Relations in Categories

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Abstract

This thesis investigates relations over a category $\mathcal{C}$ relative to an $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathcal{C}$. In order to establish the 2-category $\mathbf{Rel}(\mathcal{C})$ of relations over $\mathcal{C}$ in the first part we discuss sufficient conditions for the associativity of horizontal composition of relations, and we investigate special classes of morphisms in $\mathbf{Rel}(\mathcal{C})$. Attention is particularly devoted to the notion of mapping as defined by Lawvere. We give a significantly simplified proof for the main result of Pavlović, namely that $\mathcal{C} \simeq \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ if and only if $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$. This part also contains a proof that the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is finitely complete, and we present the results obtained by Kelly, some of them generalized, i. e., without the restrictive assumption that $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$.

The next part deals with factorization systems in $\mathbf{Rel}(\mathcal{C})$. The fact that each set-relation has a canonical image factorization is generalized and shown to yield an $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$-factorization system in $\mathbf{Rel}(\mathcal{C})$ in case $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. The setting without this condition is studied, as well. We propose a weaker notion of factorization system for a 2-category, where the commutativity in the universal property of an $(\mathcal{E}, \mathcal{M})$-factorization system is replaced by coherent 2-cells.

In the last part certain limits and colimits in $\mathbf{Rel}(\mathcal{C})$ are investigated. Coproducts exist in $\mathbf{Rel}(\mathcal{C})$ and are given as in $\mathcal{C}$ provided that $\mathcal{C}$ is extensive. However, finite (co)completeness fails. Finally we show that colimits of ω-chains do not exist in $\mathbf{Rel}(\mathcal{C})$ in general. However, it turns out that a canonical construction with a 2-categorial universal property exists if $\mathcal{C}$ has well-behaved colimits of ω-chains. For the case $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ we give a necessary and sufficient condition that forces our construction to yield colimits of ω-chains in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

Acknowledgements

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This thesis has been typeset using the typesetting system TEX of Donald E. Knuth, and using LATEX of Leslie Lamport. The commutative diagrams have been set using Kristoffer H. Rose's XY-pic.

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Contents

List of symbols

The following tables contain some of the symbols that are used most frequently in this thesis. Of course, it is far from being complete. However, it should contain all the symbols that are used more than only locally, that means more than in one section. For each symbol there is a short explanation and a reference to its definition or first place of occurrence.

Classes of morphisms

$\mathrm{Iso}(\mathcal{C})$	the class of isomorphisms of the category $\mathcal{C}$	Def. 2.1
$\mathrm{Mono}(\mathcal{C})$	class of monomorphisms of $\mathcal{C}$	Prop. 2.7
$\mathrm{Epi}(\mathcal{C})$	class of epimorphism of $\mathcal{C}$	Prop. 2.4
$\mathrm{Sect}(\mathcal{C})$	class of sections of $\mathcal{C}$ (morphisms with a left inverse)	Prop. 2.4
$\mathrm{ExtrMono}(\mathcal{C})$	class of extremal monos of $\mathcal{C}$ (A monic arrow m is called extremal if $m = fe$ with e epic implies that e is an iso.	Prop. 2.4
$\mathrm{ExtrEpi}(\mathcal{C})$	class of extremal epis of $\mathcal{C}$ (dual notion of extremal mono)	Cor. 2.5
$\mathrm{StrongEpi}(\mathcal{C})$	class of strong epis of $\mathcal{C}$ (An epimorphism is called strong if it has the unique diagonalization property w. r. t. all monos.)	Cor. 2.5
$\mathrm{RegEpi}(\mathcal{C})$	class of regular epis of $\mathcal{C}$ (An arrow is a regular epi if it is the coequalizer of a parallel pair of arrows.)	Prop. 2.7
$\mathrm{Eq}(\mathcal{B})$	class of equivalences of the 2-category $\mathcal{B}$	p. 62
$\mathcal{E}$, $\mathcal{M}$	classes of morphisms that form an $(\mathcal{E}, \mathcal{M})$-structure	Def. 2.1
$\bar{\mathcal{E}}$, $\bar{\mathcal{M}}$	classes of relations induced by the canonical factorization of relations	p. 53
Σ	abbreviation for $\mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$	Sec. 6.1

Note that we have the following chain of inclusions:

$$\mathrm{Iso}(\mathcal{C}) \subseteq \mathrm{Sect}(\mathcal{C}) \subseteq \mathrm{RegMono}(\mathcal{C}) \subseteq \mathrm{StrongMono}(\mathcal{C}) \subseteq \mathrm{ExtrMono}(\mathcal{C}).$$

By duality, the same is true for the respective classes of epimorphisms.

