Hausdorff Coalgebras

Pedro Nora (collaboration with Dirk Hofmann and Renato Neves)

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Department of Computer Science, Friedrich-Alexander University
pedro.nora@fau.de

Introduction

About me

Education (Aveiro)

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- B.Sc. in Mathematics.

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- Linux user

Switching context

- Lutz Schröder and Paul Wild. “Characteristic Logics for Behavioural Metrics via Fuzzy Lax Extensions”. In: *CONCUR* (2020)
- Dirk Hofmann and Pedro Nora. “Hausdorff coalgebras”. In: *Applied Categorical Structures* (2020)

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Tholen:

(...) Fuzzy (...)? Perhaps they should work with $\mathcal{V}$-categories.

Quantale-enriched category theory

What are categories made of?

Definition

A (small) category $\mathbf{C}$ consists of:

- a set of objects;
- for each pair x,y of objects, a set of morphisms $\mathbf{C}(x, y)$;
- for each triple x,y,z of objects, a function $\circ_{x,y,z} \colon \mathbf{C}(x, y) \times \mathbf{C}(y, z) \to \mathbf{C}(x, z)$;
- for each object x, a function $1_x \colon 1 \to \mathbf{C}(x, x)$;

such that . . .

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such that ...

Definition

A functor $F \colon \mathbf{X} \to \mathbf{Y}$ consists of:

- a function from the objects of $\mathbf{X}$ to the objects of $\mathbf{Y}$;
- for each pair of objects x, x' of $\mathbf{X}$ a function $F_{x,x'} \colon \mathbf{X}(x, x') \to \mathbf{Y}(Fx, Fx')$

such that ...

Definition

A quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a complete lattice $\mathcal{V}$ equipped with a commutative monoid structure $\otimes$, with identity k, such that, for each $u \in \mathcal{V}$,

$$u \otimes - \colon \mathcal{V} \longrightarrow \mathcal{V} \quad \text{has a right adjoint} \quad \hom(u, -) \colon \mathcal{V} \longrightarrow \mathcal{V}.$$

A thin monoidal category

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Recall

- A preordered set $(X, \leq)$ can be understood as a category where for each pair of objects x, y there is at most one morphism between x and y.

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Recall

- A preordered set $(X, \leq)$ can be understood as a category where for each pair of objects x, y there is at most one morphism between x and y.
- A quantale is a symmetric monoidal closed category.

Category

- a set of objects;

Don't get lost in translation

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Definition

Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale.

- A $\mathcal{V}$-category is a pair (X, a) consisting of a set X and a map $a\colon X \times X \longrightarrow \mathcal{V}$ satisfying

$$k \le a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \le a(x,z),$$

  for all $x, y, z \in X$.
- A $\mathcal{V}$-functor $f\colon (X,a) \longrightarrow (Y,b)$ between $\mathcal{V}$-categories is a map $f\colon X \longrightarrow Y$ such that

$$a(x,x') \le b(f(x), f(x')),$$

  for all $x, x' \in X$.
- $\mathcal{V}$-categories and $\mathcal{V}$-functors define the category $\mathcal{V}$-**Cat**.

Example

If $\mathcal{V}$ is the trivial quantale, then 1-**Cat** $\sim$ **Set**.

Example

If $\mathcal{V}$ is the trivial quantale, then $\mathbf{1}$-**Cat** $\sim$ **Set**.

Example

If $\mathcal{V}$ is the two element chain $\mathbf{2} = \{0, 1\}$ with $\otimes = \&$. Then $\mathbf{2}$-**Cat** $\sim$ **Ord**.

Example

If $\mathcal{V}$ is the trivial quantale, then **1-Cat** $\sim$ **Set**.

Example

If $\mathcal{V}$ is the two element chain $\mathbf{2} = \{0, 1\}$ with $\otimes = \&$. Then **2-Cat** $\sim$ **Ord**.

