Representing Guardedness

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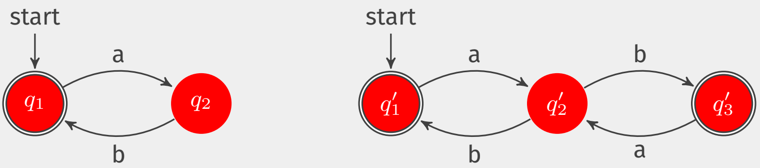
Setting the Stage: Notion of Guardedness

How do we know that automata



are equivalent?

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[1] A. Salomaa, Two Complete Axiom Systems for the Algebra of Regular Events, 1966

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PROOF "BY COINDUCTION"

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&= (ab)a(ba)^\star b + ab + 1 \\
&= (ab)(\boxed{a(ba)^\star b + 1}) + 1
\end{aligned}
$$

- This only works because $x \mapsto abx + 1$ is guarded
- $x \mapsto (a+1)x + 1$ is un-guarded and has infinitely many fixpoints

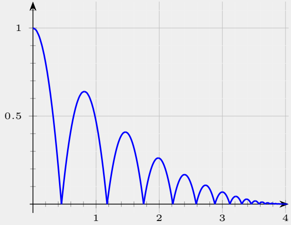
[1] A. Salomaa, Two Complete Axiom Systems for the Algebra of Regular Events, 1966

Bouncing ball is a simple Newtonian system specified by differential equation $\ddot{h} = -g$ ($g \approx 9.8$) whose solution is

$$h(t) = h_0 + v_0 t - \frac{gt^2}{2}$$

with initial values:

- $v_0 = 0$, $h_0 \neq 0$ (peak height)
- $h_0 = 0$, $v_0 \neq 0$ (zero height)



This system is progressive: every iteration consumes non-zero time (although it keeps getting smaller – Zeno behaviour)

Non-progressive (chattering) behaviour is often regarded a modelling artefact

Basic Process Algebra (BPA):

$$P, Q, \ldots ::= \checkmark \mid a \in A \mid P + Q \mid P \cdot Q$$

E.g. we can specify a 2-cell FIFO, storing bits:

$$\begin{aligned} B_0 &= \mathsf{in}_0 .\, B_1^0 + \mathsf{in}_1 .\, B_1^1 \\ B_1^i &= \mathsf{in}_0 .\, B_2^{0,i} + \mathsf{in}_1 .\, B_2^{1,i} + \mathsf{out}_i .\, B_0 && (i \in \{0,1\}) \\ B_2^{i,j} &= \mathsf{out}_j .\, B_1^i && (i, j \in \{0,1\}) \end{aligned}$$

Solutions are unique for guarded specifications. Otherwise not: $X = X$ has infinitely many solutions

We can model previous examples with monads, augmented with partially defined iteration operators

$$\frac{f\colon X \to T(Y+X)}{f^{\dagger}\colon X \to TY}$$

w.r.t. a co-Cartesian category (=category with finite coproducts)

1. $TX = \mathcal{P}(A^{\star} \times X)$
   - an automaton over n states is a Kleisli map $h\colon n \to \mathcal{P}(A^{\star} \times (1+n))$
   - languages, recognized by states: $h^{\dagger}\colon n \to \mathcal{P}(A^{\star})$
2. $TX = \mathbb{R}_{\geqslant 0} \times X + \bar{\mathbb{R}}_{\geqslant 0}$
   - $\mathbb{R}_{\geqslant 0}$ and $\bar{\mathbb{R}}_{\geqslant 0}$ model finite and infinite durations
   - For $h\colon X \to T(2 \times X) \cong T(X+X)$, $h^{\dagger}\colon X \to TX$ is a while-loop on h
3. $TX = \nu\gamma.\, \mathcal{P}_{\omega_1}(X + A \times \gamma)$ ($\nu\gamma.\, F\gamma$ is a final F-coalgebra)
   - $h\colon n \to T(\{\checkmark\} + n)$ is a system of n recursive process definitions
   - $h^{\dagger}\colon n \to \nu\gamma.\, \mathcal{P}_{\omega_1}(\{\checkmark\} + A \times \gamma)$ is a solution

Most of the time, guarded fixpoint operators are restrictions of unguarded ones. But the guarded ones are better behaved:

- Often unique, hence enable reasoning by coinduction
- If not unique, often computed as least fixponts
- Foundation-independent
- Simpler to define and to work with