Categories and 2-categories

$\mathbf{Span}(A, B)$	category of spans between objects A and B	Sec. 3.1
$\mathbf{Rel}(A, B)$	category of relations between objects A and B	Sec. 3.2
$\mathbf{Span}(\mathcal{C})$	2-category of spans over the category $\mathcal{C}$	Sec. 3.1
$\mathbf{Rel}(\mathcal{C})$	2-category of relation over the category $\mathcal{C}$	Sec. 3.3
$\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$	subcategory of maps (1-cells with a right adjoint) of $\mathbf{Rel}(\mathcal{C})$	Def. 4.2, Cor. 4.9
Set	category of sets and functions	Ex. 3.1
CAT	(2-)category of (small) categories, functors (and natural transformations)	Ex. 3.15

Top	category of topological spaces and continuous functions	Ex. 3.2
$\mathbf{Top}_1$	category T_1-spaces and continuous functions	Ex. 4.24
$\mathcal{K}$, $\mathcal{K}_p$	2-categories of finitely complete categories with a (proper) stable $(\mathcal{E}, \mathcal{M})$-structure	Sec. 6.4
Reg	full sub-2-category of regular categories in $\mathcal{K}$	Sec. 6.4
$\mathcal{RK}$, $\mathcal{RK}_p$	2-categories of 2-categories of relations over categories in $\mathcal{K}$ and $\mathcal{K}_p$ respectively	

Other symbols

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$	symbols for natural, rational, and real numbers	
$\langle a, b\rangle$	notation for the unique arrow $C : A \times B$ induced by $a : C \to A$ and $b : C \to B$	Sec. 3.1
$[c, d]$	notation for the unique arrow $A + B \to C$ induced by $c : A \to C$ and $d : B \to C$	Sec. 8.4
$\ker(f)$	kernel pair of an arrow f	Sec. 4.2
$\rightarrowtail$	notation for arrows of $\mathcal{M}$ (of an $(\mathcal{E}, \mathcal{M})$-structure) in diagrams	
$\twoheadrightarrow$	notation for arrows of $\mathcal{E}$ (of an $(\mathcal{E}, \mathcal{M})$-structure) in diagrams	
ΔA, Δ_A	constant (2-)functor	Prop. 2.8, Sec. 8.2
im	functor im : $\mathbf{Span}(A, B) \to \mathbf{Rel}(A, B)$ given by $(\mathcal{E}, \mathcal{M})$-factorizing	Sec. 3.3
$f^{\rightarrow}$, $f^{\leftarrow}$	image and inverse image of relations	Def. 3.3
$b \diamond a$	composite of the spans a and b	Sec. 3.1
δ_A	identity span given by $\langle 1_A, 1_A\rangle : A \to A \times A$	Sec. 3.1
$s \circ r$	composite of the relations r and s	Sec. 3.3
ι_A	identity relation given by image $\mathrm{im}(\delta_A)$ of an identity span	Sec. 3.4
r^o	opposite of the relation r	Sec. 3.5
$r \wedge s$	local product of the relations r and s	Sec. 3.5
$b(r)a$	notation for pointwise calculus of relations	Prop. 3.17
Γf	graph $\mathrm{im}\langle 1, f\rangle$ of an arrow f, graph functor	Def. 4.4, Sec. 4.4
$r \leq s$	simplified notation for 2-cells in $\mathbf{Rel}(\mathcal{C})$ if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$	Sec. 5.1

1 Introduction

1.1 A bit of history

Relations between sets as well as the equivalent concept of multi-valued functions have been an important tool in mathematics for a long time. The calculus of binary relations played an important role in the interaction between algebra and logic since the middle of the nineteenth century. The first adequate development of such calculi was given by de Morgan and Peirce. Their work has been taken up and systematically extended by Schröder in [24]. More than 40 years later, Tarski started with [26] the exhaustive study of relation algebras, and more generally, of Boolean algebras with operators.

