Towards Constructive Hybrid Semantics

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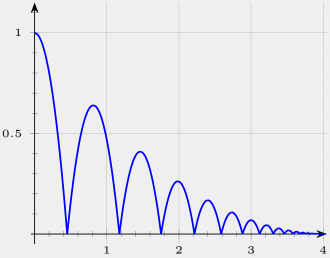
Hybrid Programming: Bouncing Ball

Bouncing ball is a simple Newtonian system specified by differential equation $\ddot{h} = -g$ ($g \approx 9.8$) whose solution is

$$h(t) = h_0 + v_0 t - \frac{gt^2}{2}$$

with initial values:

- $v_0 = 0, h_0 \neq 0$ (peak height)
- $h_0 = 0, v_0 \neq 0$ (zero height)



Features:

- deterministic
- hybrid: the velocity changes discretely at the bottom $v \mapsto -cv$, but it changes continuously in the meanwhile
- progressive: every iteration consumes non-zero time
- Zeno behaviour: the state of rest is only reachable in the limit

damping factor

Bouncing ball can be formalized in idealized language HybCore[1]:

$$
\begin{aligned}
&x := [\![(1,0)]\!]\ \textbf{while true}\\
&\{\\
&\qquad (h,v) := (x := t.\, ball(x,t)\ \&\ \mathsf{fst}\, x \geqslant 0);\\
&\qquad [\![(h,-c\,v)]\!]\\
&\}
\end{aligned}
$$

Here, $ball(a,b,t)$ is the solution of the initial value problem $\{\dot h = v, \dot v = -g, h(0) = a, v(0) = b\}$

The critical element of the semantics is Elgot iteration: $(f : X \to T(Y \uplus X)) \mapsto (f^{\dagger} : X \to TY)$ for a suitable monad T

[1]Sergey Goncharov and Renato Neves, An adequate while-language for hybrid computation (PPDP 2019)

Context and Motivation

We investigate hybrid semantics and its generalizations from the categorical and type-theoretic perspectives

Relevant preceding work:

- Modelling partiality in intentional type theory under countable choice: *James Chapman, Tarmo Uustalu, and Niccolò Veltri. Quotienting the delay monad by weak bisimilarity (ICTAC 2015)*
- Modelling partiality as a quotient inductive-inductive type: *Thorsten Altenkirch, Nils Danielsson, and Nicolai Kraus. Partiality, revisited (FOSSACS 2017)*

Here we generalize by rendering hybridness as partiality extended over time

Our main construction is implemented as a quotient inductive-inductive type in cubical Agda[2] and is available under:

`https://github.com/sergey-goncharov/hybrid-agda`

Many insights were borrowed from the preceding implementation of the partialiy monad by Danielsson (`http://www.cse.chalmers.se/~nad/listings/partiality-monad`)

[2]Andrea Vezzosi, Anders Mörtberg, and Andreas Abel. Cubical Agda: A dependently typed programming language with univalence and higher inductive types (ICFP 2019)

Outline

Hybrid semantics, non-constructively

Categorical abstraction

Characterizing classical semantics

Some details of Agda formalization

Conclusions

Hybrid Systems, Non-Constructively

Definition (Monad)

A monad T (on **Set**) is given by a Kleisli triple $(T, \eta, -^\star)$ where

- T sends sets to sets
- η is a family of morphisms $\eta_X : X \to TX$, forming monad unit
- $(-)^\star$ assigns to each $f : X \to TY$ a morphism $f^\star : TX \to TY$

satisfying the laws: $\eta^\star = \mathrm{id}$, $f^\star\,\eta = f$, $(f^\star\,g)^\star = f^\star\,g^\star$

This entails that $f : X \to TY$ and $g : Y \to TZ$ (regarded as programs) can be Kleisli composed to $g^\star f : X \to TZ$

Definition (Elgot Monad)

T with an iteration operator $(-)^\dagger : (X \to T(Y \uplus X)) \to (X \to TY)$, satisfying axioms of iteration (omitted) is called Elgot

