Unguarded Recursion on Coinductive Resumptions [1]

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Abstract

We study a model of side-effecting processes obtained by adjoining free operations to a monad modelling base effects by means of a cofree coalgebra construction; one thus arrives at what one may think of as types of non-wellfounded side-effecting trees, generalizing the infinite resumption monad. Types of this kind have received some attention in the recent literature; in particular, it has been shown that they admit guarded iteration. Here, we show that they also admit unguarded iteration, i.e. form complete Elgot monads, provided that the underlying base effect supports unguarded iteration.

Keywords: Recursion, coalgebra, coinduction, complete Elgot monad, resumptions.

1 Introduction

Following seminal work by Moggi [17], monads are widely used to represent computational effects in program semantics, and in fact in actual programming languages [28]. Their main attraction lies in the fact that they provide an interface to a generic notion of side-effect at the right level of abstraction: they subsume a wide variety of side-effects such as state, non-determinism, random, and I/O, and at the same time retain enough internal structure to support a substantial amount of generic meta-theory and programming, the latter witnessed, for example, by the monad class implemented in the Haskell basic libraries [19].

In the current work, we study a particular construction on monads motivated partly by the goal of modelling generic side-effects in the semantics of reactive processes. Specifically, given a base monad T and objects (types) a, b, we have, assuming enough structure on T and the base category, a family of final coalgebras

$$T_a^b X = \nu\gamma.\, T(X + a \times \gamma^b)$$

for each object X. These final coalgebras can be seen as arising in two ways: on the one hand, one may start from reactive processes sending messages of type a and

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receiving messages of type b (possibly terminating with results of type X), modelled as non-wellfounded a-labelled b-branching trees (with leaves labelled in X), i.e. inhabitants of $\nu\gamma.\,(X + a \times \gamma^b)$, and then add generic side-effects encapsulated by T to the model (e.g. non-determinism or access to a global shared memory). On the other hand, one may see a and b as the types of an uninterpreted side-effect $f : a \to b$ added to the base monad T, e.g. an I/O-operation (in fact, the interactive input and output monads originally considered as examples by Moggi [17] can be seen as generated by uninterpreted effects of this kind); if one wishes to model non-terminating programs that use f as well as side-effects from T, one obtains infinite trees of exactly the kind given by $T_a^b X$. The construction of $T_a^b X$ from T is an infinite version of the generalized resumption transformer introduced by Cienciarelli and Moggi [9]. It has been termed the *coalgebraic generalized resumption* transformer by Piróg and Gibbons [20] (later generalized further [21]), who show that on the Kleisli category of T, T_a^b is the free completely iterative monad generated by $T(a \times \_^b)$.

The result that T_a^b is a completely iterative monad brings us to the contribution of the current paper. Recall that complete iterativity of T_a^b means that for every morphism

$$e : X \to T_a^b(Y + X),$$

read as an equation defining the inhabitants of X, thought of as variables, as terms over the defined variables (from X) and parameters from Y, has a unique *solution*

$$e^\dagger : X \to T_a^b Y$$

in the evident sense, *provided* that e is *guarded*. The latter concept is defined in terms of additional structure of T_a^b as an *idealized monad*, which essentially allows distinguishing terms beginning with an operation from mere variables. Guardedness of e then means essentially that recursive calls can happen only under a free operation. Similar results on guarded recursion abound in the literature; for example, the fact that T_a^b admits guarded recursive definitions can also be deduced from more general results by Uustalu on parametrized monads [27].

The central result of the current paper is to remove the guardedness restriction in the above setup. That is, we show that a solution $e^\dagger : X \to T_a^b Y$ exists for *every* morphism $e : X \to T_a^b(X + Y)$. Of course, the solution is then no longer unique (for example, we admit definitions of the form $x = x$); moreover, we clearly need to make additional assumptions about T. Our result states, more precisely, that T_a^b allows for a principled *choice* of solutions $e^\dagger$ satisfying standard equational laws for recursion [25], thus making T_a^b into a *complete Elgot monad* [3] [2]. The assumption on T that we need to enable this result is that T itself is an Elgot monad (e.g. partiality, nondeterminism, or combinations of these with state), i.e. we show that *the class of Elgot monads is stable under the coinductive generalized resumption transformer*. We show moreover that the structure of T_a^b as an Elgot monad is uniquely determined as extending that of T.

[2] We vary the original definition of Elgot monads, which requires the object X of variables to be a finitely presentable object in an lfp category, by admitting unrestricted objects of variables; this change is explicitly not an important part of our contribution, and presumably not central to the technical development although we have not checked details in the finitary case.

The motivation for these results is, well, to free non-wellfounded recursive definitions from the standard guardedness constraint. Note for example that in [20], it was necessary to assume guards in all loop iterations when interpreting a while-language with actions originally proposed by Rutten [24] over a completely iterative monad. Contrastingly, given that T_a^b is a (complete) Elgot monad, one can now just write unrestricted while loops. We elaborate this example in Section 5, and recall a standard example of unguarded recursion in process algebra in Section 6.

2 Preliminaries

According to Moggi [17], a notion of computation can be formalized as a strong monad $\mathbb{T}$ over a Cartesian category $\mathbf{C}$ (i.e. a category with finite products). In order to support the constructions occurring in the main object of study, here we work in a *distributive category* $\mathbf{C}$, i.e. a category with finite products and coproducts (including a final and an initial object) and such that the natural transformation

$$X \times Y + X \times Z \xrightarrow{[\mathsf{id} \times \mathsf{inl}, \mathsf{id} \times \mathsf{inr}]} X \times (Y + Z)$$

is an isomorphism [10], whose inverse we denote $\mathsf{dist}_{X,Y,Z}$. Here we denote injections into binary coproducts by $\mathsf{inl} : A \to A + B$, $\mathsf{inr} : B \to A + B$. The projections from binary products are denoted $\mathsf{fst} : A \times B \to A$, $\mathsf{snd} : A \times B \to B$; pairing is denoted by $\langle \_, \_ \rangle$, and copairing of $f : A \to C$, $g : B \to C$ by $[f, g] : A + B \to C$. Unique morphisms $A \to 1$ into the terminal object are written $!_A$, or just $!$. We write $|\mathbf{C}|$ for the class of objects of $\mathbf{C}$. Distributivity essentially allows for using context variables in case expressions, i.e. in copairing.

We shall also require existence of certain *exponentials*, i.e. objects X^a adjoint to Cartesian products $a \times X$, which means: for any X and Y, there is an isomorphism

$$\mathsf{curry}_{X,Y} : \mathsf{Hom}_{\mathbf{C}}(X \times a, Y) \cong \mathsf{Hom}_{\mathbf{C}}(X, Y^a)$$

natural in X and Y. We write $\mathsf{uncurry}_{X,Y}$ for the inverse map $\mathsf{curry}^{-1}_{X,Y}$. Then *evaluation morphism* $\mathsf{ev}_X : X^a \times a \to X$ (natural in X) is obtained as $\mathsf{uncurry}_{X^a,X}(\mathsf{id}_{X^a})$. We commonly omit indices at natural transformations if it improves readability unless confusion arises.

Remark 2.1 The role of exponents in X^a is to capture a notion of arity of algebraic operations generating effects, e.g. $a = 2$ would correspond to binary operations such as nondeterministic choice. A more general setup would involve categories enriched over a symmetric monoidal closed category $\mathbf{V}$ whose objects are then treated as arities (and *coarities*, i.e. objects used for indexing families of operations) [13,12]. Instead of assuming existence of exponentials X^a one assumes existence of *tensors* $a \times X$ and *cotensors* X^b with $a, b \in |\mathbf{V}|$. Cotensors are adjoint to tensors in the same way as exponentials are adjoint to products with a constant object. We expect that our main results extend to this setting.

Recall that a monad $\mathbb{T}$ over $\mathbf{C}$ can be given by a *Kleisli triple* $(T, \eta, \_^\star)$ where T is an endomap of $|\mathbf{C}|$ (in the following, we always denote Kleisli triples and their functor parts by the same letter, with the former in blackboard bold), the *unit* η is a family of morphisms $\eta_X : X \to TX$, and the *Kleisli lifting* $\_^\star$ maps $f : X \to TY$ to $f^\star : TX \to TY$, subject to the equations

$$\eta^\star = \mathsf{id} \qquad\qquad f^\star \circ \eta = f \qquad\qquad (f^\star \circ g)^\star = f^\star \circ g^\star.$$

This is equivalent to the presentation in terms of an endofunctor T with natural transformations unit and multiplication. A monad is *strong* if it is equipped with a natural transformation $\tau_{X,Y} : X \times TY \to T(X \times Y)$ called *strength*, subject to a number of coherence conditions (e.g. [17]). Strength enables interpreting programs over more than one variable, and allows for internalization of the Kleisli lifting, thus legitimating expressions like $\lambda x.\,(f(x))^\star : X \to (TY \to TZ)$ for $f : X \to (Y \to TZ)$, which essentially encodes $\mathsf{curry}(\mathsf{uncurry}(f)^\star \circ \tau)$. Strength is equivalent to the monad being enriched over $\mathbf{C}$ [14]; in particular, every monad on $\mathbf{Set}$ is strong. Henceforth we shall use the term 'monad' to mean 'strong monad' unless explicitly stated otherwise.

The standard intuition for a monad $\mathbb{T}$ is to think of TX as the set of terms in some algebraic theory, with variables taken from X. In this view, the unit converts variables into terms, and a Kleisli lifting $f^\star$ applies a substitution $f : X \to TY$ to terms over X. In our setting, the 'terms' featuring here are often infinite; nevertheless, we sometimes call them *algebraic terms* for distinction from the terms in our metalanguage.

The *Kleisli category* $\mathbf{C}_\mathbb{T}$ of a monad $\mathbb{T}$ has the same objects as $\mathbf{C}$, and $\mathbf{C}$-morphisms $X \to TY$ as morphisms $X \to Y$. The identity on X in $\mathbf{C}_\mathbb{T}$ is η_X; and the *Kleisli composite* of $f : X \to TY$ and $g : Y \to TZ$ is $g^\star \circ f$. A monad $\mathbb{T}$ has rank κ if it preserves κ-filtered colimits. On $\mathbf{Set}$ this condition intuitively means that T is determined by its values on sets whose cardinality is smaller than κ.

3 Complete Elgot Monads

As indicated in the introduction, we will be interested in recursive definitions over a monad $\mathbb{T}$; abstractly, these are morphisms

$$f : X \to T(Y + X)$$

thought of as associating to each variable $x : X$ a definition $f(x)$ in the shape of an algebraic term from $T(Y + X)$, which thus employs parameters from Y as well as the defined variables from X. The latter amount to recursive calls of the definition. This notion is agnostic to what happens in the case of non-terminating recursion. For example, T might identify all non-terminating sequences of recursive calls into a single value $\bot$ signifying non-termination; at the other extreme, T might be a type of infinite trees that just records the tree of recursive calls explicitly.

To a recursive definition f as above, we wish to associate a solution

$$f^\dagger : X \to TY,$$

which amounts to a non-recursive definition of the elements of X as terms over Y only. As we do not assume any form of guardedness, this solution will in general fail to be unique. We thus require a coherent selection of solutions $f^\dagger$ for all equations f, where by coherent we mean that the selection satisfies a standard set of (quasi-)equational properties. Formally:

Definition 3.1 (Complete Elgot monads) A *complete Elgot monad* is a strong monad $\mathbb{T}$ equipped with an operator $\_^\dagger$, called *iteration*, that sends any $f : X \to T(Y+X)$ to $f^\dagger : X \to TY$ satisfying the following conditions:

- *unfolding:* $[\eta, f^\dagger]^\star \circ f = f^\dagger$;
- *naturality:* $g^\star \circ f^\dagger = ([T\,\mathsf{inl} \circ g, \eta \circ \mathsf{inr}]^\star \circ f)^\dagger$ for any $g : Y \to TZ$;
- *dinaturality:* $([\eta \circ \mathsf{inl}, h]^\star \circ g)^\dagger = [\eta, ([\eta \circ \mathsf{inl}, g]^\star \circ h)^\dagger]^\star \circ g$ for any $g : X \to T(Y+Z)$ and $h : Z \to T(Y+X)$;
- *codiagonal:* $(T[\mathsf{id}, \mathsf{inr}] \circ g)^\dagger = (g^\dagger)^\dagger$ for any $g : X \to T((Y+X)+X)$;
- *uniformity:* $f \circ h = T(\mathsf{id} + h) \circ g$ implies $f^\dagger \circ h = g^\dagger$ for any $g : Z \to T(Y+Z)$ and $h : Z \to X$.

Additionally, iteration must be compatible with strength in the following sense: for any $f : X \to T(Y+X)$, $\tau \circ (\mathsf{id} \times f^\dagger) = (T\,\mathsf{dist} \circ \tau \circ (\mathsf{id} \times f))^\dagger$.

Remark 3.2 The above definition is inspired by the axioms of parametrized uniform iterativity [25], which goes back to Bloom and Ésik [8]. Adámek et al. [3] define *Elgot monads* by means of a slightly different system of axioms: the codiagonal and dinaturality axioms are replaced with the *Bekić identity*. Both axiomatizations are however equivalent, which is essentially a result about iteration theories [8, Section 6.8]. Moreover, the iteration operator in [3] is defined only for $f : X \to T(Y+X)$ with finitely presentable X, under the assumption that $\mathbf{C}$ is locally finitely presentable; hence our use of the term 'complete Elgot monad' instead of 'Elgot monad'. We have the impression that this difference is not technically essential but have not checked details for the finitary variant of our results.

In the further development, examples of complete Elgot monads will arise either as so-called ω-continuous monads (Definition 3.3) or as extensions thereof with free operations, i.e. via the coinductive generalized resumption transformer.

If $\mathbb{T}$ supports an iteration operator $\_^\dagger$ then it is always possible to parametrize it with an additional argument to be carried over the recursion loop, i.e. we derive an operator $\_^\ddagger$ sending $f : Z \times X \to T(Y+X)$ to $f^\ddagger : Z \times X \to TY$ by

$$f^\ddagger = \big(T(\mathsf{snd} + \mathsf{id}) \circ (T\,\mathsf{dist}) \circ \tau_{Z,Y+X} \circ \langle \mathsf{fst}, f\rangle\big)^\dagger. \tag{1}$$

We call the derived operator $\_^\ddagger$ *strong iteration.*

As indicated above, an important class of examples of complete Elgot monads arises via a suitable order-enrichment of the Kleisli category.

Definition 3.3 (ω-continuous monad) An *ω-continuous* monad consists of a monad $\mathbb{T}$ and an enrichment of the Kleisli category $\mathbf{C}_\mathbb{T}$ of $\mathbb{T}$ over the category $\omega\mathbf{Cppo}$ of ω-complete partial orders with bottom and (nonstrict) continuous maps, satisfying the following conditions:

- strength is ω-continuous: $\tau \circ (\mathsf{id} \times \bigsqcup_i f_i) = \bigsqcup_i (\tau \circ (\mathsf{id} \times f_i))$;

- copairing in $\mathbf{C}_{\mathbb{T}}$ is ω-continuous in both arguments: $\big[\bigsqcup_i f_i, \bigsqcup_i g_i\big] = \bigsqcup_i [f_i, g_i]$;
- bottom elements are preserved by strength and by postcomposition in $\mathbf{C}_{\mathbb{T}}$: $\tau \circ (\mathsf{id} \times \bot) = \bot$, $f^\star \circ \bot = \bot$.

Example 3.4 Many of the standard computational monads on **Set** [17] are ω-continuous, including nontermination ($TX = X + 1$), nondeterminism ($TX = \mathcal{P}(X)$), and the nondeterministic state monad ($TX = \mathcal{P}(X \times S)^S$ for a set S of states). On ω**Cppo**, lifting ($TX = X_\bot$) and the various power domain monads are ω-continuous.

Remark 3.5 As observed by Kock [14], monad strength is equivalent to enrichment over the base category. One consequence of this fundamental fact is that if $\mathbf{C}$ is enriched over the category ω**Cpo** of bottomless ω-complete partial orders and ω-continuous maps (i.e. $\mathbf{C}$ is an **O**-category in the sense of Wand [29] and Smyth and Plotkin [26]), with the bicartesian closed structure enriched in the obvious sense, then $\mathbf{C}_{\mathbb{T}}$ is also enriched over ω**Cpo**, since T, underlying a strong monad, is an ω**Cpo**-functor (aka *locally continuous* functor [26]). Then $\mathbb{T}$ is ω-continuous in the sense of Definition 3.3 iff each $\mathsf{Hom}(X, TY)$ has a bottom element preserved by strength and postcomposition in $\mathbf{C}_{\mathbb{T}}$. This allows for incorporating numerous domain-theoretic examples by taking $\mathbf{C}$ to be a suitable category of predomains, and $\mathbb{T}$, in the simplest case, the *lifting monad* $TX = X_\bot$ (from which one builds more complex examples by the construction explored next).

If $\mathbb{T}$ is an ω-continuous monad, then the endomap

$$h \mapsto [\eta, h]^\star \circ f$$

on the hom-set $\mathsf{Hom}_{\mathbf{C}}(A, TB)$ is continuous because copairing and Kleisli composition in T are continuous, and hence has a least fixpoint by Kleene's fixpoint theorem. We can define an iteration operator by taking $f^\dagger$ to be this fixpoint; in other words, $f^\dagger$ is defined to be the smallest solution of the unfolding equation as per Definition 3.1. The verification of the remaining identities is tedious but straightforward; in summary,

Theorem 3.6 *On every ω-continuous monad, defining iteration by taking least fixpoints determines a complete Elgot monad structure.*

This result is unsurprising in the light of analogous facts known for so-called *ω-continuous theories* [8, Theorem 8.2.15, Exercise 8.2.17].

Remark 3.7 Every complete Elgot monad $\mathbb{T}$ can express *unproductive divergence* as the generic effect

$$\big(X \xrightarrow{\eta \,\circ\, \mathsf{inr}} T(Y + X)\big)^\dagger.$$

This computation never produces any effects, i.e. behaves like a deadlock. If $\mathbb{T}$ is ω-continuous, then unproductive divergence coincides with the least element of $\mathsf{Hom}(X, TY)$, for which reason we use the same symbol $\bot$ for the above morphism, but in general, there is no ordering in which unproductive divergence could be a least element.

4 The Coinductive Generalized Resumption Transformer

We proceed to recall the definition of the coinductive generalized resumption transformer [20], of which for simplicity we consider a version with only one family of free operations (rather than a whole signature or, even more generally, an arbitrary endofunctor on the base category). We then prove our main result, stability of the class of complete Elgot monads under this construction (Theorem 4.5).

Given $a, b \in |\mathbf{C}|$ such that exponentials of the form X^b exist and a monad $\mathbb{T}$, put

$$(\_)^b_a = a \times \_^b \qquad \text{and} \qquad T^b_a X = \nu\gamma.\, T(X + \gamma^b_a);$$

i.e. $T^b_a X$ is the final coalgebra of $T(X + (\_)^b_a)$, which we assume to exist. The assignment $(\_)^b_a$ is clearly a functor, i.e. applies also to morphisms. Intuitively, $T^b_a X$ is a type of possibly non-terminating computation trees, with each node consisting of a computation with side-effects specified by T that either returns a value in X or continues with one of a-many free operations each combining b-many subsequent computations. Let

$$\mathsf{out}_X : T^b_a X \to T(X + (T^b_a X)^b_a)$$

be the final coalgebra structure, and let $\mathsf{coit}(g) : Y \to T^b_a X$ denote the final morphism induced by a coalgebra $g : Y \to T(X + Y^b_a)$:

$$\begin{tikzcd} Y \arrow[rr, "\mathsf{coit}(g)"] \arrow[d, "g"'] & & T^b_a X \arrow[d, "\mathsf{out}_X"] \\ T(X + Y^b_a) \arrow[rr, "T(X+(\mathsf{coit}(g))^b_a)"'] & & T(X + (T^b_a X)^b_a). \end{tikzcd}$$

Intuitively, $\mathsf{coit}(g)$ encapsulates (in $T^b_a X$) a computation tree that begins by executing g, terminates in a leaf of type X if g does, and otherwise (co-)recursively continues to execute g, forming a new tree node for each recursive call. It is easy to verify that out_X is natural in X. By Lambek's lemma, out is a natural isomorphism. Thus, T maps into T^b_a via

$$\mathsf{ext} = T \xrightarrow{T\,\mathsf{inl}} T(\mathrm{Id} + (T^b_a)^b_a) \xrightarrow{\mathsf{out}^{-1}} T^b_a\,.$$

We record explicitly that T^b_a is a strong monad:

Lemma 4.1 *Given a monad $\mathbb{T}$ and $a, b \in |\mathbf{C}|$, T^b_a is the functorial part of a monad $\mathbb{T}^b_a$, with the monad structure characterized by the following properties.*

(i) *The unit $\eta^\nu : X \to T^b_a X$ is defined by $\mathsf{out} \circ \eta^\nu = \eta \circ \mathsf{inl}$ (i.e. $\eta^\nu = \mathsf{out}^{-1} \circ \eta \circ \mathsf{inl}$).*

(ii) *Given $f : X \to T^b_a Y$, the Kleisli lifting $f^\S : T^b_a X \to T^b_a Y$ is the unique solution of the equation $\mathsf{out} \circ f^\S = \big[\mathsf{out} \circ f, \eta \circ \mathsf{inr}(f^\S)^b_a\big]^\star \circ \mathsf{out}$.*

(iii) *Given $f : X \to T^b_a Y$, let $g = [f, \eta^\nu] : X + Y \to T^b_a Y$; then $g^\S$ is a final morphism of coalgebras, namely $g^\S = \mathsf{coit}\big([T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a) \circ \mathsf{out} \circ g, \eta \circ \mathsf{inr}]^\star \circ \mathsf{out}\big)$.*

(iv) *The strength $\tau^\nu : X \times T^b_a Y \to T^b_a(X \times Y)$ is the unique solution of $\mathsf{out} \circ \tau^\nu = T(\mathsf{id} + (\tau^\nu)^b_a) \circ T\delta \circ \tau \circ (\mathsf{id} \times \mathsf{out})$ where $\delta : X \times (Y + (T^b_a Y)^b_a) \to (X \times Y) + (X \times$*

$T_a^bY)_a^b$ is the obvious distributivity transformation:

$$\begin{tikzcd} X \times T_a^bY \arrow[r, "(T\delta)\tau(\mathsf{id}\times\mathsf{out})"] \arrow[d, "\tau^\nu"'] & T(X \times Y + (X \times T_a^bY)_a^b) \arrow[d, "T(\mathsf{id}+(\tau^\nu)_a^b)"] \\ T_a^b(X \times Y) \arrow[r, "\mathsf{out}"] & T(X \times Y + T_a^b(X \times Y)_a^b). \end{tikzcd}$$

The proof of Lemma 4.1 is facilitated by the fact that $T(X + (\_)_a^b)$ can be shown to be a *parametrized monad*, which implies that $\mathbb{T}_a^b$ is a monad [27, Theorems 3.7 and 3.9]. Alternatively, the fact that $\mathbb{T}_a^b$ is a monad can be read off directly from the results of [20]. What is new here is that we show that $\mathbb{T}_a^b$ is, in fact, strong, and hence supports an interpretation of the standard computational metalanguage [17]. This amounts to showing that the strength defined in the last item satisfies the requisite laws [17]. One fact of potentially independent interest used in the (quite involved) proof of these laws is

Lemma 4.2 *For any functor $G : \mathbf{B} \to \mathbf{C}$, $\mathsf{out}_G : T_a^bG \to T(G+(T_a^bG)_a^b)$ is the final $T(G + \mathrm{Id}_a^b)$-coalgebra in $[\mathbf{B}, \mathbf{C}]$.*

Following Uustalu [27] (and other work [20,1]), we next introduce a notion of guardedness.

