Uniform Elgot Iteration in Foundations

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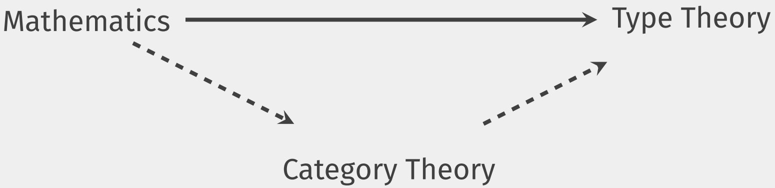
Very High-Level Message

Mathematics ⟶ Type Theory

VERY HIGH-LEVEL MESSAGE



Otherwise:

- Mathematical people are lost on the way
- Bulk of known mathematics lost in translation
- Results are type-theory specific

Category theory $\rightarrow$ neutral mathematics (term by Martin Escardó)

Context

Work on partiality monads

- Chapman, Uustalu, Veltri: Quotienting the delay monad by weak bisimilarity (coinduction + quotienting)
- Altenkirch, Danielsson, Kraus: Partiality, revisited - the partiality monad as a quotient inductive-inductive type (induction)
- Escardó, Knapp: Partial Elements and Recursion via Dominances in Univalent Type Theory (comparison + disciplined maps)

Work on iteration

- Adámek, Milius, Velebil: Elgot algebras
- Adámek, Milius, Velebil: Elgot theories: a new perspective of the equational properties of iteration

Contribution

- Generic definition of a partiality monad **K** ('K' alphabetic predecessor of 'L'), entailing
  - **K** is an equational lifting monad
  - Kleisli category of **K** enriched over pointed partial orders
  - **K** supports least fixpoint iteration
- Relation to classical mathematics via LPO categories
- Relation to constructive mathematics by quotienting delay monad

Main Idea

- Consider morphisms of the form $h\colon S \to X + S$ with
  - values X
  - states S
- Behaviour:
  - convergent: $x_0 \xrightarrow{h} x_1 \xrightarrow{h} \dots \xrightarrow{h} x_n$
  - divergent: $\bot = x_0 \xrightarrow{h} x_1 \xrightarrow{h} \dots$ (forever)

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- The generated denotations KX must include X and $\{\bot\}$
- Define KX to be free structure over X, satisfying
  - **Fixpoint:** $f^\sharp = [\mathrm{id}, f^\sharp]\, f$
  - **Uniformity:** $(\mathrm{id} + h)\; f = g\; h \;\Rightarrow\; f^\sharp = g^\sharp\; h$
    for $f\colon Y \to KX + Y$, $g\colon Z \to KX + Z$, $h\colon Y \to Z$

where iteration $(f\colon Y \to KX + Y)^\sharp\colon Y \to KX$

$\Rightarrow$ KX is a free uniform-iteration algebra over X

- By generalities, K extends to a (strong) monad $\mathbf{K}$ whose iteration operator

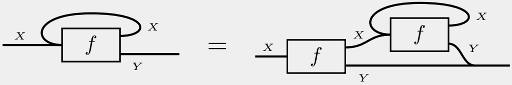
$$\frac{f\colon X \to KY + X}{f^{\sharp}\colon X \to KY}$$

- For Elgot monads, more generally

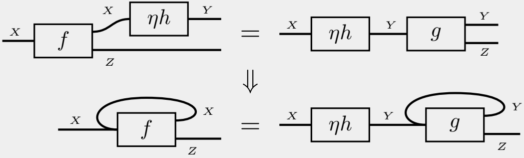
$$\frac{f\colon X \to T(Y + X)}{f^{\dagger}\colon X \to TY}$$

Axioms of Elgot Monads

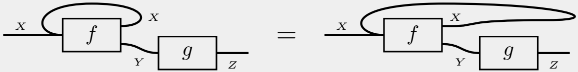
Fixpoint ($f : X \to T(Y + X)$):



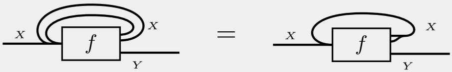
Uniformity ($f : X \to T(Z + X)$, $g : Y \to T(Z + Y)$, $h : X \to Y$):



Naturality ($f : X \to T(Y+X)$, $g : Y \to TZ$):



Codiagonal ($f : X \to T(Y + (X+X))$):



Naturality and **Codiagonal** are basically coherence laws

Free Uniform-Iteration v.s. Initial Elgot

- Elgot monads are semantic structures for implementing side-effecting while-loops
- Elgot monads properly generalize monads with $(-)^\dagger$ calculated as least fixpoints, e.g. coalgebraic resumptions

$$\nu\gamma.\, T(X + \Sigma\gamma)$$

  form an Elgot monad of generalized processes over a given Elgot monad **T**
- But (!) "initial Elgot monad" is an impredicative notion

