Tensors of Computational Effects and a Logic for Probabilistic Traces

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- **Operational Semantics** (how programs behave?)

$$s(s(0)) + s(s(0)) \rightsquigarrow s(s(0) + s(s(0))) \rightsquigarrow s(s(0 + s(s(0)))) \rightsquigarrow s(s(s(s(0))))$$

- **Denotational Semantics** (what programs denote?)

$$[\![s(s(0)) + s(s(0))]\!] = [\![s(s(0))]\!] + [\![s(s(0))]\!] = 2 + 2 = 4$$

- **Logical Semantics** (what properties programs have?)

$$\frac{\forall x.\, 0 + x \equiv x \qquad \forall x, y.\, s(x) + y \equiv s(x + y)}{s(s(0)) + s(s(0)) \equiv s(s(s(s(0))))}$$

Specifically: which program equivalences they satisfy?

$$[\![-]\!] : \text{Programs} \rightarrow \text{Meanings}$$

$$[\![-]\!] : \text{Programs} \to \text{Meanings}$$

- Programs are terms, e.g. specified with a grammar
- Meanings are given by semantic domains, forming a category
- Programs can fail, crash, hang, depend on the environment, produce different results on repeated runs
- $[\![-]\!]$ caters for the mismatch between program code and concrete observable behaviour
- One source of mismatch: computational effects $\Rightarrow$ monads

- The role of monads is twofold:
  - A monad can be used to model programs as they are written
  - A monad can be used to model the environment in which a program runs
- Monad tensor

$$\Sigma^\star \otimes \mathbf{T}$$

  can be seen as a model of interacting programs with the environment
- To use the result, we can relate it with another monad $\mathbf{T}_\Sigma$ by a monad morphism $\alpha\colon \Sigma^\star \otimes \mathbf{T} \to \mathbf{T}_\Sigma$
- This induces a semantics: $[\![-]\!]\colon \Sigma^\star \cong \Sigma^\star \otimes \mathbf{I} \to \Sigma^\star \otimes \mathbf{T} \to \mathbf{T}_\Sigma$
  - Completeness problem: is α monomorphic?
  - Definability problem: what is the image of α?

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Our main target here:

- $\Sigma^\star$ is a monad for a simple language of reading and writing
- $\mathbf{T}$ is the monad of countable distributions
- $\mathbf{T}_\Sigma$ is a monad of “probabilistic traces”

Outline

Functors and Monads

Combining Monads

Nondeterministic Traces

Probabilistic Traces

Open Problem

- $\mathcal{P}X$ — powerset of X
- $\mathcal{P}_\omega X$ — set of finite subsets
- $\mathcal{P}_{\omega 1} X$ — set of countable subsets
- $\mathcal{P}^+ X$ — powerset of X without $\emptyset$

These can be combined, e.g. $\mathcal{P}^+_\omega X$ — nonempty finite subsets

- $A^\star$ – set of finite sequences $a_1, \ldots, a_n \quad (a_i \in A)$
- A^ω – set of infinite sequences $a_1, \ldots \quad (a_i \in A)$

Functors for Languages

- We stick to a simple generic language over a signature $\Sigma = (\mathbf{c}_i\ /n_i)_{i \in I}$ (think of groups $\{e/0, \cdot/2, (-)^{-1}/1\}$)
  - The arities n_i will be countable
- Programs are well-founded terms over Σ
  = well-founded trees, labelled in Σ
- (Complete) paths in these trees, called traces, are sequences

$$(\mathbf{c}_{i_1}, k_1), (\mathbf{c}_{i_2}, k_2), \ldots, \mathbf{c}_{i_m}$$

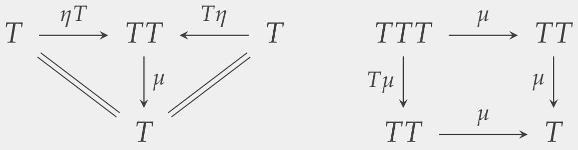
  where $i_j \in I$, $0 < k_j \leq n_j$, and $n_m = 0$
- These can be interpreted game-theoretically as interaction scenarios:
  - $\mathbf{c}_{i_1}$: a program asks a question $i \in I$, and awaits an answer from $\{1, \ldots, n_i\}$
  - k_1: the users enters k_1
  - $\vdots$
  - program prints $\mathbf{c}_{i_m}$ and finishes

