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Kleene Monads in a Short While

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- Kleene iteration is iteration of regular expressions, e.g.

$$(0+1)^* \,;\, 0 \,;\, (0+1)^*$$

- Kleene algebra is a lightweight equational theory of (Kleene) iteration, complete over formal languages
- Extremely popular, has lots of extensions: hybrid, concurrent, stateful, etc
- Kleene monads is a simple categorification of Kleene algebras
- Elgot monads: deterministic, very general, highly compositional
  - But not quite that popular ☹

So, can we combine structure (Elgot) and power (Kleene)?

Previously: Unifying Elgot iteration and while-loops

Here: Categorical notion of iteration with nondeterminism

A Kleene algebra is a structure $(S, 0, 1, +, \,;\, , (-)^*)$, where

- $(S, 0, 1, +, \,;\,)$ is an idempotent semiring:
  - $(S, 0, +)$ is a commutative ($x + y = y + x$) and idempotent ($x + x = x$) monoid
  - $(S, 1, \,;\,)$ is a monoid
  - distributive laws:

$$x \,;\, (y + z) = x \,;\, y + x \,;\, z \qquad x \,;\, 0 = 0$$
$$(x + y) \,;\, z = x \,;\, z + y \,;\, z \qquad 0 \,;\, x = 0$$

  (thus, S is partially ordered: $x \leqslant y$ iff $x + y = y$)

- Kleene iteration satisfies $x^* = 1 + x \,;\, x^*$, and

$$\frac{x \,;\, y + z \leqslant y}{x^* \,;\, z \leqslant y} \qquad \frac{x + z \,;\, y \leqslant z}{x \,;\, y^* \leqslant z}$$

  Equivalently: $x^* \,;\, z$ is a least fixpoint of $x \,;\, (-) + z$ and $z \,;\, y^*$ is a least fixpoint of $(-) \,;\, y + z$

Intuition: 0 is a deadlock, 1 is a neutral program, $;$ is sequential composition, $+$ is non-deterministic choice

USE

- Regular expressions
- Algebraic language of finite state machines and beyond
- Relational semantics of programs
- Relational reasoning and verification, e.g. via dynamic logic
- Plenty of extensions:
  - modal $\Rightarrow$ modal Kleene algebra (Struth et al.)
  - stateful $\Rightarrow$ KAT + B! (Grathwohl, Kozen, Mamouras)
  - concurrent $\Rightarrow$ concurrent Kleene algebra (Hoare et al.)
  - nominal $\Rightarrow$ nominal Kleene algebra (Kozen et al.)
  - with differential equations $\Rightarrow$ differential dynamic logic (Platzer et al.)
  - etc., etc., etc.
- decidability and completeness properties (most famously w.r.t. formal languages and relational interpretations)

A minimalist extension is Kleene algebra with tests (KAT), which adds

- another Kleene algebra B of tests
- an operation-preserving inclusion $B \hookrightarrow S$
- complementation operator $\overline{(-)}\colon B \to B$, such that

$$\overline{a} + a = \top \qquad \overline{\overline{a}} = a \qquad \overline{a + b} = \overline{a} \,;\, \overline{b} \qquad \overline{0} = 1$$

(this makes B into a Boolean algebra)

This enables encodings

Branching	(if b then p else q)	as	$b \,;\, p + \overline{b} \,;\, q$
Looping	(while b do p)	as	$(b \,;\, p)^* \,;\, \overline{b}$
Hoare triples	$\{a\}\, p\, \{b\}$	as	$a \,;\, p \,;\, b = a \,;\, p$

In particular, we can embed deterministic semantics to non-deterministic semantics

A category is a Kleene-Kozen category if it has operations

- $0\colon \mathsf{Hom}(A, B)$
- $+\colon \mathsf{Hom}(A, B) \times \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, B)$
- $(-)^*\colon \mathsf{Hom}(A, A) \to \mathsf{Hom}(A, A)$

that together with identity (1) and composition ($;$) satisfy Kleene algebra laws