This motivates a type discipline for propagating guardedness over structures

A guardedness predicate identifies for all objects X, Y, Z guarded morphisms $\mathbf{C}_\blacklozenge(X, Y, Z) \subseteq \mathbf{C}(X, Y + Z)$, such that

$$\textbf{(trv}_+\textbf{)}\ \frac{f\colon X \to Y}{\mathsf{inl}\, f\colon X \to Y \rangle Z} \qquad \textbf{(par}_+\textbf{)}\ \frac{f\colon X \to V \rangle W \qquad g\colon Y \to V \rangle W}{[f, g]\colon X + Y \to V \rangle W}$$

$$\textbf{(cmp}_+\textbf{)}\ \frac{f\colon X \to Y \rangle Z \qquad g\colon Y \to V \rangle W \qquad h\colon Z \to V + W}{[g, h]\, f\colon X \to V \rangle W}$$

where $f\colon X \to Y \rangle Z$ means $f \in \mathbf{C}_\blacklozenge(X, Y, Z)$

- A category with a guardedness predicate is called guarded
- A monad is guarded if its Kleisli category is guarded

Examples

- Every category/monad is guarded with $f\colon X \to Y \rangle Z$ iff f factors through $\mathsf{inl}\colon Y \to Y + Z$ (trivial iteration)
- Every category/monad is guarded with $\mathbf{C}_\blacklozenge(X, Y, Z) = \mathbf{C}(X, Y + Z)$ (total iteration)
- $f\colon X \to \mathcal{P}(A^\star \times (Y + Z))$ is guarded if it factors through

$$\mathcal{P}(A^\star \times Y + A^+ \times Z) \hookrightarrow \mathcal{P}(A^\star \times Y + A^\star \times Z) \cong \mathcal{P}(A^\star \times (Y + Z))$$

- $f\colon X \to \mathbb{R}_{\geqslant 0} \times (Y + Z) + \bar{\mathbb{R}}_{\geqslant 0}$ is guarded if it factors through

$$\begin{aligned}\mathbb{R}_{\geqslant 0} \times Y + \mathbb{R}_{>0} \times Z + \bar{\mathbb{R}}_{\geqslant 0} &\hookrightarrow \mathbb{R}_{\geqslant 0} \times Y + \mathbb{R}_{\geqslant 0} \times Z + \bar{\mathbb{R}}_{\geqslant 0} \\ &\cong \mathbb{R}_{\geqslant 0} \times (Y + Z) + \bar{\mathbb{R}}_{\geqslant 0}\end{aligned}$$

- $f\colon X \to \nu\gamma.\, T((Y + Z) + H\gamma)$ is guarded if it factors through

$$T(Y + H(\nu\gamma.\,\ldots)) \hookrightarrow T((Y + Z) + H(\nu\gamma.\,\ldots)) \cong \nu\gamma.\, T((Y + Z) + H\gamma)$$

Call-by-Value with Effects

Very Simple Metalanguage (VSML)

- Sorts $A, B, C, \ldots$
- Signature Σ_v of pure programs $f\colon A \to B$, and signature Σ_c of effectful programs $f\colon A \to B$
- Semantics of (Σ_v, Σ_c) w.r.t. identity-on-objects functor $J\colon \mathbf{V} \to \mathbf{C}$:
  - an object $[\![A]\!] \in |\mathbf{V}|$ to each sort A
  - a morphism $[\![f]\!] \in \mathbf{V}([\![A]\!], [\![B]\!])$ to each $f\colon A \to B \in \Sigma_v$
  - a morphism $[\![f]\!] \in \mathbf{C}([\![A]\!], [\![B]\!])$ to each $f\colon A \to B \in \Sigma_c$
- Terms in single-variable (!) context:

$$\frac{f\colon A \to B \in \Sigma_v \quad \Gamma \vdash_{\mathsf{v}} v\colon A}{\Gamma \vdash_{\mathsf{v}} f(v)\colon B} \qquad \frac{f\colon A \to B \in \Sigma_c \quad \Gamma \vdash_{\mathsf{v}} v\colon A}{\Gamma \vdash_{\mathsf{c}} f(v)\colon B}$$

$$\frac{}{x\colon A \vdash_{\mathsf{v}} x\colon A} \qquad \frac{\Gamma \vdash_{\mathsf{v}} v\colon A}{\Gamma \vdash_{\mathsf{c}} \mathsf{return}\, v\colon A} \qquad \frac{\Gamma \vdash_{\mathsf{c}} p\colon A \quad x\colon A \vdash_{\mathsf{c}} q\colon B}{\Gamma \vdash_{\mathsf{c}} x \leftarrow p;\, q\colon B}$$