- Given a signature Σ, we can adjoin variables X, and form $\Sigma \uplus X$, viewing the elements X as new constants
- $\Sigma^\star X$ denotes the set of terms over $\Sigma \uplus X$
- $X \mapsto \Sigma^\star X$ extends to an endofunctor $\Sigma^\star \colon \mathbf{Set} \to \mathbf{Set}$

Monads for Effects

Monads: Classical Perspective

- *A monad is just a monoid in the category of endofunctors (c) Saunders Mac Lane*
- More verbosely, a monad **T** consists of a functor $T\colon \mathbf{C} \to \mathbf{C}$, unit $\eta\colon \mathrm{Id} \to T$ and multiplication $\mu\colon TT \to T$



- This induces the Kleisli category $\mathbf{C}_{\mathbf{T}}$ of mophisms $X \to TY$

Example

- Identity monad $T = \mathrm{Id} \quad \Rightarrow \quad \mathbf{Set}_{\mathbf{T}} = \mathbf{Set}$
- Powerset monad $T = \mathcal{P} \quad \Rightarrow \quad \mathbf{Set}_{\mathbf{T}} = \mathbf{Rel}$
- Maybe-monad $T = \mathrm{Id} + 1 \quad \Rightarrow \quad \mathbf{Set}_{\mathbf{T}}$ — partial functions
- more generally, $T = \Sigma^{\star}$ — free monad over Σ

- Finitary monads on $\mathbf{Set}$ are equivalent to algebraic theories[1]
- That is, they have presentations in terms of operations and equations, so that
  - $TX = \Sigma X/_{\equiv}$ for some finitary (!) signature Σ
  - $\equiv$ is the equivalence relation induced by a set of equational axioms
  - $\eta_X \colon X \to TX$ forms a term from a given variable
  - $\mu_X \colon TTX \to TX$ turns a tree over trees to a tree

Example

- $T = \mathrm{Id}$: no operations, no equations
- $T = \Sigma^\star$: operations from Σ, no equations
- $T = \mathcal{P}_\omega$: operations: $\bot/0$, $\vee/2$, equations: $\bot \vee x \equiv x$, $x \vee x \equiv x$, $x \vee y \equiv y \vee x$, $x \vee (y \vee z) \equiv (x \vee y) \vee z$

- This view extends to arbitrary monads on $\mathbf{Set}$ using signatures of large arities[2] (here, we stick to countable)

[1] Lawvere, "Functorial Semantics of Algebraic Theories", 1963.
[2] Hyland et al., "Combining algebraic effects with continuations", 2007.

- In semantics, we really need strong functors and monads
- A functor $F\colon \mathbf{C} \to \mathbf{D}$ between monoidal $\mathbf{C}$ and $\mathbf{D}$ is strong if it comes with strength

$$\tau_{A,B}\colon A \otimes FB \to F(A \otimes B),$$

  subject to coherence conditions (omitted)
- A monad (T, η, μ) on $\mathbf{C}$ is strong if T and η, μ are strong
- Equivalently[3], T is strong iff it is $\mathbf{C}$-enriched (and so are η, μ)
- In $\mathbf{Set}$ every functor/monad is enriched, hence strong

[3] Kock, "Strong Functors and Monoidal Monads", 1972.