Fact 1

Kleene algebra is Kleene-Kozen category on one object

Fact 2[1]

Alternative axiomatization: laws of idempotent semirings, plus

$$1^* = 1 \qquad f^* = 1 + f \mathbin{;} f^* \qquad (f+g)^* = f^* \mathbin{;} (g \mathbin{;} f^*)^* \qquad \frac{h \mathbin{;} f = g \mathbin{;} h}{h \mathbin{;} f^* = g^* \mathbin{;} h}$$

[1] Goncharov, “Shades of Iteration: From Elgot to Kleene”, 2023.

A Kleene monad is a monad **T**, whose Kleisli category is a Kleene-Kozen category

Example (Powerset)

Powerset: **T** $= \mathcal{P}$, Kleisli morphisms $X \to \mathcal{P}Y$ = relations

Example (State Transformer)

If **T** is a Kleene monad, $(T(- \times S))^S$ yields a Kleene monad

Example (Writer Transformer)

If **T** is a (strong) Kleene monad and M is a monoid, $T(M \times \mathrm{Id})$ yields a Kleene monad. E.g. $\mathcal{P}(A^\star \times \mathrm{Id})$, "formal language monad" is Kleene

☹ This is pretty much it

Definition (Elgot monad)

A (complete) Elgot monad[2] in a category with binary coproducts (!) is a monad **T** equipped with an Elgot iteration operator

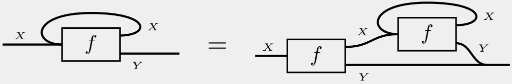
$$(-)^{\dagger}\colon \mathsf{Hom}(X, T(Y \oplus X)) \to \mathsf{Hom}(X, TY),$$

satisfying four laws: fixpoint, uniformily, naturality and codiagonal

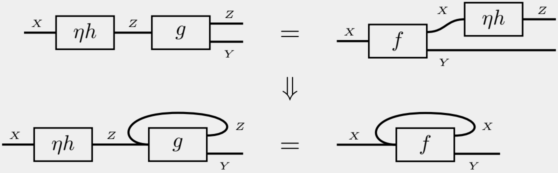
- Elgotness is robust and stable under many monad transformers
  - $T \mapsto T(M \times -)$ (writer)
  - $T \mapsto T(- \oplus E)$ (exception)
  - $T \mapsto (T(- \times S))^S$ (state)
  - $T \mapsto \nu\gamma.\, T(- \oplus H\gamma)$ (resumption)
- Laws go back to Elgot[3], except for uniformity

[2] Adámek, Milius, and Velebil, "Equational properties of iterative monads", 2010.
[3] Elgot, "Monadic Computation And Iterative Algebraic Theories", 1975.

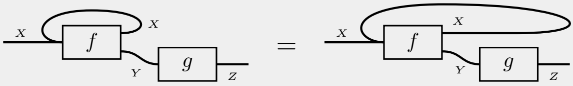
Fixpoint ($f\colon X \to T(Y \oplus X)$):



Uniformity ($f\colon X \to T(Z \oplus X)$, $g\colon Y \to T(Z \oplus Y)$, $h\colon X \to Y$):



Naturality ($f\colon X \to T(Y \oplus X)$, $g\colon Y \to TZ$):



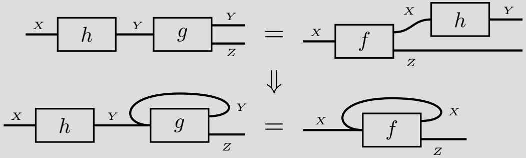
Codiagonal ($f\colon X \to T(Y \oplus (X \oplus X))$):