Definition 4.3 (Guardedness) A morphism $f : X \to T_a^b(Y + Z)$ is *guarded* if there is $u : X \to T(Y + T_a^b(Y + Z)_a^b)$ such that $\mathsf{out} \circ f = T(\mathsf{inl} + \mathsf{id}) \circ u$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & T_a^b(Y + Z) \arrow[d, "\mathsf{out}"] \\ T(Y + T_a^b(Y + Z)_a^b) \arrow[r, "T(\mathsf{inl}+\mathsf{id})"'] & T((Y + Z) + (T_a^b(Y + Z))_a^b). \end{tikzcd}$$

Guardedness of $f : X \to T_a^b(Y+Z)$ intuitively means that any call to a computation of type Z in f occurs only under a free operation, i.e. via the right summand in $T((Y + Z) + (T_a^b(Y + Z))_a^b)$. A familiar instance of this notion occurs in process algebra [7], illustrated in simplified form as follows.

Example 4.4 Let $\mathbb{T}$ be the countable powerset monad over a suitable category, i.e. $TX = \mathcal{P}_{\omega_1}X = \{Y \subseteq X \mid |Y| \le \omega\}$. The object $T_A^1X = \nu\gamma.\, \mathcal{P}_{\omega_1}(X + A \times \gamma)$ can be considered as the domain of possibly infinite countably nondeterministic processes over actions from A with final results in X. A morphism $n \to T_A^1(X+n)$ can be seen as a system of n mutually recursive process definitions; the latter is guarded in the sense of Definition 4.3 iff every recursive call of a process is preceded by an action, which coincides with the standard notion of guardedness from process algebra. We recall an example of an *unguarded* definition in this setting in Section 6.

The following result is the main technical contribution of the paper; it states essentially that iteration operators, i.e. Elgot monad structures, propagate uniquely along extensions $\mathbb{T} \to \mathbb{T}_a^b$.

Theorem 4.5 *Let $\mathbb{T}$ be a complete Elgot monad. Given $a, b \in |\mathbf{C}|$, let $\mathbb{T}_a^b$ be the monad identified in Lemma 4.1.*

(i) *There is a unique iteration operator making $\mathbb{T}_a^b$ a complete Elgot monad that extends iteration in $\mathbb{T}$ in the sense that for $f : X \to T_a^b(Y + X)$ and $g : X \to T(Y + X)$, if*

$$\mathsf{out} \circ f = (T\,\mathsf{inl}) \circ g$$

(i.e. $f = \mathsf{out}^{-1} \circ (T\,\mathsf{inl}) \circ g$) then

$$\mathsf{out} \circ f^\dagger = (T\,\mathsf{inl}) \circ g^\dagger.$$

(ii) *For any guarded morphism $f : X \to T_a^b(Y + X)$, $f^\dagger$ is the* unique *morphism satisfying the unfolding property $[\eta^\nu, f^\dagger]^\S \circ f = f^\dagger$.*

Proof. (Sketch) Uustalu already proves that guarded morphisms f have unique iterates $f^\dagger$ [27, Theorem 3.11]. The key step is then to define $f^\dagger$ for unrestricted f in a consistent manner. For $f : X \to T_a^b(Y + X)$, let $\rhd f : X \to T_a^b(Y + X)$ be the composite

$$\begin{aligned} X &\xrightarrow{w^\dagger} T(Y + T_a^b(Y + X)_a^b) \\ &\xrightarrow{T(\mathsf{inl} + \mathsf{id})} T((Y + X) + T_a^b(Y + X)_a^b) \\ &\xrightarrow{\mathsf{out}^{-1}} T_a^b(Y + X) \end{aligned}$$

(guarded by definition), where w is the composite

$$\begin{aligned} X &\xrightarrow{f} T_a^b(Y + X) \\ &\xrightarrow{\mathsf{out}} T((Y + X) + T_a^b(Y + X)_a^b) \\ &\xrightarrow{T\pi} T((Y + T_a^b(Y + X)_a^b) + X) \end{aligned}$$

with $\pi = [\mathsf{inl} + \mathsf{id}, \mathsf{inl}\,\mathsf{inr}]$. That is, $\rhd f$ makes f guarded by iterating

$$\mathsf{out} \circ f : X \to T((Y + X) + T_a^b(Y + X)_a^b)$$

(in the complete Elgot monad $\mathbb{T}$) over the middle summand of the result. It is easy to check that $\rhd f = f$ when f is guarded. We hence can define

$$f^\dagger = (\rhd f)^\dagger$$

(in $\mathbb{T}_a^b$). Further (nontrivial) calculations show that this definition indeed satisfies the axioms of complete Elgot monads.

To establish uniqueness, we first show that any morphism $f : X \to T_a^b(Y + X)$ can be decomposed into two morphisms $g : X \to T_a^b(Z + X)$ and $h : Z \to T_a^b(Y + X)$, where $Z = Y + T_a^b(Y + X)_a^b$, as

$$f = [h, \eta^\nu \circ \mathsf{inr}]^\S \circ g$$

with g *completely unguarded*, i.e. $\mathsf{out} \circ g = (T\,\mathsf{inl}) \circ g'$ for some g'; that is, we split f into a guarded part and a completely unguarded one, with iteration on the latter part being determined by the requirement that iteration on T_a^b extend iteration on T. Next we show that for any choice of Elgot monad structure $\_^\dagger$ on T_a^b,

$$f^\dagger = (h^\S \circ g^\dagger)^\dagger$$

and that

$$h^\S \circ g^\dagger = \rhd f.$$

In summary, we then obtain that $f^\dagger = (h^\S \circ g^\dagger)^\dagger = (\rhd f)^\dagger$, i.e. our previous definition of $f^\dagger$ is the only possible one with the given properties, as $\rhd f$ is guarded and therefore $(\rhd f)^\dagger$ is determined uniquely already by the unfolding property. $\square$

The following results characterize $\mathbb{T}_a^b$ within the (overlarge) category $\mathbf{CElg}(\mathbf{C})$ of complete Elgot monads over $\mathbf{C}$ and (strong) monad morphisms [16] preserving iteration in the evident sense:

Definition 4.6 A *complete Elgot monad morphism* $\xi : \mathbb{R} \to \mathbb{S}$ between complete Elgot monads $\mathbb{R}$, $\mathbb{S}$ is a morphism ξ between the underlying strong monads (i.e. $\xi \circ \eta = \eta$, $\xi \circ f^\star = (\xi \circ f)^\star \circ \xi$ for $f : X \to RY$, and $\xi \circ \tau = \tau \circ (\mathsf{id} \times \xi)$) additionally satisfying

$$(\xi \circ g)^\dagger = \xi \circ g^\dagger$$

for $g : X \to R(Y + X)$.

Lemma 4.7 *The natural transformation* $\mathsf{ext} : \mathbb{T} \to \mathbb{T}_a^b$ *is a complete Elgot monad morphism.*

Theorem 4.8 *Suppose that* $\mathbf{CElg}(\mathbf{C})$ *has an initial object* $\mathbb{L}$*. Then*

(i) $\mathbb{L}_a^b$ *is the free complete Elgot monad over the signature functor* $(\_)_a^b : \mathbf{C} \to \mathbf{C}$*;*

(ii) *For any complete Elgot monad* $\mathbb{T}$, $\mathbb{T}_a^b$ *is the coproduct of* $\mathbb{T}$ *and* $\mathbb{L}_a^b$ *in* $\mathbf{CElg}(\mathbf{C})$*, with left injection* $\mathsf{ext} : \mathbb{T} \to \mathbb{T}_a^b$ *(in particular,* ext *is a morphism in* $\mathbf{CElg}(\mathbf{C})$*).*

The crucial step in proving Theorem 4.8 is the following statement, which is interesting in its own right.

Lemma 4.9 *Let* $a, b \in |\mathbf{C}|$ *and let* $\mathbb{T}$, $\mathbb{S}$ *be two complete Elgot monads. Given a complete Elgot monad morphism* $\rho : \mathbb{T} \to \mathbb{S}$ *and a Kleisli morphism* $u : a \to Sb$*, the transformation* $\zeta^\dagger : T_a^b \to S$ *with* ζ *defined componentwise as the composite*

$$T_a^b X \xrightarrow{\mathsf{out}} T(X + a \times (T_a^b X)^b) \xrightarrow{[\eta \circ \mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr} \circ f)u(x)]^\star \circ \rho} S(X + T_a^b X)$$

extends to a complete Elgot monad morphism. Conversely, any $\xi : \mathbb{T}_a^b \to \mathbb{S}$ *induces* $\xi\,\mathsf{ext} : \mathbb{T} \to \mathbb{S}$ *and*

$$a \xrightarrow{\mathsf{out}^{-1} \circ \eta \circ \mathsf{inr} \circ \langle \mathsf{id}, \lambda\_.\, \eta\rangle} T_a^b b \xrightarrow{\xi_b} Sb.$$

These two passages are mutually inverse and thus witness a bijection between complete Elgot monad morphisms $\mathbb{T}_a^b \to \mathbb{S}$ *and pairs consisting of Kleisli morphisms* $a \to Sb$ *and complete Elgot monad morphisms* $\mathbb{T} \to \mathbb{S}$*.*

The existence and the exact shape of the initial complete Elgot monad $\mathbb{L}$ mentioned in Theorem 4.8 depend on the properties of $\mathbf{C}$. Recall that $\mathbf{C}$ is *hyperextensive* [2] if it has countable coproducts that are disjoint and universal (i.e. stable under pullbacks), and coproduct injections are, as subobjects, closed under countable disjoint unions. Examples include **Set**, ω**Cpo**, and complete metric spaces as well as all presheaf categories.

Theorem 4.10 *Let* $\mathbf{C}$ *be hyperextensive. Then the monad* $\mathbb{L}$ *given by* $LX = X+1$ *is* ω*-continuous. Equipped with the arising complete Elgot monad structure according to Theorem 3.6,* $\mathbb{L}$ *is the initial complete Elgot monad over* $\mathbf{C}$*.*

Proof. The base category $\mathbf{C}$ is, a fortiori, extensive; in any extensive category, $\mathbb{L}$ is the partial map classifier for partial morphisms whose domains are coproduct injections. Thus, the Kleisli category of $\mathbb{L}$ inherits orderings on its hom-sets from the extension ordering on partial functions; the fact that coproduct injections are closed under unions in $\mathbf{C}$ then guarantees that these orderings are ω-complete (note that any ascending chain of coproduct injections qua subobjects can, using universality of coproducts, be transformed into a disjoint union of coproduct injections). Using the properties of hyperextensive categories, one can show that this induces an ω**Cppo**-enrichment of $\mathbf{C}_{\mathbb{L}}$ that satisfies all additional conditions imposed in Definition 3.3.

To see initiality, note that any complete Elgot monad $\mathbb{T}$ for any $X \in |\mathbf{C}|$ possesses a global element $\bot_X = \delta_X^\dagger : 1 \to TX$ where $\delta_X = \eta \circ \mathsf{inr} : 1 \to T(X+1)$. It follows by naturality of iteration that $\bot_X$ is actually natural in X. Moreover, $\bot$ is preserved by complete Elgot monad morphisms. It is easy to see that $\xi_X = [\eta, \bot_X]$ yields a complete Elgot monad morphism $\xi : \mathbb{L} \to \mathbb{T}$. On the other hand it is the only such because for any other complete Elgot monad morphism $\theta : \mathbb{L} \to \mathbb{T}$ one would have $\theta \circ \mathsf{inl} = \theta \circ \eta = \eta = \xi \circ \mathsf{inl}$ and $\theta \circ \mathsf{inr} = \theta \circ \bot = \bot = \xi \circ \mathsf{inr}$ implying $\theta = \xi$. ◻

5 Example: Unrestricted While Loops

We use a simple while-language with actions proposed by Rutten, given by the grammar

$$P, Q ::= A \mid P; Q \mid \mathsf{if}\, b\, \mathsf{then}\, P\, \mathsf{else}\, Q \mid \mathsf{while}\, b\, \mathsf{do}\, P$$

and, following Piróg and Gibbons [20], interpreted in the Kleisli category of a monad $\mathbb{M}$. Here, A ranges over atomic actions interpreted as Kleisli morphisms $[\![A]\!] : n \to Mn$ for some fixed object n, and b over atomic predicates, interpreted as Kleisli morphisms $[\![b]\!] : n \to M(1+1)$ (with the left-hand summand read as 'false'). We say that A is of *output type* if $[\![A]\!]$ has the form $(M\,\mathsf{fst}) \circ \tau \circ \langle \mathsf{id}_n, p\rangle$ for some $p : n \to M1$, and of *input type* if $[\![A_i]\!]$ factors through $! : n \to 1$. Sequential composition $P; Q$ is interpreted as Kleisli composition $[\![Q]\!]^\star \circ [\![P]\!]$, and

$$[\![\mathsf{if}\, b\, \mathsf{then}\, P\, \mathsf{else}\, Q]\!] = [[\![Q]\!] \circ \mathsf{fst}, [\![P]\!] \circ \mathsf{fst}]^\star \circ M\,\mathsf{dist} \circ \tau \circ \langle \mathsf{id}, [\![b]\!]\rangle.$$

The key point, of course, is the interpretation of the while loop, given in the presence of iteration $\_^\dagger$ by

$$[\![\mathsf{while}\, b\, \mathsf{do}\, P]\!] = \big([(M\,\mathsf{inl}) \circ \eta \circ \mathsf{fst}, (M\,\mathsf{inr}) \circ [\![P]\!] \circ \mathsf{fst}]^\star \circ M\,\mathsf{dist} \circ \tau \circ \langle \mathsf{id}, [\![b]\!]\rangle\big)^\dagger. \quad (2)$$

It has been observed by Piróg and Gibbons that if one instantiates $\mathbb{M}$ with a completely iterative monad, one needs to guard every iteration of the while loop, i.e.

change the semantics of while to be

$$[\![\mathsf{while}\ b\ \mathsf{do}\ P]\!] = \Big([(M\,\mathsf{inl})\circ\eta\circ\mathsf{fst}, (M\,\mathsf{inr})\circ[\![P]\!]\circ\mathsf{fst}]^\star\circ M\,\mathsf{dist}\circ\tau\circ\langle\mathsf{id},[\![b]\!]\rangle\circ\gamma\Big)^\dagger$$

where $\gamma : n \to Mn$ is guarded, as otherwise the iteration may fail to be defined. If we instantiate $\mathbb{M}$ with an Elgot monad, such as $\mathbb{T}_a^b$ for an Elgot monad $\mathbb{T}$, then the guard is unnecessary, i.e. we can stick to the original semantics (2). As an example, consider a simple-minded form of processes that input and output symbols from n and have side effects specified by $\mathbb{T}$; i.e. we work in $\mathbb{M} = (\mathbb{T}_n^1)_1^n$ meaning to use the adjoined free effects $1 \to n$ to capture output and those of type $n \to 1$ to capture input. We assume an atomic action *write* that outputs a symbol from n, and an atomic action *read* that inputs a symbol. We interpret *write* as being of output type, i.e. by $[\![\mathit{write}]\!] = (M\,\mathsf{fst})\circ\tau\circ\langle\mathsf{id}_n, w\rangle$ where

$$w = \mathsf{ext}\circ\mathsf{out}^{-1}\circ\eta\circ\mathsf{inr}\circ\langle\mathsf{id}_n, \eta^\nu\circ !_n\rangle : n \to (T_n^1)_1^n(1)$$

(η^ν being the unit of $\mathbb{T}_n^1$), while *read* is of input type, i.e. $[\![\mathit{read}]\!] = r\circ !_n$ where

$$r = \mathsf{out}^{-1}\circ\eta^\nu\circ\mathsf{inr}\circ\langle\mathsf{id}_1, r_0\rangle : 1 \to (T_n^1)_1^n n$$

and $r_0 : 1 \to (T_n^1)_1^n(n)^n$ is obtained by currying $\eta^M\circ\mathsf{snd} : 1\times n \to (T_n^1)_1^n(n)$ (η^M being the unit of $\mathbb{M}$). Moreover, assume a basic predicate b whose interpretation is largely irrelevant to the example as long as it may take both truth values; for example, b might just pick a truth value non-deterministically or at random, depending on the nature of the base monad $\mathbb{T}$. Consider the program

$$\mathit{read};\mathsf{while}\ \mathit{true}\ \mathsf{do}\ \mathsf{if}\ b\ \mathsf{then}\ \mathit{skip}\ \mathsf{else}\ \mathit{write}$$

where *skip* is an atomic action interpreted as $[\![\mathit{skip}]\!] = \eta_n^M : n \to Mn$. It is possible for the loop to not perform any write operations, as b might happen to always pick the left-hand branch; that is, the loop body fails to be guarded. Since M is an Elgot monad and not just completely iterative, the semantics of the loop is defined (by (2)) nonetheless.

6 Example: Simple Process Algebra

It is shown in [5, Theorem 5.7.3] that a simple process algebra BSP featuring finite choice and action prefixing can express all countable transition systems if unguarded recursion is allowed. The idea of the proof is to introduce variables X_{ik} for $i, k \in \mathbb{N}$ representing the k-th transition of the i-th state, with X_{i0} representing the i-th state itself, and (unguarded) recursive equations

$$X_{ik} = b_{ik}.X_{j(i,k),0} + X_{i,k+1} \tag{3}$$

where the k-th transition of the i-th state performs action b_{ik} and reaches the $j(i,k)$-th state. (It is then stated explicitly that the use of unguarded recursion is essential.) To model this phenomenon using the coinductive generalized resumption transformer, we take $\mathbb{T} = \mathcal{P}$, the powerset monad on **Set** (an Elgot monad by Theorem 3.6) and an operation *act* with interpretation $[\![act]\!] = \mathsf{out}^{-1} \circ \eta \circ \mathsf{inr} \circ \langle \mathsf{id}_a, \eta^\nu !_a \rangle : a \to T_a^1 1$, where a is the type of actions. That is, we regard (unbounded) nondeterminism as part of the base effect, and add action prefixing via coinductive generalized resumptions. Then the definition (3) is represented by the map $g = \mathsf{out}^{-1} \circ f : \mathbb{N} \times \mathbb{N} \to T_a^1(\mathbb{N} \times \mathbb{N}) \cong T_a^1(0 + \mathbb{N} \times \mathbb{N})$ with $f : \mathbb{N} \times \mathbb{N} \to T((\mathbb{N} \times \mathbb{N}) + a \times T_a^1(\mathbb{N} \times \mathbb{N})$ (eliding the exponent 1) given by

$$f(i,k) = \{\mathsf{inr}(b_{ik}, \eta^\nu(j(i,k),0)), \mathsf{inl}(i,k+1)\}.$$

Again, our result that $\mathbb{T}_a^1$ is an Elgot monad guarantees that this equation has a solution $g^\dagger$; the choice $\_^\dagger$ of solutions in $\mathbb{T}_a^1$ is uniquely determined as extending the usual structure of $\mathbb{T} = \mathcal{P}$ as an Elgot monad via taking least fixed points.