Example

Consider the quantales based on the extended real half line $\overleftarrow{[0, \infty]}$ ordered by the "greater or equal" relation $\geqslant$ and

- the tensor product given by addition $+$, denoted by $\overleftarrow{[0, \infty]}_{+}$;
- or with $\otimes = \max$, denoted as $\overleftarrow{[0, \infty]}_{\wedge}$.

Then $\overleftarrow{[0, \infty]}_{+}$-**Cat** $\sim$ **Met** is the category of (generalised) metric spaces and non-expansive maps and $\overleftarrow{[0, \infty]}_{\wedge}$-**Cat** $\sim$ **UMet** is the category of (generalised) ultrametric spaces and non-expansive maps.

Example

If $\mathcal{V}$ is the quantale $\overleftarrow{[0,1]}_{\oplus}$ given by the unit interval $[0,1]$ with the "greater or equal" relation $\geqslant$ and the tensor $u \oplus v = \min\{1, u+v\}$. Then $\overleftarrow{[0,1]}_{\oplus}$-**Cat** $\sim$ **BMet** is the category of (generalised) bounded-by-one metric spaces and non-expansive maps.

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Alert

Do not forget to reverse the order!

Definition

Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale, and X, Y and Z sets.

- A $\mathcal{V}$-relation from X to Y, $X \nrightarrow Y$, is a map $X \times Y \to \mathcal{V}$.
- For $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$, the composite $s \cdot r\colon X \nrightarrow Z$ is calculated pointwise by
$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$
for every $x \in X$ and $z \in Z$.
- Sets and $\mathcal{V}$-relations define the category $\mathcal{V}$-**Rel**.

Quantale-valued relations

Definition

Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale, and X, Y and Z sets.

- A $\mathcal{V}$-relation from X to Y, $X \relbar\joinrel\mapsto Y$, is a map $X \times Y \to \mathcal{V}$.
- For $r \colon X \relbar\joinrel\mapsto Y$ and $s \colon Y \relbar\joinrel\mapsto Z$, the composite $s \cdot r \colon X \relbar\joinrel\mapsto Z$ is calculated pointwise by
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for every $x \in X$ and $z \in Z$.
- Sets and $\mathcal{V}$-relations define the category $\mathcal{V}$-**Rel**.

Remark

The structure of a $\mathcal{V}$-category is a reflexive and transitive $\mathcal{V}$-relation. That is, for a $\mathcal{V}$-category (X, a), $1_X \leq a$ and $a \cdot a \leq a$.

Coalgebras of strict liftings

Strict functorial liftings

Problem

Given an endofunctor F on a category **A** and a faithful functor $U\colon \mathbf{X} \to \mathbf{A}$, study a "lifting" of F to an endofunctor $\overline{F}$ on **X**. In a strict sense, by "lifting" we mean that the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\overline{F}} & \mathbf{X} \\ \downarrow & & \downarrow \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{A}. \end{array}$$

commutes.

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$$\begin{array}{ccc} \mathbf{Met} & \xrightarrow{\overline{F}} & \mathbf{Met} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow[F]{} & \mathbf{Set}. \end{array}$$

commutes[a].

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[a] Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Coalgebraic Behavioral Metrics". In: *Logical Methods in Computer Science* **14**.(3) (2018), pp. 1860–5974.

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$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbf{Cat} & \xrightarrow{\overline{F}} & \mathcal{V}\text{-}\mathbf{Cat} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow[F]{} & \mathbf{Set}. \end{array}$$

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[a] Adriana Balan, Alexander Kurz, and Jiří Velebil. "Extending set functors to generalised metric spaces". In: *Logical Methods in Computer Science* **15**.(1) (2019).

Lax extension

Consider first a lax extension $\widehat{\mathsf{F}}\colon \mathcal{V}\text{-}\mathbf{Rel} \to \mathcal{V}\text{-}\mathbf{Rel}$ of the functor F:[a]

1. $r \leq r' \implies \widehat{\mathsf{F}}r \leq \widehat{\mathsf{F}}r'$,
2. $\widehat{\mathsf{F}}s \cdot \widehat{\mathsf{F}}r \leq \widehat{\mathsf{F}}(s \cdot r)$,
3. $\mathsf{F}f \leq \widehat{\mathsf{F}}(f)$ and $(\mathsf{F}f)^\circ \leq \widehat{\mathsf{F}}(f^\circ)$.