- Semantics:
  - $[\![x\colon A \vdash_{\mathsf{v}} x\colon A]\!] = \mathsf{id}$
  - $[\![\Gamma \vdash_{\mathsf{v}} f(v)\colon B]\!] = [\![f]\!]\, [\![\Gamma \vdash_{\mathsf{v}} v\colon A]\!]$
  - $[\![\Gamma \vdash_{\mathsf{c}} f(v)\colon B]\!] = [\![f]\!]\, J[\![\Gamma \vdash_{\mathsf{v}} v\colon A]\!]$
  - $[\![\Gamma \vdash_{\mathsf{c}} \mathsf{return}\, v\colon A]\!] = J[\![\Gamma \vdash_{\mathsf{v}} v\colon A]\!]$
  - $[\![\Gamma \vdash_{\mathsf{c}} x \leftarrow p;\, q\colon B]\!] = [\![x\colon A \vdash_{\mathsf{c}} q\colon B]\!]\, [\![\Gamma \vdash_{\mathsf{c}} p\colon A]\!]$

Variable contexts can normally carry a list of variables $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$, e.g. variable introduction now looks like this:

$$\frac{}{x_1 : A_1, \ldots, x_n : A_n \vdash_{\mathsf{v}} x_i : A_i}$$

This yields fine-grained call-by-value (FGCBV)[2] metalanguage. It can be interpreted over a Freyd category:

- $\mathbf{V}$ is a category with finite products
- action $\mathbf{V} \times \mathbf{C} \to \mathbf{C}$ of $\mathbf{V}$ on $\mathbf{C}$
- $J : \mathbf{V} \to \mathbf{C}$ is an identity-on-objects functor, preserving the action

[2] P. Levy, J. Power, H. Thielecke, Modelling Environments in Call-By-Value Programming Languages, 2002

Strong Monads

Originally, Moggi[3] interpreted call-by-value over strong monads

- A functor $F\colon \mathbf{C} \to \mathbf{D}$ between monoidal $\mathbf{C}$ and $\mathbf{D}$ is strong if there is (natural in A, B) strength $\tau_{A,B}\colon A \otimes FB \to F(A \otimes B)$, such that

$$\begin{array}{ccc} I \otimes FX & \cong & FX \\ {\scriptstyle\tau}\downarrow & & \| \\ F(I \otimes X) & \cong & FX \end{array} \qquad \begin{array}{ccccc} (X \otimes Y) \otimes FZ & & \xrightarrow{\quad\tau\quad} & & F((X \otimes Y) \otimes Z) \\ \wr\| & & & & \wr\| \\ X \otimes (Y \otimes FY) & \xrightarrow{X \otimes \tau} & X \otimes F(Y \otimes Z) & \xrightarrow{\tau} & F(X \otimes (Y \otimes Z)) \end{array}$$

- A monad (T, η, μ) on $\mathbf{C}$ is strong if T is strong additionally η, μ are strong:

$$\begin{array}{ccc} X \otimes Y & = & X \otimes Y \\ {\scriptstyle X \otimes \eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ X \otimes TY & \xrightarrow{\tau} & T(X \otimes Y) \end{array} \qquad \begin{array}{ccccc} X \otimes TTY & \xrightarrow{\tau} & T(X \otimes TY) & \xrightarrow{T\tau} & TT(X \otimes Y) \\ {\scriptstyle X \otimes \mu}\downarrow & & & & \downarrow{\scriptstyle\mu} \\ X \otimes TY & & \xrightarrow{\quad\tau\quad} & & T(X \otimes Y) \end{array}$$

Theorem ([4])

In monoidal closed categories, strength is equivalent to enrichment

[3] E. Moggi, Notions of Computation and Monads, 1991

[4] A. Kock, Strong Functors and Monoidal Monads, 1972

If we want to implement higher order:

$$\frac{\Gamma, x\colon A \vdash_{\mathsf{c}} p\colon B}{\Gamma \vdash_{\mathsf{v}} \lambda x.\, p\colon A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{v}} f\colon A \to B \qquad \Gamma \vdash_{\mathsf{v}} v\colon A}{\Gamma \vdash_{\mathsf{c}} f\, v\colon B}$$

we need to have a semantics $[\![A \to B]\!] = U([\![A]\!], [\![B]\!])$, such that

$$\mathbf{C}(J(X \times A), B) \cong \mathbf{V}(X, U(A, B))$$

naturally in A

Theorem ([5])