Duration Semantics and Evolution Semantics

We distinguish

- Duration semantics: $TX = \mathbb{R}_+ \times X \cup \bar{\mathbb{R}}_+$ – a computation either converges in finite time with a value in X, or diverges in possibly infinite time ($\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$)
- Evolution semantics: $TX = S^{[0,\mathbb{R}_+)} \times X \cup S^{[0,\bar{\mathbb{R}}_+)}$ where $S^{[0,\mathbb{R}_+)} = \Sigma_{d:\,\mathbb{R}_+}\,[0,d) \to S$ is the space of finite trajectories and $S^{[0,\bar{\mathbb{R}}_+)} = \Sigma_{d:\,\bar{\mathbb{R}}_+}\,[0,d) \to S$ is the space of possibly infinite trajectories over S

Idea for Abstraction: Vault to general monoids instead of concrete $\mathbb{R}_+$, $S^{[0,\mathbb{R}_+)}$

Generalized Writer Monad

Fix a (not necessarily commutative) monoid $(\mathbb{M}, +, 0)$

A monoid $\mathbb{M}$-module is a set $\mathbb{E}$ equipped with a map $\triangleright : \mathbb{M} \times \mathbb{E} \to \mathbb{E}$, subject to the laws

$$0 \triangleright e = e \qquad (m + n) \triangleright e = m \triangleright (n \triangleright e)$$

Every monoid-module pair $(\mathbb{M}, \mathbb{E})$ induces the generalized writer monad: $T = \mathbb{M} \times (-) \cup \mathbb{E}$

For example, with $\mathbb{M} = \mathbb{E} = 1$ we obtain the maybe monad $(-) \cup \{\bot\}$, which is incidentally an Elgot monad

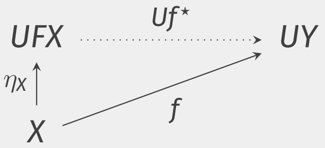
Problem: Elgotness relies on the law of excluded middle!

Categorical Construction

Free Objects

We extend previous construction of partiality monad ($\mathbb{M} = 1$) [3]

Recall the concept of free object: Given a functor $U : \mathbf{C} \to \mathbf{Set}$, an object FX is free on X if there is $(\eta_X : X \to UFX)_X$ and for every $f : X \to UY$ there is unique $f^\star : FX \to Y$ such that



Standard facts:

- All free objects FX exist iff F is a left adjoint to U
- If all free objects exist then $(UF, \eta, (-)^\star)$ constitutes a monad

[3] Thorsten Altenkirch, Nils Danielsson, and Nicolai Kraus. Partiality, revisited – the partiality monad as a quotient inductive-inductive type

We assume $\mathbb{M}$ to be ordered with 0 being the bottom and right monotone $+$

Examples: 1, $\mathbb{N}$, $\mathbb{Q}_+$, $\mathbb{R}_+$, $S^{[0,\mathbb{R}_+)}$, S^* (for last two $+$ is neither commutative, nor left monotone)

Definition (Complete $\mathbb{M}$-Modules)

An ordered $\mathbb{M}$-module is additionally equipped with a partial order $\sqsubseteq$ and $\bot$, such that

$$\frac{}{\bot \sqsubseteq x} \qquad \frac{x \sqsubseteq y}{a \triangleright x \sqsubseteq a \triangleright y} \qquad \frac{a \leqslant b}{a \triangleright \bot \sqsubseteq b \triangleright \bot}$$

An ordered $\mathbb{M}$-module is complete if for any directed $(s_i)_i$ on $\mathbb{E}$ there is a least upper bound $\bigsqcup_i s_i$ and

$$\bigsqcup_i a \triangleright s_i \sqsubseteq a \triangleright \bigsqcup_i s_i$$

Theorem

Let $\mathbf{Alg}_{\widetilde{L}}$ *be the category of complete* $\mathbb{M}$*-modules and* $U\colon \mathbf{Alg}_{\widetilde{L}} \to \mathbf{Set}$ *be the obvious forgetful functor*

1. *All free objects w.r.t.* U *exist, yielding a monad* $\widetilde{L}$
2. $\widetilde{L}$ *is enriched over directed complete partial orders, and moreover, Kleisli composition is strict on both sides*
3. $\widetilde{L}$ *is an Elgot monad with the iteration operator* $(f\colon X \to \widetilde{L}(Y \uplus X))^{\dagger}$ *calculated as a least fixed point of the map* $[\eta, -]^{\star} f\colon (X \to \widetilde{L}Y) \to (X \to \widetilde{L}Y)$

This is true both classically and constructively, thanks to quotient inductive-inductive types[4]

[4]Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, Nicolai Kraus, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types.