Categorial generalizations of calculi of relations have been playing a role in many works for quite a while, too. Traditionally the relation $R \subseteq A \times B$ defined set theoretically is substituted by a monomorphism $r : R \to A \times B$ in a category $\mathcal{C}$, where, moreover, r often lies in a special class $\mathcal{M}$ of monomorphisms belonging to a pullback stable $(\mathcal{E}, \mathcal{M})$-factorization system of the category $\mathcal{C}$. The first categorial treatment of relations is due to MacLane (cf. [18]). He axiomatizes additive relations between (left) modules over a fixed ring. His results appear at about the same time as the axiomatization of relations in Abelian categories of Puppe (cf. [23] and [4] for a more extensive treatment). The notion of relations relative to a factorization system first appears in [15], still with slightly distorted terminology. It is fully developed with the introduction of the *bicategory* **Rel**($\mathcal{C}$) of relations over a category $\mathcal{C}$ for the first time in [20]. However, both of these papers impose conditions on the $(\mathcal{E}, \mathcal{M})$-factorizations system of $\mathcal{C}$, namely $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ and additionally $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$ in the first paper. It seems that most of the later investigations on relations, like for example, in [14] and [10], always use one or both of these assumptions. But as the interest of theoretical computer science in relations grew, these conditions became a great obstacle for considering certain important examples.

In his important work [21], Pavlović shows how to obtain a reasonable theory avoiding all assumptions on $\mathcal{E}$ and $\mathcal{M}$ other than necessary. Admittedly, this had to be done at the cost of making the proofs quite involved, and finally it resulted in an even more general treatment of relations relative to regular fibrations in the sequel [22] of [21] by the same author.

Meanwhile, the work of Freyd and Scedrov (cf. [8]) led to an axiomatization of relations over regular categories, the so-called *allegories*. Some authors have used this setting to investigate relations further. A very recent example of this is [27]. Here we shall mainly stick to relations relative to an $(\mathcal{E}, \mathcal{M})$-factorization system.

1.2 About this thesis

This thesis starts by recalling a few basic facts about $(\mathcal{E}, \mathcal{M})$-factorization structures in categories. In the third section the results of Pavlović ([21]) and Jayewardene and Wyler ([10]) will be used to define the 2-category **Rel**($\mathcal{C}$) of relations over an $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$. We shall investigate two sufficient conditions for associativity of the horizontal compositions of relations. One of these is that $\mathcal{E}$ is stable under pullback. The other one is a weaker condition. These conditions are known to be equivalent in case $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$. As a new result we add that they are necessary for the associativity in this case.

The fourth section is devoted to the notion of a mapping as defined by Lawvere, i. e. the class of 1-cells with a right adjoint. Of great interest is the question under what circumstances the category $\mathcal{C}$ can be recovered via the isomorphism

$$\mathcal{C} \simeq \mathbf{Map}(\mathbf{Rel}(\mathcal{C})).$$

The answer to this is that these categories are isomorphic precisely when $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$. In principle this section presents the main result of [21]. However, it was possible to significantly simplify the proofs and to remove an error in the argument for the main result and its technical lemma. Parts of the credit for this has to go to Pavlović himself since his more general main result of [22], which characterizes maps, has a very easy proof in our setting, as we shall show here. The last part of the section adopts a proof from the theory of allegories (cf. [8]) and shows that the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is finitely complete; hence, we generalize a similar result of Jayewardene and Wyler (cf. [10]), and of Kelly (cf. [14]) respectively.

In the fifth section we investigate other important classes of special relations in the setting of [10], i. e., where $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, and, in the second part of the section, with the additional condition that $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. That section presents the result of [10] and some of [25].