Then, the functor $\mathsf{F}\colon \mathbf{Set} \to \mathbf{Set}$ admits a natural lifting to $\mathcal{V}$-**Cat**[b]: the functor $\overline{\mathsf{F}}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ sends a $\mathcal{V}$-category (X, a) to $(\mathsf{F}X, \widehat{\mathsf{F}}a)$.

---

[a] Gavin J. Seal. "Canonical and op-canonical lax algebras". In: *Theory and Applications of Categories* **14**.(10) (2005), pp. 221–243.

[b] Walter Tholen. "Ordered topological structures". In: *Topology and its Applications* **156**.(12) (2009), pp. 2148–2157.

Remark

One advantage of this type of lifting is that allows us to use the calculus of $\mathcal{V}$-relations.

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Proposition

$\overline{\mathsf{F}}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ *preserves initial* $\mathcal{V}$*-functors.*

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Proposition

$\overline{\mathsf{F}}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ *preserves initial* $\mathcal{V}$*-functors.*

Proof.

Let $f\colon (X,a) \to (Y,b)$ be a $\mathcal{V}$-functor with $a = f^\circ \cdot b \cdot f$. Then $\widehat{\mathsf{F}}a = \mathsf{F}f^\circ \cdot \widehat{\mathsf{F}}b \cdot \mathsf{F}f$. $\square$

The Wasserstein lifting

One possible way to construct lax extensions based on a (lax) $\mathbb{T}$-algebra structure $\xi\colon \mathsf{T}\mathcal{V} \to \mathcal{V}$ is devised in[a]: for every $\mathcal{V}$-relation $r\colon X \times Y \to \mathcal{V}$ and for all $\mathfrak{x} \in \mathsf{T}X$ and $\mathfrak{y} \in \mathsf{T}Y$,

$$\widehat{\mathsf{T}}r(\mathfrak{x},\mathfrak{y}) = \bigvee \left\{ \xi \cdot \mathsf{T}r(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in \mathsf{T}(X \times Y), \mathsf{T}\pi_1(\mathfrak{w}) = \mathfrak{x}, \mathsf{T}\pi_2(\mathfrak{w}) = \mathfrak{y} \right\}.$$

[a] Dirk Hofmann. “Topological theories and closed objects”. In: *Advances in Mathematics* **215**.(2) (2007), pp. 789–824.

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Remark

The lax extension $\widehat{\mathsf{T}}$ preserves the involution on $\mathcal{V}$-**Rel**, that is, $\widehat{\mathsf{T}}(r^\circ) = (\widehat{\mathsf{T}}r)^\circ$ for all $\mathcal{V}$-relations $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

Definition

A functor $\mathsf{U}\colon \mathbf{X} \to \mathbf{A}$ is called topological whenever each cone $\mathcal{C} = (A \to \mathsf{U}X_i)_{i \in I}$ in $\mathbf{A}$ admits a U-initial lifting. In this case we say that X is topological over $\mathbf{A}$.

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Example

- The category **Top** is topological over **Set**.

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Proposition

If $\mathbf{X}$ *is a topological over a category* $\mathbf{A}$*, then* $\mathbf{X}$ *has limits of shape* $\mathbf{I}$ *if and only if* $\mathbf{A}$ *has limits of shape* $\mathbf{I}$.

Notation

For a functor $\mathsf{F}\colon \mathbf{A} \to \mathbf{A}$ and $\mathbf{A}$-morphisms $\psi\colon A \to \widetilde{A}$ and $\sigma\colon \mathsf{F}\widetilde{A} \to \widetilde{A}$, let $\psi^{\Diamond}\colon \mathsf{F}A \to \widetilde{A}$ denote the composite $\mathsf{F}A \xrightarrow{\mathsf{F}\psi} \mathsf{F}\widetilde{A} \xrightarrow{\sigma} \widetilde{A}$.