7 Related Work

The above results benefit from extensive previous work on monad-based axiomatic iteration. In particular we draw on the concept of *Elgot monad* studied by Adámek et al. [3]; the construction of the free Elgot monad over a functor [4] is strongly related to Theorem 4.8.i, and we do not claim this result as a contribution of this paper. There is extensive literature on solutions of (co)recursive program schemes [6,1,15,11,20,21], from which our present work differs primarily in that we do not restrict to guarded systems of equations. In particular, as mentioned in the introduction, Piróg and Gibbons [20] actually work with the same monad transformer, the coinductive generalized resumption transformer. Piróg and Gibbons [21, Corollary 4.6] also prove a characterization of the coinductive generalized resumption transformer as taking coproducts of monads similar to our Theorem 4.8.ii; but again, this takes place in a different category, that is, in completely iterative monads (admitting guarded recursive definitions) rather than complete Elgot monads (admitting unrestricted corecursive definitions). One consequence of this is that the second summand in our coproduct result is a free Elgot monad and not a free completely iterative monad over $a \times \_^b$, and hence has a built-in notion of divergence. Technically, results on T_a^b being a completely iterative monad are incomparable to our result on T_a^b being a complete Elgot monad – we prove a stronger recursion scheme for T_a^b but need to assume that T is an Elgot monad, while T_a^b is completely iterative without any assumptions on T.

We construct solutions of unguarded recursive equations from solutions of guarded recursive equations, for the latter relying crucially on results by Uustalu on guarded recursion over parametrized monads [27], which in particular has allowed us to make do without idealized monads.

The axiomatic treatment of iteration via Elgot monads is essentially dual to the axiomatic treatment of recursion by Simpson and Plotkin [25], who work in a category $\mathbf{D}$ with a *parametrized uniform recursion operator* $\mathsf{Hom}_{\mathbf{D}}(Y \times X, X) \to \mathsf{Hom}_{\mathbf{D}}(Y, X)$ and a subcategory $\mathbf{S}$ of *strict* functions in $\mathbf{D}$. Given a distributive category $\mathbf{C}$ equipped with a complete Elgot monad, we can take $\mathbf{S} = \mathbf{C}^{op}$ and $\mathbf{D} = \mathbf{C}_{\mathbb{T}}^{op}$. Then the iteration operator over $\mathbf{C}_{\mathbb{T}}$ sending $f : X \to T(Y + X)$ to $f^\dagger : X \to TY$ induces precisely a parametrized uniform recursion operator for the pair $(\mathbf{D}, \mathbf{S})$ in the sense of Simpson and Plotkin.

8 Conclusions and Future Work

We have developed semantic foundations for non-wellfounded side-effecting recursive definitions, in the form of iteration, specifically for recursive definitions over the so-called *coinductive generalized resumption transformer* that constructs from a base monad T the monad $\nu\gamma.\, T(\_ + a \times \gamma^b)$. While previous work on the same monad transformer was focussed on guarded corecursive definitions, in the framework of completely iterative monads, we work in the setting of (complete) Elgot monads, which admit unrestricted recursive definitions. As the core results, we have established that

- T_a^b is a complete Elgot monad if T is a complete Elgot monad;
- the structure of T_a^b as a complete Elgot monad is uniquely determined as extending the one of T;
- if the underlying category $\mathbf{C}$ admits an initial complete Elgot monad L (typically $L = \_ + 1$) then $T_a^b \cong T + L_a^b$ in the category of complete Elgot monads on $\mathbf{C}$.

In particular this requires proving the equational laws of complete Elgot monads for the solution operator that we construct on T_a^b. Ongoing work is concerned with a formal verification of our results, which are technically quite involved, in the Coq proof assistant; a preliminary version can be found at `https://git8.cs.fau.de/redmine/projects/corque`.

Besides the fact that applying the coinductive resumption monad transformer to a complete Elgot monad $\mathbb{T}$ again yields a complete Elgot monad $\mathbb{T}_a^b$, the resulting object obviously has a richer structure provided by the adjoined free operations. One topic for further investigation is to identify (and possibly axiomatize) this structure. Future work is concerned with using this structure for programming definitions of free operations as morphisms $T_a^b X \to TX$ in a similar spirit as in the paradigm of *handling algebraic effects* [23]. In conjunction with iteration this actually produces a recursion operator more expressive than iteration. This however requires going beyond the first-order setting of this paper (which was sufficient for iteration), as call-by-value recursion is known to be an inherently higher-order concept.

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A Appendix: Omitted Proofs

Proof of Theorem 3.6

Since copairing and Kleisli composition are continuous, the endomap $h \mapsto [\eta, h]^\star f$ over a hom-set $\mathsf{Hom}_{\mathbf{C}}(A, TB)$ is also continuous and hence the least fixpoint of it exists by Kleene fixpoint theorem. We take this fixpoint as the definition of $f^\dagger$.

Let us verify the axioms of Elgot monads one by one. To that end we employ the following uniformity rule for continuous functionals [22]:

$$\frac{UF = GU \qquad U(\bot) = \bot}{U(\mu F) = \mu G} \tag{A.1}$$

- *Unfolding.* This holds by definition.
- *Naturality.* In (A.1) take $F(u) = [\eta, u]^\star f$, $G(u) = [\eta, u]^\star[(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f$ and $U(h) = g^\star u$. By definition, $U(\bot) = \bot$, $\mu F = f^\dagger$, $\mu G = ([\eta, u]^\star[(T\,\mathsf{inl})g, \eta\,\mathsf{inr}])^\dagger$. Then we have

  $$U(F(u)) = g^\star[\eta, u]^\star f = [\eta, g^\star u]^\star[(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f = G(U(u)).$$

  Therefore, by (A.1), $g^\star f^\dagger = U(\mu F) = \mu G = ([(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f)^\dagger$.
- *Dinaturality.* Let us denote $s = [\eta\,\mathsf{inl}, h]^\star g$ and $t = [\eta\,\mathsf{inl}, g]^\star h$. The identity in question is then $s^\dagger = [\eta, t^\dagger]^\star g$. Observe that

  $$\begin{aligned}
  [\eta, t^\dagger]^\star g &= \big[\eta, [\eta, t^\dagger]^\star[\eta\,\mathsf{inl}, g]^\star h\big]^\star g && \text{(unfolding)}\\
  &= \big[\eta, [\eta, [\eta, t^\dagger]^\star g]^\star h\big]^\star g\\
  &= \big[\eta, [\eta, t^\dagger]^\star g\big]^\star[\eta\,\mathsf{inl}, h]^\star g,
  \end{aligned}$$

  i.e. $[\eta, t^\dagger]^\star g$ satisfies the unfolding identity for $s^\dagger$, therefore $s^\dagger \sqsubseteq [\eta, t^\dagger]^\star g$. By symmetry we obtain $t^\dagger \sqsubseteq [\eta, s^\dagger]^\star h$ and therefore

  $$\begin{aligned}
  [\eta, t^\dagger]^\star g &\sqsubseteq \big[\eta, [\eta, s^\dagger]^\star h\big]^\star g && \text{(continuity)}\\
  &= [\eta, s^\dagger]^\star[\eta\,\mathsf{inl}, h]^\star g\\
  &= s^\dagger
  \end{aligned}$$

  We have thus shown the identity $s^\dagger = [\eta, t^\dagger]^\star g$ by mutual inclusion.
- *Codiagonal.* Recall that we are claiming that

  $$(T(\mathsf{id} + \nabla)g)^\dagger = (g^\dagger)^\dagger$$

  for $g : A \to T((B + A) + A)$. We first show that $(g^\dagger)^\dagger$ is a fixpoint of the functional defining the left-hand side as a least fixpoint, thus proving $\sqsubseteq$. That is, we have to show that

  $$[\eta, (g^\dagger)^\dagger]^\star T(\mathsf{id} + \nabla)g = (g^\dagger)^\dagger. \tag{A.2}$$

We proceed as follows:

$$\begin{aligned}
(g^\dagger)^\dagger &= [\eta, (g^\dagger)^\dagger]^\star g^\dagger && \text{(unfolding)}\\
&= [\eta, (g^\dagger)^\dagger]^\star [\eta, g^\dagger]^\star g && \text{(unfolding)}\\
&= [[\eta, (g^\dagger)^\dagger], [\eta, (g^\dagger)^\dagger]^\star g^\dagger]^\star g\\
&= [[\eta, (g^\dagger)^\dagger], (g^\dagger)^\dagger]^\star g && \text{(unfolding)}\\
&= [\eta, (g^\dagger)^\dagger \nabla]^\star g\\
&= [\eta, (g^\dagger)^\dagger]^\star\, T(\mathsf{id} + \nabla) g.
\end{aligned}$$

For the converse inequality, continuity allows us to use fixpoint induction. Recall that the right hand side is the least fixed point of the functional $F : (A \to TB) \to (A \to TB)$ defined by

$$F(f) = [\eta, f]^\star g^\dagger,$$

and hence the supremum of the chain $(F^i(\bot))_{i\in\mathbb{N}}$. We show by induction on i that all members of this chain are below $(T(\mathsf{id} + \nabla) \circ g)^\dagger$, with trivial induction base. So let $f \sqsubseteq (T(\mathsf{id} + \nabla) \circ g)^\dagger$. We have to show

$$[\eta, f]^\star g^\dagger \sqsubseteq (T(\mathsf{id} + \nabla) g)^\dagger.$$

We establish this by a second fixpoint induction on the definition of $g^\dagger$, again with trivial induction base. So assume that $[\eta, f]^\star r \sqsubseteq (T(\mathsf{id} + \nabla) g)^\dagger$, with $r : A \to T(B + A)$; we have to show that

$$[\eta, f]^\star [\eta, r]^\star g \sqsubseteq (T(\mathsf{id} + \nabla) g)^\dagger.$$

We calculate as follows:

$$\begin{aligned}
[\eta, f]^\star [\eta, r]^\star g &= [\eta, f, [\eta, f]^\star r]^\star g\\
&\sqsubseteq [\eta, f, (T(\mathsf{id} + \nabla) g)^\dagger]^\star g && \text{(inner IH)}\\
&\sqsubseteq [\eta, (T(\mathsf{id} + \nabla) g)^\dagger, (T(\mathsf{id} + \nabla) g)^\dagger]^\star g && \text{(outer IH)}\\
&= [\eta, (T(\mathsf{id} + \nabla) g)^\dagger] T(\mathsf{id} + \nabla) g\\
&= (T(\mathsf{id} + \nabla) g)^\dagger. && \text{(unfolding)}
\end{aligned}$$

- *Uniformity.* Let $f : A \to T(X + A)$, $g : B \to T(X + B)$, $G(u) = [\eta, u]^\star g$, $F(u) = [\eta, u]^\star f$, and $U(u) = u^\star h$. Then $U(\bot) = \bot$ and

$$\begin{aligned}
UF(u) &= ([\eta, u]^\star f)^\star h\\
&= [\eta, u]^\star (f^\star h)\\
&= [\eta, u]^\star [\eta\, \mathsf{inl}, (T\, \mathsf{inr}) h]^\star g\\
&= \Big[[\eta, u]^\star \eta\, \mathsf{inl}, [\eta, u]^\star (T\, \mathsf{inr}) h\Big]^\star g\\
&= \Big[[\eta, u]\, \mathsf{inl}, u^\star h\Big] g\\
&= [\eta, u^\star h] g\\
&= GU(u).
\end{aligned}$$

Therefore by (A.1), $(f^\dagger)^\star h = U\mu F = \mu G = g^\dagger$.

In the following, we will use the axioms of strength as in [16]:

$$\mathsf{snd} = T\mathsf{snd}\tau \tag{STR1}$$
$$(T\alpha)\tau = \tau(\mathsf{id}\times\tau)\alpha \tag{STR2}$$
$$\tau(\mathsf{id}\times\eta) = \eta \tag{STR3}$$
$$(\tau(\mathsf{id}\times f))^\star\tau = \tau(\mathsf{id}\times f^\star) \tag{STR4}$$

where $\alpha : (X \times Y) \times TC \to X \times (Y \times TC)$ is the associativity isomorphism of products.

To prove compatibility of strength and iteration, we proceed by first showing

$$((T\,\mathsf{dist})\tau(\mathsf{id}\times f))^\dagger \sqsubseteq \tau(\mathsf{id}\times f^\dagger).$$

First observe that, for any $g : A \to TB$,

$$\begin{tikzcd}
C\times(B+A) \arrow[r, shift left, "\mathsf{dist}"] \arrow[d, "\mathsf{id}\times[\eta,g]"'] & C\times B + C\times A \arrow[l, shift left, "\mathsf{dist}^{-1}"] \arrow[d, "{[\eta,\tau(\mathsf{id}\times g)]}"] \\
C\times TB \arrow[r, "\tau"] & T(C\times B).
\end{tikzcd} \tag{DST}$$

This is easily checked componentwise starting from $C \times B + C \times A$ and using the fact that by definition $\mathsf{dist}^{-1} = [\mathsf{id}\times\mathsf{inl}, \mathsf{id}\times\mathsf{inr}]$. Then we have

$$\begin{aligned}
&\tau(\mathsf{id}\times f^\dagger)\\
&= \tau(\mathsf{id}\times[\eta, f^\dagger]^\star f)\\
&= \tau(\mathsf{id}\times[\eta, f^\dagger]^\star)(\mathsf{id}\times f)\\
&= (\tau(\mathsf{id}\times[\eta, f^\dagger]))^\star\tau(\mathsf{id}\times f) && \text{(STR4)}\\
&= ([\eta, \tau(\mathsf{id}\times f^\dagger)]\,\mathsf{dist})^\star\tau(\mathsf{id}\times f) && \text{(DST)}\\
&= [\eta, \tau(\mathsf{id}\times f^\dagger)]^\star(T\,\mathsf{dist})\tau(\mathsf{id}\times f).
\end{aligned}$$

Therefore, $\tau(\mathsf{id}\times f^\dagger)$ is a fixed point of the functional defining $((T\,\mathsf{dist})\tau(\mathsf{id}\times f))^\dagger$ as a least fixpoint and the inequality above holds. The converse inequality,

$$\tau(\mathsf{id}\times f^\dagger) \sqsubseteq ((T\,\mathsf{dist}\,\tau)(\mathsf{id}\times f))^\dagger,$$

is shown by fixpoint induction as above for the codiagonal. The base case is trivial with

$$\tau\langle\mathsf{fst}, \bot\,\mathsf{snd}\rangle = \tau\langle\mathsf{fst}, \bot\,\mathsf{fst}\rangle = \tau\langle\mathsf{id}, \bot\rangle\,\mathsf{fst} = \bot\,\mathsf{fst} = \bot.$$

Assume now that $\tau(\mathsf{id}\times g) \sqsubseteq (T\,\mathsf{dist}\,\tau(\mathsf{id}\times f))^\dagger$. We can then calculate

$$\begin{aligned}
&\tau(\mathsf{id}\times[\eta, g]^\star f)\\
&= \tau(\mathsf{id}\times[\eta, g]^\star)(\mathsf{id}\times f)\\
&= (\tau(\mathsf{id}\times[\eta, g]))^\star\tau(\mathsf{id}\times f) && \text{(STR4)}
\end{aligned}$$

$$
\begin{aligned}
&= ([\eta, \tau(\mathsf{id} \times g)]\,\mathsf{dist})^\star \tau(\mathsf{id} \times f) && \text{(DST)}\\
&\sqsubseteq ([\eta, (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger]\,\mathsf{dist})^\star \tau(\mathsf{id} \times f)\\
&= [\eta, (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger]^\star T\,\mathsf{dist}\,\tau(\mathsf{id} \times f)\\
&= (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger
\end{aligned}
$$

which completes the proof. □

We extend the statement of Lemma 4.2:

Lemma A.1 *If $T_a^b X$ exists for each X, then T_a^b is a functor and $\mathsf{out} : T_a^b \to T(\mathrm{Id} + (T_a^b)_a^b)$ is a natural transformation. For any functor $G : \mathbf{B} \to \mathbf{C}$, $\mathsf{out}_G : T_a^b G \to T(G + (T_a^b G)_a^b)$ is the final $T(G + \mathrm{Id}_a^b)$-coalgebra in $[\mathbf{B}, \mathbf{C}]$.*

Proof. Functoriality follows from T_a^b carrying a monad structure, as shown in the proof of Theorem 4.5 independently of this lemma. Now $T_a^b f = (\eta^\nu f)^\S$, so by the description of § we have

$$
\begin{aligned}
\mathsf{out}\, T_a^b f &= [\mathsf{out}\, \eta^\nu f, \eta^\nu\, \mathsf{inr}(T_a^b f)_a^b]^\star\, \mathsf{out}\\
&= [\eta\, \mathsf{inl}\, f, \eta\, \mathsf{inr}(T_a^b f)_a^b]^\star\, \mathsf{out}\\
&= T[\mathsf{inl}\, f, \mathsf{inr}(T_a^b f)_a^b]\, \mathsf{out}\\
&= T(f + (T_a^b f)_a^b)\, \mathsf{out},
\end{aligned}
$$

i.e. out is natural.

To show finality, let $\beta : F \to T(G + a \times F^b)$ be a natural transformation. We define $f : F \to T_a^b G$ componentwise by the equation

$$\mathsf{out}\, f_X = T(GX + a \times f_X^b)\beta_X$$

using finality of $\mathsf{out} : T_a^b GX \to T(GX + (T_a^b GX)_a^b)$. We have to show that f is natural (uniqueness is clear). So let $g : X \to Y$; we have to show $f_Y Fg = T_a^b G g f_X$. Note that we have a $T(GY + (\_)_a^b)$-coalgebra

$$T(Gg + \mathsf{id})\beta_X : FX \to T(GX + (FX)_a^b) \to T(GY + (FX)_a^b);$$

we show that both $f_Y Fg$ and $(T_a^b Gg) f_X$ are final coalgebra morphisms into $T_a^b GY$ for $T(Gg + \mathsf{id})\beta_X$. On the one hand, we have

$$
\begin{aligned}
\mathsf{out}\, f_Y Fg &= T(GY + (f_Y)_a^b)\beta_Y Fg && \text{(definition of } f_Y)\\
&= T(GY + (f_Y)_a^b) T(Gg + (Fg)_a^b)\beta_X && \text{(naturality of } \beta)\\
&= T(GY + (f_Y Fg)_a^b) T(Gg + id)\beta_X.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathsf{out}(T_a^b Gg) f_X &= T(Gg + (T_a^b Gg)_a^b)\, \mathsf{out}\, f_X && \text{(naturality of } \mathsf{out})\\
&= T(Gg + (T_a^b Gg)_a^b) T(GX + (f_X)_a^b)\beta_X && \text{(definition of } f_X)\\
&= T(GY + ((T_a^b Gg) f_X)_a^b) T(Gg + \mathsf{id})\beta_X.
\end{aligned}
$$

□

Proof of Lemma 4.1

$T(X + (\_)^b_a)$ can be shown to be a *parametrized monad*, which implies that T^b_a underlies a monad $\mathbb{T}^b_a$, with unit and Kleisli lifting uniquely characterized by the corresponding equations [27, Theorems 3.7 and 3.9]. What is missing is to show that $\mathbb{T}^b_a$ is a strong monad, as we need here.

Let us first show the identity

$$g^{\S} = \mathsf{coit}\big([T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out}\big), \tag{A.3}$$

By definition, $g^{\S}$ is the unique morphism making the following diagram commute:

$$\begin{tikzcd} T^b_a(X+Y) \arrow[r, "g^{\S}"] \arrow[d, "\mathsf{out}"'] & T^b_a Y \arrow[d, "\mathsf{out}"] \\ T(X+Y+T^b_a(X+Y)^b_a) \arrow[r, "{[\mathsf{out}\,g,\,\eta\,\mathsf{inr}({g^{\S}}^b_a)]^\star}"] & T(Y + (T^b_a Y)^b_a) \end{tikzcd}$$

We then have on the one hand,

$$\begin{aligned} \mathsf{out}\,g^{\S} &= [\mathsf{out}[f,\eta^\nu],\ \eta\,\mathsf{inr}({g^{\S}}^b_a)]^\star\,\mathsf{out} && \text{(definition of } \S) \\ &= [[\mathsf{out}\,f, \mathsf{out}\,\eta^\nu], \eta\,\mathsf{inr}({g^{\S}}^b_a)]^\star\,\mathsf{out} \\ &= [[\mathsf{out}\,f, \eta\,\mathsf{inl}], \eta\,\mathsf{inr}({g^{\S}}^b_a)]^\star\,\mathsf{out} && \text{(definition of } \eta^\nu) \end{aligned}$$

and also on the other hand,

$$\begin{aligned} &T(\mathsf{id} + {g^{\S}}^b_a)\ [T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out} \\ &\quad = [T(\mathsf{id} + (g^{\S}T^b_a\,\mathsf{inr})^b_a)[\mathsf{out}\,f, \mathsf{out}\,\eta^\nu], \eta\,\mathsf{inr}(g^{\S})^b_a]^\star\,\mathsf{out} \\ &\quad = [[\mathsf{out}\,f, \eta\,\mathsf{inl}], \eta\,\mathsf{inr}({g^{\S}}^b_a)]^\star\,\mathsf{out}, \end{aligned}$$

i.e. indeed $g^{\S}$ satisfies the characteristic property of the final morphism (A.3).