Naturality and **Codiagonal** are basically coherence laws

Theorem

A category $\mathbf{C}$ is a Kleene-Kozan category if

1. *$\mathbf{C}$ is enriched over bounded join-semilattices and strict join-preserving morphisms*
2. *$\mathbf{C}$ supports Elgot iteration that satisfies fixpoint, naturality, codiagonal and strong uniformity:*



where h is strict: $h\,;0 = 0$

3. **T** *satisfies the law* $(\mathsf{inl} + \mathsf{inr})^\dagger = 1$*, equivalently* $1^* = 1$

What we want:

- Single-category notion of iteration
- Robustness under monad transformers
- Nondeterminism
- $0 \,;\, f = 0$
- $(f + g) \,;\, h = f \,;\, h + g \,;\, h$

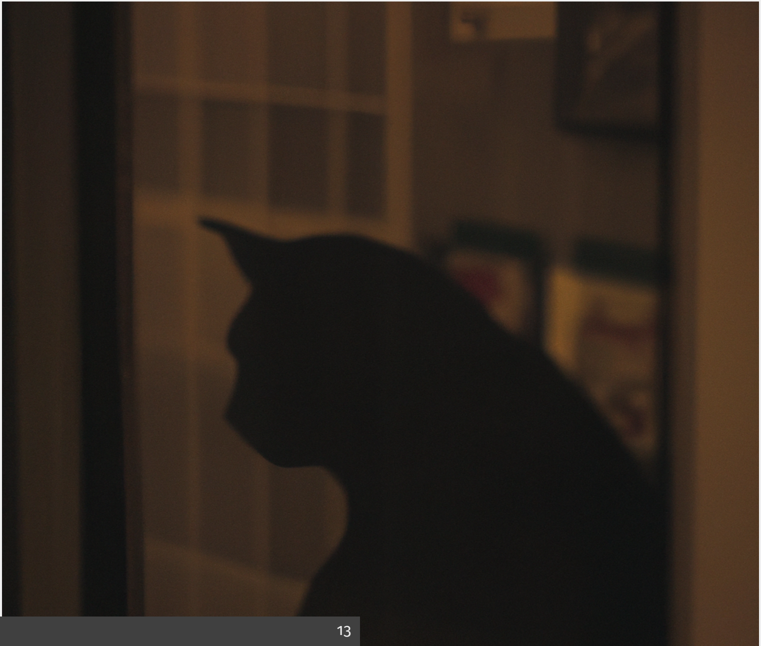
What we do not want:

- $f \,;\, 0 = 0$ *(because of* $\mathsf{raise}\ \mathsf{e}_1 = \mathsf{raise}\ \mathsf{e}_1 \,;\, 0 = \mathsf{raise}\ \mathsf{e}_2 \,;\, 0 = \mathsf{raise}\ \mathsf{e}_2$*)*
- $f \,;\, (g + h) = f \,;\, g + f \,;\, h$ *(because of process algebra semantics)*
- Coincidence of deadlock and divergence *(think of* $\mathcal{P}(\mathrm{Id} + 1)$*)*
- $1^* = 1$ *(again* $\mathcal{P}(\mathrm{Id} + 1)$*, but there are others)*

Problems

- Problem with coproducts
  - Can we make do without coproducts, sticking the the principle: “kleene-iteration-algebra” is a single-object “kleene-iteration-category”?
- Problem with tests
  - Instead of coproducts, Kleene algebra models control with tests. What are tests in our case?
- Problem with uniformity
  - Uniform w.r.t. what we need?