The following are equivalent:

- $\mathbf{C}(J(X \times A), B) \cong \mathbf{V}(X, U(A, B))$ *for some* $U\colon \mathbf{V} \times \mathbf{C} \to \mathbf{V}$*, naturally in* A
- *Presheaves* $\mathbf{C}(J(X \times (\text{–})), B)\colon \mathbf{V}^{\mathsf{op}} \to \mathbf{Set}$ *are representable*
- $\mathbf{C}$ *is isomorphic to a Kleisli category of a strong monad* $\mathbf{T}$ *on* $\mathbf{V}$ *and all exponentials* $(TB)^A$ *exist*

[5] Essentially: P. Levy, J. Power, H. Thielecke, Modelling Environments in Call-By-Value Programming Languages, 2002

If we do not care about strength, we have a simpler characterization

Theorem

Given id-on-objects functor $J \colon \mathbf{V} \to \mathbf{C}$, the following are equivalent:

- *J is a left adjoint*
- *Presheaves $\mathbf{C}(J(-), B) \colon \mathbf{V}^{\mathrm{op}} \to \mathbf{Set}$ are representable*
- *$\mathbf{C}$ is isomorphic to a Kleisli category of a monad*

and then:

Theorem

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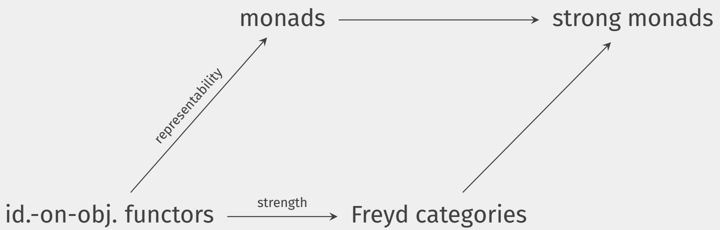
- *$\mathbf{C}$ is isomorphic to a Kleisli category of a strong monad*


monads
strong monads
representability
id.-on-obj. functors
strength
Freyd categories

Call-by-Value Meets Guardedness

- We stick to guarded $\mathbf{C}$, and identity-on-objects $J\colon \mathbf{V} \to \mathbf{C}$ strictly preserving finite coproducts
- Modify the type system:

$$\frac{}{\emptyset\ \ \Diamond\text{-type}} \qquad \frac{A\ \ \text{sort}}{A\ \ \Diamond\text{-type}} \qquad \frac{A\ \ \Diamond\text{-type} \qquad B\ \ \Diamond\text{-type}}{A + B\ \ \Diamond\text{-type}}$$

$$\frac{A\ \ \Diamond\text{-type}}{\blacklozenge A\ \ \text{type}} \qquad \frac{A\ \ \Diamond\text{-type}}{A\ \ \text{type}} \qquad \frac{A\ \ \text{type} \qquad B\ \ \text{type}}{A + B\ \ \text{type}}$$

So, e.g. $A + (\blacklozenge B + \blacklozenge C)$ indicates that we are guarded in B and in C

- Let $\trianglelefteqslant$ be partial order on types, generated by the rules:

$$\frac{}{\blacklozenge A \trianglelefteqslant A} \qquad \frac{A \trianglelefteqslant C \qquad B \trianglelefteqslant D}{A + B \trianglelefteqslant C + D}$$

- This allow us to compute "guarded" $[\![A]\!]_{\blacklozenge}$ and "un-guarded" $[\![A]\!]_{\Diamond}$ part of every type A, and $[\![A \trianglelefteqslant B]\!] \in \mathbf{V}([\![A]\!]_{\Diamond}, [\![B]\!]_{\Diamond}) \times \mathbf{V}([\![A]\!]_{\blacklozenge}, [\![B]\!]_{\Diamond} + [\![B]\!]_{\blacklozenge})$

- We interpret

$$[\![x\colon A \vdash_{\mathsf{v}} v\colon B]\!] \in \mathbf{V}([\![A]\!]_{\lozenge}, [\![B]\!]_{\lozenge}) \times \mathbf{V}([\![A]\!]_{\blacklozenge}, [\![B]\!]_{\lozenge} + [\![B]\!]_{\blacklozenge}),$$
$$[\![x\colon A \vdash_{\mathsf{c}} p\colon B]\!] \in \mathbf{C}_{\blacklozenge}([\![A]\!]_{\lozenge}, [\![B]\!]_{\lozenge}, [\![B]\!]_{\blacklozenge}) \times \mathbf{C}([\![A]\!]_{\blacklozenge}, [\![B]\!]_{\lozenge} + [\![B]\!]_{\blacklozenge}).$$