Commutativity

- A strong monad is commutative if

$$
\begin{array}{ccccc}
TX \otimes TY & \xrightarrow{\tau'} & T(X \otimes TY) & \xrightarrow{T\tau} & TT(X \otimes Y) \\
\tau \big\downarrow & & & & \\
T(TX \otimes Y) & & & & \big\downarrow \mu \\
T\tau' \big\downarrow & & & & \\
TT(X \otimes Y) & & \xrightarrow{\qquad\qquad\mu\qquad\qquad} & & T(X \otimes Y)
\end{array}
$$

- In terms of theories: every two operations commute:

$$g(f(x_1^1, \ldots, x_n^1), \ldots, f(x_1^m, \ldots, x_n^m)) \equiv f(g(x_1^1, \ldots, x_1^m), \ldots, g(x_n^1, \ldots, x_n^m))$$

Call it the commutativity law

Combining Monads

- Sum of monads is their coproduct in the category of monads
- On the level of theories, to form $\mathbf{T} + \mathbf{S}$, we
  - join signatures
  - take all axioms of $\mathbf{T}$
  - take all axioms of $\mathbf{S}$
- For signatures of bounded arities, we thus always can form $\mathbf{T} + \mathbf{S}$

- A monad **R** is a tensor product of two strong monads **T** and **S** under monad morphisms $\alpha\colon \mathbf{T} \to \mathbf{R}$, $\beta\colon \mathbf{S} \to \mathbf{R}$ if

$$
\begin{array}{ccccc}
TX \otimes SY & \xrightarrow{\tau'} & T(X \otimes SY) & \xrightarrow{T\tau} & TS(X \otimes Y) \\
\tau\big\downarrow & & & & \big\downarrow(R\beta)(\alpha S) \\
S(TX \otimes Y) & & & & RR(X \otimes Y) \\
S\tau'\big\downarrow & & & & \big\downarrow\mu \\
ST(X \otimes Y) & \xrightarrow{(R\alpha)(\beta S)} & RR(X \otimes Y) & \xrightarrow{\mu} & R(X \otimes Y)
\end{array}
$$

  and **R** is universal with this property
- On the level of theories, $\mathbf{T} \otimes \mathbf{S}$ is obtained from $\mathbf{T} + \mathbf{S}$ by adding all the commutativity laws

$$g(f(x_1^1,\dots,x_n^1),\dots,f(x_1^m,\dots,x_n^m)) \equiv f(g(x_1^1,\dots,x_1^m),\dots,g(x_n^1,\dots,x_n^m))$$

  for f from **T** and g from **S**

- Often, **T** $\otimes$ **S** can be characterized directly
  - Let $(W, 0, 1, +, \cdot)$ be a semiring ($\approx$ ring without subtraction)
  - Semiring W-modules are sets, equipped with a left action of W
  - Free W-modules form a monad $\mathcal{S}_{W,\omega}$
  - Elements of $\mathcal{S}_{W,\omega}X$ are formal sums $w_1 \cdot x_1 + \ldots + w_k \cdot x_k$ ($w_i \in W, x_i \in X$)

Theorem (Freyd[4])

For any finitary monad **T**, **T** $\otimes \mathcal{S}_{W,\omega}$ *exists and is again a free semiring-module monad*

[4] Freyd, “Algebra valued functors in general and tensor products in particular”, 1966.

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Theorem (Freyd[4])

For any finitary monad $\mathbf{T}$, $\mathbf{T} \otimes \mathcal{S}_{W,\omega}$ *exists and is again a free semiring-module monad*

Proof Idea.

For any f/n of $\mathbf{T}$, the commutativity law entails

$$f(x_1, \ldots, x_n) \equiv f_1(x_1) + \ldots + f_n(x_n)$$

where $f_i(x) = f(0, \ldots, x, \ldots, 0)$. These f_i are the semiring actions of the new monad! □

[4] Freyd, "Algebra valued functors in general and tensor products in particular", 1966.