We proceed to prove that $\mathbb{T}^b_a$ is strong. We define the strength τ^ν as the unique final coalgebra morphism shown in the following diagram:

$$\begin{tikzcd} X \times T^b_a Y \arrow[r, "(T\delta)\tau(\mathsf{id}\times\mathsf{out})"] \arrow[d, "\tau^\nu"'] & T(X\times Y + (X\times T^b_a Y)^b_a) \arrow[d, "T(\mathsf{id}+(\tau^\nu)^b_a)"] \\ T^b_a(X\times Y) \arrow[r, "\mathsf{out}"] & T(X\times Y + T^b_a(X\times Y)^b_a) \end{tikzcd}$$

That is, τ^ν is the unique solution of equation $\mathsf{out}\,\tau^\nu = T(\mathsf{id} + (\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out})$. By Lemma A.1, τ^ν is a natural transformation. Let us check the axioms of strength from [16].

- (STR$_1$) The identity $\mathsf{snd} = (T^b_a\,\mathsf{snd})\tau^\nu$ follows from $T^b_a\langle !, \mathsf{id}\rangle\,\mathsf{snd} = \tau^\nu$, since obviously $\mathsf{snd} = (T^b_a\,\mathsf{snd})T^b_a\langle !, \mathsf{id}\rangle\,\mathsf{snd}$. Since τ^ν is uniquely defined by the corresponding characteristic identity, it suffices to show that $T^b_a\langle !, \mathsf{id}\rangle\,\mathsf{snd}$ satisfies the same identity. Indeed,

$$\begin{aligned}
&T\Big(\mathsf{id} + (T_a^b\langle !, \mathsf{id}\rangle\,\mathsf{snd})_a^b\Big)(T\delta)\tau(\mathsf{id}\times\mathsf{out})\\
=\;& T\Big(\langle !, \mathsf{id}\rangle\,\mathsf{snd} + (T_a^b\langle !, \mathsf{id}\rangle\,\mathsf{snd})_a^b\Big)(T\delta)\tau(\mathsf{id}\times\mathsf{out})\\
=\;& T\Big(\big(\langle !, \mathsf{id}\rangle + (T_a^b\langle !, \mathsf{id}\rangle)_a^b\big)\,\mathsf{snd}\Big)\tau(\mathsf{id}\times\mathsf{out})\\
=\;& T\Big(\langle !, \mathsf{id}\rangle + (T_a^b\langle !, \mathsf{id}\rangle)_a^b\Big)\,\mathsf{out}\,\mathsf{snd}\\
=\;& \mathsf{out}(T_a^b\langle !, \mathsf{id}\rangle)\,\mathsf{snd}\,.
\end{aligned}$$

- (STR$_2$) In order to prove that $(T_a^b\alpha)\tau^\nu = \tau^\nu(\mathsf{id}\times\tau^\nu)\alpha : (X\times Y)\times T_a^b\underline{C} \to T_a^b((X\times Y)\times\underline{C})$ where $\alpha : (X\times Y)\times T_a^b\underline{C} \to X\times(Y\times T_a^b\underline{C})$ is the obvious associativity isomorphism, we show that $(T_a^b\alpha^{-1})\tau^\nu(\mathsf{id}\times\tau^\nu)\alpha$ satisfies the identity characterizing τ^ν, i.e.

$$\mathsf{out}\,T_a^b\alpha^{-1}\tau^\nu(\mathsf{id}\times\tau^\nu)\alpha = T(\mathsf{id} + (T_a^b\alpha^{-1}\tau^\nu(\mathsf{id}\times\tau^\nu)\alpha)_a^b)T\delta\tau(\mathsf{id}\times\mathsf{out}).$$

We calculate, transforming the left hand side,

$$\begin{aligned}
&\mathsf{out}\,T_a^b\alpha^{-1}\tau^\nu(\mathsf{id}\times\tau^\nu)\alpha\\
&\quad= T(\alpha^{-1} + (T_a^b\alpha^{-1})_a^b)\,\mathsf{out}\,\tau^\nu(\mathsf{id}\times\tau^\nu)\alpha && \text{(naturality of } \mathsf{out})\\
&\quad= T(\alpha^{-1} + (T_a^b\alpha^{-1})_a^b)\\
&\qquad T(\mathsf{id} + (\tau^\nu)_a^b)(T\delta)\tau(\mathsf{id}\times\mathsf{out})(\mathsf{id}\times\tau^\nu)\alpha && \text{(definition of } \tau^\nu)\\
&\quad= T(\alpha^{-1} + (T_a^b\alpha^{-1}\tau^\nu)_a^b)\\
&\qquad (T\delta)\tau(\mathsf{id}\times(T(\mathsf{id} + (\tau^\nu)_a^b)(T\delta)\tau(\mathsf{id}\times\mathsf{out})))\alpha && \text{(definition of } \tau^\nu)\\
&\quad= T(\alpha^{-1} + (T_a^b\alpha^{-1}\tau^\nu)_a^b)\\
&\qquad (T\delta)T(\mathsf{id}\times((\mathsf{id} + (\tau^\nu)_a^b)\delta))\tau(\mathsf{id}\times(\tau(\mathsf{id}\times\mathsf{out})))\alpha. && \text{(naturality of } \tau)
\end{aligned}$$

We continue to transform the last part of the term:

$$\begin{aligned}
&\tau(\mathsf{id}\times(\tau(\mathsf{id}\times\mathsf{out})))\alpha\\
&\;= \tau(\mathsf{id}\times\tau)(\mathsf{id}\times(\mathsf{id}\times\mathsf{out}))\alpha\\
&\;= \tau(\mathsf{id}\times\tau)\alpha((\mathsf{id}\times\mathsf{id})\times\mathsf{out}) && \text{(naturality of } \alpha)\\
&\;= T\alpha\tau(\mathsf{id}\times\mathsf{out}) && (\tau \text{ strength})
\end{aligned}$$

(contracting a product of identities into an identity in the last step). Summing up, it remains to show that

$$\begin{aligned}
&T(\alpha^{-1} + (T_a^b\alpha^{-1}\tau^\nu)_a^b)T\delta T(\mathsf{id}\times((\mathsf{id} + (\tau^\nu)_a^b)\delta))T\alpha\tau(\mathsf{id}\times\mathsf{out})\\
&= T(\mathsf{id} + (T_a^b\alpha^{-1}\tau^\nu(\mathsf{id}\times\tau^\nu)\alpha)_a^b)T\delta\tau(\mathsf{id}\times\mathsf{out}).
\end{aligned}$$

This will follow once we show that

$$\delta(\mathsf{id}\times((\mathsf{id} + (\tau^\nu)_a^b)\delta))\alpha = (\alpha + ((\mathsf{id}\times\tau^\nu)\alpha)_a^b)\delta.$$

We calculate

$$\delta(\mathsf{id}\times((\mathsf{id} + (\tau^\nu)_a^b)\delta))\alpha$$

$$
\begin{aligned}
&= \delta(\mathsf{id}\times(\mathsf{id}+(\tau^\nu)^b_a))(\mathsf{id}\times\delta)\alpha \\
&= (\mathsf{id}\times\mathsf{id}+(\mathsf{id}\times\tau^\nu)^b_a)\delta(\mathsf{id}\times\delta)\alpha && \text{(naturality of } \delta) \\
&= (\mathsf{id}\times\mathsf{id}+(\mathsf{id}\times\tau^\nu)^b_a)(\alpha+\alpha^b_a)\delta \\
&= (\alpha+((\mathsf{id}\times\tau^\nu)\alpha)^b_a)\delta.
\end{aligned}
$$

Here, we use the obvious identity

$$\delta(\mathsf{id}\times\delta)\alpha = (\alpha+\alpha^b_a)\delta$$

in the last step, which is easily proved by reasoning in the internal language of a bicartesian closed category.

- ($\textsc{str}_3$) In order to obtain the identity $\tau^\nu(\mathsf{id}\times\eta^\nu) = \eta^\nu$, we show that the left hand side satisfies the characteristic equation for η^ν, i.e. $\mathsf{out}\,\tau^\nu(\mathsf{id}\times\eta^\nu) = \eta\,\mathsf{inl}$. Indeed,

$$
\begin{aligned}
\mathsf{out}\,\tau^\nu(\mathsf{id}\times\eta^\nu) &= T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out})(\mathsf{id}\times\eta^\nu) && \text{(definition of } \tau^\nu) \\
&= T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\eta)(\mathsf{id}\times\mathsf{inl}) && \text{(definition of } \eta^\nu) \\
&= T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)\eta(\mathsf{id}\times\mathsf{inl}) && (\textsc{str}_3 \text{ for } \tau) \\
&= T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)T(\mathsf{id}\times\mathsf{inl})\eta && \text{(naturality of } \eta) \\
&= T(\mathsf{id}+(\tau^\nu)^b_a)(T\,\mathsf{inl})\eta \\
&= \eta\,\mathsf{inl}\,.
\end{aligned}
$$

- ($\textsc{str}_4$) Given $f : X \to T^b_a Z$, we show that $(\tau^\nu(\mathsf{id}\times f))^\S\tau^\nu = \tau^\nu(\mathsf{id}\times f^\S)$, which generalizes the corresponding identity in [18] under $f = \mathsf{id}$. Let $g = [f, \eta^\nu]$ and let us show first that $(\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu = \tau^\nu(\mathsf{id}\times g^\S)$. This implies the identity for f as follows:

$$
\begin{aligned}
(\tau^\nu(\mathsf{id}\times f))^\S\tau^\nu &= (\tau^\nu(\mathsf{id}\times g)(\mathsf{id}\times\mathsf{inl}))^\S\tau^\nu \\
&= (\tau^\nu(\mathsf{id}\times g))^\S T^b_a(\mathsf{id}\times\mathsf{inl})\tau^\nu \\
&= (\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu(\mathsf{id}\times T^b_a\,\mathsf{inl}) && \text{(naturality of } \tau^\nu) \\
&= \tau^\nu(\mathsf{id}\times g^\S)(\mathsf{id}\times T^b_a\,\mathsf{inl}) && (\textsc{str}_4 \text{ for } g \text{ and } \tau^\nu) \\
&= \tau^\nu(\mathsf{id}\times g^\S T^b_a\,\mathsf{inl}) \\
&= \tau^\nu(\mathsf{id}\times f^\S).
\end{aligned}
$$

By Lemma 4.1 and by definition of τ^ν, both $g^\S$ and τ^ν are final morphisms from suitable coalgebras. By composing the corresponding commutative squares we obtain the following diagram:

$$\begin{tikzcd}
X \times T^b_a Y \arrow[r, "\mathsf{id} \times [T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out}"] \arrow[d, "\mathsf{id} \times g^\S"'] & X \times T(Z + T^b_a Y^b_a) \arrow[r, "(T\delta)\tau"] \arrow[d, "\mathsf{id} \times T(\mathsf{id} + g^{\S}{}^b_a)"] & T(X \times Z + (X \times T^b_a Y)^b_a) \arrow[dddl, bend left, "T(\mathsf{id} + (\tau^\nu(\mathsf{id} \times g^\S))^b_a)"] \\
X \times T^b_a Z \arrow[r, "\mathsf{id} \times \mathsf{out}"] \arrow[dr, "(T\delta)\tau(\mathsf{id} \times \mathsf{out})"'] \arrow[dd, "\tau^\nu"'] & X \times T(Z + T^b_a Z) \arrow[d, "(T\delta)\tau"] & \\
 & T(X \times Z + (X \times T^b_a Z)^b_a) \arrow[d, "T(\mathsf{id} + (\tau^\nu)^b_a)"] & \\
T^b_a(X \times Z) \arrow[r, "\mathsf{out}"'] & T^b_a(X \times Z + T^b_a(X \times Z)^b_a) &
\end{tikzcd}$$

from which we conclude that

$$\tau^\nu(\mathsf{id} \times g^\S) = \mathsf{coit}\big((T\delta)\tau(\mathsf{id} \times [T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out})\big).$$

We will be done once we show that also $(\tau^\nu(\mathsf{id} \times g))^\S\tau^\nu$ is a morphism from the same coalgebra to the final one, i.e. the identity

$$\begin{aligned}
&\mathsf{out}(\tau^\nu(\mathsf{id} \times g))^\S\tau^\nu \\
&= T(\mathsf{id} + ((\tau^\nu(\mathsf{id} \times g))^\S\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id} \times [T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out}).
\end{aligned} \tag{A.4}$$

Let us denote $T(\mathsf{id} + (T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g$ by h and show that the following diagram commutes:

$$\begin{tikzcd}
T(X \times (Y + (T^b_a Y)^b_a)) \arrow[r, "T\delta"] \arrow[d, "(\tau(\mathsf{id} \times [h, \eta\,\mathsf{inr}]))^\star"'] & T(X \times Y + (X \times T^b_a Y)^b_a) \arrow[d, "{[(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}]^\star}"] \\
T(X \times (Z + (T^b_a Y)^b_a)) \arrow[r, "T\delta"] & T(X \times Z + (X \times T^b_a Y)^b_a)
\end{tikzcd} \tag{A.5}$$

Note that δ can explicitly be given by expression

$$\delta(x, e) = [\lambda y.\,\mathsf{inl}\langle x, y\rangle, \lambda\langle z, c\rangle.\ \mathsf{inr}\langle z, \lambda v.\ \langle x, c(v)\rangle\rangle]e.$$

Therefore,

$$\begin{aligned}
&[(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}]^\star(T\delta) \\
&\quad = ([(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}]\delta)^\star \\
&\quad = (\lambda\langle x, e\rangle.\,[(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}][\lambda y.\,\mathsf{inl}\langle x, y\rangle, \lambda\langle z, c\rangle.\ \mathsf{inr}\langle z, \lambda v.\ \langle x, c(v)\rangle\rangle]e)^\star \\
&\quad = (\lambda\langle x, e\rangle.\,[\lambda y.(T\delta)\tau(x, h(y)), \lambda\langle z, c\rangle.\,\eta\,\mathsf{inr}\langle z, \lambda v.\ \langle x, c(v)\rangle\rangle]e)^\star \\
&\quad = (\lambda\langle x, e\rangle.\,[\lambda y.(T\delta)\tau(x, h(y)), \lambda y.\eta\delta(x, \mathsf{inr}\,y)]e)^\star \\
&\quad = (\lambda\langle x, e\rangle.\,[\lambda y.(T\delta)\tau(x, h(y)), \lambda y.(T\delta)\tau(x, \eta\,\mathsf{inr}(y))]e)^\star \\
&\quad = (\lambda\langle x, e\rangle.\,(T\delta)\tau(x, [h, \eta\,\mathsf{inr}]e))^\star
\end{aligned}$$

$$= (T\delta)(\tau(\mathsf{id}\times[h,\eta\,\mathsf{inr}]))^\star.$$

Now, we obtain (A.4) as follows:

$$\begin{aligned}
&T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)\\
&\quad(T\delta)\tau(\mathsf{id}\times[T(\mathsf{id}+(T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g,\eta\,\mathsf{inr}]^\star\,\mathsf{out})\\
&= T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)\\
&\quad[(T\delta)\tau(\mathsf{id}\times T(\mathsf{id}+(T^b_a\,\mathsf{inr})^b_a)\,\mathsf{out}\,g),\eta\,\mathsf{inr}]^\star(T\delta)\,\mathsf{out}) && \text{(A.5)}\\
&= T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)\\
&\quad[(T\delta)\tau(\mathsf{id}\times T(\mathsf{id}+(T^b_a\,\mathsf{inr})^b_a))(\mathsf{id}\times\mathsf{out}\,g),\eta\,\mathsf{inr}]^\star\\
&\quad(T\delta)\tau(\mathsf{id}\times\mathsf{out})\\
&= T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)\\
&\quad[(T\delta)T(\mathsf{id}\times(\mathsf{id}+(T^b_a\,\mathsf{inr})^b_a))\tau(\mathsf{id}\times\mathsf{out}\,g),\eta\,\mathsf{inr}]^\star\\
&\quad(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && \text{(naturality of } \tau)\\
&= T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)\\
&\quad[T(\mathsf{id}+(\mathsf{id}\times T^b_a\,\mathsf{inr})^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out}\,g),\eta\,\mathsf{inr}]^\star\\
&\quad(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && \text{(naturality of } \delta)\\
&= [T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu(\mathsf{id}\times T^b_a\,\mathsf{inr}))^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out}\,g),\\
&\quad\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)]^\star(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && \text{(naturality of } \eta)\\
&= [T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times g\,\mathsf{inr}))^\S\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out}\,g),\\
&\quad\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)]^\star(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && \text{(naturality of } \tau^\nu)\\
&= [T(\mathsf{id}+((\tau^\nu(\mathsf{id}\times\eta^\nu))^\S\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out}\,g),\\
&\quad\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)]^\star(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && (\text{since } g=[f,\eta^\nu])\\
&= [T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out}\,g),\\
&\quad\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)]^\star(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && (\textsc{str}_3\ \textsc{for}\tau^\nu)\\
&= [T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out})(\mathsf{id}\times g),\\
&\quad\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)]^\star(T\delta)\tau(\mathsf{id}\times\mathsf{out})\\
&= [\mathsf{out}\,\tau^\nu(\mathsf{id}\times g),\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu)^b_a)]^\star(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && \text{(definition of } \tau^\nu)\\
&= [\mathsf{out}\,\tau^\nu(\mathsf{id}\times g),\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S)^b_a)]^\star\\
&\quad T(\mathsf{id}+(\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id}\times\mathsf{out})\\
&= [\mathsf{out}\,\tau^\nu(\mathsf{id}\times g),\eta\,\mathsf{inr}((\tau^\nu(\mathsf{id}\times g))^\S)^b_a)]^\star\,\mathsf{out}\,\tau^\nu && \text{(definition of } \tau^\nu)\\
&= \mathsf{out}(\tau^\nu(\mathsf{id}\times g))^\S\tau^\nu. && \text{(definition of } \S)
\end{aligned}$$

We have thus shown all properties ($\textsc{str}_1$)–($\textsc{str}_4$) and the proof is completed.□

Proof of Theorem 4.5

We tackle Claim i and first show existence, i.e. we define an iteration operator on $\mathbb{T}^b_a$ and show that it satisfies the axioms for complete Elgot monads and is consistent with iteration in $\mathbb{T}$. Along the way we will establish also Claim ii.

Our notion of guardedness coincides with that of Uustalu [27], who shows that guarded morphisms f have unique iterates $f^\dagger$. For any $f : X \to T^b_a(Y+X)$ (possibly not guarded) let $\rhd f : X \to T^b_a(Y+X)$ be the morphism $\mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})w^\dagger$ where w is the composed morphism

$$X \xrightarrow{f} T^b_a(Y+X) \xrightarrow{\mathsf{out}} T((Y+X)+T^b_a(Y+X)^b_a)$$
$$\xrightarrow{(T\pi)} T((Y+T^b_a(Y+X)^b_a)+X)$$

with $\pi = [[\mathsf{inl}\,\mathsf{inl}, \mathsf{inr}], \mathsf{inl}\,\mathsf{inr}]$. Obviously, $\rhd f$ is guarded by definition. We now define $f^\dagger = (\rhd f)^\dagger$. To make sure that this definition is consistent we check that $\rhd f = f$ whenever f is guarded. Suppose, $\mathsf{out}\, f = T(\mathsf{inl}+\mathsf{id})u$. Then $f = \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})u$ and therefore

$$\begin{aligned}
\rhd f &= \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger \\
&= \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,\mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})u)^\dagger \\
&= \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)T(\mathsf{inl}+\mathsf{id})u)^\dagger \\
&= \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})(T\,\mathsf{inl}\, u)^\dagger \\
&= \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})u \\
&= f.
\end{aligned}$$

We are left to check the axioms of Elgot monads (Definition 3.1).