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The Kantorovich lifting

Consider a category $\mathbf{X}$ equipped with a topological functor $\mathsf{U}\colon \mathbf{X} \to \mathbf{A}$, and an $\mathbf{X}$-object $\widetilde{X}$ whose underlying set $\mathsf{U}\widetilde{X}$ carries the structure $\sigma\colon \mathsf{F}\mathsf{U}\widetilde{X} \to \mathsf{U}\widetilde{X}$ of a F-algebra. Then $(\psi^{\lozenge}\colon \mathsf{F}\mathsf{U}X \to \mathsf{U}\widetilde{X})_{\psi\in\mathbf{X}(X,\widetilde{X})}$ is a U-structured cone, and we define $\overline{\mathsf{F}}X$ to be the domain of the initial lift of this cone.

Theorem

1. *The construction of the previous slide defines a strict lifting* $\overline{\mathsf{F}}\colon \mathbf{X} \to \mathbf{X}$ *of the functor* $\mathsf{F}\colon \mathbf{A} \to \mathbf{A}$.
2. *For every* $\psi\colon X \to \widetilde{X}$ *in* $\mathbf{X}$, $\psi^{\lozenge}$ *is an* $\mathbf{X}$*-morphism* $\psi^{\lozenge}\colon \overline{\mathsf{F}}X \to \widetilde{X}$. *In particular,* $\sigma = 1_{\widetilde{X}}{}^{\lozenge}$ *is an* $\mathbf{X}$*-morphism* $\sigma\colon \overline{\mathsf{F}}\widetilde{X} \to \widetilde{X}$.
3. *If* $\widetilde{X}$ *is injective with respect to initial morphisms, then* $\overline{\mathsf{F}}\colon \mathbf{X} \to \mathbf{X}$ *preserves initial morphism.*
4. *Let* $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ *be a natural transformation such that* $\sigma_{\mathsf{G}} \cdot \alpha_{\widetilde{X}} = \sigma_{\mathsf{F}}$. *Then* α *lifts to a natural transformation between the corresponding* $\mathbf{X}$*-functors.*
5. *If* $\mathsf{F} = \mathsf{T}$ *is part of a monad* $\mathbb{T} = (\mathsf{T}, m, e)$ *on* $\mathbf{A}$ *and* $\sigma\colon \mathsf{T}|\widetilde{X}| \to |\widetilde{X}|$ *is a* $\mathbb{T}$*-algebra, then* $\mathbb{T}$ *lifts naturally to a monad* $\bar{\mathbb{T}} = (\bar{\mathsf{T}}, m, e)$ *on* $\mathbf{X}$.

Limits in categories of coalgebras of strict lifting

Theorem

Consider the following commutative diagram of functors.

$$
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\overline{F}} & \mathbf{X} \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathbf{A} & \xrightarrow[F]{} & \mathbf{A}
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1. *If* $\overline{\mathsf{F}}$ *has a fix-point, then so has* F. *Hence, if* F *does not have a fix-point, then neither does* $\overline{\mathsf{F}}$.

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2. *If* $\mathsf{U}\colon \mathbf{X} \to \mathbf{A}$ *is topological, then so is* $\mathsf{U}\colon\ \mathsf{CoAlg}(\overline{\mathsf{F}}) \to \mathsf{CoAlg}(\mathsf{F})$.

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1. *If* $\overline{\mathsf{F}}$ *has a fix-point, then so has* F*. Hence, if* F *does not have a fix-point, then neither does* $\overline{\mathsf{F}}$*.*
2. *If* $\mathsf{U}\colon \mathbf{X} \to \mathbf{A}$ *is topological, then so is* $\mathsf{U}\colon \ \mathsf{CoAlg}(\overline{\mathsf{F}}) \to \mathsf{CoAlg}(\mathsf{F})$*.*

   In particular, the category $\mathsf{CoAlg}(\overline{\mathsf{F}})$ *has limits of shape I if and only if* $\mathsf{CoAlg}(\mathsf{F})$ *has limits of shape I.*

Hausdorff Coalgebras

Previous work

Dirk Hofmann, Renato Neves, and Pedro Nora. “Limits in categories of Vietoris coalgebras”. In: *Mathematical Structures in Computer Science* **29**.(4) (2019), pp. 552–587

Kripke polynomial functors

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Definition

We call a functor Kripke polynomial whenever it belongs to the smallest class of endofunctors on **Set** that contains the identity functor, all constant functors and is closed under composition with a powerset functor, products and sums of functors.

