Computational Adequacy and Choice: How Deep is the Rabbit Hole?

Sergey Goncharov
FAU Erlangen-Nürnberg

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SETTING THE STAGE:
Adequacy in the Abstract
Classically:

- **Operational semantics** defines a reduction relation $\rightarrow$ on terms.
- **Denotational semantics** defines domains $DX$ and maps $\llbracket \cdot \rrbracket$ from programs producing values of type $X$, to $DX$.
- **Soundness**: If $t \rightarrow^* v$ for a value $v$ then $\llbracket t \rrbracket = \llbracket v \rrbracket$.
- **(Computational) Adequacy**: if $\llbracket t \rrbracket = \llbracket v \rrbracket$ then there is a reduction $t \rightarrow^* v$.

Call this scenario “Coarse-Grained Adequacy“. There are many variations.
Let the reduction relation $\rightarrow$ be labelled, and let $t \rightarrow_{a_1 \ldots a_n} v$ iff $t \rightarrow_{a_1} t' \rightarrow_{a_2} \ldots \rightarrow_{a_n} v$

Let $DX$ be the final coalgebra $\nu\gamma. (X + A \times \gamma)$

**Fine-Grained Adequacy:** if $\llbracket t \rrbracket = (w, v)$ then there is a reduction $t \rightarrow^w_v$

**Questions:**

- Generic categorical\(^1\) machinery for coarse-grained adequacy
- Generic categorical machinery for fine-grained adequacy
- Generic connection between them

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\(^1\)“Generic categorical” = at least includes Top
Language of programs:

\[ p, q ::= a \mid p; q \mid \text{if } c \text{ then } p \text{ else } q \]

where \(a\) ranges over atomic programs, and \(c\) over conditions

Operational semantics (w.r.t. \(a : S \rightarrow S\), \(c : S \rightarrow 2\)):

\[
\begin{align*}
\langle \sigma, a \rangle \downarrow a(\sigma) & \quad \langle \sigma, p \rangle \rightarrow \langle \sigma', p' \rangle \\
\langle \sigma, p; q \rangle & \rightarrow \langle \sigma', p'; q \rangle \\
\langle \sigma, p \rangle & \downarrow \sigma' \\
\langle \sigma, p; q \rangle & \rightarrow \langle \sigma', q \rangle
\end{align*}
\]

\[
\begin{align*}
\text{if } c(\sigma) = \text{true} & \quad \langle \sigma, \text{if } c \text{ then } p \text{ else } q \rangle \rightarrow \langle \sigma, p \rangle \\
\text{if } c(\sigma) = \text{false} & \quad \langle \sigma, \text{if } c \text{ then } p \text{ else } q \rangle \rightarrow \langle \sigma, q \rangle
\end{align*}
\]

Fine-grained soundness and adequacy: \(\langle \sigma, p \rangle \rightarrow^n \langle \sigma', p' \rangle \downarrow \sigma''\) iff \([p](\sigma) = (n, \sigma'')\) where the semantics \([p] : S \rightarrow S \times \mathbb{N}\)
Observe that we are indeed too fine-grained, i.e. if $p$ terminates in $n$ steps, (if true then $p$ else $q$) terminates in $n + 1$ steps, hence no operational equivalence:

$$p \sim \text{ if true then } p \text{ else } q$$

We can either prove coarse-grained adequacy, or (better!) quotient the denotational domain $S \times \mathbb{N}$ by weak bisimilarity:

$$\triangleright p \approx p \quad p \approx \triangleright p \quad p \approx q \quad \triangleright p \approx \triangleright q$$

This is generic, for

$$S \times \mathbb{N} \xrightarrow{\triangleright} S \times \mathbb{N} \xrightarrow{\text{id}} S \times \mathbb{N} \xrightarrow{\text{fst}} S$$

is a co-equalizer. In fact, this can be taken as a definition of the natural number object (suggested by Freyd (with $S = 1$))
Define fine-grained semantics (e.g. using GSOS)

This produces a monad $D$, so that $DX$ collects denotations of programs with terminal values in $X$

Introduce a notion of weak bisimilarity $\approx$ on $D$

Obtain $\tilde{D}$ as a quotient of $D$

Is it possible that $\tilde{D}$ is a monad?