$\widetilde{L}$ VIA DIRECTED COMPLETION

Let $\mathbb{M}_X = \mathbb{M} \times (X \uplus \{\bot\})$, and define $\triangleright_X$, $\sqsubseteq_X$ with rules:

$$\frac{}{a \triangleright_X (b, p) = (a + b, p)} \qquad \frac{}{(a, \mathrm{inl}\, p) \sqsubseteq_X (a, \mathrm{inl}\, p)} \qquad \frac{a \leqslant b}{(a, \mathrm{inr}\, \bot) \sqsubseteq_X (b, p)}$$

Lemma

For any set X, $(\mathbb{M}_X, \triangleright_X, (0, \mathrm{inr}\, \bot), \sqsubseteq_X)$ *is an ordered* $\mathbb{M}$*-module.*

Intuitively, we are interested in limits of directed sequences over $\mathbb{M}_X$, e.g.

- convergent: $(1, \mathrm{inr}\, \bot) \sqsubseteq_X \ldots \sqsubseteq_X (n, \mathrm{inr}\, \bot) \sqsubseteq_X (n+1, \mathrm{inl}\, 0)$
- divergent: $(1, \mathrm{inr}\, \bot) \sqsubseteq_X \ldots \sqsubseteq_X (1, \mathrm{inr}\, \bot) \sqsubseteq_X \ldots$
- Zeno: $(1/2, \mathrm{inr}\, \bot) \sqsubseteq_X \ldots \sqsubseteq_X (n/(n+1), \mathrm{inr}\, \bot) \sqsubseteq_X \ldots$

Directed Completion

- $(s_i)_i \lesssim_X (t_i)_i \quad \text{if} \quad \forall i : \mathbb{N}.\, \exists j : \mathbb{N}.\, s_i \sqsubseteq_X t_j$
- $s \sim_X t \quad \text{if} \quad s \lesssim_X t \quad \text{and} \quad t \lesssim_X s$
- $\tilde{\mathbb{M}}_X$ is the quotient under $\sim_X$ of the set of directed sequences over $\mathbb{M}_X$ and write $[s_i]_i$ instead of $[(s_i)_i]_\sim$

Theorem (Only Classically!)

$(\tilde{\mathbb{M}}_X, \triangleright_X, \bot_X, \lessapprox_X, \tilde{\tilde{\bigvee}}_X)$ is a free complete $\mathbb{M}$-module on X with

$$[s]_\sim \lessapprox_X [t]_\sim \ \textit{if}\ s \lesssim_X t \qquad a \triangleright_X [s_i]_i = [a \triangleright_X s_i]_i$$

$$\bot_X = [(0, \mathrm{inr}\, \bot)]_i \qquad \overset{\approx}{\bigvee}_{X\, i} [s_{i,j}]_j = [s_{\pi_1^{-1}(i), \pi_2^{-1}(i)}]_i$$

where $\pi(x, y) = \frac{1}{2}(x + y)(x + y + 1) + x$

Cantor pairing function

Let us write $\tilde{\mathbb{M}}_\emptyset$ as $\tilde{\mathbb{M}}$

Theorem (Only Classically!)

- *$\tilde{L}X$ and $\tilde{\mathbb{M}}_X$ are isomorphic as complete $\mathbb{M}$-modules*
- *$\tilde{L}X \cong \mathbb{M} \times X \cup \tilde{\mathbb{M}}$ – generalized writer monad over $(\mathbb{M}, \tilde{\mathbb{M}})$*

Examples:

- For $\mathbb{M} = 1$, $\tilde{L}X = X \cup \{\bot\}$
- For $\mathbb{M} = \mathbb{N}$, $\tilde{L}X = \mathbb{N} \times X \cup \bar{\mathbb{N}}$
- **But** for $\mathbb{M} = \mathbb{R}_+$, $\tilde{L}X = \mathbb{R}_+ \times X \cup \tilde{\mathbb{R}}_+$ where $\tilde{\mathbb{R}}_+ \cong \mathbb{R}_+ \uplus (\mathbb{R}_+ \setminus \{0\}) \uplus \{\infty\} \cong \bar{\mathbb{R}}_+ \uplus (\mathbb{R}_+ \setminus \{0\})$, because of Zeno behaviour!