Picture of Cat



Problem with uniformity

Call maps w.r.t. which we want to be uniform tight

- Smallest candidate (Elgot monads): maps that factor through monad unit
- Largest candidate (Kleene monads): all maps
- Unless we demand tight maps to be strict ($f\,;0 = 0$), we can easily show that non-trivial exception monads fail uniformity

So, tight maps must be somewhere between

- The class generated from coproduct injections by copairing
- The class of all strict maps

Fact

There is an Elgot monad **T**, uniform w.r.t. strict morphisms, such that its exception monad transform is not

Call a category non-deterministic if the hom-sets are semi-lattices under $(0, +)$, and

$$0 \,;\, f = 0, \qquad (g + h) \,;\, f = g \,;\, f + h \,;\, f$$

In such a category, call f linear if for all g, h

$$f \,;\, 0 = 0, \qquad f \,;\, (g + h) = f \,;\, g + f \,;\, h$$

If there are binary coproducts, require coproduct injections to be linear

Theorem

Let **T** *be an Elgot monad with non-deterministic Kleisli category, and uniform w.r.t. linear maps.*

- *Every exception monad transform of* **T** *is uniform w.r.t. linear maps*
- *If* **T** *has the property* $f \,;\, T\,! = 0 \implies f = 0$ *then* $\nu\gamma.\, T(-\oplus H\gamma)$ *is uniform w.r.t. linear maps*

Problem with Tests

To model control with coproducts, it suffices to use decisions, i.e. morphisms of type $X \to X \oplus X$, e.g.

$$\frac{d \in \mathbf{C}(X, X \oplus X) \quad f, g \in \mathbf{C}(X, Y)}{\underline{\text{if}}\ d\ \underline{\text{then}}\ f\ \underline{\text{else}}\ g = d\,;[g, f] \in \mathbf{C}(X, Y)}$$

Theorem

Let $\mathbf{C}$ be a non-deterministic category with binary coproducts; for every X, let $\mathbf{C}^\flat(X) \subseteq \mathbf{C}(X, X)$ be a Boolean algebra under $;$ and $+$

1. *Maps $\Diamond\colon \mathbf{C}^\flat(X) \to \mathbf{C}(X, X \oplus X)$, $?\colon \mathbf{C}(X, X \oplus X) \to \mathbf{C}^\flat(X)$:*

   $$\Diamond b = \bar{b}\,;\mathsf{inl} + b\,;\mathsf{inr} \qquad\qquad d? = d\,;[0, 1]$$

   form a retraction
2. *Every d in the image of $\Diamond$ is linear*
3. $(\underline{\text{if}}\ e\ \underline{\text{then}}\ \mathsf{inr}\ \underline{\text{else}}\ d)? = d? + e?$, $(\underline{\text{if}}\ e\ \underline{\text{then}}\ d\ \underline{\text{else}}\ \mathsf{inl})? = e?\,;d?$
4. *For every d in the image of $\Diamond$,* $(\underline{\text{if}}\ d\ \underline{\text{then}}\ \mathsf{inl}\ \underline{\text{else}}\ \mathsf{inr})? = \overline{d?}$

Proposition

Let

- $\mathbf{C}$ be a non-deterministic category
- $\mathbf{D}$ be a wide subcategory of $\mathbf{C}$ with coproducts preserved by inclusion
- $\mathbf{C}^\flat(X) \subseteq \mathbf{D}(X, X)$ for all X
  - form Boolean algebras under $;$ and $+$
  - contain $[\mathsf{inl}, 0]$ and $[0, \mathsf{inr}]$ whenever $X = X_1 \oplus X_2$.

Then $\mathbf{C}$ supports uniform w.r.t. $\mathbf{D}$ Elgot iteration iff it supports

$$\frac{b \in \mathbf{C}^\flat(X) \qquad f \in \mathbf{C}(X, X)}{\mathsf{while}\, b\, \mathsf{do}\, f \in \mathbf{C}(X, X)}$$

such that

$$\mathsf{while}\, b\, \mathsf{do}\, f = \mathsf{if}\ b\ \mathsf{then}\ f\,;(\mathsf{while}\, b\, \mathsf{do}\, f)\ \mathsf{else}\ 1$$

$$\mathsf{while}\,(b \vee c)\,\mathsf{do}\, f = (\mathsf{while}\, b\, \mathsf{do}\, f)\,;\mathsf{while}\, c\, \mathsf{do}\,(f\,;\mathsf{while}\, b\, \mathsf{do}\, f)$$

$$\frac{u\,;\bar{b} = \bar{c}\,;v \qquad u\,;b\,;f = c\,;g\,;u}{u\,;\mathsf{while}\, b\, \mathsf{do}\, f = (\mathsf{while}\, c\, \mathsf{do}\, g)\,;v} \qquad (u, v \in \mathbf{D})$$

