Tensors of Computational Effects and a Logic for Probabilistic Traces

Sergey Goncharov

Theoretical Computer Science, FAU Erlangen-Nürnberg
[joint work with Paul Blain Levy and Nathan Bowler]

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Operational Semantics (how programs behave?)

\[ s(s(0)) + s(s(0)) \leadsto s(s(0) + s(s(0))) \]
\[ \leadsto s(s(0 + s(s(0)))) \leadsto s(s(s(s(0)))) \]

Denotational Semantics (what programs denote?)

\[ \llbracket s(s(0)) + s(s(0)) \rrbracket = \llbracket s(s(0)) \rrbracket + \llbracket s(s(0)) \rrbracket = 2 + 2 = 4 \]

Logical Semantics (what properties programs have?)

\[ \forall x. 0 + x \equiv x \quad \forall x, y. s(x) + y \equiv s(x + y) \]
\[ s(s(0)) + s(s(0)) \equiv s(s(s(s(0)))) \]

Specifically: which program equivalences they satisfy?
Denotational Semantics: Term ≠ Meaning

\[ \llbracket - \rrbracket : \text{Programs} \rightarrow \text{Meanings} \]
Denotational Semantics: Term ≠ Meaning

[←] : Programs → Meanings

- Programs are terms, e.g. specified with a grammar
- Meanings are given by semantic domains, forming a category
- Programs can fail, crash, hang, depend on the environment, produce different results on repeated runs
- [←] caters for the mismatch between program code and concrete observable behaviour
- One source of mismatch: computational effects ⇒ monads
The role of monads is twofold:
▶ A monad can be used to model programs as they are written
▶ A monad can be used to model the environment in which a program runs

Monad tensor

\[ \Sigma^* \otimes T \]

can be seen as a model of interacting programs with the environment

To use the result, we can relate it with another monad \( T_\Sigma \) by a monad morphism \( \alpha : \Sigma^* \otimes T \rightarrow T_\Sigma \)

This induces a semantics: \([\cdot] : \Sigma^* \cong \Sigma^* \otimes I \rightarrow \Sigma^* \otimes T \rightarrow T_\Sigma\)

▶ Completeness problem: is \( \alpha \) monomorphic?
▶ Definability problem: what is the image of \( \alpha \)?
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- Completeness problem: is \( \alpha \) monomorphic?
- Definability problem: what is the image of \( \alpha \)?

Our main target here:

- \( \Sigma^* \) is a monad for a simple language of reading and writing
- \( T \) is the monad of countable distributions
- \( T_\Sigma \) is a monad of “probabilistic traces”
Functors and Monads

Combining Monads

Nondeterministic Traces

Probabilistic Traces

Open Problem
A Little Dictionary

- $\mathcal{P}X$ — powerset of $X$
- $\mathcal{P}\omega X$ — set of finite subsets
- $\mathcal{P}\omega_1 X$ — set of countable subsets
- $\mathcal{P}^+ X$ — powerset of $X$ without $\emptyset$

These can be combined, e.g. $\mathcal{P}^+_\omega X$ — nonempty finite subsets

- $A^*$ — set of finite sequences $a_1, \ldots, a_n$ ($a_i \in A$)
- $A^\omega$ — set of infinite sequences $a_1, \ldots$ ($a_i \in A$)
FUNCTORS FOR LANGUAGES
We stick to a simple generic language over a signature
\[ \Sigma = (c_i / n_i)_{i \in I} \] (think of groups \( \{ e/0, \cdot/2, (-)^{-1}/1 \} \))

- The arities \( n_i \) will be countable

Programs are well-founded terms over \( \Sigma \)
= well-founded trees, labelled in \( \Sigma \)

(Complete) paths in these trees, called traces, are sequences

\[ (c_{i_1}, k_1), (c_{i_2}, k_2), \ldots, c_{i_m} \]

where \( i_j \in I, 0 < k_j \leq n_j, \) and \( n_m = 0 \)

These can be interpreted game-theoretically as interaction scenarios:

- \( c_{i_1} \): a program asks a question \( i \in I, \) and awaits an answer from \( \{1, \ldots, n_i\} \)
- \( k_1 \): the users enters \( k_1 \)

- 
- program prints \( c_{i_m} \) and finishes
Given a signature $\Sigma$, we can adjoin variables $X$, and form $\Sigma \uplus X$, viewing the elements $X$ as new constants.