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Definition

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Remark

The class of Kripke polynomial functors is well-behaved in regard to the existence of limits in their respective categories of coalgebras — if the powerset functor is submitted to certain cardinality restrictions.

Recall

- The terminal coalgebra for $F\colon \mathbf{C} \to \mathbf{C}$ is a fix-point of F. [a]

[a] Joachim Lambek. "A fixpoint theorem for complete categories". In: *Mathematische Zeitschrift* **103**.(2) (1968), pp. 151–161.

Recall

- The terminal coalgebra for $F \colon \mathbf{C} \to \mathbf{C}$ is a fix-point of F.
- The power-set functor $P \colon \mathbf{Set} \to \mathbf{Set}$ does not have a fix-point; hence P does not admit a terminal coalgebra. [a]

[a] Georg Cantor. "Über eine elementare Frage der Mannigfaltigkeitslehre". In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* **1** (1891), pp. 75–78.

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- The finite power-set functor $\mathsf{P}_{\mathrm{fin}}\colon \mathbf{Set} \to \mathbf{Set}$ admits a terminal coalgebra (for instance, because $\mathsf{P}_{\mathrm{fin}}$ is finitary). [a]

---

[a] Michael Barr. "Terminal coalgebras in well-founded set theory". In: *Theoretical Computer Science* **114**.(2) (1993), pp. 299–315.

A BRIEF HISTORY OF "POWERFULL" FUNCTORS

Recall

- The terminal coalgebra for $F: \mathbf{C} \to \mathbf{C}$ is a fix-point of F.
- The power-set functor $P: \mathbf{Set} \to \mathbf{Set}$ does not have a fix-point; hence P does not admit a terminal coalgebra.
- The finite power-set functor $P_{\mathrm{fin}}: \mathbf{Set} \to \mathbf{Set}$ admits a terminal coalgebra (for instance, because P_{fin} is finitary).
- *Somehow more general*: the Vietoris functor $V: \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a terminal coalgebra[a]

[a] Here: V preserves codirected limits. This result appears as an exercise in Ryszard Engelking. *General topology*. 2nd ed. Vol. 6. Sigma Series in Pure Mathematics. Berlin: Heldermann Verlag, 1989. viii + 529.

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- *Somehow more general*: the Vietoris functor $V\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a terminal coalgebra (and the same is true for $V\colon \mathbf{PosComp} \to \mathbf{PosComp}$).[a][b]

[a] Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

[b] Dirk Hofmann, Renato Neves, and Pedro Nora. “Limits in categories of Vietoris coalgebras”. In: *Mathematical Structures in Computer Science* **29** (4) (2019), pp. 552–587.

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- *Somehow more general*: the Vietoris functor $\mathsf{V}\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a terminal coalgebra (and the same is true for $\mathsf{V}\colon \mathbf{PosComp} \to \mathbf{PosComp}$).
- *A bit more general*: the compact Vietoris functor $\mathsf{V}_{\mathrm{c}}\colon \mathbf{Top} \to \mathbf{Top}$ admits a terminal coalgebra.

A brief history of "powerfull" functors

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- The finite power-set functor $P_{\mathrm{fin}}\colon \mathbf{Set} \to \mathbf{Set}$ admits a terminal coalgebra (for instance, because P_{fin} is finitary).
- *Somehow more general*: the Vietoris functor $V\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a terminal coalgebra (and the same is true for $V\colon \mathbf{PosComp} \to \mathbf{PosComp}$).
- *A bit more general*: the compact Vietoris functor $V_{\mathrm{c}}\colon \mathbf{Top} \to \mathbf{Top}$ admits a terminal coalgebra.
- *A bit surprising(?)*: Also the lower Vietoris functor $V\colon \mathbf{Top} \to \mathbf{Top}$ admits a terminal coalgebra.