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  - Let $(W, 0, 1, +, \cdot)$ be a semiring ($\approx$ ring without subtraction)
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Theorem (Freyd[4])

For any finitary monad $\mathbf{T}$, $\mathbf{T} \otimes \mathcal{S}_{W,\omega}$ exists and is again a free semiring-module monad

- Since $\mathcal{P}_\omega = \mathcal{S}_{\{0,1\},\omega}$, for finitary Σ we obtain an exact presentation of $\Sigma^\star \otimes \mathcal{P}_\omega$:
  - $\Sigma^\star \otimes \mathcal{P}_\omega \cong \mathcal{S}_{W,\omega} \cong \mathcal{P}_\omega(\overline{\Sigma}^\star \times \mathrm{Id})$ where $W = \mathcal{P}_\omega(\overline{\Sigma}^\star)$ is the free idempotent semiring over $\overline{\Sigma} = \{(\mathbf{c}_i, k_i) \mid \mathbf{c}_i / n_i \in \Sigma, 0 < k_i \leq n_i\}$
  - $\mathcal{P}_\omega(\overline{\Sigma}^\star \times X) =$ finite subsets of $\overline{\Sigma}^\star \times X$
- Countable non-determinism is treated analogously

[4] Freyd, “Algebra valued functors in general and tensor products in particular”, 1966.

Nondetermistic Traces

From now on, assume that $\Sigma = \{*/2\}$

- Let $\mathcal{S}_W$ be the monad of ω-complete semiring W-modules
  That is, we allow for countable sums $\sum_{i\in\mathbb{N}} t_i = t_1 + t_2 + \dots$
- Call $\mathcal{P}_{\omega 1}(\overline{\Sigma}^\star \times \mathrm{Id}) \cong \mathcal{S}_{\mathcal{P}\overline{\Sigma}^\star}$ the non-deterministic trace monad induced by Σ
- With $\Sigma = \{*/2\}$, $\overline{\Sigma} \cong 2$, $\mathcal{P}_{\omega 1}(\overline{\Sigma}^\star \times X)$ consists of countable subsets of $\{b_1. \dots b_n. x \mid b_i \in \{0,1\}, x \in X\}$
- We thus obtain trace semantics of non-deterministic programs

$$[\![-]\!] \colon \Sigma^\star \otimes \mathcal{P}_\omega \to \mathcal{P}_{\omega 1}(\overline{\Sigma}^\star \times \mathrm{Id})$$

Example: $(x * y) \vee (x * (y * z)) \quad \mapsto \quad \{0.x, 1.y, 10.y, 11.z\}$

✔ Completeness: $[\![-]\!]$ is monomorphic
✔ Definability: the image of $[\![-]\!]$ consists precisely of finite sets of traces

- Freyd's proof crucially relies on existence of the neutral element in semiring, equivalently, the empty set in $\mathcal{P}_\omega$
- The commutativity law entails

$$b_1 \cdots b_m \cdot \emptyset \equiv \emptyset \qquad\qquad b_i \in \{0,1\}$$

which is arguably a modelling artefact
- Let $\mathcal{P}^+_\omega$ be the monad of non-empty (!) finite powersets
What is $\Sigma^\star \otimes \mathcal{P}^+_\omega$?
- Again, we define trace semantics

$$[\![-]\!] \colon \Sigma^\star \otimes \mathcal{P}^+_\omega \to \mathcal{P}_{\omega 1}(\overline{\Sigma}^\star \times X)$$

Theorem (Completeness[5])

$[\![-]\!]$ is monomorphic: $[\![p]\!] = [\![q]\!]$ implies $p \equiv q$

Proof Idea.

Assume $[\![p]\!] = [\![q]\!]$

- Let $p \leq q$ if $p \vee q \equiv q$. Then $p \equiv q$ iff $p \leq q$ and $q \leq p$
- Call $p \vee q$ an interpolant of p, q
- It suffices to show that $p \equiv p \vee q$ ($p \vee q \equiv q$ by symmetry)
- Proof of $p \equiv p \vee q$ is constructed by induction on the strucure of q □

[5] Bowler, Levy, and Plotkin, "Initial Algebras and Final Coalgebras Consisting of Nondeterministic Finite Trace Strategies", 2018.