$\Sigma^* X$ denotes the set of terms over $\Sigma \uplus X$.

$X \mapsto \Sigma^* X$ extends to an endofunctor $\Sigma^* : \text{Set} \to \text{Set}$.
MONADS FOR EFFECTS
A monad is just a monoid in the category of endofunctors
(c) Saunders Mac Lane

More verbosely, a monad $T$ consists of a functor $T : C \to C$, unit $\eta : \text{Id} \to T$ and multiplication $\mu : TT \to T$

This induces the Kleisli category $C_T$ of morphisms $X \to TY$

Example

- Identity monad $T = \text{Id} \Rightarrow \text{Set}_T = \text{Set}$
- Powerset monad $T = \mathcal{P} \Rightarrow \text{Set}_T = \text{Rel}$
- Maybe-monad $T = \text{Id} + 1 \Rightarrow \text{Set}_T$ — partial functions
- more generally, $T = \Sigma^* \rightarrow$ free monad over $\Sigma$
Finitary monads on $\text{Set}$ are equivalent to algebraic theories\(^1\)

That is, they have **presentations** in terms of operations and equations, so that

- $TX = \Sigma X/\equiv$ for some finitary (!) signature $\Sigma$
- $\equiv$ is the equivalence relation induced by a set of equational axioms
- $\eta_X : X \to TX$ forms a term from a given variable
- $\mu_X : TTX \to TX$ turns a tree over trees to a tree

### Example

- $T = \text{Id}$: no operations, no equations
- $T = \Sigma^*$: operations from $\Sigma$, no equations
- $T = \mathcal{P}_\omega$: operations: $\bot/0, \lor/2$, equations: $\bot \lor x \equiv x$, $x \lor x \equiv x$, $x \lor y \equiv y \lor x$, $x \lor (y \lor z) \equiv (x \lor y) \lor z$

This view extends to arbitrary monads on $\text{Set}$ using signatures of **large** arities\(^2\) (here, we stick to countable)

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\(^1\)Lawvere, “Functorial Semantics of Algebraic Theories”, 1963.

In semantics, we really need strong functors and monads

A functor \( F : \mathcal{C} \to \mathcal{D} \) between monoidal \( \mathcal{C} \) and \( \mathcal{D} \) is strong if it comes with strength

\[
\tau_{A,B} : A \otimes FB \to F(A \otimes B),
\]

subject to coherence conditions (omitted)

A monad \((T, \eta, \mu)\) on \( \mathcal{C} \) is strong if \( T \) and \( \eta, \mu \) are strong

Equivalently\(^3\), \( T \) is strong iff it is \( \mathcal{C} \)-enriched (and so are \( \eta, \mu \))

In \( \text{Set} \) every functor/monad is enriched, hence strong

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A strong monad is **commutative** if

$$ TX \otimes TY \xrightarrow{\tau'} T(X \otimes TY) \xrightarrow{T\tau} TT(X \otimes Y) $$

In terms of theories: every two operations commute:

$$ g(f(x_1^1, \ldots, x_n^1), \ldots, f(x_1^m, \ldots, x_n^m)) \equiv f(g(x_1^1, \ldots, x_1^m), \ldots, g(x_n^1, \ldots, x_n^m)) $$

Call it the **commutativity law**
COMBINING MONADS
Sum of monads is their **coproduct** in the category of monads.