A triple $(\mathbf{C}, \mathbf{C}^\ell, \mathbf{C}^\flat)$ is Kleene-iteration category with tests (KiCT) if

- $\mathbf{C}$ is a non-deterministic category
- $\mathbf{C}^\ell$ is a wide subcategory of tight morphisms, which are all linear
- $\mathbf{C}^\flat$ is a family $(\mathbf{C}^\flat(X) \subseteq \mathbf{C}^\ell(X, X))_{X \in |\mathbf{C}|}$ of tests where every $\mathbf{C}^\flat(X)$ is a Boolean algebra under $\,;\,$ and $+$
- for every $X \in |\mathbf{C}|$ there is $(-)^* \colon \mathbf{C}(X, X) \to \mathbf{C}(X, X)$ such that

$$(*\textbf{-Fix})\quad f^* = 1 + f \,;\, f^* \qquad\qquad (*\textbf{-Or})\quad (f + g)^* = f^* \,;\, (g \,;\, f^*)^*$$

$$(*\textbf{-Uni})\quad \frac{u \,;\, f = g \,;\, u}{u \,;\, f^* = g^* \,;\, u} \qquad (u \in \mathbf{C}^\ell)$$

PLUS (!) an unidentified set of principles like (*-**Uni**)

Fact 1

A Kleene algebra with tests is precisely a single-object Kleene-iteration category with tests, such that all morphisms are linear and $1^* = 1$

Fact 2

Let $\mathbf{C}$, $\mathbf{C}^\ell$ and $\mathbf{C}^\flat$ be as follows:

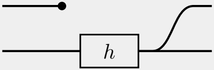
- $\mathbf{C}$ is a non-deterministic category with binary coproducts (!)
- $\mathbf{C}^\ell$ is a wide subcategory of $\mathbf{C}$ with binary coporoducts, consisting of linear morphisms only, such that the inclusion preserves coproducts
- For every $X \in |\mathbf{C}|$, $\mathbf{C}^\flat(X) \subseteq \mathbf{C}^\ell(X, X)$
  - forms a Boolean algebra under $;$ and $+$
  - contains $[\mathsf{inl}, 0]$ and $[0, \mathsf{inr}]$ whenever $X = X_1 \oplus X_2$

Then $(\mathbf{C}, \mathbf{C}^\ell, \mathbf{C}^\flat)$ is a KiCT iff $\mathbf{C}$ supports Elgot iteration uniform w.r.t. $\mathbf{C}^\ell$

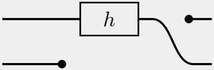
- Unfortunately, the coproduct problem is deeper
- For example, we seem to require coproducts to prove identities like

$$h^* = (h\,;(h+1))^*$$

- Basically, because we can go 2d and instantiate f and g in $(f+g)^* = f^*\,;(g\,;f^*)^*$ with



and



- An ultimate yardstick for a correct axiomatization could be a completeness result

Towards Completeness

Kozen's Completeness

- Kozen[4] showed completeness of Kleene algebra w.r.t. regular events (=regular languages)
- In fact, he did more: he showed that for a fixed signature Σ, regular languages **Reg**(Σ) is a free Kleene algebra on Σ, hence

$$[\![p]\!] = [\![q]\!] \implies \vdash p = q$$

where $[\![p]\!]$ is the language, generated by p

- Idea of the proof: given $[\![p]\!] = [\![q]\!]$, arrange a series of equivalent non-deterministic automata $A_1, \ldots, A_n$, such that

$$\begin{array}{ccccccc} A_1 & \sim & A_2 & \sim & A_3 & \cdots & A_n \\ \uparrow\downarrow & & \downarrow & & \downarrow & & \downarrow \\ p = p_1 & \sim & p_2 & \sim & p_3 & \cdots & p_n = q \end{array}$$