- $[\![\underbrace{x * y \vee x' * y'}_{p}]\!] = \{0.x, 1.y, 0.x', 1.y'\} = [\![\underbrace{x' * y \vee x * y'}_{q}]\!]$
- $$\begin{aligned} p \vee q &\equiv x * y \vee x' * y' \vee x' * y \vee x * y' \\ &\equiv (x \vee x' \vee x' \vee x) * (y \vee y' \vee y \vee y') \\ &\equiv (x \vee x') * (y \vee y') \\ &\equiv (x * y) \vee (x' * y') \\ &\equiv p \end{aligned}$$
- Crucially, the proof works for arbitrary signatures and countable non-determinism $\Rightarrow$ it does not just ammount to propagating joins downwards, e.g. doing this for

  $$x * y \vee (x * y) * y \vee ((x * y) * y) * y \vee \ldots$$

  would not produce a vaild (i.e. well-founded) term (!)

Still, under finite non-determinism, we can normalize terms by propagating $\vee$ downwards, which helps us to characterize the image of $[\![-]\!]\colon \Sigma^\star \otimes \mathcal{P}^+_\omega \to \mathcal{P}_{\omega 1}(\overline{\Sigma}^\star \times \mathrm{Id})$

Theorem (Definability)

The image of $[\![-]\!]\colon \Sigma^\star \otimes \mathcal{P}^+_\omega \to \mathcal{P}_{\omega 1}(\overline{\Sigma}^\star \times \mathrm{Id})$ *consists of such subsets* S *of* $2^\star \times X$ *that*

1. *For any* $w \in 2^\star$, $S \cap \{w.\,x \mid x \in X\}$ *is finite*
2. *For any* $w \in 2^\star$, *if* $w0u.\,x \in S$ *with some* $u \in 2^\star$ *and* $x \in X$ *then* $w1u'.\,x' \in S$ *with some* $u' \in 2^\star$ *and* $x' \in X$
3. *There is no infinite sequence* $w \in 2^\omega$, *such* $u.\,x \in S$ *for infinitely many finite prefixes* u *of* w

- Condition (3) ensures well-foundedness, and fails for countable non-determinism
- Definability under countable non-determinism is thus more complicated

Probabilistic Traces

- Denote the monad of finitary probability distributions $\mathcal{D}_\omega$
  - $\mathcal{D}_\omega X = \{d\colon X \to [0,1] \mid \sum_{x \in X} d(x) = 1, \mathrm{supp}\, d < \omega\}$
  - unit $\eta(x)$ is the Dirac distribution centered at x
  - $\mathcal{D}_\omega$ is a submonad of $\mathcal{S}_{[0,1],\omega}$, from which it inherits the structure
- Algebraically, $\mathcal{D}_\omega X$ are free barycentric algebras over $\{+_p/2 \mid p \in (0,1)\}$:

$$\begin{aligned} x +_p x &= x \\ x +_p y &= y +_{1-p} x \\ (x +_p y) +_q z &= x +_{\frac{p}{p+q-pq}} (y +_{p+q-pq} z) \end{aligned}$$

Definition (Probabilistic Trace Monad)

We call $\mathcal{S}_{[0,1]}(\overline{\Sigma}^\star \times \mathrm{Id})$ the probabilistic trace monad

Previous techniques can be adapted to obtain completeness and definability for $[\![-]\!]\colon \Sigma^\star \otimes \mathcal{D}_\omega \to \mathcal{S}_{[0,1]}(\overline{\Sigma}^\star \times \mathrm{Id})$

- $\mathcal{D}X = \{d\colon X \to [0,1] \mid \sum_{x \in X} d(x) = 1\}$ is the monad of countable distributions
- Signature of $\mathcal{D}$ consists of countable sums, which build $\sum_{i \in I} p_i \cdot t_i$ from terms $(t_i)_{i \in I}$ where $\sum_{i \in I} p_i = 1$ (axioms omitted)
- Commutativity law takes form

$$\Big(\sum_{i \in I} p_i \cdot s_i\Big) * \Big(\sum_{i \in I} p_i \cdot t_i\Big) \equiv \sum_{i \in I} p_i \cdot t_i * s_i$$