- For example, variable introduction incorporates weakening:

$$\frac{A \trianglelefteq B}{x\colon A \vdash_{\mathsf{v}} x\colon B}$$

So, $[\![x\colon A \vdash_{\mathsf{v}} x\colon B]\!] = [\![A \trianglelefteq B]\!]$

- We then can type guarded iteration:

$$\frac{\Gamma \vdash_{\mathsf{c}} p\colon A \qquad x\colon A \vdash_{\mathsf{c}} q\colon B + \blacklozenge A}{\Gamma \vdash_{\mathsf{c}} \mathsf{iter}\, x \leftarrow p;\, q\colon B}$$

Definition

Given $J\colon \mathbf{V} \to \mathbf{C}$, as before and guarded $\mathbf{C}$, we call the guardedness predicate $\mathbf{C}_\blacklozenge$, (*J*-)representable if for all $A, B \in |\mathbf{C}|$ the presheaves

$$\mathbf{C}_\blacklozenge(J(-), A, B)\colon \mathbf{V}^{\mathsf{op}} \to \mathbf{Set}$$

are representable

Note that $\mathbf{C}_\blacklozenge(X, A, \emptyset) \cong \mathbf{C}(X, A)$, hence

Lemma

If $\mathbf{C}_\blacklozenge$ is representable, J is a left adjoint. In this case, $\mathbf{C}$ is a Kleisli category of some monad on $\mathbf{V}$

Recall, that a bifunctor $\# \colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ is a parametrized monad[6] if

- Every $(\text{–}) \# X$ is a monad
- Every $(\text{–}) \# f$ is a monad morphism

Theorem (Main)

$\mathbf{C}_\blacklozenge$ *is representable iff*

1. *There is a parametrized monad* $\#$ *on* $\mathbf{V}$, *such that* $\mathbf{C}$ *is isomorphic to the Kleisli category of* $(\text{–}) \# \emptyset$
2. *There is a family of monics* $\epsilon_{X,Y} \colon X \# Y \to (X + Y) \# \emptyset$ *natural in* X, Y, *such every* $\epsilon_{-,X} \colon \text{–} \# X \to (\text{–} + X) \# \emptyset$ *is a monad morphism*
3. *There is natural* $\zeta_{X,Y} \colon (X \# Y) \# ((X + Y) \# \emptyset) \to X \# Y$ *such that*

$$\begin{array}{ccccc} (X \# Y) \# ((X + Y) \# \emptyset) & & \xrightarrow{\quad\zeta_{X,Y}\quad} & & X \# Y \\ \Big\downarrow{\scriptstyle \epsilon_{X \# Y, (X+Y) \# \emptyset}} & & & & \Big\downarrow{\scriptstyle \epsilon_{X,Y}} \\ (X \# Y + (X + Y) \# \emptyset) \# \emptyset & \xrightarrow{[\epsilon_{X,Y}, \mathsf{id}] \# \emptyset} & ((X + Y) \# \emptyset) \# \emptyset & \xrightarrow{\mu_{X+Y,\emptyset}} & (X + Y) \# \emptyset \end{array}$$

Then, up to isomorphism, $f \colon X \to Y \rangle Z$ *iff* f *factors through* ϵ

[6] T. Uustalu, Generalizing Substitution, 2003

- Trivial guardedness corresponds to $X \# Y = TX$, i.e. $X \# Y$ does not depend on the parameter
- Total guardedness corresponds to $X \# Y = T(X + Y)$ (exception monad transformer)
- Automata: $X \# Y = \mathcal{P}(A^\star \times X + A^+ \times Y)$
- Hybrid systems: $X \# Y = \mathbb{R}_{\geqslant 0} \times X + \mathbb{R}_{>0} \times Y + \bar{\mathbb{R}}_{\geqslant 0}$
- Generalized processes: $X \# Y = T(X + H(\nu\gamma.\, T((X + Y) + H\gamma)))$

If J is not a left adjoint, guardedeness is not representable. But this is boring. Is there other counterexamples?