- Key technical step: for any Kleene algebra A, matrices $\mathbf{Mtx}_n(A)$ of size $n \times n$ over A again form a Kleene algebra

[4] Kozen, "A completeness theorem for Kleene algebras and the algebra of regular events", 1994.

Given a Kleene-Kozen category $\mathbf{C}$, let $\mathbf{Mtx}(\mathbf{C})$ be as follows:

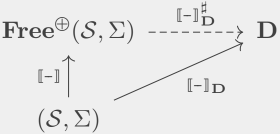
- Objects are non-empty lists $\langle A_1, \ldots, A_n \rangle$ of objects of $\mathbf{C}$
- a morphism $f \colon \langle A_1, \ldots, A_n \rangle \to \langle B_1, \ldots, B_m \rangle$ in $\mathbf{Mtx}(\mathbf{C})$ is given by a family $\langle f_{i,j} \colon A_i \to B_j \rangle_{i \leqslant n, j \leqslant m}$ of morphisms in $\mathbf{C}$
- the identity morphism over $\langle A_1, \ldots, A_n \rangle$ is the family $\langle \delta_{i,j} \rangle_{i,j \leqslant n}$ where $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$
- morphism composition as for matrices: given $f \colon \langle A_1, \ldots, A_n \rangle \to \langle B_1, \ldots, B_m \rangle$ and $g \colon \langle B_1, \ldots, B_m \rangle \to \langle C_1, \ldots, C_k \rangle$,

$$(f \mathbin{;} g)_{i,j} = f_{i,1} \mathbin{;} g_{1,j} + \ldots + f_{i,m} \mathbin{;} g_{m,j}$$

Fact

$\mathbf{Mtx}(\mathbf{C})$ is a Kleene-Kozen category with strict binary coproducts, and $\mathbf{C}$ fully embeds to it

- Fix
  - Set of sorts $\mathcal{S}$, and
  - Signature Σ of symbols, together with types of the form $A \to B$
- These can be interpreted in any Kleene-Kozen category $\mathbf{D}$: $[\![A]\!] \in |\mathbf{D}|$, $[\![f\colon A \to B]\!] \in \mathbf{D}([\![A]\!], [\![B]\!])$
- Free Kleene-Kozen category with coproducts $\mathbf{Free}^{\oplus}(\mathcal{S}, \Sigma)$ over $(\mathcal{S}, \Sigma)$ is characterized by the universal properly:



- Objects of $\mathbf{Free}^{\oplus}(\mathcal{S}, \Sigma)$ are lists of sorts $\langle A_1, \ldots, A_n \rangle$ from $\mathcal{S}$
- A morphism from $\langle A_1, \ldots, A_n \rangle$ to $\langle B_1, \ldots, B_m \rangle$ is a tuple $\langle t_i^{A_i, B_1, \ldots, B_m} \rangle_{i \leqslant n}$ of rational trees, specified with the grammar:

  $$t^{A, B_1, \ldots, B_m} ::= \left\langle o \subseteq \{i \in \mathbb{N} \mid A = B_i\}, \langle t^{C, B_1, \ldots, B_m} \rangle_{f\colon A \to C \in \mathcal{S}} \right\rangle$$