On the level of theories, to form \( T + S \), we

- join signatures
- take all axioms of \( T \)
- take all axioms of \( S \)

For signatures of bounded arities, we thus always can form \( T + S \).
Combining Monads: Tensors

A monad $R$ is a tensor product of two strong monads $T$ and $S$ under monad morphisms $\alpha : T \to R$, $\beta : S \to R$ if

$$
\begin{align*}
&TX \otimes SY \xrightarrow{\tau'} T(X \otimes SY) \xrightarrow{T\tau} TS(X \otimes Y) \\
&\tau \downarrow \\
&S(TX \otimes Y) \\
&S\tau' \downarrow \\
&ST(X \otimes Y) \xrightarrow{(R\alpha)(\beta S)} RR(X \otimes Y) \xrightarrow{\mu} R(X \otimes Y)
\end{align*}
$$

and $R$ is universal with this property

On the level of theories, $T \otimes S$ is obtained from $T + S$ by adding all the commutativity laws

$$
\begin{align*}
g(f(x_1^1, \ldots, x_1^n), \ldots, f(x_m^1, \ldots, x_m^n)) &\equiv f(g(x_1^1, \ldots, x_1^m), \ldots, g(x_m^1, \ldots, x_m^n))
\end{align*}
$$

for $f$ from $T$ and $g$ from $S$
Often, $T \otimes S$ can be characterized directly

- Let $(W, 0, 1, +, \cdot)$ be a semiring ($\approx$ ring without subtraction)
- Semiring $W$-modules are sets, equipped with a left action of $W$
- Free $W$-modules form a monad $S_{W,\omega}$
- Elements of $S_{W,\omega}X$ are formal sums $w_1 \cdot x_1 + \ldots + w_k \cdot x_k$ ($w_i \in W, x_i \in X$)

Theorem (Freyd$^4$)

For any finitary monad $T$, $T \otimes S_{W,\omega}$ exists and is again a free semiring-module monad

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**Theorem (Freyd$^4$)**

*For any finitary monad $T$, $T \otimes S_{W,\omega}$ exists and is again a free semiring-module monad*

**Proof Idea.**

For any $f/n$ of $T$, the commutativity law entails

$$f(x_1, \ldots, x_n) \equiv f_1(x_1) + \ldots + f_n(x_n)$$

where $f_i(x) = f(0, \ldots, x, \ldots, 0)$. These $f_i$ are the semiring actions of the new monad!

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**Theorem (Freyd⁴)**

For any finitary monad $T$, $T \otimes S_{W,\omega}$ exists and is again a free semiring-module monad

- Since $P_\omega = S_{\{0,1\},\omega}$, for finitary $\Sigma$ we obtain an exact presentation of $\Sigma^* \otimes P_\omega$:
  - $\Sigma^* \otimes P_\omega \cong S_{W,\omega} \cong P_\omega(\Sigma^* \times \text{Id})$ where $W = P_\omega(\Sigma^*)$ is the free idempotent semiring over $\overline{\Sigma} = \{(c_i, k_i) \mid c_i / n_i \in \Sigma, 0 < k_i \leq n_i\}$
  - $P_\omega(\Sigma^* \times X) = \text{finite subsets of } \Sigma^* \times X$

- Countable non-determinism is treated analogously

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Nondeterministic Traces
From now on, assume that $\Sigma = \{*/2\}$
Trace Semantics

Let $S_W$ be the monad of $\omega$-complete semiring $W$-modules. That is, we allow for countable sums $\sum_{i \in \mathbb{N}} t_i = t_1 + t_2 + \ldots$

Call $P_{\omega 1}(\bar{\Sigma}^* \times \text{Id}) \cong S_{P\bar{\Sigma}^*}$ the non-deterministic trace monad induced by $\Sigma$.

With $\Sigma = \{*/2\}$, $\bar{\Sigma} \cong 2$, $P_{\omega 1}(\bar{\Sigma}^* \times X)$ consists of countable subsets of $\{b_1, \ldots, b_n. x \mid b_i \in \{0, 1\}, x \in X\}$.