Theorem

Suppose, every morphism in $\mathbf{V}$ *factorizes as a regular epic, followed by a monic. Let* $\mathbf{T}$ *be a guarded monad on* $\mathbf{V}$*. Then a family of monos* $(\epsilon_{X,Y} : X \# Y \hookrightarrow T(X+Y))_{X,Y \in |\mathbf{V}|}$ *extends to a guarded parametrized monad iff*

- *every* $\epsilon_{X,Y}$ *is the* *largest* *guarded subobject of* $T(X+Y)$
- *for every* $f : X \to T(Y+Z)$ *and a regular epic* $g : X' \to X$*, if* $f\,g$ *is guarded then* f *is guarded*

Example

In the category of topological spaces, let $f : X \to Y \rangle Z$ if Z is compact. This is not Id-representable, since there is no "largest compact subspace of X" (e.g. $X = \mathbb{N}$ under cofinite topology)

Making Monads Strong (Again)

Most sophisticated case:

- $J\colon \mathbf{V} \to \mathbf{C}$ is a Freyd category
- $\mathbf{V}$ and $\mathbf{C}$ are co-Cartesian and J strictly preserves coproducts

Theorem

Presheaves

$$\mathbf{C}_{\blacklozenge}(J(- \times A), B, C)\colon \mathbf{V}^{\mathsf{op}} \to \mathbf{Set}$$

are representable iff

- *There is a guarded parametrized monad* $\#$ *on* $\mathbf{V}$
- *The monad* $(-) \# \emptyset$ *is strong*
- $\mathbf{C}$ *is isomorphic to the Kleisli category of* $(-) \# \emptyset$
- *Exponentials* $(B \# C)^A$ *exist*

Proof Idea.

$\mathbf{C}_{\blacklozenge}(J(- \times A), B, C) \cong \mathbf{V}(- \times A, B \# C) \cong \mathbf{V}(-, (B \# C)^A)$ □

Let additionally

- $\mathbf{V}$ is distributive i.e. $A \times (B + C) \cong A \times B + A \times C$
- the action of $\mathbf{V}$ on $\mathbf{C}$ preserves guardedness: $f \in \mathbf{V}(A, B)$, $g \in \mathbf{C}_{\blacklozenge}(X, Y, Z)$,

$$A \times X \xrightarrow{f \oslash g} B \times (Y + Z) \cong B \times Y + B \times Z$$

is in $\mathbf{C}_{\blacklozenge}(A \times X, B \times Y, B \times Z)$

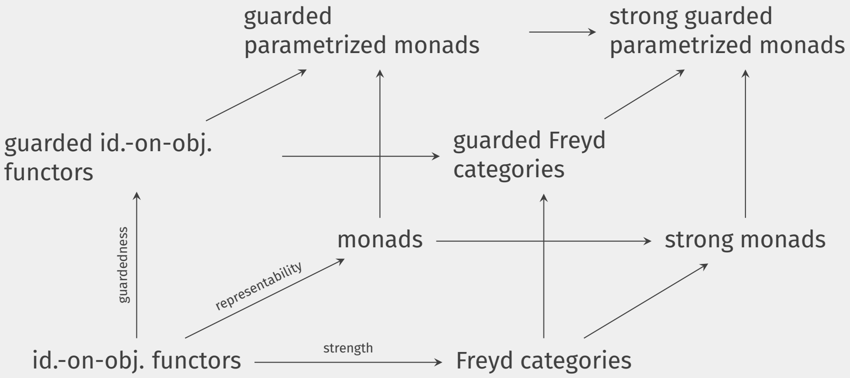
We then call J guarded Freyd category

Proposition

If $\mathbf{C}_{\blacklozenge}$ is J-representable then the induced $\#$ becomes equipped with strength:

$$X \times (Y \# Z) \to (X \times Y) \# (X \times Z)$$

Call such $\#$ strong guarded parametrized monad


guarded
parametrized monads
strong guarded
parametrized monads
guarded id.-on-obj.
functors
guarded Freyd
categories
monads
strong monads
guardedness
representability
id.-on-obj. functors
strength
Freyd categories

- Computational metalanguage a la Moggi w.r.t. guarded parametrized monads?
- Combine guardedness with more advanced structures (e.g. effect handling)
- Dualize: representation of guarded recursion by comonads
  - It is known already that (guarded) recursion is categorically dual to (guarded) iteration
  - What are instances of comonadic guarded recursion?
  - Representing recursion on casual streams/course-of-value recursion
- Implementation