Limited Principle of Omniscience

Our Assumptions

We work in a category **C** that

- has finite products
- has finite coproducts
- is extensive (coproducts are stable under pullbacks, coproduct injections are disjoint, pullbacks of coproduct injections exist)
- natural number object $\mathbb{N}$ exists and stable
- exponentials $X^{\mathbb{N}}$ exist

Examples: Classical and non-classical set theories, toposes, pretoposes, importantly also topological spaces **Top**

- Delay monad $D = \nu\gamma.\,(- + \gamma)$ is co-generated by

$$\frac{x : X}{\mathsf{now}\ x : DX} \qquad\qquad \frac{\sigma : DX}{\triangleright\,\sigma : DX}$$

- DX consists of precisely those infinite streams over $X \cup \{\bot\}$, which contain an element of X not more than in one place
- We call **C** an LPO[1] category if the embedding of $\mathbb{N}$ to $\bar{\mathbb{N}} = D\mathbf{1}$ is a coproduct injection

Proposition: **C** is LPO iff $DX \cong X \times \mathbb{N} + \mathbf{1}$

[1] LPO=Limited Principle of Omniscience

In any LPO category

- **K** is the maybe monad $(- + 1)$
- **K** is the initial Elgot monad
- **K** is enriched over pointed ω-complete partial orders
- KX is free pointed ω-complete partial orders on X

Examples: ZF (without choice), nominal sets, any presheaf topos, Lawvere's ETCS, **Cpo** (in classical **Set**), ...

Non-Examples: Top, (non-pathological) realizability toposes, Johnstone's topological topos, ΠW-pretoposes of h-sets and setoids, ...

Beyond LPO

Properties of $\mathbf{K}$

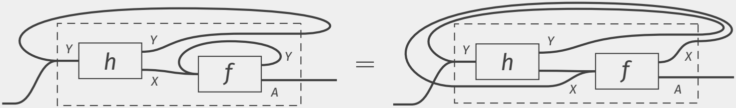
Theorem: If every KX exists and stable then

1. $\mathbf{K}$ is strong, commutative and satisfies the law

$$\begin{array}{ccc} KX & \xrightarrow{K\Delta} & K(X \times X) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle K(\eta \times \mathsf{id})} \\ KX \times KX & \xrightarrow{\tau} & K(KX \times X) \end{array}$$

   $\Rightarrow$ $\mathbf{K}$ is an equational lifting monad
2. Kleisli Category of $\mathbf{K}$ is a restriction category
   $\Rightarrow$ Kleisli Category of $\mathbf{K}$ is order enriched
3. Kleisli composition and strength moreover respect the order and the bottom element $\bot$ (divergence)
4. $f^\sharp$ is an internal order-limit of finite approximations (Kleene fixpoint theorem)

- **D** can be equipped with an iteration operator, which adds delays, i.e. $f^\sharp = [\mathsf{id}, \triangleright f^\sharp]\, f$
- **D** has a complete axiomatization in terms of $(-)^\sharp$, consisting of three axioms of Elgot algebras: **Fixpoint**, **Uniformity** and **Compositionality**[2]
- We can think of **K** as **D** with removed delays
- Then **Compositionality**



holds for **K** (non-trivially from Kleene's fixpoint property)

[2] Adámek, Milius, Velebil: Elgot algebras

Quotienting Delay Monad

- Following previous work[3], consider weak bisimilarity over DX:

$$\frac{\sigma \approx \mathsf{now}\ x}{\triangleright\, \sigma \approx \mathsf{now}\ x} \qquad \frac{\mathsf{now}\ x \approx \sigma}{\mathsf{now}\ x \approx \triangleright\, \sigma} \qquad \frac{\triangleright\, \sigma \approx \triangleright\, \sigma'}{\sigma \approx \sigma'}$$

- We generically model it with the coequalizer:

$$D(X \times \mathbb{N}) \underset{D\,\mathsf{fst}}{\overset{\iota^\star}{\rightrightarrows}} DX \xrightarrow{\rho_X} \tilde{D}X \qquad (*)$$

$\Rightarrow \rho_X\, \triangleright = \rho_X$, but this seems to be weaker than $(*)$

Theorem: The following are equivalent:

1. $(*)$ is preserved by D
2. $\tilde{D}$ extends to a strong monad and ρ to a strong monad morphism
3. $\tilde{D}X \cong KX$ and ρ_X is iteration preserving

[3] Chapman, Uustalu, Veltri: Quotienting the delay monad by weak bisimilarity

- A stronger property than preservation by D is preservation by $(-)^{\mathbb{N}}$ – this is a version of the axiom of countable choice
- If additionally $\mathbf{C}$ is an exact category (e.g. a pretopos) then $\tilde{D}$ is an Elgot monad

Further Work

- Implementation in Agda (for which notion of category?)
- Concrete natural example of **K** not being (initial) Elgot monad
- Relation between initial Elgot monad and "free pointed ω-cpo monads"
- Other recursion call trees (e.g. $\nu\gamma. - + \gamma \times \gamma$) and other semantics (e.g. hybrid[4])

[4] Diezel, Goncharov: Towards Constructive Hybrid Semantics