We thus obtain trace semantics of non-deterministic programs

$$\llbracket \cdot \rrbracket : \Sigma^* \otimes P_{\omega} \rightarrow P_{\omega 1}(\bar{\Sigma}^* \times \text{Id})$$

Example: $(x \ast y) \lor (x \ast (y \ast z)) \mapsto \{0.x, 1.y, 10.y, 11.z\}$

- Completeness: $\llbracket \cdot \rrbracket$ is monomorphic
- Definability: the image of $\llbracket \cdot \rrbracket$ consists precisely of finite sets of traces
Freyd’s proof crucially relies on existence of the neutral element in semiring, equivalently, the empty set in $\mathcal{P}_\omega$

The commutativity law entails

$$b_1 \cdots b_m \cdot \emptyset \equiv \emptyset$$

which is arguably a modelling artefact

Let $\mathcal{P}_\omega^+$ be the monad of non-empty (!) finite powersets

What is $\Sigma^* \otimes \mathcal{P}_\omega^+$?

Again, we define trace semantics

$$\llbracket - \rrbracket : \Sigma^* \otimes \mathcal{P}_\omega^+ \rightarrow \mathcal{P}_{\omega^1}(\Sigma^* \times X)$$
**Theorem (Completeness)**

$[\vdash]$ is monomorphic: $[p] = [q]$ implies $p \equiv q$

**Proof Idea.**

Assume $[p] = [q]$

- Let $p \leq q$ if $p \lor q \equiv q$. Then $p \equiv q$ iff $p \leq q$ and $q \leq p$
- Call $p \lor q$ an interpolant of $p, q$
- It suffices to show that $p \equiv p \lor q$ ($p \lor q \equiv q$ by symmetry)
- Proof of $p \equiv p \lor q$ is constructed by induction on the structure of $q$

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Examples

\[ [x \ast y \lor x' \ast y'] = \{0.x, 1.y, 0.x', 1.y'\} = [x' \ast y \lor x \ast y'] \]

\[ p \lor q \equiv x \ast y \lor x' \ast y' \lor x' \ast y \lor x \ast y' \]
\[ \equiv (x \lor x' \lor x' \lor x) \ast (y \lor y' \lor y \lor y') \]
\[ \equiv (x \lor x') \ast (y \lor y') \]
\[ \equiv (x \ast y) \lor (x' \ast y') \]
\[ \equiv p \]

Crucially, the proof works for arbitrary signatures and countable non-determinism \( \Rightarrow \) it does not just amount to propagating joins downwards, e.g. doing this for

\[ x \ast y \lor (x \ast y) \ast y \lor ((x \ast y) \ast y) \ast y \lor \ldots \]

would not produce a valid (i.e. well-founded) term (!)
Still, under finite non-determinism, we can normalize terms by propagating $\lor$ downwards, which helps us to characterize the image of $[\_] : \Sigma^* \otimes \mathcal{P}_{\omega}^+ \rightarrow \mathcal{P}_{\omega_1}(\Sigma^* \times \text{Id})$

**Theorem (Definability)**

The image of $[\_] : \Sigma^* \otimes \mathcal{P}_{\omega}^+ \rightarrow \mathcal{P}_{\omega_1}(\Sigma^* \times \text{Id})$ consists of such subsets $S$ of $2^* \times X$ that

1. For any $w \in 2^*$, $S \cap \{w. x \mid x \in X\}$ is finite
2. For any $w \in 2^*$, if $w0u. x \in S$ with some $u \in 2^*$ and $x \in X$ then $w1u'. x' \in S$ with some $u' \in 2^*$ and $x' \in X$
3. There is no infinite sequence $w \in 2^\omega$, such $u. x \in S$ for infinitely many finite prefixes $u$ of $w$

- Condition (3) ensures well-foundedness, and fails for countable non-determinism
- Definability under countable non-determinism is thus more complicated
Probabilistic Traces
Denote the monad of finitary probability distributions $\mathcal{D}_\omega$

- $\mathcal{D}_\omega X = \{ d: X \to [0, 1] \mid \sum_{x \in X} d(x) = 1, \text{supp} d < \omega \}$
- unit $\eta(x)$ is the Dirac distribution centered at $x$
- $\mathcal{D}_\omega$ is a submonad of $\mathcal{S}_{[0,1],\omega}$, from which it inherits the structure