Conservatively Complete Modules

A conservatively complete $\mathbb{M}$-module additionally satisfies

$$\bigsqcup_i a_i \triangleright \bot = \left(\bigvee_i a_i\right) \triangleright \bot$$

whenever $\bigvee_i a_i$ exists

- Again, we obtain an Elgot monad $\bar{L}$ by the same argument
- There is $\bar{\mathbb{M}}_X$ – an analogue of $\tilde{\mathbb{M}}_X$
- Again, $\bar{L}X \cong \mathbb{M} \times X \cup \bar{\mathbb{M}}$
- $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$, $\overline{S^{[0,\mathbb{R}_+)}} \cong S^{[0,\bar{\mathbb{R}}_+)}$, etc

Analogously, we introduce free conservatively complete $\mathbb{M}$-modules $\bar{L}X$ on X, and obtain an Elgot monad $\bar{L}$

MLTT and HoTT

In Martin-Löf Type Theory:

Formula	Type
$\top$	$\mathbf{1}$
$\bot$	$\mathbf{0}$
$A \wedge B$	$A \times B$
$A \vee B$	$A \uplus B$
$\forall x \to A(x)$	$\prod_{x\,:\,I} A(x)$
$\exists[\, x \,]\, A(x)$	$\sum_{x\,:\,I} A(x)$

Some examples:

- A is a (mere) proposition: IsProp A = $\forall\ (x\ y : A) \to x \equiv y$
- A is a set: IsSet A = $\forall\ (x\ y : A) \to$ IsProp $(x \equiv y)$
- A is decidable: IsDec A = $A \vee \neg A$

In HoTT and in Cubical Agda (which implements HoTT), we can express properly more, e.g. propositional truncation as a quotient inductive type:

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data ||_|| (A : Set ℓ) : Set ℓ where
  |_| : A → || A ||
  ||||-prop : IsProp || A ||
```

Axiom of Countable Choice:

$$AC\omega\ \{\ell\} = \forall\ (P : \mathbb{N} \to \mathsf{Set}\ \ell) \to (\forall\ n \to \|\ P\ n\ \|) \to \|\ (\forall\ n \to P\ n)\ \|$$

Implementing $\tilde{L}$ and $\bar{L}$ requires more sophisticated quotient inductive-inductive types

Notions of Completeness

Chains, intensionally and extensionally directed sequences:

$$\mathsf{Inc}\ \sigma = \forall\ (n : \mathbb{N}) \to \sigma\ n \leqslant \sigma\ (\mathsf{suc}\ n)$$
$$\mathsf{Dir}\ \sigma = \forall\ (n\ m : \mathbb{N}) \to \exists[\ k\ ]\ (\sigma\ n \leqslant \sigma\ k \wedge \sigma\ m \leqslant \sigma\ k)$$
$$||\mathsf{Dir}||\ \sigma = \forall\ (n\ m : \mathbb{N}) \to ||\ \exists[\ k\ ]\ (\sigma\ n \leqslant \sigma\ k \wedge \sigma\ m \leqslant \sigma\ k)\ ||$$

Theorem

Let (a), (b) and (c) stand for completeness of a fixed set A w.r.t. ||Dir||, Dir and Inc correspondingly. Then

- *(a) ⇒ (b) ⇒ (c)*
- *(b) ⇒ (a) under countable choice*
- *(c) ⇒ (a) under the decidability of* $\leqslant$ *on A (i.e. under* $\forall\ (x\ y : A) \to \mathit{IsDec}\ (x \leqslant y)$*)*

Our $\tilde{L}$ and $\bar{L}$ are based on intensional completeness

Further Work

Further Work

- Implement the concept of free objects in cubical Agda
- Is it possible to rebase $\widetilde{L}$ and $\bar{L}$ on extensional completeness (currently based on intensional completeness)? Which approach would be the right one if both are possible?
- Implement classical characterizations of $\widetilde{L}$ and $\bar{L}$ in cubical Agda (possible under countable choice?)
- Further flavours of hybrid semantics, e.g. non-deterministic