- *Unfolding.* For any $f : X \to T^b_a(Y+X)$ we have

$$\begin{aligned}
f^\dagger &= [\eta^\nu, f^\dagger]^\S \rhd f && \text{(definition of } \dagger) \\
&= [\eta^\nu, f^\dagger]^\S\,\mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger && \text{(definition of } \rhd) \\
&= \mathsf{out}^{-1}\Big[[\eta\,\mathsf{inl}, \mathsf{out}\, f^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, f^\dagger]^{\S^b_a}\Big]^\star \\
&\quad\; T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger && \text{(definition of } \S) \\
&= \mathsf{out}^{-1}\Big[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\,[\eta^\nu, f^\dagger]^{\S^b_a}\Big]^\star((T\pi)\,\mathsf{out}\, f)^\dagger \\
&= \mathsf{out}^{-1}\, T(\mathsf{id}+[\eta^\nu, f^\dagger]^{\S^b_a})((T\pi)\,\mathsf{out}\, f)^\dagger
\end{aligned}$$

  and thus we obtain the following intermediate equation:

$$\mathsf{out}\, f^\dagger = T\Big(\mathsf{id}+[\eta^\nu, f^\dagger]^{\S^b_a}\Big)((T\pi)\,\mathsf{out}\, f)^\dagger. \tag{A.6}$$

  Now, continuing the above calculation we obtain

$$\begin{aligned}
f^\dagger &= \mathsf{out}^{-1}\, T(\mathsf{id}+[\eta^\nu, f^\dagger]^{\S^b_a})((T\pi)\,\mathsf{out}\, f)^\dagger \\
&= \mathsf{out}^{-1}\, T(\mathsf{id}+[\eta^\nu, f^\dagger]^{\S^b_a})[\eta, ((T\pi)\,\mathsf{out}\, f)^\dagger]^\star(T\pi)\,\mathsf{out}\, f && \text{(unfolding)} \\
&= \mathsf{out}^{-1}[T(\mathsf{id}+[\eta^\nu, f^\dagger]^{\S^b_a})\eta, \mathsf{out}\, f^\dagger]^\star(T\pi)\,\mathsf{out}\, f && \text{(A.6)} \\
&= \mathsf{out}^{-1}[\eta(\mathsf{id}+[\eta^\nu, f^\dagger]^{\S^b_a}), \mathsf{out}\, f^\dagger]^\star(T\pi)\,\mathsf{out}\, f && \text{(naturality of } \eta) \\
&= \mathsf{out}^{-1}\Big[[\eta\,\mathsf{inl}, \mathsf{out}\, f^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, f^\dagger]^{\S^b_a}\Big]^\star\,\mathsf{out}\, f \\
&= \mathsf{out}^{-1}\Big[\mathsf{out}[\eta^\nu, f^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, f^\dagger]^{\S^b_a}\Big]^\star\,\mathsf{out}\, f && \text{(definition of } \eta^\nu)
\end{aligned}$$

$$= [\eta^\nu, f^\dagger]^\S \,\mathsf{out}^{-1}\,\mathsf{out}\, f \qquad \text{(naturality of §)}$$
$$= [\eta^\nu, f^\dagger]^\S f.$$

- *Naturality.* Assume that $h : X \to T_a^b(Y + X)$ is guarded and show that so is $[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S h$ for any $g : Y \to Z$. Let u be such that $\mathsf{out}\, h = T(\mathsf{inl} + \mathsf{id})u$ and let $w = [(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]$. Then

$$\begin{aligned}
\mathsf{out}[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S h &= \left[\mathsf{out}\, w, \eta\,\mathsf{inr}(w^\S)_a^b\right]^\star \mathsf{out}\, h \\
&= \left[\mathsf{out}\, w, \eta\,\mathsf{inr}(w^\S)_a^b\right]^\star T(\mathsf{inl} + \mathsf{id})u \\
&= \left[\mathsf{out}\, w\,\mathsf{inl}, \eta\,\mathsf{inr}(w^\S)_a^b\right]^\star u \\
&= \left[\mathsf{out}(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}(w^\S)_a^b\right]^\star u \\
&= \left[T(\mathsf{inl} + T_a^b\,\mathsf{inl}_a^b)\,\mathsf{out}\, g, \eta\,\mathsf{inr}(w^\S)_a^b\right]^\star u \\
&= T(\mathsf{inl} + \mathsf{id})\left[T(\mathsf{id} + T_a^b\,\mathsf{inl}_a^b)\,\mathsf{out}\, g, \eta\,\mathsf{inr}(w^\S)_a^b\right]^\star u.
\end{aligned}$$

Now, since $t = [(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f$ is guarded, it is the unique fixpoint of the map

$$t \mapsto [\eta^\nu, t]^\S[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f.$$

However, on the other hand,

$$\begin{aligned}
&[\eta^\nu, g^\S f^\dagger]^\S[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f \\
&\quad= [g, g^\S f^\dagger]^\S \rhd f \\
&\quad= [g, g^\S (\rhd f)^\dagger]^\S \rhd f \\
&\quad= g^\S[\eta^\nu, (\rhd f)^\dagger]^\S \rhd f \\
&\quad= g^\S f^\dagger
\end{aligned}$$

and therefore $t^\dagger = g^\S f^\dagger$. It remains to show that $[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f = \rhd[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S f$. Since $\rhd f$ is guarded by definition, we know by the calculation above that $[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f$ is guarded and therefore

$$[(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\S \rhd f = \rhd[(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\S \rhd f.$$

To finish the proof, we calculate

$$\rhd[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f = \mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}[(T_a^b\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S \rhd f)^\dagger.$$

Further transforming the dagger expression in the previous term yields

$$\begin{aligned}
&((T\pi)\,\mathsf{out}[(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\S \rhd f)^\dagger \\
&= ((T\pi)[\mathsf{out}\, w, \eta\,\mathsf{inr}(w^\S)_a^b]^\star T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger)^\dagger \\
&= ((T\pi)[\mathsf{out}(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}(w^\S)_a^b]^\star((T\pi)\,\mathsf{out}\, f)^\dagger)^\dagger \\
&= (([(T\,\mathsf{inl})(T\pi)[\mathsf{out}(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}(w^\S)_a^b], \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\, f)^\dagger)^\dagger && \text{(nat. for } \mathbb{T}) \\
&= (T[\mathsf{id}, \mathsf{inr}][(T\,\mathsf{inl})(T\pi)[\mathsf{out}(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}(w^\S)_a^b], \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\, f)^\dagger && \text{(codiag. for } \mathbb{T}) \\
&= ([[(T\pi)\,\mathsf{out}(T_a^b\,\mathsf{inl})g, \eta\,\mathsf{inr}], (T\pi)\eta\,\mathsf{inr}(w^\S)_a^b]^\star\,\mathsf{out}\, f)^\dagger
\end{aligned}$$

$$= ((T\pi)[[\mathsf{out}(T^b_a\,\mathsf{inl})g, \eta\,\mathsf{inl}\,\mathsf{inr}], \eta\,\mathsf{inr}(w^\S)^b_a]^\star\,\mathsf{out}\,f)^\dagger$$
$$= ((T\pi)[\mathsf{out}[(T^b_a\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}], \eta\,\mathsf{inr}(w^\S)^b_a]^\star\,\mathsf{out}\,f)^\dagger$$
$$= ((T\pi)\,\mathsf{out}[(T^b_a\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^\S f)^\dagger$$

and therefore

$$\rhd[(T^b_a\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\S \rhd f = \rhd[(T^b_a\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\S f.$$

- *Dinaturality.* Given $g : X \to T^b_a(Y+Z)$ and $h : Z \to T^b_a(Y+X)$, let $s = [\eta^\nu\,\mathsf{inl}, h]^\S g : X \to T^b_a(Y+X)$, $t = [\eta^\nu\,\mathsf{inl}, g]^\S h : Z \to T^b_a(Y+Z)$, $w = [\eta^\nu, t^\dagger]^\S g : X \to T^b_a Y$. The idea is to show the identity

$$[\eta^\nu, w]^\S \rhd s = [\eta^\nu, t^\dagger]^\S [\eta^\nu\,\mathsf{inl}, \rhd t]^\S g \tag{A.7}$$

from which we will be able to obtain that

$$\begin{aligned} w &= [\eta^\nu, t^\dagger]^\S g \\ &= [\eta^\nu, [\eta^\nu, t^\dagger]^\S \rhd t]^\S g && \text{(unfolding)} \\ &= [\eta^\nu, t^\dagger]^\S [\eta^\nu\,\mathsf{inl}, \rhd t]^\S g \\ &= [\eta^\nu, w]^\S \rhd s, && \text{(A.7)} \end{aligned}$$

i.e. that w satisfies the recursive equation uniquely identifying $s^\dagger$ and hence $w = s^\dagger$. Let

$$\mathbf{p} = T\big((\mathsf{id} + [\eta^\nu\,\mathsf{inl}, h]^{\S}{}^b_a) + \mathsf{id}\big) : T\big(Y + T^b_a(Y+Z) + Z\big) \to T\big(Y + T^b_a(Y+X) + Z\big),$$
$$\mathbf{q} = T\big((\mathsf{id} + [\eta^\nu\,\mathsf{inl}, g]^{\S}{}^b_a) + \mathsf{id}\big) : T\big(Y + T^b_a(Y+X) + X\big) \to T\big(Y + T^b_a(Y+Z) + X\big)$$

and observe that

$$\begin{aligned} ((T\pi)\,\mathsf{out}\,s)^\dagger &= \Big((T\pi)\big[\mathsf{out}[\eta^\nu\,\mathsf{inl}, h], \eta\,\mathsf{inr}[\eta^\nu\,\mathsf{inl}, h]^{\S}{}^b_a\big]^\star\,\mathsf{out}\,g\Big)^\dagger \\ &= \Big(\big[\eta\,\mathsf{inl}(\mathsf{id} + [\eta^\nu\,\mathsf{inl}, h]^{\S}{}^b_a), ((T\pi)\,\mathsf{out}\,h)\big]^\star((T\pi)\,\mathsf{out}\,g)\Big)^\dagger \\ &= ([\eta\,\mathsf{inl}, ((T\pi)\,\mathsf{out}\,h)]^\star \mathbf{p}((T\pi)\,\mathsf{out}\,g))^\dagger. \end{aligned}$$

An analogous calculation applies to $((T\pi)\,\mathsf{out}\,t)^\dagger$ and hence we obtain

$$((T\pi)\,\mathsf{out}\,s)^\dagger = ([\eta\,\mathsf{inl}, ((T\pi)\,\mathsf{out}\,h)]^\star \mathbf{p}((T\pi)\,\mathsf{out}\,g))^\dagger, \tag{A.8}$$
$$((T\pi)\,\mathsf{out}\,t)^\dagger = ([\eta\,\mathsf{inl}, ((T\pi)\,\mathsf{out}\,g)]^\star \mathbf{q}((T\pi)\,\mathsf{out}\,h))^\dagger. \tag{A.9}$$

Let us calculate the left-hand side of (A.7):

$$\begin{aligned} &\mathsf{out}[\eta^\nu, w]^\S \rhd s \\ &\quad= \big[\mathsf{out}[\eta^\nu, w], \eta\,\mathsf{inr}\,[\eta^\nu, w]^{\S}{}^b_a\big]^\star\,\mathsf{out} \rhd s \\ &\quad= \big[\mathsf{out}[\eta^\nu, w], \eta\,\mathsf{inr}\,[\eta^\nu, w]^{\S}{}^b_a\big]^\star T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\,s)^\dagger \\ &\quad= T\big(\mathsf{id} + [\eta^\nu, w]^{\S}{}^b_a\big)((T\pi)\,\mathsf{out}\,s)^\dagger \end{aligned}$$

$$
\begin{aligned}
&= T\big(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a}\big)([\eta\,\mathsf{inl}, ((T\pi)\,\mathsf{out}\,h)]^\star \mathbf{p}((T\pi)\,\mathsf{out}\,g))^\dagger \qquad (A.8)\\
&= T\big(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a}\big)[\eta, ([\eta\,\mathsf{inl}, \mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger]^\star \mathbf{p}((T\pi)\,\mathsf{out}\,g)\\
&= T\big(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a}\big)[\eta(\mathsf{id} + [\eta^\nu\,\mathsf{inl}, h]^{\S}{}^{b}_{a}),\\
&\qquad ([\eta\,\mathsf{inl}, \mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger]^\star((T\pi)\,\mathsf{out}\,g)\\
&= [\eta(\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}),\\
&\qquad T(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a})([\eta\,\mathsf{inl}, \mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger]^\star((T\pi)\,\mathsf{out}\,g),
\end{aligned}
$$

where for the last step, note that

$$
\begin{aligned}
&[\eta^\nu, [\eta^\nu, w]^{\S} h]^{\S}\\
&\quad = [\eta^\nu, [\eta^\nu, [\eta^\nu, t^\dagger]^{\S} g]^{\S} h]^{\S}\\
&\quad = [\eta^\nu, [\eta^\nu, t^\dagger]^{\S}[\eta^\nu\,\mathsf{inl}, g]^\star h]^{\S}\\
&\quad = [\eta^\nu, t^\dagger]^{\S}.
\end{aligned}
$$

Now, let us calculate the right-hand side of (A.7):

$$
\begin{aligned}
&\mathsf{out}[\eta^\nu, t^\dagger]^{\S}[\eta^\nu\,\mathsf{inl}, \rhd t]^{\S} g\\
&\quad = \big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star\, \mathsf{out}[\eta^\nu\,\mathsf{inl}, \rhd t]^{\S} g\\
&\quad = \big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star [\mathsf{out}[\eta^\nu\,\mathsf{inl}, \rhd t], \eta\,\mathsf{inr}\,[\eta^\nu\,\mathsf{inl}, \rhd t]^{\S}{}^{b}_{a}]^\star\, \mathsf{out}\,g\\
&\quad = \big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star [\eta(\mathsf{id} + [\eta^\nu\,\mathsf{inl}, \rhd t]^{\S}{}^{b}_{a}), \mathsf{out} \rhd t]^\star((T\pi)\,\mathsf{out}\,g)\\
&\quad = [\eta(\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}), \big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star\, \mathsf{out} \rhd t]^\star((T\pi)\,\mathsf{out}\,g).
\end{aligned}
$$

We have thus reduced (A.7) to

$$
T(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a})([\eta\,\mathsf{inl}, \mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star(T\pi)\,\mathsf{out}\,h)^\dagger = \big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star\, \mathsf{out} \rhd t.
$$

Then, on the one hand

$$
\begin{aligned}
&T(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a})([\eta\,\mathsf{inl}, \mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger\\
&= (T((\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a}) + \mathsf{id})[\eta\,\mathsf{inl}, \mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger\\
&= ([\eta\,\mathsf{inl}(\mathsf{id} + [\eta^\nu, w]^{\S}{}^{b}_{a}), T((\mathsf{inl} + [\eta^\nu, w]^{\S}{}^{b}_{a}) + \mathsf{id})\mathbf{p}((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger\\
&= ([\eta\,\mathsf{inl}(\,\mathrm{d} + [\eta^\nu, w]^{\S}{}^{b}_{a}), T((\mathsf{inl} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}) + \mathsf{id})((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger\\
&= (T((\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}) + \mathsf{id})[\eta\,\mathsf{inl}(\mathsf{id} + [\eta^\nu\,\mathsf{inl}, g]^{\S}{}^{b}_{a}), ((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger\\
&= T(\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a})([\eta\,\mathsf{inl}(\mathsf{id} + [\eta^\nu\,\mathsf{inl}, g]^{\S}{}^{b}_{a}), ((T\pi)\,\mathsf{out}\,g)]^\star((T\pi)\,\mathsf{out}\,h))^\dagger\\
&= T(\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a})([\eta\,\mathsf{inl}, ((T\pi)\,\mathsf{out}\,g)]^\star \mathbf{q}((T\pi)\,\mathsf{out}\,h))^\dagger
\end{aligned}
$$

and also on the other hand

$$
\begin{aligned}
&\big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star\, \mathsf{out} \rhd t\\
&\qquad = \big[\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,[\eta^\nu, t^\dagger]^{\S}{}^{b}_{a}\big]^\star T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\,t)^\dagger
\end{aligned}
$$

$$= T(\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a})((T\pi)\,\mathsf{out}\,t)^\dagger$$
$$= T(\mathsf{id} + [\eta^\nu, t^\dagger]^{\S}{}^{b}_{a})([\eta\,\mathsf{inl}, ((T\pi)\,\mathsf{out}\,g)]^\star \mathbf{q}((T\pi)\,\mathsf{out}\,h))^\dagger. \qquad \text{(A.9)}$$

The proof is thus completed.

- *Codiagonal.* Let $g : X \to T^b_a(Y + X + X)$. We shall show below that

$$\rhd(T^b_a[\mathsf{id}, \mathsf{inr}] \rhd g) = \rhd (T^b_a[\mathsf{id}, \mathsf{inr}]g). \qquad \text{(A.10)}$$

Since $T^b_a[\mathsf{id}, \mathsf{inr}]^\dagger g$ is the unique fixpoint of the map

$$\gamma \mapsto [\eta^\nu, \gamma]^\S \rhd T^b_a[\mathsf{id}, \mathsf{inr}]g$$

we will be done once we show that $(g^\dagger)^\dagger$ is also a fixpoint of the same map, i.e.

$$(g^\dagger)^\dagger = \big[\eta^\nu, (g^\dagger)^\dagger\big]^\S \rhd \big(T^b_a[\mathsf{id}, \mathsf{inr}]g\big). \qquad \text{(A.11)}$$

We denote by $\pi : (Y + X) + X \to (Y + X) + X$ the morphism swapping two last components of the coproduct. We consider the following three cases.

(i) $T^b_a[\mathsf{id}, \mathsf{inr}]g$ *is guarded.* Then we obtain (A.11) directly as follows

$$\begin{aligned}
(g^\dagger)^\dagger &= [\eta^\nu, (g^\dagger)^\dagger]^\S g^\dagger && \text{(unfolding)}\\
&= [\eta^\nu, (g^\dagger)^\dagger]^\S [\eta^\nu, g^\dagger]^\S g && \text{(unfolding)}\\
&= \big[[\eta^\nu, (g^\dagger)^\dagger], [\eta^\nu, (g^\dagger)^\dagger]^\S g^\dagger\big]^\S g\\
&= \big[[\eta^\nu, (g^\dagger)^\dagger], (g^\dagger)^\dagger\big]^\S g && \text{(unfolding)}\\
&= [\eta^\nu, (g^\dagger)^\dagger]^\S (T^b_a[\mathsf{id}, \mathsf{inr}])g\\
&= [\eta^\nu, (g^\dagger)^\dagger]^\S \rhd \big((T^b_a[\mathsf{id}, \mathsf{inr}])g\big).
\end{aligned}$$

(ii) $(T^b_a\pi)g$ *is guarded.* E.g. let $(T^b_a\pi)g = \mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})u$. Then $T^b_a[\mathsf{id}, \mathsf{inr}] \rhd g$ is also guarded, which is certified by the following calculation:

$$\begin{aligned}
&T^b_a[\mathsf{id}, \mathsf{inr}] \rhd g\\
&\quad = T^b_a[\mathsf{id}, \mathsf{inr}] \rhd ((T^b_a\pi)\,\mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})u)\\
&\quad = T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})\big((T\pi)\,\mathsf{out}(T^b_a\pi)\,\mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})u\big)^\dagger\\
&\quad = T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})\big((T\pi)T(\pi + (T^b_a\pi)^b_a)T(\mathsf{inl} + \mathsf{id})u\big)^\dagger\\
&\quad = T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})\big(T((\mathsf{inl} + \mathsf{id}) + \mathsf{id})(T\pi)T(\mathsf{id} + (T^b_a\pi)^b_a)u\big)^\dagger\\
&\quad = T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})T(\mathsf{inl} + \mathsf{id})\big((T\pi)T(\mathsf{id} + (T^b_a\pi)^b_a)u\big)^\dagger\\
&\quad = T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1} T(\mathsf{inl}\,\mathsf{inl} + \mathsf{id})\big((T\pi)T(\mathsf{id} + (T^b_a\pi)^b_a)u\big)^\dagger\\
&\quad = \mathsf{out}^{-1} T([\mathsf{id}, \mathsf{inr}] + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a)T(\mathsf{inl}\,\mathsf{inl} + \mathsf{id})\big((T\pi)T(\mathsf{id} + (T^b_a\pi)^b_a)u\big)^\dagger\\
&\quad = \mathsf{out}^{-1} T(\mathsf{inl} + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a)\big((T\pi)T(\mathsf{id} + (T^b_a\pi)^b_a)u\big)^\dagger\\
&\quad = \mathsf{out}^{-1} T(\mathsf{inl} + \mathsf{id})T(\mathsf{id} + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a)\big((T\pi)T(\mathsf{id} + (T^b_a\pi)^b_a)u\big)^\dagger.
\end{aligned}$$

The proof of (A.11) now runs as follows:

$$
\begin{aligned}
(g^\dagger)^\dagger &= ((\rhd g)^\dagger)^\dagger && \text{(definition of } \dagger\text{)}\\
&= [\eta^\nu, ((\rhd g)^\dagger)^\dagger]^\S \rhd \big((T^b_a[\mathsf{id}, \mathsf{inr}]) \rhd g\big) && \text{(Clause (i))}\\
&= [\eta^\nu, (g^\dagger)^\dagger]^\S \rhd \big((T^b_a[\mathsf{id}, \mathsf{inr}])g\big). && \text{(A.10)}
\end{aligned}
$$