Completeness issue becomes really hard
Example: $[\![p]\!] = [\![q]\!]$ where

$$p = \tfrac{1}{2}a * c + \tfrac{1}{4}b * (a * c) + \tfrac{1}{8}b * (b * (a * c)) + \tfrac{1}{16}b * (b * (b * (a * c))) + \cdots$$
$$q = \tfrac{1}{2}b * c + \tfrac{1}{4}a * (b * c) + \tfrac{1}{8}a * (a * (b * c)) + \tfrac{1}{16}a * (a * (a * (b * c))) + \cdots$$

and then in fact $p \equiv q$:

$$\begin{aligned} p &\equiv \tfrac{1}{4}a * c + \tfrac{2}{8}(\tfrac{1}{2}a * c + \tfrac{1}{2}b * (a * c)) + \tfrac{3}{16}(\tfrac{1}{3}a * c + \tfrac{1}{3}b * (a * c) + \tfrac{1}{3}b * (b * (a * c))) + \cdots \\ &\equiv \tfrac{1}{4}a * c + \tfrac{2}{8}(\tfrac{1}{2}b * c + \tfrac{1}{2}a * (a * c)) + \tfrac{3}{16}(\tfrac{1}{3}b * c + \tfrac{1}{3}b * (b * c) + \tfrac{1}{3}a * (a * (a * c))) + \cdots \\ &\equiv \tfrac{1}{4}a * c + \tfrac{1}{8}a * (a * c) + \tfrac{1}{16}a * (a * (a * c)) + \cdots + \\ &\quad\ \tfrac{1}{4}b * c + \tfrac{1}{8}b * (b * c) + \tfrac{1}{16}b * (b * (b * c)) + \cdots \end{aligned}$$

and symmetrically for q

- We generally cannot fully normalize by the commutativity law

$$\sum_{i \in I} p_i \cdot (t_i * s_i) \rightsquigarrow (\sum_{i \in I} p_i \cdot t_i) * (\sum_{i \in I} p_i \cdot s_i)$$

  since this often produces non-well-founded trees
- If $[\![s]\!] = [\![t]\!]$, we also failed to find interpolating $r_{\{s,t\}}$, such that $s \equiv r_{\{s,t\}}$

Still, we have

Theorem (Completeness)

$[\![-]\!] \colon \Sigma^\star \otimes \mathcal{D} \to \mathcal{S}_{[0,1]}(\overline{\Sigma}^\star \times \mathrm{Id})$ *is monomorphic*

Theorem (Definability)

The image of $[\![-]\!]\colon \Sigma^\star \otimes \mathcal{D} \to \mathcal{S}_{[0,1]}(\overline{\Sigma}^\star \times \mathrm{Id})$ *consists exactly of such* $h\colon 2^* \times X \to [0,1]$ *that for every* $(w_1 w_2 \ldots) \in 2^\omega$,

$$\sum\{h(w_1 \ldots w_n.\, x) \mid n \in \mathbb{N}, x \in X\} = 1$$

Example

$$[\![\tfrac{1}{2}x + \tfrac{1}{4}x * x + \tfrac{1}{8}(x * x) * x + \ldots]\!] = \{\tfrac{1}{2}x, \tfrac{1}{2}1.x, \tfrac{1}{4}0.x, \tfrac{1}{8}01.x, \tfrac{1}{8}00.x, \ldots\}$$

- Our completeness proof works for any finitary signatures
- It crucially relies on the fact that terms can be normalized by pushing infinite sums upwards using

$$(\sum_{i\in I} p_i \cdot t_i) * (\sum_{j\in J} q_j \cdot t_j) \rightsquigarrow \sum_{i\in I, j\in J} (p_i \cdot q_j) \cdot t_i * s_j$$

- With countable signatures, this is not possible, e.g.

$$\mathbf{c}\,(x_0, \tfrac{1}{2}x_0 + \tfrac{1}{2}x_1, \tfrac{1}{3}x_0 + \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2, \ldots)$$

  there is no normal form in this sense
- So, do we have completeness in this case? We still do not know

Thank you for your **Attention**!