The sixth section further investigates the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ of mappings in $\mathbf{Rel}(\mathcal{C})$. Kelly proved in [14] that in the setting where $\mathcal{C}$ has a so-called *proper stable* $(\mathcal{E}, \mathcal{M})$-factorization system, so that $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ and $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ is stable under pullback, there is an isomorphism

$$\mathbf{Rel}(\mathcal{C}) \simeq \mathbf{Rel}(\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))).$$

Kelly's proof of this is presented here, and we analyze where the conditions on $\mathcal{M}$ and $\mathcal{E}$ are used. It turns out, that some of the results in [14] do not need these conditions or only $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$.

In the seventh section we turn our attention to factorization systems in $\mathbf{Rel}(\mathcal{C})$. It is well-known that any relation between sets factorizes through its image when considered as multivalued function. We shall show that this can be generalized. In fact, the factorization gives rise to classes $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$ so that we obtain an $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$-factorization system in $\mathbf{Rel}(\mathcal{C})$ provided that $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. Moreover, even without this condition a weaker 2-categorial universal property still holds for the canonical factorization. However, the question whether there is an $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$-factorization system in $\mathcal{B} := \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ such that $\mathbf{Rel}(\mathcal{C}) \simeq \mathbf{Rel}(\mathcal{B})$ without any condition on $\mathcal{M}$ remains open.

In the last two sections we investigate limits and colimits in the ordinary as well as in the 2-category $\mathbf{Rel}(\mathcal{C})$. We shall show that the existence of well-behaved coproducts in $\mathcal{C}$ implies the existence of coproducts in $\mathbf{Rel}(\mathcal{C})$. More precisely, if $\mathcal{C}$ is an *extensive* category (cf. [6]), then the coproducts in $\mathbf{Rel}(\mathcal{C})$ are given as in $\mathcal{C}$. Moreover, $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is closed under coproducts in $\mathbf{Rel}(\mathcal{C})$. Unfortunately, $\mathbf{Rel}(\mathcal{C})$ is not finitely (co)complete in general. As open problems we leave the questions whether the coproducts in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ are extensive and whether $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is cocomplete if $\mathcal{C}$ is so.

Last, but not least, we shed some light on colimits of ω-chains. These are of particular interest especially in theoretical computer science, because they allow the iterative construction of initial algebras of ω-cocontinuous functors.

Initial algebras can be used as a model for recursively specified data types. (cf. [19]). Being able to construct initial algebras in a category of relations yields a powerful tool for the specification of non-deterministic problems, for example optimization problems (cf. [2]).

Unfortunately, the desired colimits do not exist in general in $\mathbf{Rel}(\mathcal{C})$. However, if we impose certain conditions on $\mathcal{C}$, then there is a canonical construction with a weaker (2-categorial) universal property. As in the case of coproducts, these sufficient conditions simply say that colimits of ω-chains in $\mathcal{C}$ must exist and be well-behaved. To be precise, colimits of ω-chains in $\mathcal{C}$ have to be universal and they need to commute with pullbacks. It is somewhat unfortunate that the construction seems to force us to deal only with monic relations, so that $\mathcal{M}$ has to consist of monomorphism. For maps, however, this is not a problem at all because it is automatically true. Moreover, in case $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ our canonical construction yields colimits of ω-chain in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. It remains an open problem though, whether the condition $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ is necessary for this result.

But now let us begin our treatment of relations in categories by quickly recalling the basics about $(\mathcal{E}, \mathcal{M})$-factorization systems.

2 $(\mathcal{E}, \mathcal{M})$-structured categories

The $(\mathcal{E}, \mathcal{M})$-structured categories discussed in this section give the appropriate environment to derive a calculus of relations in categories. Therefore the most important results about $(\mathcal{E}, \mathcal{M})$-structured categories used in this thesis are listed here. The definitions and theorems are all standard. The proofs are almost all omitted. They can be found for example in [1].