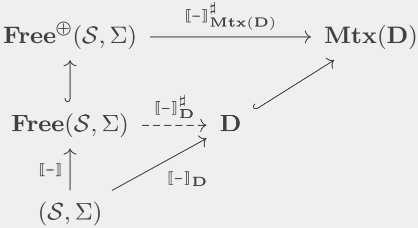
  where “rational” means:
  - infinite
  - number of subtrees are finite
- Intuitively, a rational tree $t^{A, B_1, \ldots, B_m}$ represents the behaviour of a deterministic automaton with m kinds of acceptance, with states classified by sorts, such that an action from q_1 of sort A to q_2 of sort B must be some $f\colon A \to B \in \Sigma$

Constructing $\mathbf{Free}^{\oplus}(\mathcal{S}, \Sigma)$

$$t^{A,B_1,\dots,B_m} ::= \left\langle o \subseteq \{i \in \mathbb{N} \mid A = B_i\}, \langle t^{C,B_1,\dots,B_m} \rangle_{f\colon A \to C \in \mathcal{S}} \right\rangle$$

- Sum of rational trees is computed (co-)recursively, by joining the o's
- 0 is the tree with all the o's empty
- $;$ is a bit tricky: it is like substitution, combined with $+$
- $(–)^{\dagger}$ is defined as a least fixpoint (iteration laws follow)
- To define $[\![–]\!]^{\sharp}_{\mathbf{D}} \colon \mathbf{Free}^{\sharp}(\mathcal{S}, \Sigma) \to \mathbf{D}$ we need to fold trees into finite expressions and use uniformity to show that $[\![–]\!]^{\sharp}_{\mathbf{D}}$ is structure preserving

- Define $\mathbf{Free}(\mathcal{S},\Sigma) \hookrightarrow \mathbf{Free}^{\oplus}(\mathcal{S},\Sigma)$ as a full subcategory on $\mathcal{S}$
- Let $\mathbf{D}$ be any Kleene-Kozen category, which interprets $(\mathcal{S},\Sigma)$ by $[\![-]\!]_{\mathbf{D}}$
- The composition of $[\![-]\!]^{\sharp}_{\mathbf{Mtx}(\mathbf{D})}$ with $\mathbf{Free}(\mathcal{S},\Sigma) \hookrightarrow \mathbf{Free}^{\oplus}(\mathcal{S},\Sigma)$ factors though the inclusion of $\mathbf{D}$ to $\mathbf{Mtx}(\mathbf{D})$, yielding $[\![-]\!]^{\sharp}_{\mathbf{D}} : \mathbf{Free}(\mathcal{S},\Sigma) \to \mathbf{D}$:



The general problem of organizing the totality of valid laws of iteration is solved by one of few known feats:

- by assuming linearity globaly (Kozen)
- by relying on coproducts (Bloom and Ésik)
- "coalgebraic" approach[5], relying on uniqueness of (some) fixpoints

The case of general Kleene-iteration categories with tests is open

[5] Salomaa, "Two Complete Axiom Systems for the Algebra of Regular Events", 1966.

Questions?

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Monads

Definition (Monad)

A monad **T** (on a category $\mathbf{C}$) is given by a Kleisli triple $(T, 1, \text{–}^\star)$ where

- $T\colon |\mathbf{C}| \to |\mathbf{C}|$
- 1 is a family of morphisms $1_X : X \to TX$, forming monad unit
- $(\text{–})^\star$ assigns to each $f\colon X \to TY$ a morphism $f^\star\colon TX \to TY$

satisfying the laws: $1^\star = 1$, $f^\star\, 1 = f$, $(f^\star\, g)^\star = f^\star\, g^\star$

This entails that

- T is a functor, 1 is a natural transformation
- $f\colon X \to TY$ and $g\colon Y \to TZ$ (regarded as programs) can be Kleisli composed to $f \,;\, g = g \cdot f = g^\star f\colon X \to TZ$

By varying **T** we obtain various 'generalized programs' $f\colon X \to TY$ while programs of the form $1f$ can be seen as 'pure programs'

Example: $T =$ powerset $\Rightarrow$ generalized programs = non-deterministic programs, pure programs = deterministic programs = functions