**Spoiler:** probably not, unless we rely on some weak form of the axiom of choice

Why it is a problem? Because some fundamental models of computation (prominently MLTT and HoTT) do not assume any form of choice, even very weak ones
Generally, either

- Coarse-grained adequacy fails
- Target denotational model not compositional (not a monad)
- There is some choice
Towards Semantics of Non-termination
1. Automata

Let $O$ be a (possibly infinite) set of output symbols. An automaton with outputs in $O$ is a pair $S = (S, \alpha)$ consisting of a set $S$ of states and a transition function $\alpha : S \rightarrow O + S$. The transition function $\alpha$ specifies for a state $s$ in $S$ either an output $o$ in $O$ or a next state $s'$ in $S$. The intuition is that in the first case, the computation is terminating, with observable output $o$; in the second case, the computation takes one step and will continue from the new state $s'$. We shall sometimes write $s \Downarrow o$ if $\alpha(s) = o \in O$, and $s \xrightarrow{s'} s'$ if $\alpha(s) = s' \in S$. If $S$ is clear from the context, we shall simply write $s \xrightarrow{s'}$.

This type of automaton is sometimes referred to as Elgot machine, because of the prominent role similar such structures play in the work of Elgot (cf. [3]).
**Adding While-Loops**

- \(p, q ::= a \mid p \mid q \mid \text{if } c \text{ then } p \text{ else } q \mid \text{while } c \text{ do } p\)
- In the operational semantics add:

\[
c(\sigma) = \text{true} \\
\langle \sigma, \text{while } c \text{ do } p \rangle \rightarrow \langle \sigma, p; \text{while } c \text{ do } p \rangle
\]

\[
c(\sigma) = \text{false} \\
\langle \sigma, \text{while } c \text{ do } p \rangle \downarrow \sigma
\]

- A fine-grained semantics can be defined using (now) standard machinery over \(DX = \nu \gamma. X + \gamma\)
- Weak bisimilarity can be defined as before
- Rutten relies on \(DX \simeq X \times \mathbb{N} + 1\), which entails \(\tilde{DX} \simeq X + 1\)

**Rest of the talk:** analysis of situations when it need not be so
Work on **partiality monads**

- Chapman, Uustalu, Veltri: Quotienting the delay monad by weak bisimilarity (coinduction + quotienting)
- Altenkirch, Danielsson, Kraus: Partiality, revisited - the partiality monad as a quotient inductive-inductive type (induction)
- Escardó, Knapp: Partial Elements and Recursion via Dominances in Univalent Type Theory (comparison + disciplined maps)

Work on **iteration**

- Adámek, Milius, Velebil: Elgot algebras
- Adámek, Milius, Velebil: Elgot theories: a new perspective of the equational properties of iteration
ITERATION-BASED NOTION OF PARTIALITY
Our Assumptions

We work in a category $\mathcal{C}$ that

- has finite products
- has finite coproducts
- is extensive (coproducts are stable under pullbacks, coproduct injections are disjoint, pullbacks of coproduct injections exist)
- natural number object $\mathbb{N}$ exists and stable
- exponentials $X^\mathbb{N}$ exist

Examples: Classical and non-classical set theories, toposes, pretoposes, importantly also topological spaces $\text{Top}$
Consider pairs \((A, s : A \rightarrow A)\) where

- \(A \in |\mathcal{C}|\)
- \(s\) is the successor function
- These algebras form a category, and for every \(X \in |\mathcal{C}|\) there a free object over \(X\), which is precisely the initial algebra

\[
\mu\gamma. X + \gamma \cong X \times \mathbb{N}
\]

Interpretation: \(X \times \mathbb{N}\) contains only terminating computations, and remembers number of steps, needed for termination;

\[
\triangleright : X \times \mathbb{N} \xrightarrow{[\text{id} \times \text{suc}]} X \times \mathbb{N}
\]

is the delay operator
Let us augment our algebras \((A, s : A \rightarrow A)\) with means for iterating computations.

Consider morphisms of the form \(h : S \rightarrow A + S\) with
- values in \(A\)
- states in \(S\)

An iteration operator (needed to interpret while-loops!) sends \(h\) to \(h^# : S \rightarrow A\) and must satisfy laws
- **Fixpoint**: \(f^# = [\text{id}, s f^#] f\)
- **Uniformity**: \((\text{id} + h) f = g h \Rightarrow f^# = g^# h\)

for \(f : Y \rightarrow A + Y, g : Z \rightarrow A + Z, h : Y \rightarrow Z\)

Call triples \((A, s : A \rightarrow A, (-)^#)\) guarded uniform-iteration algebras.

**Conjecture**: The free object over \(X\) is a final coalgebra \(\nu \gamma. X + \gamma\).
(Guarded) Elgot Algebras

- The conjecture becomes true after adding one more axiom, called **Compositionality**.
- Guarded uniform-iteration algebras satisfying **Compositionality** are called **Elgot algebras**\(^2\) and can be defined more generally, so that

\[
\frac{f : X \to A + HX}{f^\# : X \to A}
\]

for arbitrary endofunctor \(H : C \to C\), and an \(H\)-algebra \(s : HA \to A\).