Definition 2.1. *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. $(\mathcal{E}, \mathcal{M})$ is called a* factorization structure for morphisms *in $\mathcal{C}$ and $\mathcal{C}$ is called* $(\mathcal{E}, \mathcal{M})$-structured *provided that*

1. *each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition with isomorphisms, i.e.*
   - *if $e \in \mathcal{E}$, $h \in \mathrm{Iso}(\mathcal{C})$, and if he exists, then $he \in \mathcal{E}$,*
   - *if $m \in \mathcal{M}$, $h \in \mathrm{Iso}(\mathcal{C})$, and if mh exists, then $mh \in \mathcal{M}$,*
2. *$\mathcal{C}$ has $(\mathcal{E}, \mathcal{M})$-factorizations of morphisms; i. e., each morphism f in $\mathcal{C}$ has a factorization $f = me$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and*
3. *$\mathcal{C}$ has the unique $(\mathcal{E}, \mathcal{M})$-diagonalization property; i. e., for each commutative square*

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[m]{} & D \end{array} \qquad (1)$$

   with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exist a unique diagonal $d : B \to C$ such that $de = f$ and $md = g$.

Note that if $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured, then $\mathcal{C}^{\mathrm{op}}$ is $(\mathcal{M}, \mathcal{E})$-structured.

Proposition 2.2. *If $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured, then the following hold:*

1. *$\mathcal{E} \cap \mathcal{M} = \mathrm{Iso}(\mathcal{C})$,*
2. *each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition,*
3. *$\mathcal{E}$ and $\mathcal{M}$ determine each other via the diagonalization property; i. e., a morphism m belongs to $\mathcal{M}$ if and only if for each commutative square of the form (1) with $e \in \mathcal{E}$ there is a diagonal[1].*

Proposition 2.3. *If fg and f are both in $\mathcal{M}$, then g is in $\mathcal{M}$.*

Proposition 2.4. *In an $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$ with products of pairs of objects the following are equivalent:*

1. *$\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$,*
2. *$\mathrm{ExtrMono}(\mathcal{C}) \subseteq \mathcal{M}$,*
3. *$\mathrm{Sect}(\mathcal{C}) \subseteq \mathcal{M}$,*
4. *for each $\mathcal{C}$-object A the diagonal $\delta_A = \langle 1_A, 1_A \rangle : A \to A \times A$ belongs to $\mathcal{M}$,*

[1]Note that uniqueness of the diagonal is not necessary here.

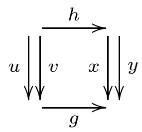
5. $fg \in \mathcal{M}$ *implies that* $g \in \mathcal{M}$,

6. $fe \in \mathcal{M}$ *and* $e \in \mathcal{E}$ *imply that* $e \in \mathrm{Iso}(\mathcal{C})$.

Corollary 2.5. *If* $\mathcal{C}$ *is* $(\mathcal{E}, \mathrm{Mono}(\mathcal{C}))$*-structured, and has binary products, then* $\mathcal{E} = \mathrm{StrongEpi}(\mathcal{C}) = \mathrm{ExtrEpi}(\mathcal{C})$.

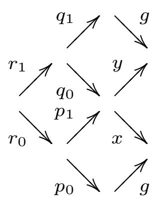
The following results 2.6 and 2.7, which will be needed in Section 6 are taken from [14]. Parts of them can also be found in [12].

Let $\mathcal{C}$ be a category with pullbacks, so that the strong epimorphisms coincide with the extremal ones. Recall that the *pullback of a pair* x, y along a morphism g is the limit



(2)

of the diagram given by x, y and g. Since it is formed by taking three pullbacks



and setting $h = p_1 r_0 = q_0 r_1$, $u = p_0 r_0$, and $v = q_1 r_1$, it follows that h is epimorphic if every pullback of g is epimorphic.

Lemma 2.6. *If* mg *has the same kernel-pair as* g*, and every pullback of* g *is epimorphic, then* m *is monomorphic.*

Proof. It is easy to see that the pullback u, v of the kernel-pair x, y of m along g as in (2) is the kernel-pair of mg. By hypothesis, this is the kernel of g. Thus $xh = gu = gv = yh$, and therefore $x = y$ since h is epic, whence m is monic. □

Now recall that in a category with pullbacks an epimorphism is regular, precisely when it is the coequalizer of its kernel-pair.