Definition

Let $f\colon (X,a)\to(Y,b)$ be a $\mathcal{V}$-functor.

1. For every $A\subseteq X$, put $\uparrow^{a} A=\{y\in X\mid k\le\bigvee_{x\in A}a(x,y)\}$.

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3. We consider the $\mathcal{V}$-category $\mathrm{H}X = \{A \subseteq X \mid A \text{ is increasing}\}$, equipped with $\mathrm{H}a(A,B) = \bigwedge_{y\in B} \bigvee_{x\in A} a(x,y)$, for all $A, B \in \mathrm{H}X$.

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4. The map $\mathsf{H}f\colon \mathsf{H}(X,a)\longrightarrow\mathsf{H}(Y,b)$ sends an increasing subset $A\subseteq X$ to $\uparrow^b f(A)$.
5. The functor H is part of a Kock–Zöberlein monad $\mathbb{H}=(\mathsf{H},w,h)$ on $\mathcal{V}$-**Cat**.

$$h_X\colon X\longrightarrow \mathsf{H}X,\qquad w_X\colon \mathsf{H}\mathsf{H}X\longrightarrow\mathsf{H}X.$$

$$x\longmapsto\uparrow x\qquad\qquad \mathcal{A}\longmapsto\bigcup\mathcal{A}$$

Theorem

Let X be a partially ordered set. Then there is no embedding $\varphi\colon \mathrm{Up}(X) \to X$. [a]

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[a] Robert P. Dilworth and Andrew M. Gleason. "A generalized Cantor theorem". In: *Proceedings of the American Mathematical Society* **13**.(5) (1962), pp. 704–705.

About the upset funtor

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The upset functor $\mathrm{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$ *does not admit a terminal coalgebra.*

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Corollary

The upset functor $\mathrm{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$ *does not admit a terminal coalgebra.*

Remark

The category $\mathsf{CoAlg}(\mathrm{Up})$ has equalisers.

Coalgebras for the Hausdorff functor

Theorem

Let $\mathcal{V}$ be a non-trivial quantale and (X, a) be a $\mathcal{V}$-category. There is no embedding of type $\mathsf{H}(X, a) \to (X, a)$.

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In particular, the (non-symmetric) Hausdorff functor on **Met** does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces.

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Remark

In particular, the (non-symmetric) Hausdorff functor on **Met** does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces. Passing to the symmetric version of the Hausdorff functor does not remedy the situation.

Adding topology

Extending the Ultrafilter monad

We assume that $\mathcal{V}$ is a completely distributive quantale, then

$$\xi\colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}, \quad \mathfrak{v} \longmapsto \bigwedge_{A\in\mathfrak{v}} \bigvee A$$

is the structure of an $\mathbb{U}$-algebra on $\mathcal{V}$ (the Lawson topology).

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Here:

$$\mathsf{U}a(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A,B} \bigvee_{x,y} a(x, y), \qquad (X, a) \longmapsto (\mathsf{U}X, \mathsf{U}a).$$

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Its algebras are $\mathcal{V}$-categories equipped with a *compatible* compact Hausdorff topology [a][b]; we call them $\mathcal{V}$-categorical compact Hausdorff spaces, and denote the corresponding Eilenberg–Moore category by $\mathcal{V}$-**CatCH**.

---

[a] Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

[b] Walter Tholen. “Ordered topological structures”. In: *Topology and its Applications* **156**.(12) (2009), pp. 2148–2157.