Algebraically, $\mathcal{D}_\omega X$ are free barycentric algebras over $\{+\frac{p}{2} \mid p \in (0, 1)\}$:

\[
\begin{align*}
  x + \frac{p}{2} x &= x \\
  x + p y &= y + 1 - p x \\
  (x + p y) + q z &= x + p \frac{p}{p + q - pq} (y + p + q - pq z)
\end{align*}
\]

**Definition (Probabilistic Trace Monad)**

We call $\mathcal{S}_{[0,1]}(\Sigma^* \times \text{Id})$ the probabilistic trace monad

Previous techniques can be adapted to obtain completeness and definability for $\llbracket - \rrbracket: \Sigma^* \otimes \mathcal{D}_\omega \to \mathcal{S}_{[0,1]}(\Sigma^* \times \text{Id})$
\( DX = \{d: X \to [0, 1] | \sum_{x \in X} d(x) = 1\} \) is the monad of countable distributions.

Signature of \( D \) consists of countable sums, which build \( \sum_{i \in I} p_i \cdot t_i \) from terms \( (t_i)_{i \in I} \) where \( \sum_{i \in I} p_i = 1 \) (axioms omitted).

Commutativity law takes form

\[
(\sum_{i \in I} p_i \cdot s_i) \ast (\sum_{i \in I} p_i \cdot t_i) \equiv \sum_{i \in I} p_i \cdot t_i \ast s_i
\]
Completeness issue becomes really hard

**Example:** \([p] = [q]\) where

\[
p = \frac{1}{2} a \ast c + \frac{1}{4} b \ast (a \ast c) + \frac{1}{8} b \ast (b \ast (a \ast c)) + \frac{1}{16} b \ast (b \ast (b \ast (a \ast c))) + \cdots
\]

\[
q = \frac{1}{2} b \ast c + \frac{1}{4} a \ast (b \ast c) + \frac{1}{8} a \ast (a \ast (b \ast c)) + \frac{1}{16} a \ast (a \ast (a \ast (b \ast c))) + \cdots
\]

and then in fact \(p \equiv q:\)

\[
p \equiv \frac{1}{4} a \ast c + \frac{2}{8} (\frac{1}{2} a \ast c + \frac{1}{2} b \ast (a \ast c)) + \frac{3}{16} \left(\frac{1}{3} a \ast c + \frac{1}{3} b \ast (a \ast c) + \frac{1}{3} b \ast (b \ast (a \ast c))\right) + \cdots
\]

\[
\equiv \frac{1}{4} a \ast c + \frac{2}{8} (\frac{1}{2} b \ast c + \frac{1}{2} a \ast (a \ast c)) + \frac{3}{16} \left(\frac{1}{3} b \ast c + \frac{1}{3} b \ast (b \ast c) + \frac{1}{3} a \ast (a \ast (a \ast c))\right) + \cdots
\]

\[
\equiv \frac{1}{4} a \ast c + \frac{1}{8} a \ast (a \ast c) + \frac{1}{16} a \ast (a \ast (a \ast c))) + \cdots +
\]

\[
\frac{1}{4} b \ast c + \frac{1}{8} b \ast (b \ast c) + \frac{1}{16} b \ast (b \ast (b \ast c))) + \cdots
\]

and symmetrically for \(q\)
We generally cannot fully normalize by the commutativity law

\[ \sum_{i \in I} p_i \cdot (t_i \ast s_i) \rightsquigarrow (\sum_{i \in I} p_i \cdot t_i) \ast (\sum_{i \in I} p_i \cdot s_i) \]

since this often produces non-well-founded trees

If \([s] = [t]\), we also failed to find interpolating \(r\{s,t\}\), such that \(s \equiv r\{s,t\}\)

Still, we have

**Theorem (Completeness)**

\([\rightarrow] : \Sigma^* \otimes \mathcal{D} \to S_{[0,1]}(\Sigma^* \times \text{Id}) \text{ is monomorphic} \)
Theorem (Definability)

The image of \([-\]: \Sigma^* \otimes D \to S_{[0,1]}(\Sigma^* \times \text{Id})\) consists exactly of such \(h: 2^* \times X \to [0,1]\) that for every \((w_1w_2\ldots) \in 2^\omega\),

\[
\sum\{h(w_1\ldots w_n.x) \mid n \in \mathbb{N}, x \in X\} = 1
\]