(iii) *g is guarded.* Let $h = (T^b_a\pi) \rhd (T^b_a\pi)g$. It is easy to see that h is guarded. We use the following identity

$$
\rhd g^\dagger = [\eta^\nu, g^\dagger]^\S h \tag{A.12}
$$

whose proof runs as follows. Let $g = \mathsf{out}^{-1}T(\mathsf{inl}+\mathsf{id})u$ for some u and observe that $\pi\,\mathsf{inl} = (\mathsf{inl}+\mathsf{id})$. We apply out to the right-hand side of the equation,

$$
\begin{aligned}
&\mathsf{out}[\eta^\nu, g^\dagger]^\S (T^b_a\pi) \rhd (T^b_a\pi)g\\
&= [\mathsf{out}[\eta^\nu, g^\dagger], \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star\\
&\qquad \mathsf{out}(T^b_a\pi)\,\mathsf{out}^{-1}T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}(T^b_a\pi)g)^\dagger\\
&= [\mathsf{out}[\eta^\nu, g^\dagger]\pi\,\mathsf{inl}, \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S(T^b_a\pi))^b_a]^\star((T\pi)\,\mathsf{out}(T^b_a\pi)g)^\dagger\\
&= [\mathsf{out}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S(T^b_a\pi))^b_a]^\star\\
&\qquad ((T\pi)T(\pi + (T^b_a\pi)^b_a)\,\mathsf{out}\,g)^\dagger\\
&= ([(T\,\mathsf{inl})[\mathsf{out}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S(T^b_a\pi))^b_a], \eta\,\mathsf{inr}]^\star\\
&\qquad (T\pi)T(\pi + (T^b_a\pi)^b_a)\,\mathsf{out}\,g)^\dagger && \text{(naturality)}\\
&= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S(T^b_a\pi))^b_a], \eta\,\mathsf{inr}]^\star\\
&\qquad (T\pi)T(\pi + (T^b_a\pi)^b_a)\,\mathsf{out}\,g)^\dagger\\
&= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}], \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S(T^b_a\pi))^b_a]^\star\\
&\qquad T(\pi + (T^b_a\pi)^b_a)\,\mathsf{out}\,g)^\dagger && \text{(defn. } (T\pi)\text{)}\\
&= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}]\pi, \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S\\
&\qquad (T^b_a\pi)(T^b_a\pi))^b_a]^\star T(\mathsf{inl}+\mathsf{id})u)^\dagger && (g\text{ guarded})\\
&= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}]\pi\,\mathsf{inl}, \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star u)^\dagger\\
&= ([[\eta\,\mathsf{inl}\,\mathsf{inl}\,\mathsf{inl}, \eta\,\mathsf{inr}], \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star u)^\dagger\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}), \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star u)^\dagger.
\end{aligned}
$$

On the other hand, applying out to the left-hand side yields the same result:

$$
\begin{aligned}
&\mathsf{out} \rhd (g^\dagger)\\
&= T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}(g^\dagger))^\dagger\\
&= ([(T\,\mathsf{inl})\eta(\mathsf{inl}+\mathsf{id}), \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}(g^\dagger))^\dagger && \text{(naturality)}\\
&= ([[\eta\,\mathsf{inl}\,\mathsf{inl}\,\mathsf{inl}, \mathsf{inl}\,\mathsf{inr}], \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}[\eta^\nu, g^\dagger]^\S g)^\dagger\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}), \eta\,\mathsf{inl}\,\mathsf{inr}]^\star[\mathsf{out}[\eta^\nu, g^\dagger], \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star\,\mathsf{out}\,g)^\dagger && \text{(defn. } (T\pi)\text{)}\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}), \eta\,\mathsf{inl}\,\mathsf{inr}]^\star[[\eta\,\mathsf{inl}, \mathsf{out}\,g^\dagger], \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star\\
&\qquad T(\mathsf{inl}+\mathsf{id})u)^\dagger && (g\text{ guarded})
\end{aligned}
$$

$$= ([\eta(\mathsf{inl}\,\mathsf{inl} + \mathsf{id}), \eta\,\mathsf{inl}\,\mathsf{inr}]^\star[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star u)^\dagger$$
$$= ([\eta(\mathsf{inl}\,\mathsf{inl} + \mathsf{id}), \eta\,\mathsf{inl}\,\mathsf{inr}([\eta^\nu, g^\dagger]^\S)^b_a]^\star u)^\dagger.$$

Then the goal can be obtained as follows. First, observe the following:

$$\begin{aligned}
(g^\dagger)^\dagger &= ([\eta^\nu, g^\dagger]^\S h)^\dagger && \text{(A.12)}\\
&= ([[\eta^\nu\,\mathsf{inl}, \eta^\nu\,\mathsf{inr}], g^\dagger]^\S h)^\dagger\\
&= ([[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}]^\S \rhd T^b_a\pi\, g)^\dagger && \text{(defn. } \pi)\\
&= ((T^b_a[\mathsf{id}, \mathsf{inr}])[[\eta^\nu\,\mathsf{inl}\,\mathsf{inl}, T^b_a\,\mathsf{inl}\, g^\dagger], \eta\,\mathsf{inr}]^\S \rhd T^b_a\pi\, g)^\dagger\\
&= (([[\eta^\nu\,\mathsf{inl}\,\mathsf{inl}, T^b_a\,\mathsf{inl}\, g^\dagger], \eta\,\mathsf{inr}]^\S \rhd T^b_a\pi\, g)^\dagger)^\dagger && \text{(Clause (ii))}\\
&= (([T^b_a\,\mathsf{inl}[\eta^\nu\,\mathsf{inl}, g^\dagger], \eta\,\mathsf{inr}]^\S \rhd T^b_a\pi\, g)^\dagger)^\dagger\\
&= ([\eta^\nu\,\mathsf{inl}, g^\dagger]^\S(\rhd T^b_a\pi\, g)^\dagger)^\dagger && \text{(naturality)}\\
&= ([\eta^\nu\,\mathsf{inl}, g^\dagger]^\S(T^b_a\pi\, g)^\dagger)^\dagger. && \text{(defn. } \dagger)\\
&= [\eta^\nu, ([\eta^\nu\,\mathsf{inl}, g^\dagger]^\S(T^b_a\pi\, g)^\dagger)^\dagger]^\S[\eta^\nu\,\mathsf{inl}, g^\dagger]^\S(T^b_a\pi\, g)^\dagger && \text{(unfolding)}\\
&= [\eta^\nu, (g^\dagger)^\dagger]^\S[\eta^\nu\,\mathsf{inl}, g^\dagger]^\S(T^b_a\pi\, g)^\dagger\\
&= [\eta^\nu, [\eta^\nu, (g^\dagger)^\dagger]^\S g^\dagger]^\S(T^b_a\pi\, g)^\dagger\\
&= [\eta^\nu, (g^\dagger)^\dagger]^\S(T^b_a\pi\, g)^\dagger. && \text{(unfolding)}
\end{aligned}$$

It is easy to see that $(T^b_a\pi\, g)^\dagger$ is guarded, and hence, by the previous calculation, $(g^\dagger)^\dagger = ((T^b_a\pi\, g)^\dagger)^\dagger$. Finally, by Clause (ii), $((T^b_a\pi\, g)^\dagger)^\dagger = ((T^b_a[\mathsf{id}, \mathsf{inr}])T^b_a\pi\, g)^\dagger = ((T^b_a[\mathsf{id}, \mathsf{inr}])g)^\dagger$.

(iv) *g is unrestricted.* Then,

$$\begin{aligned}
(g^\dagger)^\dagger &= ((\rhd g)^\dagger)^\dagger\\
&= ((T^b_a[\mathsf{id}, \mathsf{inr}]) \rhd g)^\dagger && \text{(Clause (iii))}\\
&= (\rhd(T^b_a[\mathsf{id}, \mathsf{inr}]) \rhd (T^b_a\pi)g)^\dagger\\
&= (\rhd(T^b_a[\mathsf{id}, \mathsf{inr}])(T^b_a\pi)g)^\dagger && \text{(A.10)}\\
&= ((T^b_a[\mathsf{id}, \mathsf{inr}])g)^\dagger
\end{aligned}$$

and we are done.

It remains to prove (A.10):

$$\begin{aligned}
&\rhd T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1}T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger\\
=\;& \mathsf{out}^{-1}T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, T^b_a[\mathsf{id}, \mathsf{inr}]\,\mathsf{out}^{-1}T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger)^\dagger
\end{aligned}$$

Let us transform the part after $\mathsf{out}^{-1}T(\mathsf{inl} + \mathsf{id})$ further:

$$\begin{aligned}
&((T\pi)T([\mathsf{id}, \mathsf{inr}] + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a)T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger)^\dagger\\
=\;& ((T\pi)T(\mathsf{id} + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a)((T\pi)\,\mathsf{out}\, g)^\dagger)^\dagger\\
=\;& (([T\,\mathsf{inl}\,\pi\eta(\mathsf{id} + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a), \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\, g)^\dagger)^\dagger && \text{(naturality)}\\
=\;& ([(T\pi)\eta(\mathsf{id} + T^b_a[\mathsf{id}, \mathsf{inr}]^b_a), (T\pi)\eta\,\mathsf{inl}\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\, g)^\dagger && \text{(codiagonal)}\\
=\;& ((T\pi)[[\eta\,\mathsf{inl}, \eta\,\mathsf{inl}\,\mathsf{inr}], \eta\,\mathsf{inr}\, T^b_a[\mathsf{id}, \mathsf{inr}]^b_a]^\star\,\mathsf{out}\, g)^\dagger
\end{aligned}$$

$$= ((T\pi)[\mathsf{out}(\eta^\nu[\mathsf{id},\mathsf{inr}]), \eta\,\mathsf{inr}\, T^b_a[\mathsf{id},\mathsf{inr}]^b_a]^\star\, \mathsf{out}\, g)^\dagger$$
$$= ((T\pi)\,\mathsf{out}(T^b_a[\mathsf{id},\mathsf{inr}])g)^\dagger$$

Therefore,

$$\begin{aligned} & \rhd T^b_a[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{-1}T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger \\ = {} & \mathsf{out}^{-1}T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}(T^b_a[\mathsf{id},\mathsf{inr}])g)^\dagger \\ = {} & \rhd T^b_a[\mathsf{id},\mathsf{inr}]g. \end{aligned}$$

- *Uniformity.* First, show uniformity under the assumption that g is guarded. Suppose $fh = T^b_a(\mathsf{id}+h)g$. It is then sufficient to verify that $f^\dagger h$ satisfies the unfolding identity for g. Indeed,

$$\begin{aligned} f^\dagger h = {} & [\eta^\nu, f^\dagger]^\S fh \\ = {} & [\eta^\nu, f^\dagger]^\S T^b_a(\mathsf{id}+h)g \\ = {} & [\eta^\nu, f^\dagger h]^\S g. \end{aligned}$$

Now consider the general case. Suppose, again we have $fh = T^b_a(\mathsf{id}+h)g$. We prove the following auxiliary identity:

$$((T\pi)\,\mathsf{out}\, f)^\dagger h = T\big(\mathsf{id}+T^b_a(\mathsf{id}+h)^b_a\big)((T\pi)\,\mathsf{out}\, g)^\dagger. \tag{A.13}$$

Observe that

$$\begin{aligned} ((T\pi)\,\mathsf{out}\, f)h &= (T\pi)\,\mathsf{out}\, T^b_a(\mathsf{id}+h)g \\ &= (T\pi)T(\mathsf{id}+h+T^b_a(\mathsf{id}+h)^b_a)\,\mathsf{out}\, g \\ &= T(\mathsf{id}+h)T((\mathsf{id}+T^b_a(\mathsf{id}+h)^b_a)+\mathsf{id})((T\pi)\,\mathsf{out}\, g), \end{aligned}$$

from which by uniformity of the iteration operator of $\mathbb{T}$, we obtain

$$((T\pi)\,\mathsf{out}\, f)^\dagger h = \big(T((\mathsf{id}+T^b_a(\mathsf{id}+h)^b_a)+\mathsf{id})((T\pi)\,\mathsf{out}\, g)\big)^\dagger.$$

After transforming the right hand side by naturality of the iteration operator of $\mathbb{T}$ we arrive at (A.13).
Next we prove that $(\rhd f)h = T^b_a(\mathsf{id}+h) \rhd g$.

$$\begin{aligned} (\rhd f)h = {} & \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger h && \text{(definition of } \rhd) \\ = {} & \mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})T(\mathsf{id}+T^b_a(\mathsf{id}+h)^b_a)((T\pi)\,\mathsf{out}\, g)^\dagger && \text{(A.13)} \\ = {} & \mathsf{out}^{-1}\, T((\mathsf{id}+h)+T^b_a(\mathsf{id}+h)^b_a)T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger \\ = {} & T^b_a(\mathsf{id}+h)\,\mathsf{out}^{-1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger && \text{(Lemma A.1)} \\ = {} & T^b_a(\mathsf{id}+h) \rhd g. && \text{(definition of } \rhd) \end{aligned}$$

As we have shown before, for guarded g uniformity holds, and therefore $f^\dagger h = (\rhd f)^\dagger h = (\rhd g)^\dagger = g^\dagger$.

- *Compatibility of strength with iteration*, i.e. $\tau^\nu(\mathsf{id}\times f^\dagger) = ((T^b_a\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f))^\dagger$. Let f be guarded with $\mathsf{out}\, f = T(\mathsf{inl}+\mathsf{id})u$. Then, $f' = (T^b_a\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f)$ is also guarded with $\mathsf{out}\, f' = T(\mathsf{inl} + \mathsf{id})T(\mathsf{id} + ((T_a^b\,\mathsf{dist})\tau^\nu)_a^b)(T\delta)\tau(\mathsf{id} \times u)$ where δ is as in Lemma 4.1 (besides guardedness of f, the proof of this equation uses naturality of out and the definitions of τ and dist). The following calculation shows that $\tau^\nu(\mathsf{id} \times f^\dagger)$ satisfies the unfolding property for $((T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f)))^\dagger$:

$$
\begin{aligned}
&\tau^\nu(\mathsf{id} \times f^\dagger)\\
={}& \tau^\nu(\mathsf{id} \times [\eta^\nu, f^\dagger]^\S f)\\
={}& \tau^\nu(\mathsf{id} \times [\eta^\nu, f^\dagger]^\S)(\mathsf{id} \times f)\\
={}& (\tau^\nu(\mathsf{id} \times [\eta^\nu, f^\dagger]))^\S \tau^\nu(\mathsf{id} \times f) && (\textsc{str}_4)\\
={}& ([\eta^\nu, \tau^\nu(\mathsf{id} \times f^\dagger)]\,\mathsf{dist})^\S \tau^\nu(\mathsf{id} \times f) && (\textsc{dst})\\
={}& [\eta^\nu, \tau^\nu(\mathsf{id} \times f^\dagger)]^\S (T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f)\\
={}& [\eta^\nu, \tau^\nu(\mathsf{id} \times f^\dagger)]^\S \rhd (T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f),
\end{aligned}
$$

and hence $\tau^\nu(\mathsf{id} \times f^\dagger)$ and $((T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f))^\dagger$ are equal.

The general case (when f is not necessarily guarded) reduces to the considered one by means of equation

$$
(T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times \rhd f) = \rhd((T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f)), \tag{A.14}
$$

as follows:

$$
\begin{aligned}
\tau^\nu(\mathsf{id} \times f^\dagger) &= \tau^\nu(\mathsf{id} \times (\rhd f)^\dagger) && (\text{definition of } \_^\dagger)\\
&= ((T\,\mathsf{dist})\tau^\nu(\mathsf{id} \times \rhd f))^\dagger\\
&= (\rhd(T_a^b\,\mathsf{dist}\,\tau^\nu(\mathsf{id} \times f)))^\dagger && (\text{A.14})\\
&= (T_a^b\,\mathsf{dist}\,\tau^\nu(\mathsf{id} \times f))^\dagger. && (\text{definition of } \_^\dagger)
\end{aligned}
$$

We show (A.14) by establishing commutativity of the following diagram where $Q = C \times B + C \times A$:

$$
\begin{tikzcd}
& C \times A \arrow[dl, "((T\pi)\,\mathsf{out}(T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f))^\dagger"'] \arrow[dr, "\mathsf{id} \times ((T\pi)\,\mathsf{out}\, f)^\dagger"] & \\
T(C \times B + (T_a^b Q)_a^b) \arrow[d, "T(\mathsf{inl} + \mathsf{id})"] & & C \times T(B + (T_a^b(B + A))_a^b) \arrow[ll, "T(\mathsf{id} + (T_a^b\,\mathsf{dist})_a^b)T(\mathsf{id} + (\tau^\nu)_a^b)(T\delta)\tau"'] \arrow[d, "\mathsf{id} \times T(\mathsf{inl} + \mathsf{id})"] \\
T(Q + (T_a^b Q)_a^b) \arrow[dr, "\mathsf{out}^{-1}"'] & & C \times T((B + A) + (T_a^b(B + A))_a^b) \arrow[ll, "T(\mathsf{dist} + (T_a^b\,\mathsf{dist})_a^b)T(\mathsf{id} + (\tau^\nu)_a^b)(T\delta)\tau"'] \arrow[dl, "(T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times \mathsf{out}^{-1})"] \\
& T_a^b Q &
\end{tikzcd}
$$

The bottom triangle commutes as follows:

$$
\begin{aligned}
&(T_a^b\,\mathsf{dist})\tau^\nu(\mathsf{id} \times \mathsf{out}^{-1})\\
={}& (T_a^b\,\mathsf{dist})\,\mathsf{out}^{-1} T(\mathsf{id} + (\tau^\nu)_a^b)(T\delta)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times \mathsf{out}^{-1})
\end{aligned}
$$

$$\begin{aligned}
&= \mathsf{out}^{-1}\mathsf{out}(T_a^b\,\mathsf{dist})\,\mathsf{out}^{-1}T(\mathsf{id}+(\tau^\nu)_a^b)(T\delta)\tau\\
&= \mathsf{out}^{-1}T(\mathsf{dist}+(T_a^b\,\mathsf{dist})_a^b)T(\mathsf{id}+(\tau^\nu)_a^b)T\delta\tau.
\end{aligned}$$

The middle square commutes by properties of strength and the morphisms dist and δ:

$$\begin{aligned}
&T(\mathsf{dist}+(T_a^b\,\mathsf{dist}\,\tau^\nu)_a^b)(T\delta)\tau(\mathsf{id}\times T(\mathsf{inl}+\mathsf{id}))\\
={}&T(\mathsf{dist}+(T_a^b\,\mathsf{dist}\,\tau^\nu)_a^b)(T\delta)T(\mathsf{id}\times(\mathsf{inl}+\mathsf{id}))\tau && (\text{STR}_4,\ \text{STR}_3)\\
={}&T(\mathsf{dist}+(T_a^b\,\mathsf{dist}\,\tau^\nu)_a^b)T((\mathsf{id}\times\mathsf{inl})+\mathsf{id})(T\delta)\tau\\
={}&T(\mathsf{dist}(\mathsf{id}\times\mathsf{inl})+\mathsf{id})T(\mathsf{id}+(T_a^b\,\mathsf{dist}\,\tau^\nu)_a^b)(T\delta)\tau\\
={}&T(\mathsf{inl}+\mathsf{id})T(\mathsf{id}+(T_a^b\,\mathsf{dist}\,\tau^\nu)_a^b)(T\delta)\tau.
\end{aligned}$$

This leaves us with the top triangle. Let $\rho = (\mathsf{id}+(T_a^b\,\mathsf{dist}\,\tau^\nu)_a^b)\delta$.

$$\begin{aligned}
&(T\rho)\tau(\mathsf{id}\times((T\pi)\,\mathsf{out}\,f)^\dagger)\\
={}&(T\rho)(T\,\mathsf{dist}\,\tau(\mathsf{id}\times((T\pi)\,\mathsf{out}\,f)))^\dagger && (\flat)\\
={}&(T(\rho+\mathsf{id})(T\,\mathsf{dist})\tau(\mathsf{id}\times((T\pi)\,\mathsf{out}\,f)))^\dagger && (\text{naturality})\\
={}&(T(\rho+\mathsf{id})(T\,\mathsf{dist})T(\mathsf{id}\times\pi)\tau(\mathsf{id}\times\mathsf{out})(\mathsf{id}\times f))^\dagger && (\text{STR}_4)\\
={}&((T\pi)T(\mathsf{dist}+\mathsf{id})(T\rho)\tau(\mathsf{id}\times\mathsf{out})(\mathsf{id}\times f))^\dagger\\
={}&((T\pi)T(\mathsf{dist}+(T_a^b\,\mathsf{dist})_a^b)\,\mathsf{out}\,\tau^\nu(\mathsf{id}\times f))^\dagger\\
={}&((T\pi)\,\mathsf{out}\,T_a^b\,\mathsf{dist}\,\tau^\nu(\mathsf{id}\times f))^\dagger.
\end{aligned}$$

In ($\flat$), we used compatibility of strength with iteration for $\mathbb{T}$. This concludes the proof of the existence part of Claim i.

It remains to show uniqueness. The proof proceeds as follows. We first show that any morphism $f : X \to T_a^b(Y+X)$ can be decomposed by means of two morphisms $g : X \to T_a^b(Z+X)$ and $h : Z \to T_a^b(Y+X)$, where $Z = Y + T_a^b(Y+X)_a^b$, as

$$f = [h, \eta^\nu\,\mathsf{inr}]^\S g \tag{A.15}$$

with g *completely unguarded*, i.e. $\mathsf{out}\,g = (T\,\mathsf{inl})g'$ for some g'. Next we show that

$$f^\dagger = (h^\S g^\dagger)^\dagger \tag{A.16}$$

and that

$$h^\S g^\dagger = \rhd f. \tag{A.17}$$

In summary, we obtain that $f^\dagger = (h^\S g^\dagger)^\dagger = (\rhd f)^\dagger$. The following proofs of (A.16) and (A.17) do not depend on the concrete definition of $\_^\dagger$ on $\mathbb{T}_a^b$ but only use its abstract properties as an iteration operator of a complete Elgot monad and compatibility with the underlying iteration operator for $\mathbb{T}$. Hence, the identity $f^\dagger = (\rhd f)^\dagger$ would be valid for any other such operator, but since $(\rhd f)^\dagger$ is uniquely defined all of them must be unique.