- **Compositionality** is no longer admissible if \(H \neq \text{Id}\). e.g. \(H = \text{Id}\)\(^2\).

\(^2\)Adámek, Milius, and Velebil, “Elgot Algebras”.
Summarized:

<table>
<thead>
<tr>
<th></th>
<th>Finite Behaviour</th>
<th>Infinite Behaviour</th>
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<tbody>
<tr>
<td>Guarded (Fine-Grained)</td>
<td>$\text{Id} \times \mathbb{N}$</td>
<td>$\nu\gamma. (- + \gamma)$</td>
</tr>
<tr>
<td>Unguarded (Coarse-Grained)</td>
<td>$\text{Id}$</td>
<td>$?$</td>
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</table>

$? = \text{Free (Unguarded) Uniform-Iteration (or Elgot?) Algebras!}$
Define $KX$ to be a free structure over $X$, satisfying

- **Fixpoint:** $f^\# = [\text{id}, f^\#] \ f$
- **Uniformity:** $(\text{id} + h) \ f = g \ h \Rightarrow f^\# = g^\# \ h$
  for $f : Y \to KX + Y$, $g : Z \to KX + Z$, $h : Y \to Z$

By generalities, $K$ extends to a (strong) monad $\mathbf{K}$ whose iteration operator

$$f : X \to KY + X \quad \Rightarrow \quad f^\# : X \to KY$$
**Elgot Monads**

**Fixpoint** \( (f : X \rightarrow T(Y + X)) : \)

\[
\begin{align*}
  f \quad f \\
  x \quad x \\
  y \\
  \end{align*}
\]

**Uniformity** \( (f : X \rightarrow T(Z + X), g : Y \rightarrow T(Z + Y), h : X \rightarrow Y) : \)

\[
\begin{align*}
  f \quad \eta h \\
  x \quad x \\
  z \\
  y \\
  \end{align*}
\]

\[
\begin{align*}
  \eta h \quad g \\
  x \quad y \\
  z \\
  \\
  \eta h \quad g \\
  x \quad y \\
  \end{align*}
\]

\[
\begin{align*}
  \eta h \quad g \\
  x \quad y \\
  \end{align*}
\]

\[
\begin{align*}
  \eta h \quad g \\
  x \quad y \\
  \end{align*}
\]
Elgot Monads (Cont’d)

Naturality \((f : X \to T(Y + X), g : Y \to TZ)\):

\[
\begin{align*}
  \begin{array}{c}
    x \\
    \overrightarrow{f} \\
    y \\
    \overleftarrow{g} \\
    z \\
  \end{array} & = & \begin{array}{c}
    x \\
    \overrightarrow{f} \\
    y \\
    \overleftarrow{g} \\
    z \\
  \end{array}
\end{align*}
\]

Codiagonal \((f : X \to T(Y + (X + X)))\):

\[
\begin{align*}
  \begin{array}{c}
    x \\
    \overrightarrow{f} \\
    y \\
  \end{array} & = & \begin{array}{c}
    x \\
    \overrightarrow{f} \\
    y \\
  \end{array}
\end{align*}
\]

Naturality and Codiagonal are basically coherence laws
Elgot monads are semantic structures for implementing side-effecting while-loops

Elgot monads properly generalize monads with \((-)\) calculated as least fixpoints, e.g. coalgebraic resumptions

$$\nu \gamma. T(X + \Sigma \gamma)$$

form an Elgot monad of generalized processes over a given Elgot monad \(T\)

But (!) “initial Elgot monad” is an impredicative notion but Elgot algebra is not
LIMITED PRINCIPLE OF OMNISCIENCE
Delay monad $D = \nu \gamma. (\neg + \gamma)$ is co-generated by

$$\begin{align*}
x : X \\
\text{now } x : DX
\end{align*}$$

$$\begin{align*}
\sigma : DX \\
\downarrow \sigma : DX
\end{align*}$$

$DX$ consists of precisely those infinite streams over $X \cup \{\bot\}$, which contain an element of $X$ not more than once.

We call $C$ an LPO\textsuperscript{3} category if the embedding of $\mathbb{N}$ to $\tilde{\mathbb{N}} = D1$ is a coproduct injection.