Example

\[
\left[\frac{1}{2}x + \frac{1}{4}x \ast x + \frac{1}{8}(x \ast x) \ast x + \ldots\right] = \left\{\frac{1}{2}x, \frac{1}{2}1.x, \frac{1}{4}0.x, \frac{1}{8}01.x, \frac{1}{8}00.x, \ldots\right\}
\]
Our completeness proof works for any finitary signatures. It crucially relies on the fact that terms can be normalized by pushing infinite sums upwards using

\[(\sum_{i \in I} p_i \cdot t_i) \ast (\sum_{j \in J} q_j \cdot t_j) \leadsto \sum_{i \in I, j \in J} (p_i \cdot q_j) \cdot t_i \ast s_j\]

With countable signatures, this is not possible, e.g.

\[c(x_0, \frac{1}{2}x_0 + \frac{1}{2}x_1, \frac{1}{3}x_0 + \frac{1}{3}x_1 + \frac{1}{3}x_2, \ldots)\]

there is no normal form in this sense.

So, do we have completeness in this case? We still do not know.
THANK YOU FOR YOUR ATTENTION!
References


**Definition (Kleisli Triple)**

A Kleisli triple is a triple of the form $(T, \eta, -*)$ where

- $T$ sends sets to sets
- $\eta$ is a family of morphisms $\eta_X : X \to TX$, forming monad unit
- $(-)^*$ assigns to each $f : X \to TY$ a morphism $f^* : TX \to TY$

satisfying the laws: $\eta^* = \text{id}$, $f^* \eta = f$, $(f^* g)^* = f^* g^*$

This entails that $f : X \to TY$ and $g : Y \to TZ$ can be Kleisli composed to $g^* f : X \to TZ$, which yields Kleisli category $\text{Set}_T$

**Example**

- **Identity monad** $T = \text{Id} \quad \Rightarrow \quad \text{Set}_T = \text{Set}$
- **Powerset monad** $T = \mathcal{P} \quad \Rightarrow \quad \text{Set}_T = \text{Rel}$
- **Maybe-monad** $T = \text{Id} + 1 \quad \Rightarrow \quad \text{Set}_T$ – partial functions
- more generally, $T = \Sigma^* \quad \Rightarrow \quad \text{free monad over } \Sigma$
A functor $F: C \to D$ between monoidal $C$ and $D$ is **strong** if it comes with strength

$$\tau_{A,B}: A \otimes FB \to F(A \otimes B),$$

such that

$$I \otimes FX \cong FX$$

$$\tau\downarrow \cong \downarrow$$

$$F(I \otimes X) \cong FX$$

$$(X \otimes Y) \otimes FZ \overset{\tau}{\longrightarrow} F((X \otimes Y) \otimes Z)$$

$$\text{X} \otimes (Y \otimes FY) \overset{X \otimes \tau}{\longrightarrow} X \otimes F(Y \otimes Z) \overset{\tau}{\longrightarrow} F(X \otimes (Y \otimes Z))$$
A monad \((T, \eta, \mu)\) on \(C\) is strong if \(T\) and \(\eta, \mu\) are strong.
• Equivalently, $\Sigma$ is an endofunctor $\Sigma : \text{Set} \to \text{Set}$: $\Sigma X = \coprod_{i \in I} X^{n_i}$

• We can form the category of $\Sigma$-algebras with morphisms

$\Sigma A \xrightarrow{a} A$
$\Sigma h \downarrow \quad \downarrow h$
$\Sigma B \xrightarrow{b} B$

• Free object $\Sigma^* X$ over a set $X$ in this category is the same as an initial $(\Sigma + X)$-algebra

▶ $\Sigma^*$ extends to a functor
▶ $\Sigma^* \emptyset$ is carried by well-founded $\Sigma$-terms
▶ $\Sigma^* X$ is carried by well-founded $\Sigma$-terms with variables from $X$