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Programming Perspective

Definition (Kleisli Triple)

A Kleisli triple is a triple of the form $(T, \eta, -^\star)$ where

- T sends sets to sets
- η is a family of morphisms $\eta_X \colon X \to TX$, forming monad unit
- $(-)^\star$ assigns to each $f \colon X \to TY$ a morphism $f^\star \colon TX \to TY$

satisfying the laws: $\eta^\star = \mathrm{id}$, $f^\star \eta = f$, $(f^\star g)^\star = f^\star g^\star$

This entails that $f \colon X \to TY$ and $g \colon Y \to TZ$ can be Kleisli composed to $g^\star f \colon X \to TZ$, which yields Klesili category $\mathbf{Set}_\mathbf{T}$

Example

- Identity monad $T = \mathrm{Id} \quad\Rightarrow\quad \mathbf{Set}_\mathbf{T} = \mathbf{Set}$
- Powerset monad $T = \mathcal{P} \quad\Rightarrow\quad \mathbf{Set}_\mathbf{T} = \mathbf{Rel}$
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- more generally, $T = \Sigma^\star$ — free monad over Σ

Strong Functors

A functor $F\colon \mathbf{C} \to \mathbf{D}$ between monoidal $\mathbf{C}$ and $\mathbf{D}$ is strong if it comes with strength

$$\tau_{A,B}\colon A \otimes FB \to F(A \otimes B),$$

such that

$$\begin{array}{ccc}
I \otimes FX & \cong & FX \\
{\scriptstyle\tau}\big\downarrow & & \big\| \\
F(I \otimes X) & \cong & FX
\end{array}$$

$$\begin{array}{ccccc}
(X \otimes Y) \otimes FZ & & \xrightarrow{\qquad\qquad\tau\qquad\qquad} & & F((X \otimes Y) \otimes Z) \\
\wr\| & & & & \wr\| \\
X \otimes (Y \otimes FY) & \xrightarrow{X \otimes \tau} & X \otimes F(Y \otimes Z) & \xrightarrow{\tau} & F(X \otimes (Y \otimes Z))
\end{array}$$

Strong Monads

A monad (T, η, μ) on $\mathbf{C}$ is strong if T and η, μ are strong

$$\begin{array}{ccc} X \otimes Y & = & X \otimes Y \\ {\scriptstyle X\otimes\eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ X \otimes TY & \xrightarrow{\tau} & T(X \otimes Y) \end{array}$$

$$\begin{array}{ccccc} X \otimes TTY & \xrightarrow{\tau} & T(X \otimes TY) & \xrightarrow{T\tau} & TT(X \otimes Y) \\ {\scriptstyle X\otimes\mu}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ X \otimes TY & & \xrightarrow{\qquad\quad\tau\quad\qquad} & & T(X \otimes Y) \end{array}$$

Algebras

- Equivalently, Σ is an endofunctor $\Sigma\colon \mathbf{Set} \to \mathbf{Set}$:
  $\Sigma X = \coprod_{i \in I} X^{n_i}$
- We can form the category of Σ-algebras with morphisms

$$\begin{array}{ccc} \Sigma A & \xrightarrow{a} & A \\ {\scriptstyle \Sigma h}\downarrow & & \downarrow{\scriptstyle h} \\ \Sigma B & \xrightarrow{b} & B \end{array}$$

- Free object $\Sigma^\star X$ over a set X in this category is the same as an initial $(\Sigma + X)$-algebra
  - $\Sigma^\star$ extends to a functor
  - $\Sigma^\star \emptyset$ is carried by well-founded Σ-terms
  - $\Sigma^\star X$ is carried by well-founded Σ-terms with variables from X