Let $g = \mathsf{out}^{-1}(T\mathsf{inl})((T\pi)\,\mathsf{out}\,f)$, which is, by definition, completely unguarded, and let $h = \mathsf{out}^{-1}\eta(\mathsf{inl}+\mathsf{id})$.

Then the proof of (A.15) runs as follows:

$$
\begin{aligned}
& [h, \eta^\nu \, \mathsf{inr}]^\S g \\
&= [\mathsf{out}^{\text{-}1}\eta(\mathsf{inl} + \mathsf{id}), \eta^\nu \, \mathsf{inr}]^\S g \\
&= [\mathsf{out}^{\text{-}1}\eta(\mathsf{inl} + \mathsf{id}), \mathsf{out}^{\text{-}1}\eta \, \mathsf{inl} \, \mathsf{inr}]^\S g && \text{(Lemma 4.1)} \\
&= (\mathsf{out}^{\text{-}1}\eta[(\mathsf{inl} + \mathsf{id}), \mathsf{inl} \, \mathsf{inr}])^\S g \\
&= (\mathsf{out}^{\text{-}1}\eta\pi)^\S g \\
&= \mathsf{out}^{\text{-}1}[\eta\pi, \eta \, \mathsf{inr}((\mathsf{out}^{\text{-}1}\eta\pi)^\S)^b_a]^\star \, \mathsf{out} \, g && \text{(Lemma 4.1)} \\
&= \mathsf{out}^{\text{-}1}(\eta[\pi, \mathsf{inr}((\mathsf{out}^{\text{-}1}\eta\pi)^\star)^b_a])^\star (T\mathsf{inl})(T\pi) \, \mathsf{out} \, f \\
&= \mathsf{out}^{\text{-}1}(\eta\pi)^\star (T\pi) \, \mathsf{out} \, f \\
&= \mathsf{out}^{\text{-}1}(T\pi) \; (T\pi) \, \mathsf{out} \, f \\
&= f.
\end{aligned}
$$

where $\pi = [[\mathsf{inl}\,\mathsf{inl}, \mathsf{inr}], \mathsf{inl}\,\mathsf{inr}]$. Equation (A.16) can be shown as follows:

$$
\begin{aligned}
& (h^\S g^\dagger)^\dagger \\
&= \big((T^b_a\mathsf{inl} \; h, \eta^\nu \, \mathsf{inr}]^\S g)^\dagger\big)^\dagger && \text{(naturality)} \\
&= \big(T^b_a[\mathsf{id}, \mathsf{inr}][(T^b_a\mathsf{inl})h, \eta^\nu \, \mathsf{inr}]^\S g\big)^\dagger && \text{(codiagonal)} \\
&= \big([h, T^b_a[\mathsf{id}, \mathsf{inr}]\eta^\nu \, \mathsf{inr}]^\S g\big)^\dagger \\
&= \big([h, \eta^\nu \, \mathsf{inr}]^\S g\big)^\dagger \\
&= f^\dagger.
\end{aligned}
$$

Finally, we prove (A.17):

$$
\begin{aligned}
& h^\S g^\dagger \\
&= (\mathsf{out}^{\text{-}1}\eta(\mathsf{inl} + \mathsf{id}))^\S g^\dagger && \text{(definition of } \_^\S) \\
&= \mathsf{out}^{\text{-}1}[\eta(\mathsf{inl} + \mathsf{id}), \eta \, \mathsf{inr}(h^\S)^b_a]^\star \, \mathsf{out} \, g^\dagger \\
&= \mathsf{out}^{\text{-}1}[\eta(\mathsf{inl} + \mathsf{id}), \eta \, \mathsf{inr}(h^\S)^b_a]^\star (T\mathsf{inl})((T\pi) \, \mathsf{out} \, f)^\dagger && (g \text{ compl. ung.}) \\
&= \mathsf{out}^{\text{-}1}T(\mathsf{inl} + \mathsf{id})((T\pi) \, \mathsf{out} \, f)^\dagger \\
&= \rhd f.
\end{aligned}
$$

This finishes the proof. □

Lemma A.2 *Kleisli composition of a complete Elgot monad $\mathbb{T}$ can be characterized in terms of iteration as follows:*

$$
g^\star f = [T(\mathsf{inr}\,\mathsf{inr})f, T(\mathsf{inl}) \; g]^\dagger \, \mathsf{inl} \tag{A.18}
$$

Proof.

$$
\begin{aligned}
& [T(\mathsf{inr}\,\mathsf{inr})f, T(\mathsf{inl})g]^\dagger \, \mathsf{inl} \\
&= [\eta, [T(\mathsf{inr}\,\mathsf{inr})f, T(\mathsf{inl})g]^\dagger]^\star T(\mathsf{inr}\,\mathsf{inr})f \\
&= ([\eta, [T(\mathsf{inr}\,\mathsf{inr})f, T(\mathsf{inl})g]^\dagger] \, \mathsf{inr}\,\mathsf{inr})^\star f
\end{aligned}
$$

$$
\begin{aligned}
&= ([T(\mathsf{inr}\,\mathsf{inr})f, T(\mathsf{inl})g]^\dagger\,\mathsf{inr})^\star f \\
&= ([\eta, [T(\mathsf{inr}\,\mathsf{inr})f, T(\mathsf{inl})\ g]^\dagger]^\star T(\mathsf{inl})\ g)^\star f \\
&= g^\star f
\end{aligned}
$$

□

Proof of Lemma 4.7

Proof. Let us verify the identities (A.21) from left to right.

- Compatibility of ext with unit is a straightforward consequence of Lemma 4.1: $\mathsf{ext}\,\eta = \mathsf{out}^{-1}(T_\Delta\mathsf{inl})\eta = \mathsf{out}^{-1}\eta\,\mathsf{inl} = \eta^\nu$.
- In order to show compatibility of ext with Kleisli star we call the definition of the latter from Lemma 4.1:

$$
\begin{aligned}
(\mathsf{ext}\,g)^\S\,\mathsf{ext} &= (\mathsf{out}^{-1}(T\mathsf{inl})g)^\S\,\mathsf{out}^{-1}(T\mathsf{inl}) \\
&= \mathsf{out}^{-1}\big[\mathsf{out}\,\mathsf{out}^{-1}(T\mathsf{inl})g, \eta\,\mathsf{inr}((\mathsf{ext}\,g)^\S)^b_a\big]^\star(T\mathsf{inl}) \\
&= \mathsf{out}^{-1}((T\mathsf{inl})g)^\star \\
&= \mathsf{out}^{-1}(T\mathsf{inl})g^\star \\
&= \mathsf{ext}\,g^\star.
\end{aligned}
$$

- Recall the distributivity transformation $\delta : A \times (B + C^b_a) \to A \times B + (A \times C)^b_a$ from Lemma 4.1. Then by the corresponding definition of τ^ν,

$$
\begin{aligned}
\tau^\nu(\mathsf{id} \times \mathsf{ext}) &= \mathsf{out}^{-1}T(\mathsf{id} + (\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id} \times \mathsf{out}\,\mathsf{ext}) \\
&= \mathsf{out}^{-1}T(\mathsf{id} + (\tau^\nu)^b_a)(T\delta)\tau(\mathsf{id} \times T\mathsf{inl}) \\
&= \mathsf{out}^{-1}T(\mathsf{id} + (\tau^\nu)^b_a)(T\mathsf{inl})\tau \\
&= \mathsf{out}^{-1}(T\mathsf{inl})\tau \\
&= \mathsf{ext}\,\tau.
\end{aligned}
$$

- Since $\mathsf{out}(\mathsf{ext}\,g) = (T^b_a\mathsf{inl})g$, then by Theorem 4.5, $\mathsf{out}(\mathsf{ext}\,g)^\dagger = (T\,\mathsf{inl})g^\dagger$, from which the last identity in (A.21) follows by composition with out^{-1} on the left.

□

Proof of Lemma 4.9

Let $\xi = \zeta^\dagger$. Suppose $u : a \to Sb$ and $\rho : \mathbb{T} \to \mathbb{S}$ induce ξ as in the statement of the lemma, assume for the time being that ξ is indeed a complete Elgot monads morphism, and let us verify that $\xi_b\,\mathsf{out}^{-1}\eta\,\mathsf{inr}\langle\mathsf{id}, \lambda\_.\,\eta\rangle = u$. Let $w = [\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\,S(\mathsf{inr}\,f)u(x)]^\star\rho$. Then

$$
\begin{aligned}
&\xi_b\,\mathsf{out}^{-1}\eta\,\mathsf{inr}\langle\mathsf{id}, \lambda\_.\,\eta\rangle \\
&= (w\,\mathsf{out})^\dagger\,\mathsf{out}^{-1}\eta\,\mathsf{inr}\langle\mathsf{id}, \lambda\_.\,\eta\rangle \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star w\,\mathsf{out}\,\mathsf{out}^{-1}\eta\,\mathsf{inr}\langle\mathsf{id}, \lambda\_.\,\eta\rangle \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\,S(\mathsf{inr}\,f)u(x)]^\star\rho\eta\,\mathsf{inr}\langle\mathsf{id}, \lambda\_.\,\eta\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= [\eta, (w\,\mathsf{out})^\dagger]^\star[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\,f)u(x)]\,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\,\eta\rangle \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star(\lambda\langle x, f\rangle.\, S(\mathsf{inr}\,f)u(x))\langle \mathsf{id}, \lambda\_.\,\eta\rangle \\
&= [\eta, \xi_b]^\star S(\mathsf{inr}\,\eta)u \\
&= (\xi_b\eta)^\star u \\
&= \eta^\star u \\
&= u.
\end{aligned}
$$

Suppose now that $\xi : \mathbb{T}_a^b \to S$ is a morphism of complete Elgot monads. Let $\rho = \xi\,\mathsf{ext}$ (which is a complete Elgot monad morphism by Lemma 4.7) and let $u = \xi_b\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\,\eta\rangle$. Then

$$
\begin{aligned}
&\big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\,f)u(x)]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\,f)\xi_b\,\mathsf{out}^{-1}\,\mathsf{inl}\,\eta\langle x, \eta\rangle]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, \xi T_a^b(\mathsf{inr}\,f)\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\langle x, \eta\rangle]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, \xi\,\mathsf{out}^{-1}\,T(\mathsf{inr}\,f + (T_a^b(\mathsf{inr}\,f))_a^b)\eta\,\mathsf{inr}\langle x, \eta\rangle]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, \xi\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}(T_a^b(\mathsf{inr}\,f))_a^b\langle x, \eta\rangle]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, \xi\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\langle x, T_a^b(\mathsf{inr}\,f)\eta\rangle]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, \xi\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\langle x, \eta\,\mathsf{inr}\,f\rangle]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big([\eta\,\mathsf{inl}, \xi\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\star\rho\,\mathsf{out}\big)^\dagger \\
&= \big((\xi[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})])^\star\xi\,\mathsf{ext}\,\mathsf{out}\big)^\dagger \\
&= \big(\xi[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\S\,\mathsf{ext}\,\mathsf{out}\big)^\dagger \\
&= \xi\big([\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\S\,\mathsf{ext}\,\mathsf{out}\big)^\dagger.
\end{aligned}
$$

To finish the calculation we have to verify that the latter iteration term is equal to the identity. Note that the term under the iteration operator is guarded. Hence, it suffices to show that id satisfies the corresponding characteristic equation for iteration, i.e. that

$$[\eta, \mathsf{id}]^\S[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\S\,\mathsf{ext}\,\mathsf{out} = \mathsf{id}\,.$$

Note that we can rephrase the description of Kleisli binding in $\mathbb{T}_a^b$ (Lemma 4.1) to

$$f^\S\,\mathsf{out}^{-1} = \mathsf{out}^{-1}\big[\mathsf{out}\,f, \eta\,\mathsf{inr}(f^\S)_a^b\big]^\star \tag{A.19}$$

for $f : X \to T_a^b Y$. We have

$$
\begin{aligned}
&[\eta, \mathsf{id}]^\S[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\S\,\mathsf{ext}\,\mathsf{out} \\
&= [\eta, [\eta, \mathsf{id}]^\S\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\S\,\mathsf{ext}\,\mathsf{out} \\
&= [\eta, \mathsf{out}^{-1}[\mathsf{out}[\eta, \mathsf{id}], \eta\,\mathsf{inr}\,([\eta, \mathsf{id}]^\S)_a^b]^\star\eta\,\mathsf{inr}(\mathsf{id}\times\eta\,\mathsf{inr})]^\S\,\mathsf{ext}\,\mathsf{out} \\
&= [\eta, \mathsf{out}^{-1}\eta\,\mathsf{inr}(\mathsf{id}\times\mathsf{id})]^\S\,\mathsf{ext}\,\mathsf{out} \\
&= [\eta, \mathsf{out}^{-1}\eta\,\mathsf{inr}]^\S\,\mathsf{out}^{-1}T\,\mathsf{inl}\,\mathsf{out}
\end{aligned}
$$

$$\begin{aligned}
&= \mathsf{out}^{-1}\big[\mathsf{out}[\eta, \mathsf{out}^{-1}\eta\,\mathsf{inr}], \eta\,\mathsf{inr}([\eta, \mathsf{out}^{-1}\eta\,\mathsf{inr}]^{\S})^{b}_{a}\big]^{\star} T\,\mathsf{inl}\,\mathsf{out} && \text{(by (A.19))}\\
&= \mathsf{out}^{-1}(\mathsf{out}[\eta, \mathsf{out}^{-1}\eta\,\mathsf{inr}])^{\star}\,\mathsf{out}\\
&= \mathsf{out}^{-1}([\mathsf{out}\,\eta, \eta\,\mathsf{inr}])^{\star}\,\mathsf{out}\\
&= \mathsf{out}^{-1}([\eta\,\mathsf{inl}, \eta\,\mathsf{inr}])^{\star}\,\mathsf{out}\\
&= \mathsf{out}^{-1}\mathsf{out}\\
&= \mathsf{id}\,.
\end{aligned}$$

We are left to show that any $\xi_X : T^b_a X \to SX$ induced by $u : a \to Sb$ and $\rho : \mathbb{T} \to \mathbb{S}$ is a morphism of complete Elgot monads, that is, ξ_X is natural in X and satisfies identities (A.21).

Let us first argue naturality of ξ. Let $\xi = w^{\dagger}$. We have

$$\begin{aligned}
wT^b_a f &= [\eta\,\mathsf{inl}, \lambda\langle x, g\rangle.S(\mathsf{inr}\,g)u(x)]\rho\,\mathsf{out}\,T^b_a f\\
&= [\eta\,\mathsf{inl}, \lambda\langle x, g\rangle.S(\mathsf{inr}\,g)u(x)]\rho T(f + (T^b_a f)^b_a)\,\mathsf{out}\\
&= [\eta\,\mathsf{inl}\,f, \lambda\langle x, g\rangle.S(\mathsf{inr}\,T^b_a f g)u(x)]\rho\,\mathsf{out}\\
&= S(f + T^b_a f)w
\end{aligned}$$

and thus

$$\begin{tikzcd}
T^b_a X \arrow[r, "w"] \arrow[d, "T^b_a f"] & S(X + T^b_a X) \arrow[r, "S(f+\mathsf{id})"] \arrow[d, "S(f+T^b_a f)"'] & S(Y + T^b_a X) \arrow[dl, "S(\mathsf{id}+T^b_a f)"] \\
T^b_a Y \arrow[r, "w"] & S(Y + T^b_a Y) &
\end{tikzcd}$$

commutes. Therefore, the lower triangle in the following diagram commutes by uniformity of the iteration operator:

$$\begin{tikzcd}
T^b_a X \arrow[r, "\xi"] \arrow[d, "T^b_a f"] \arrow[dr, "(S(f+\mathsf{id})w)^{\dagger}"] & SX \arrow[d, "Sf"] \\
T^b_a Y \arrow[r, "\xi"] & SY
\end{tikzcd}$$

The upper triangle commutes by naturality of the iteration operator:

$$\begin{aligned}
Sf\xi &= (\eta f)^{\star} w^{\dagger}\\
&= ([S\,\mathsf{inl}\;\eta f, \eta\,\mathsf{inr}]^{\star} w)^{\dagger}\\
&= ([\eta\,\mathsf{inl}\,f, \eta\,\mathsf{inr}]^{\star} w)^{\dagger}\\
&= (S(f + \mathsf{id})w)^{\dagger}
\end{aligned}$$

The equation $\xi\eta = \eta$ can be shown as follows:

$$\begin{aligned}
&\xi\eta\\
&= [\eta, \xi]^{\star}[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle\,.\,S(\mathsf{inr}\,f)(u(x))]^{\star}\rho\,\mathsf{out}\,\eta\\
&= [\eta, \xi]^{\star}[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle\,.\,S(\mathsf{inr}\,f)(u(x))]^{\star}\rho\,\mathsf{out}\;\mathsf{out}^{-1}\eta\,\mathsf{inl}\\
&= [\eta, \xi]^{\star}\eta\,\mathsf{inl}
\end{aligned}$$

$$= \eta.$$

Compatibility of ξ with Kleisli star follows from Lemma A.2 and compatibility of ξ with iteration, which we argue later:

$$\begin{aligned}\xi g^{\S} f &= \xi[T_a^b(\mathsf{inr}\,\mathsf{inr})f, T_a^b\mathsf{inl} g]^{\dagger}\,\mathsf{inl}\\ &= [S(\mathsf{inr}\,\mathsf{inr})\xi f, S\mathsf{inl}\xi g]^{\dagger}\,\mathsf{inl}\\ &= (\xi g)^{\star}(\xi f).\end{aligned}$$

To show compatibility of ξ with strength, consider the following diagram:

$$\begin{tikzcd} A \times T_a^b X \arrow[rr, "\mathsf{id} \times \rho\,\mathsf{out}"] \arrow[d, "\tau"] & & A \times S(X + (T_a^b X)_a^b) \arrow[d, "S\delta\tau"] \\ T_a^b(A \times X) \arrow[d, "v\rho\,\mathsf{out}"] & & S((A \times X) + (A \times T_a^b X)_a^b) \arrow[d, "v"] \\ S((A \times X) + T_a^b(A \times X)) & & S((A \times X) + (A \times T_a^b X)) \arrow[ll, "S(\mathsf{id} + \tau)"'] \end{tikzcd}$$

where $v = [\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\,f)u(x)]^{\star}$, i.e. $\xi = (v\rho\,\mathsf{out})^{\dagger}$. It is easy to show that this commutes by expanding $\mathsf{out}\,\tau$, using the fact that ρ is a complete monad morphism and observing that

$$\begin{aligned}S(\mathsf{id} + \tau)v &= [\eta\,\mathsf{inl}, S(\mathsf{id} + \tau)\lambda\langle x, h\rangle\,.\,S(\mathsf{inr}\,h)(u(x))]^{\star}\\ &= [\eta\,\mathsf{inl}, \lambda\langle x, h\rangle\,.\,S(\mathsf{inr}\,\tau h)(u(x))]^{\star}\\ &= [\eta\,\mathsf{inl}, \lambda\langle x, h\rangle\,.\,S(\mathsf{inr}\,h)(u(x))(\mathsf{id} \times \tau^b)]^{\star}\\ &= vS(\mathsf{id} + \tau_a^b).\end{aligned}$$

Thus, by uniformity,

$$\xi\tau = (v(S\delta)\tau(\mathsf{id} \times \rho\,\mathsf{out}))^{\dagger}.$$

On the other hand, by compatibility of strength with iteration, we have

$$\begin{aligned}&\tau(\mathsf{id} \times \xi)\\ &= ((S\mathsf{dist})\tau(\mathsf{id} \times v\ \rho\,\mathsf{out}))^{\dagger}\\ &= (S\mathsf{dist})(\tau(\mathsf{id} \times \underline{v}))^{\star}\tau(\mathsf{id} \times \rho\,\mathsf{out})\end{aligned}$$

where $\underline{v} = [\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\,f)u(x)]$, i.e. $\underline{v}^{\star} = v$. Therefore, to prove the identity in question, we need to show that the following diagram commutes:

$$\begin{tikzcd} S(A \times (X + (T_a^b X)_a^b)) \arrow[r, "S\delta"] \arrow[d, "(\tau(\mathsf{id} \times \underline{v}))^{\star}"] & S((A \times X) + (A \times T_a^b X)_a^b) \arrow[d, "v"] \\ S(A \times (X + T_a^b X)) \arrow[r, "S\mathsf{dist}"] & S((A \times X) + (A \times T_a^b X)), \end{tikzcd}$$

or, differently put, $\underline{v}\ \delta = (S\mathsf{dist})\tau(\mathsf{id} \times \underline{v})$, which, as a morphism out of a coproduct, decomposes into two equations:

On the one hand,

$$\begin{aligned}
&(S\mathsf{dist})\tau(\mathsf{id}\times\underline{v})(\mathsf{id}\times\mathsf{inl})\\
&=(S\mathsf{dist})\tau(\mathsf{id}\times\eta\,\mathsf{inl})\\
&=(S\mathsf{dist})\eta\,\mathsf{inl}\\
&=\eta\,\mathsf{inl}\\
&=\underline{v}\,\mathsf{inl}\\
&=\underline{v}\;\delta(\mathsf{id}\times\mathsf{inl}).
\end{aligned}$$

On the other hand, after simplifying, we get

$$\begin{aligned}
&((S\mathsf{dist})\tau(\mathsf{id}\times\underline{v})(\mathsf{id}\times\mathsf{inr}))(x,(z,c))\\
&=(S\mathsf{inr})\tau(\mathsf{id}\times\lambda\langle x,f\rangle.Sf(u(x)))(x,(z,c))\\
&=S\mathsf{inr}(\tau(x,(Sc)(u(z))))
\end{aligned}$$

as well as

$$\begin{aligned}
&\underline{v}(\delta(x,\mathsf{inr}(z,c)))\\
&=\underline{v}(\mathsf{inr}(z,\lambda v.(x,c(v))))\\
&=S(\mathsf{inr}(\lambda v.(x,c(v))))(u(z))
\end{aligned}$$

(for the first step, recall the explicit lambda-expression for δ). Identity of these expressions follows by the fact that we are working in a bicartesian closed base category, which allows us to give an explicit lambda-expression for the strength, namely $\tau=\lambda\langle a,b\rangle.S(\lambda c.(a,c))(b)$.