**Proposition:** $C$ is LPO iff $DX \cong X \times \mathbb{N} + 1$

\textsuperscript{3}LPO=Limited Principle of Omniscience
LPO and non-LPO Categories

- LPO categories: classical set theories, presheaf toposes, (classical) dcpos, nominal sets, (classical) metric spaces
- Non-LPO categories: some Grothendieck toposes, most realizability toposes, intuitionistic and constructive set theories, \textbf{Top}

Why LPO = some choice?

It entails Escardó’s \textbf{very week countable choice}: any epic

\[
A + B \times (A + A') \xrightarrow{[\text{inl, snd}]} A + A' \cong \mathbb{N}
\]

has a section, which is an instance of (weak) countable choice
In any LPO category

- **K** is the **maybe monad** (– +1)
- **K** is the initial Elgot monad
- **K** is enriched over pointed $\omega$-complete partial orders
- **KX** is free pointed $\omega$-complete partial orders on **X**
BEYOND LPO
A monad is an **equational lifting monad** if it is strong, commutative and satisfies the law

\[
\begin{align*}
TX & \xrightarrow{T\Delta} T(X \times X) \\
\Delta \downarrow & \Downarrow T(\eta \times \text{id}) \\
TX \times TX & \xrightarrow{\tau} T(KX \times X)
\end{align*}
\]

In terms of “do-notation“:

\[
\begin{align*}
x \leftarrow p; y \leftarrow q; \eta\langle x, y \rangle & = y \leftarrow q; x \leftarrow p; \eta\langle x, y \rangle \\
x \leftarrow p; \eta\langle x, \eta x \rangle & = x \leftarrow p; \eta\langle x, p \rangle
\end{align*}
\]

Then the Kleisli category is a **restriction category**, with

\[
\begin{align*}
f : X \rightarrow TY \\
\text{dom } f : X \rightarrow TX
\end{align*}
\]

satisfying axioms of domains of definiteness

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\(^4\)Bucalo, Führmann, and Simpson, “An equational notion of lifting monad”.

\(^5\)Cockett and Lack, “Restriction categories I: categories of partial maps”.
**Theorem:** If every $KX$ exists and stable then

1. $K$ is an **equational lifting monad**
2. Kleisli Category of $K$ is a **restriction category**
   - $\Rightarrow$ Kleisli Category of $K$ is order enriched
3. Kleisli composition and strength moreover respect the order and the bottom element $\perp$ (divergence)
4. $f^\#$ is an internal order-limit of finite approximations
   - ($\approx$ Kleene fixpoint theorem)
Compositionality

holds for $\mathbf{K}$ (non-trivially from Kleene’s fixpoint property)

Hence, initial Uniform-iteration algebra = initial unguarded Elgot algebra
Following previous work\(^6\), consider weak bisimilarity over \(DX\):

\[
\begin{align*}
\sigma & \approx \text{now } x \\
\triangleright \sigma & \approx \text{now } x \\
\text{now } x & \equiv \sigma \\
\triangleright \sigma & \equiv \triangleright \sigma' \\
\sigma & \equiv \sigma'
\end{align*}
\]

We generically model it with the coequalizer:

\[
\begin{array}{c}
D(X \times \mathbb{N}) \\
\xrightarrow{\nu^*} \\
\xrightarrow{D\text{fst}} \\
\xrightarrow{\rho_X} \\
\tilde{DX}
\end{array}
\]

\[
\Rightarrow \rho_X \triangleright = \rho_X, \text{ but this seems to be weaker than } (\ast)
\]

**Theorem:** The following are equivalent:

1. (\ast) is preserved by \(D\)
2. \(\tilde{D}\) extends to a strong monad and \(\rho\) to a strong monad morphism
3. \(\tilde{DX} \simeq KX\) and \(\rho_X\) is iteration preserving

\(^6\)Chapman, Uustalu, Veltri: Quotienting the delay monad by weak bisimilarity
A stronger property than preservation by $D$ is preservation by $(-)^\mathbb{N}$ – this is a version of the axiom of countable choice.

If additionally $\mathbf{C}$ is an exact category (e.g. a pretopos) then $\tilde{D}$ is an Elgot monad.

Open Questions: Is it initial? Is it order-enriched?
Implementation in Agda (for which notion of category?)

Concrete natural examples of discrepancy of $K$ and $\tilde{D}$

Concrete natural examples of $K$ not being (initial) Elgot monad

Relation between the initial Elgot monad and “free pointed $\omega$-cpo monads”

Other recursion call trees (e.g. $\nu \gamma . - + \gamma \times \gamma$) and other semantics (e.g. hybrid$^7$)

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$^7$Diezel, Goncharov: Towards Constructive Hybrid Semantics