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Theorem

For an ordered set $(X, \leq)$ and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon (\mathsf{U}X, \mathsf{U}{\leq}) \to (X, \leq)$ *is monotone.*

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(ii) $G_{\leq} \subseteq X\times X$ *is closed.*

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Theorem

For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon \mathsf{U}(X, a) \to (X, a)$ *is a $\mathcal{V}$-functor.*

(ii) $a\colon (X, \alpha) \times (X, \alpha) \to (\mathcal{V}, \xi_{\leq})$ *is continuous.*

Towards "Urysohn"

Lemma

Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space and $A, B \subseteq X$ so that $A \cap B = \varnothing$, A is increasing and compact in $(X, \alpha_{\leq})^{\mathrm{op}}$ and B is compact in $(X, \alpha_{\leq})$. Then there exists some $u \ll k$ so that, for all $x \in A$ and $y \in B$, $u \nleq a(x, y)$.

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Corollary

For every compact subset $A \subseteq X$ of $(X, \alpha_{\leq})^{\mathrm{op}}$, $\uparrow^{a} A = \uparrow^{\leq} A$. In particular, for every closed subset $A \subseteq X$ of (X, α), $\uparrow^{a} A = \uparrow^{\leq} A$.

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Theorem (Nachbin)

Let $A \subseteq X$ be closed and decreasing and $B \subseteq X$ be closed and increasing with $A \cap B = \varnothing$. Then there exist $V \subseteq X$ open and co-increasing and $W \subseteq X$ open and co-decreasing with

$$A \subseteq V, \quad B \subseteq W, \quad V \cap W = \varnothing.$$

The Hausdorff monad (again)

Definition

For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$\mathsf{H}X = \{A \subseteq X \mid A \text{ is closed and increasing}\}$$

with the restriction of the Hausdorff structure to $\mathsf{H}X$ and the hit-and-miss topology (Vietoris topology). That is, the topology generated by the sets

$$V^{\lozenge} = \{A \in \mathsf{H}X \mid A \cap V \neq \varnothing\} \qquad (V \text{ open, co-increasing})$$

and

$$W^{\square} = \{A \in \mathsf{H}X \mid A \subseteq W\} \qquad (W \text{ open, co-decreasing}).$$

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Proposition

For every $\mathcal{V}$-categorical compact Hausdorff space X, $\mathrm{H}X$ is a $\mathcal{V}$-categorical compact Hausdorff space.

Compare with:

For a compact metric space, the Hausdorff metric induces the Vietoris topology.

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The construction above defines a functor $\mathrm{H}\colon \mathcal{V}\text{-}\mathbf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathbf{CatCH}$.

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In fact, we obtain a Kock–Zöberlein monad.

Coalgebras for Hausdorff functors

Proposition

The diagrams of functors commutes.

$$
\begin{array}{ccc}
\mathbf{OrdCH} & \xrightarrow{\mathsf{H}} & \mathbf{OrdCH} \\
\downarrow\uparrow & & \downarrow\uparrow \\
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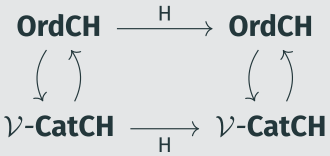

Proposition

*The Hausdorff functor on $\mathcal{V}$-**CatCH** preserves codirected initial cones with respect to the forgetful functor $\mathcal{V}$-**CatCH** $\to$ **CompHaus**.*

Theorem

*The Hausdorff functor $\mathsf{H}\colon \mathcal{V}$-**CatCH** $\to \mathcal{V}$-**CatCH** preserves codirected limits.*

Coalgebras for the Hausdorff functor

Theorem

For $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{CatCH} \to \mathcal{V}\text{-}\mathbf{CatCH}$*, the forgetful functor* $\mathsf{CoAlg}(\mathsf{H}) \to \mathcal{V}\text{-}\mathbf{CatCH}$ *is comonadic. Moreover,* $\mathcal{V}\text{-}\mathbf{CatCH}$ *has equalisers and is therefore complete.*

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Theorem

The category of coalgebras of a Hausdorff polynomial functor on $\mathcal{V}$-**CatCH** *is (co)complete.*

Definition

We call a functor Hausdorff polynomial whenever it belongs to the smallest class of endofunctors on $\mathcal{V}$-**Cat** that contains the identity functor, all constant functors and is closed under composition with H, products and sums of functors.