Proposition 2.7. *Suppose that* $\mathcal{C}$ *is a category in which pullbacks of extremal epimorphisms are epimorphic. Then*

$$\mathrm{ExtrEpi}(\mathcal{C}) = \mathrm{RegEpi}(\mathcal{C})$$

if $\mathcal{C}$ *either*

(a) admits coequalizers, or

(b) is finitely complete and $(\mathrm{ExtrEpi}, \mathrm{Mono})$*-structured.*

Proof. (a) If e is an extremal epi and g the coequalizer of e's kernel-pair, then $e = mg$ for some morphism m. By Lemma 2.6, m is monic, and therefore, by extremality of e, an iso, whence e is a coequalizer of its kernel-pair.

(b) Let e be an extremal epi again, let k_0, k_1 be its kernel-pair, and let f be a morphism with $fk_0 = fk_1$. Factorize $\langle e, f\rangle = \langle m, n\rangle g$. Clearly the kernel-pair of $mg = e$ coincides with that of $\langle e, f\rangle$, and since $\langle m, n\rangle$ is monic with that of g. Hence, by Lemma 2.6, m is monic and, by extremality of e, an iso. So $f = ng = nm^{-1}e$, which shows that e is regular. □

Note that (b) implies that a finitely complete (ExtrEpi, Mono)-structured category $\mathcal{C}$ is regular as soon as extremal epimorphisms are stable under pullback. From (a) we get that in every $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ and $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ stable under pullback (also called a category with a *proper* and *stable* factorization system) the extremal epimorphisms coincide with the regular ones, because $\mathrm{ExtrEpi}(\mathcal{C}) \subseteq \mathcal{E}$, by the dual of Proposition 2.4.

Proposition 2.8. *In any $(\mathcal{E}, \mathcal{M})$-structured category the class $\mathcal{M}$ (as a full subcategory of $\mathcal{C}^2$) is closed under all limits.*

Proof. Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$-structured and $\mathcal{D}$ be a category. Further let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{D} \to \mathcal{C}$ be diagrams and let $\lambda : \Delta A \to F$ and $\nu : \Delta B \to G$ respectively be their limits. Finally, let $\mu : F \to G$ be a natural transformation which is pointwise in $\mathcal{M}$.

We must show that the unique arrow $f : A \to B$ that makes the diagram

$$\begin{array}{ccc} \Delta A & \xrightarrow{\Delta f} & \Delta B \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ F & \xrightarrow[\mu]{} & G \end{array} \tag{3}$$

commutative lies in $\mathcal{M}$. In order to see this $(\mathcal{E}, \mathcal{M})$-factorize $f = me$. By the diagonalization property there is a unique arrow $d_i : E \to F_i$ for all $i \in \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{e} & E \\ {\scriptstyle\lambda_i}\downarrow & \swarrow{\scriptstyle d_i} & \downarrow{\scriptstyle\nu_i m} \\ F_i & \xrightarrow[\mu_i]{} & G_i\,. \end{array} \tag{4}$$

Note that, by uniqueness, the d_i form a cone $d : \Delta E \to F$.

Therefore there exists a unique arrow $h : E \to A$ such that $\lambda \cdot \Delta h = d$. Then since $\lambda = d \cdot \Delta e$ and since λ is a mono source, $he = 1_A$. To see that $eh = 1_E$, recall that $f = me$ is an $(\mathcal{E}, \mathcal{M})$-factorization and consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & E \\ {\scriptstyle e}\downarrow & \swarrow{\scriptstyle eh} & \downarrow{\scriptstyle m} \\ E & \xrightarrow[m]{} & A\,, \end{array}$$

where the lower right triangle commutes since

$$\nu_i fh = \mu_i \lambda_i h = \mu_i d_i = \nu_i m$$

by diagram (3) and diagram (4).

This proves that m is an isomorphism, whence $f \in \mathcal{M}$. □

Corollary 2.9. *The class $\mathcal{E}$ is closed under all colimits in any $(\mathcal{E}, \mathcal{M})$-structured category.*

Proposition 2.10. *If $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured then $\mathcal{M}$ is stable under the formation of pullbacks, i. e. given a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[m]{} & C \end{array},$$

then $q \in \mathcal{M}$ if $m \in \mathcal{M}$.

