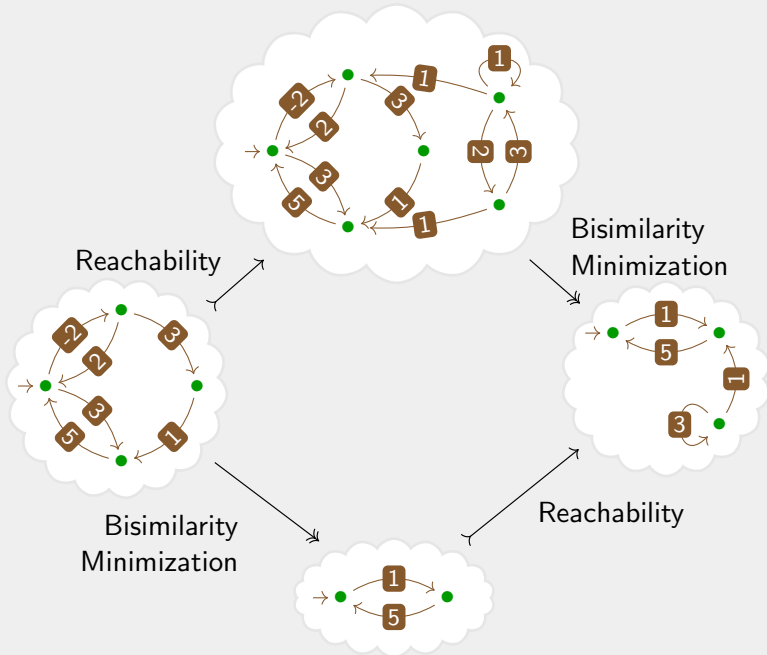
$$\mathcal{K} = \text{Coalg}_I(F) \quad F: \text{Set} \rightarrow \text{Set}$$

If F preserves inverse images, then reachability and bisimilarity minimization can be performed in any order.

Inverse image preservation

- $FX = 2 \times X^A$ preserves inverse images
- $FX = \mathcal{P}X$ preserves inverse images
- $FX = \mathbb{R}^{(X)}$ does not preserve inverse images, because the monoid $(\mathbb{R}, +, 0)$ is not positive.





References



Jiří Adámek, Stefan Milius, Lawrence S. Moss, Lurdes Sousa. “Well-pointed Coalgebras”. In: **Log. Methods Comput. Sci.** 9.3:2 (2013), 51 pp.



H. P. Gumm. **Algemeine Algebra. Mit einem Anhang über Universelle Coalgebra von H. P. Gumm.** Ed. by Thomas Ihringer. Vol. 10. Berliner Studienreihe zur Mathematik. Heldermann Verlag, 2003.



H. Peter Gumm. “On Minimal Coalgebras”. In: **Applied Categorical Structures** 16.3 (June 2008), pp. 313–332. DOI: 10.1007/s10485-007-9116-1. URL: <https://doi.org/10.1007/s10485-007-9116-1>.

Simple Quotient in $\text{Coalg}_I(F)$

Nothing new in $\text{Coalg}_I(F)$ for \rightarrow

The forgetful functor

$$\text{Coalg}_I(F) \longrightarrow \text{Coalg}(F)$$

preserves and reflects simple coalgebras and simple quotients.

Proof

For every pointed coalgebra (C, c, i) , we have isomorphic categories:

$$(C, c, i)/\text{Coalg}_I(F) \cong (C, c)/\text{Coalg}(F)$$