Finally, we are left to show that

$$\xi f^{\dagger}=(\xi f)^{\dagger} \tag{A.20}$$

for any $f:X\to T_a^b(Y+X)$. For the sake of brevity let us denote $\lambda\langle x,f\rangle.\,(Sf)(u(x))$ by ev_u. Then $\xi=([\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out})^{\dagger}$.

First, we argue that w.l.o.g. f may be taken to be guarded. Assuming that $\xi(\rhd f)^{\dagger}=(\xi\rhd f)^{\dagger}$, since by definition $\xi(\rhd f)^{\dagger}=\xi f^{\dagger}$, we can deduce (A.20) from the equality $(\xi\rhd f)^{\dagger}=(\xi f)^{\dagger}$. To show the latter, consider the morphism w given by the composition

$$X\xrightarrow{\mathsf{out}\,f}T((Y+X)+T_a^b(Y+X)_a^b)\xrightarrow{[\eta[\mathsf{inl}\,\mathsf{inl},\mathsf{inr}],(S\,\mathsf{inl})\xi^{\star}\,\mathsf{ev}_u]^{\star}\rho}S((Y+X)+X).$$

Now, on the one hand

$$\begin{aligned}
(S[\mathsf{id},\mathsf{inr}]w)^{\dagger}&=([\eta[\mathsf{inl},\mathsf{inr}],\xi^{\star}\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\,f)^{\dagger}\\
&=([\eta,\xi^{\star}\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\,f)^{\dagger}\\
&=([\eta,\xi]^{\star}[\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\,f)^{\dagger}\\
&=(\xi f)^{\dagger}
\end{aligned}$$

and on the other hand, by naturality of $\_^{\dagger}$,

$$\begin{aligned}
(w^{\dagger})^{\dagger} &= \big(([[\eta\,\mathsf{inl}\,\mathsf{inl},\eta\,\mathsf{inr}], S\,\mathsf{inl}\,\xi^{\star}\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\,f)^{\dagger}\big)^{\dagger}\\
&= \big(([(S\,\mathsf{inl})[\eta\,\mathsf{inl},\xi^{\star}\,\mathsf{ev}_u],\eta\,\mathsf{inr}]^{\star}[[\eta\,\mathsf{inl}\,\mathsf{inl},\eta\,\mathsf{inr}],\eta\,\mathsf{inl}\,\mathsf{inr}]^{\star}\rho\,\mathsf{out}\,f)^{\dagger}\big)^{\dagger}\\
&= \big([\eta\,\mathsf{inl},\xi^{\star}\,\mathsf{ev}_u]^{\star}(\rho(T\pi)\,\mathsf{out}\,f)^{\dagger}\big)^{\dagger}\\
&= \big([\eta,\xi^{\star}\,\mathsf{ev}_u]^{\star}S(\mathsf{inl}+\mathsf{id})(\rho(T\pi)\,\mathsf{out}\,f)^{\dagger}\big)^{\dagger}\\
&= \big([\eta,\xi^{\star}\,\mathsf{ev}_u]^{\star}\rho T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,f)^{\dagger}\big)^{\dagger}\\
&= \big([\eta,\xi^{\star}\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\rhd f\big)^{\dagger}\\
&= ([\eta,\xi]^{\star}[\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\rhd f)^{\dagger}\\
&= (\xi\rhd f)^{\dagger}.
\end{aligned}$$

Therefore, we obtain the equality of $(\xi f)^{\dagger}$ and $(\xi\rhd f)^{\dagger}$ by the codiagonal property of $\_^{\dagger}$. We thus proceed under the assumption that f is guarded, i.e. $\mathsf{out}\,f = T(\mathsf{inl}+\mathsf{id})g$ for some $g : X\to T(Y+T_a^b(Y+X)_a^b)$.

We introduce the following morphism w,

$$\begin{aligned}
&T_a^b(Y+X)\xrightarrow{\mathsf{out}} T((Y+X)+T_a^b(Y+X)_a^b)\\
&\xrightarrow{\rho[[\eta\,\mathsf{inl}\,\mathsf{inl},[\eta\,\mathsf{inl}\,\mathsf{inl},(T\,\mathsf{inl}\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}g],(T\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}} S((Y+T_a^b(Y+X))+T_a^b(Y+X)).
\end{aligned}$$

Then, on the one hand, using dinaturality,

$$\begin{aligned}
(w^{\dagger})^{\dagger} &= ((\rho[\eta\,\mathsf{inl}\,\mathsf{inl},[\eta\,\mathsf{inl}\,\mathsf{inl},(T\,\mathsf{inl}\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}g,(T\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\,\mathsf{out})^{\dagger})^{\dagger}\\
&= ([\eta\,\mathsf{inl},[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho g]^{\star}([\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out})^{\dagger})^{\dagger}\\
&= ([\eta\,\mathsf{inl},[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho g]^{\star}\xi)^{\dagger}\\
&= [\eta,([\eta\,\mathsf{inl},\xi]^{\star}[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho g)^{\dagger}]^{\star}\xi\\
&= [\eta,([\eta,\xi]^{\star}[\eta\,\mathsf{inl}\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho g)^{\dagger}]^{\star}\xi\\
&= [\eta,([\eta,\xi]^{\star}[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^{\star}\rho\,\mathsf{out}\,f)^{\dagger}]^{\star}\xi\\
&= [\eta,(\xi f)^{\dagger}]^{\star}\xi
\end{aligned}$$

and hence $(w^{\dagger})^{\dagger}\eta^{\nu}\,\mathsf{inr} = (\xi f)^{\dagger}$. Let us furthermore introduce the following morphism t:

$$\begin{aligned}
T_a^b(Y+X) &\xrightarrow{\mathsf{out}} T((Y+X)+T_a^b(Y+X)_a^b)\\
&\xrightarrow{[[\eta\,\mathsf{inl},g],\eta\,\mathsf{inr}]^{\star}} T(Y+T_a^b(Y+X)_a^b)\\
&\xrightarrow{T(\mathsf{inl}+(\eta^{\nu}\,\mathsf{inr})_a^b)} T((Y+T_a^b(Y+X))+T_a^b(Y+T_a^b(Y+X))_a^b)\\
&\xrightarrow{\mathsf{out}^{-1}} T_a^b(Y+T_a^b(Y+X)).
\end{aligned}$$

Recall that guardedness of f means that $\mathsf{out}\,f$ factors through $X\to T(Y+T_a^b(Y+X)_a^b)$. Observe that t satisfies a stronger assumption. Let us call f *strongly guarded* if there is $h : X\to T(Y+X_a^b)$ such that $\mathsf{out}\,f = T(\mathsf{inl}+(\eta^{\nu}\,\mathsf{inr})_a^b)h$. The morphism

t is strongly guarded by definition. Furthermore,

$$\begin{aligned}
\xi t &= [\eta,\xi]^\star[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho[[\eta\,\mathsf{inl}\,\mathsf{inl},T(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)g],\eta\,\mathsf{inr}(\eta^\nu\,\mathsf{inr})^b_a]^\star\,\mathsf{out}\\
&= [\eta,\xi]^\star[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^\star[[\eta\,\mathsf{inl}\,\mathsf{inl},S(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)\rho g],\eta\,\mathsf{inr}(\eta^\nu\,\mathsf{inr})^b_a]^\star\rho\,\mathsf{out}\\
&= [\eta,\xi]^\star[[\eta\,\mathsf{inl}\,\mathsf{inl},[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^\star S(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)\rho g],S(\mathsf{inr}\,\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\\
&= [\eta,\xi]^\star[[\eta\,\mathsf{inl}\,\mathsf{inl},[\eta\,\mathsf{inl}\,\mathsf{inl},S(\mathsf{inr}\,\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho g],S(\mathsf{inr}\,\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl},[\eta\,\mathsf{inl},\xi^\star S(\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho g],\xi^\star S(\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl},[\eta\,\mathsf{inl},(S\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho g],(S\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\\
&= S[\mathsf{id},\mathsf{inr}]w
\end{aligned}$$

Let us assume for the moment that $\xi t^\dagger = (\xi t)^\dagger$ for strongly guarded t. Then, by the above calculations, $(\xi f)^\dagger = (w^\dagger)^\dagger\eta^\nu\,\mathsf{inr} = (S[\mathsf{id},\mathsf{inr}]w)^\dagger\eta^\nu\,\mathsf{inr} = (\xi t)^\dagger\eta^\nu\,\mathsf{inr} = \xi t^\dagger(\eta^\nu\,\mathsf{inr})$. In order to show that the right-hand side is equal to $\xi f^\dagger$ we prove that $t^\dagger = [\eta^\nu,f^\dagger]^\S$, for then $\xi t^\dagger(\eta^\nu\,\mathsf{inr}) = \xi[\eta^\nu,f^\dagger]^\S(\eta^\nu\,\mathsf{inr}) = \xi f^\dagger$. Since t is guarded, it suffices to show that $[\eta^\nu,f^\dagger]^\S$ satisfies the unfolding law for $t^\dagger$. It is easy to verify that $\mathsf{out}\,f^\dagger = T(\mathsf{id}+([\eta^\nu,f^\dagger]^\S)^b_a)g$. Then we have

$$\begin{aligned}
\mathsf{out}[\eta^\nu,f^\dagger]^\S &= [\mathsf{out}[\eta^\nu,f^\dagger],\eta\,\mathsf{inr}([\eta^\nu,f^\dagger]^\S)^b_a]^\star\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl},\mathsf{out}\,f^\dagger],\eta\,\mathsf{inr}([\eta^\nu,f^\dagger]^\S)^b_a]^\star\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl},T(\mathsf{id}+([\eta^\nu,f^\dagger]^\S)^b_a)g],\eta\,\mathsf{inr}([\eta^\nu,f^\dagger]^\S)^b_a]^\star\,\mathsf{out}
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
t^\dagger &= [\eta^\nu,t^\dagger]^\S\,\mathsf{out}^{-1}[[\eta\,\mathsf{inl}\,\mathsf{inl},T(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)g],\eta\,\mathsf{inr}(\eta^\nu\,\mathsf{inr})^b_a]^\star\,\mathsf{out}\\
&= \mathsf{out}^{-1}[\mathsf{out}[\eta^\nu,t^\dagger],\eta\,\mathsf{inr}([\eta^\nu,t^\dagger]^\S)^b_a]^\star\\
&\qquad [[\eta\,\mathsf{inl}\,\mathsf{inl},T(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)g],\eta\,\mathsf{inr}(\eta^\nu\,\mathsf{inr})^b_a]^\star\,\mathsf{out}\\
&= \mathsf{out}^{-1}[\eta\,\mathsf{inl},T(\mathsf{id}+(t^\dagger)^b_a)g,\eta\,\mathsf{inr}(t^\dagger)^b_a]^\star\,\mathsf{out}\,.
\end{aligned}$$

Hence, indeed, $[\eta^\nu,f^\dagger]^\S = t^\dagger$.

Finally, let us show (A.20) with strongly guarded f. Suppose, h is such that $\mathsf{out}\,f = T(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)h$. Recall that $\xi = ([\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out})^\dagger$. By uniformity, it suffices to show that

$$[\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\,f^\dagger = S(\mathsf{id}+f^\dagger)\xi f.$$

On the one hand,

$$\begin{aligned}
&[\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\,f^\dagger\\
&= [\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}[\eta^\nu,f^\dagger]^\S f\\
&= [\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star[\mathsf{out}[\eta^\nu,f^\dagger],\eta\,\mathsf{inr}([\eta^\nu,f^\dagger]^\S)^b_a]^\star\rho\,\mathsf{out}\,f\\
&= [\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star[\mathsf{out}[\eta^\nu,f^\dagger],\eta\,\mathsf{inr}([\eta^\nu,f^\dagger]^\S)^b_a]^\star S(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)\rho h\\
&= [\eta\,\mathsf{inl},S(\mathsf{inr})\,\mathsf{ev}_u]^\star[\eta\,\mathsf{inl},\eta\,\mathsf{inr}(f^\dagger)^b_a]^\star\rho h\\
&= [\eta\,\mathsf{inl},S(\mathsf{inr}\,f^\dagger)\,\mathsf{ev}_u]^\star\rho h.
\end{aligned}$$

And on the other hand,

$$\begin{aligned}
S(\mathsf{id}+f^\dagger)\xi f &= S(\mathsf{id}+f^\dagger)[\eta,\xi]^\star[\eta\,\mathsf{inl}, S(\mathsf{inr})\,\mathsf{ev}_u]^\star\rho\,\mathsf{out}\, f\\
&= S(\mathsf{id}+f^\dagger)[\eta,\xi]^\star[\eta\,\mathsf{inl}, S(\mathsf{inr})\,\mathsf{ev}_u]^\star\rho T(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)h\\
&= S(\mathsf{id}+f^\dagger)[\eta,\xi]^\star[\eta\,\mathsf{inl}, S(\mathsf{inr})\,\mathsf{ev}_u]^\star S(\mathsf{inl}+(\eta^\nu\,\mathsf{inr})^b_a)\rho h\\
&= S(\mathsf{id}+f^\dagger)[\eta,\xi]^\star[\eta\,\mathsf{inl}\,\mathsf{inl}, S(\mathsf{inr}\,\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho h\\
&= S(\mathsf{id}+f^\dagger)[\eta\,\mathsf{inl}, \xi^\star S(\eta^\nu\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho h\\
&= S(\mathsf{id}+f^\dagger)[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\,\mathsf{ev}_u]^\star\rho h\\
&= [\eta\,\mathsf{inl}, S(\mathsf{inr}\, f^\dagger)\,\mathsf{ev}_u]^\star\rho h.
\end{aligned}$$

This finishes the proof of Lemma 4.9. □

Proof of Theorem 4.8

The (overlarge) category of complete Elgot monads if formed by (strong) complete Elgot monads and (strong) complete Elgot monad morphisms. The latter are the usual (strong) monad morphisms [16] preserving iteration. Summarized, a complete Elgot monad morphism is a natural transformation $\xi : T \to S$ satisfying the following identities:

$$\xi\eta = \eta \qquad \xi f^\star = (\xi f)^\star\xi \qquad \xi\tau = \tau(\mathsf{id}\times\xi) \qquad (\xi g)^\dagger = \xi g^\dagger \tag{A.21}$$

with $f : X \to TY$ and $g : X \to T(Y + X)$.

The proof of the theorem relies crucially on Lemma 4.9. Additionally, we need the following.

Lemma A.3 *Let $f : X \to T(Y + X)$. Then $[\eta, f^\dagger]^\star = (T(\mathsf{id}+f))^\dagger$.*

Proof. Consider the following trivially commuting diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"] & T(Y+X) \arrow[d, "T(\mathsf{id}+f)"] \\
T(Y+X) \arrow[r, "T(\mathsf{id}+f)"] & T(Y+T(Y+X))
\end{tikzcd}$$

By uniformity, this implies $f^\dagger = (T(\mathsf{id}+f))^\dagger f$. Therefore $[\eta, f^\dagger]^\star = [\eta, (T(\mathsf{id}+f))^\dagger f]^\star = [\eta, (T(\mathsf{id}+f))^\dagger]^\star T(\mathsf{id}+f) = (T(\mathsf{id}+f))^\dagger$ and we are done. □

We proceed with the proof of Theorem 4.8.

- The fact that $\mathbb{L}^b_a$ is a complete Elgot monad follows from the assumption and Theorem 4.5. We have to show that for any complete Elgot monad $\mathbb{S}$ equipped with an algebraic operation $\alpha : S^b \to S^a$ there is a unique monad morphism $\xi : \mathbb{L}^b_a \to \mathbb{S}$ compatible with the corresponding algebraic operation $\beta : (L^b_a)^b \to (L^b_a)^a$, i.e. $\xi^a\beta = \alpha\xi^b$ where

$$\beta_X(f : b \to L^b_a X)(x : A) = \mathsf{out}^{-1}\mathsf{inr}\langle x, f\rangle.$$

Recall that algebraic operations dually correspond to generic effects [22], i.e. α induces a Kleisli morphism $u : a \to Sb$. By Lemma 4.9, u induces a monad morphism $\xi : \mathbb{L}_a^b \to \mathbb{S}$. According to Lemma 4.9, $u : a \to Sb$ is now representable as the composition

$$a \xrightarrow{\mathsf{out}^{-1}\,\mathsf{inl}\,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\,\eta\rangle} L_a^b b \xrightarrow{\xi_b} Sb,$$

which exactly means the ξ takes β to α. On the other hand, any other morphism $\theta : \mathbb{L}_a^b \to \mathbb{S}$ for which u decomposes as above with ξ replaced by θ, corresponds to u under the bijection of Lemma 4.9, and hence such θ is identically ξ.

- By Lemma 4.9 (with $\mathbb{S} = \mathbb{T}_a^b$, $\mathbb{T} = \mathbb{L}$), the Kleisli morphism

$$a \xrightarrow{\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\,\eta\rangle} T_a^b b$$

induces a monad morphism $\xi : \mathbb{L}_a^b \to \mathbb{T}_a^b$. We next show that $\mathbb{T}_a^b$ is the coproduct of $\mathbb{T}$ and $\mathbb{L}_a^b$ with ξ and $\mathsf{ext} : \mathbb{T} \to \mathbb{T}_a^b$ being coproduct injections. Let $\mathbb{R}$ be a complete Elgot monad and let $\rho : \mathbb{T} \to \mathbb{R}$, $\theta : \mathbb{L}_a^b \to \mathbb{R}$ be two complete Elgot monad morphisms. We have to prove that that there is a unique $\kappa : \mathbb{T}_a^b \to \mathbb{R}$ such that

$$\rho = \kappa\,\mathsf{ext} \qquad\qquad \theta = \kappa\xi. \tag{A.22}$$

By Lemma 4.9, there is a Kleisli morphism $u : a \to Rb$ induced by θ. Again, by Lemma 4.9, the pair u, ρ induces a monad morphism $\mathbb{T}_a^b \to \mathbb{R}$ which we take as κ. Let us show the left part of (A.22):

$$\begin{aligned}
\kappa\,\mathsf{ext} &= \big([\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\, f)u(x)]^\star \rho\,\mathsf{out}\big)^\dagger\,\mathsf{ext}\\
&= [\eta, \kappa]^\star[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\, f)u(x)]^\star \rho\,\mathsf{out}\,\mathsf{out}^{-1} T\,\mathsf{inl}\\
&= [\eta, \kappa]^\star[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\, f)u(x)]^\star \rho\, T\,\mathsf{inl}\\
&= [\eta, \kappa]^\star[\eta\,\mathsf{inl}, \lambda\langle x, f\rangle.\, S(\mathsf{inr}\, f)u(x)]^\star (R\,\mathsf{inl})\rho\\
&= [\eta, \kappa]^\star (R\,\mathsf{inl})\rho\\
&= \rho.
\end{aligned}$$

In order to show the right-hand side of (A.22), observe that by Lemma 4.9 both side of the equation in question are completely identified by the corresponding Kleisli morphism $a \to Rb$. For ρ, such morphism is by definition u. Let us calculate the corresponding morphism for the right hand-side to prove that it is also u:

$$\kappa_b\xi_b\,\mathsf{out}^{-1}\eta\,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\,\eta\rangle = \kappa_b\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\,\eta\rangle = u.$$

Finally, we show that κ satisfying (A.22) is unique. Suppose, κ' is another such. By Lemma 4.9, κ' induces $u' : a \to Rb$ and $\rho' : \mathbb{T} \to \mathbb{R}$. We will be done once we show that $u = u'$ and $\rho = \rho'$. On the one hand, by definition,

$\rho' = \kappa' \,\mathsf{ext} = \rho$. One the other hand,

$$
\begin{aligned}
u &= \theta_b \,\mathsf{out}^{\text{-}1} \eta \,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\, \eta\rangle \\
&= \kappa'_b \xi_b \,\mathsf{out}^{\text{-}1}\, \eta \,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\, \eta\rangle \\
&= \kappa'_b \,\mathsf{out}^{\text{-}1}\, \eta \,\mathsf{inr}\langle \mathsf{id}, \lambda\_.\, \eta\rangle \\
&= u
\end{aligned}
$$

and thus we are done. $\square$