3 Spans and Relations

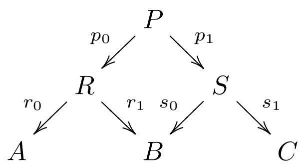
The introduction into relations given in this section and the discussion on maps in Section 4 in principle follow the work of Jayewardene and Wyler (cf. [10]) and Pavlović (cf. [21]) respectively. However, the proofs presented here may differ from the original papers and the material is presented in a different order. Moreover, some of the results have been refined here and others are new. Different proofs and refined or new results will be indicated where they appear. Now let us begin by introducing spans.

3.1 Spans

Let $\mathcal{C}$ be a finitely complete category. For objects A and B of $\mathcal{C}$ we form the category $\mathbf{Span}(A, B) \simeq \mathcal{C}/A \times B$ consisting of equivalence classes of isomorphic objects of $\mathcal{C}/A \times B$. Hence, an object of $\mathbf{Span}(A, B)$ can be represented by a pair of arrows $r = \langle r_0, r_1 \rangle : R \to A \times B$, which will be denoted by $r : A \nrightarrow B$. We will also take the freedom and refer to such an arrow as a *span*, instead of considering the equivalence classes.

Arrows in $\mathbf{Span}(A, B)$ are denoted by $\alpha : r \to s$, for spans $r : R \to A \times B$ and $s : S \to A \times B$; hence, α is represented by an arrow $\alpha_0 : R \to S$ such that $r = s\alpha_0$.

To compose spans $r : A \nrightarrow B$ and $s : B \nrightarrow C$ horizontally, one forms a pullback as shown in the next diagram



(5)

and puts $s \diamond r :\simeq \langle r_0 p_0, s_1 p_1 \rangle$. This composition is clearly functorial and associative, up to isomorphism at the level of morphisms in $\mathcal{C}$, hence strictly so at the level of equivalence classes. The identities are represented by the diagonals

$$\delta_A = \langle 1_A, 1_A \rangle : A \to A \times A.$$

It is easy to check that we obtain a 2-category $\mathbf{Span}(\mathcal{C})$. Its 0-cells are the objects of $\mathcal{C}$, its 1-cells are the spans and its 2-cells are the arrows of the hom-categories $\mathbf{Span}(A, B)$.

Example 3.1. Consider the bicategory $\mathbf{Span}(\mathbf{Set})$. A function $f : X \to Y$ in $\mathbf{Set}$ can be viewed as a functor $F : Y \to \mathbf{Set}$ assigning to each object of the discrete category Y its preimage under f. Hence, a span $r = \langle r_0, r_1 \rangle$ can be represented by a set-valued matrix

$$M = \left(r^{-1}(a, b)\right)_{\substack{a \in A \\ b \in B}}.$$

Composition of $M : A \nrightarrow B$ and $N : B \nrightarrow C$ then resembles matrix multiplication:

$$(N \circ M)_{ac} = \sum_{b \in B} M_{ab} \times N_{bc}.$$

The identity on A is of course represented by the identity matrix whose entries are singleton sets containing the elements of A along the diagonal, and whose other entries are empty.

Let us close this section with the remark that a calculus of spans, and more generally a calculus of relations as presented below, does not depend on the existence of binary products in $\mathcal{C}$. Assuming that they exist forces $\mathcal{C}$ to be finitely complete, since the existence of pullbacks is required to define horizontal composition. However, dropping the existence requirement for binary products in $\mathcal{C}$ greatly increases the technical effort necessary to discuss relations. Therefore we will always assume that $\mathcal{C}$ has binary products, and that it is therefore finitely complete.

3.2 Relations relative to a factorization system

Let $\mathcal{C}$ be any $(\mathcal{E}, \mathcal{M})$-structured category. For an object X we denote by $\mathcal{M}/X$ the full subcategory of $\mathcal{C}/X$ formed by the arrows of $\mathcal{M}$. By considering equivalence classes of isomorphic elements of $\mathcal{M}/X$ we get the equivalent category $\mathrm{sub}(X)$ of $\mathcal{M}$-*subobjects* of X. Rather than using these equivalence classes we shall refer to elements of $\mathcal{M}/X$ as $\mathcal{M}$-subobjects in lieu of the equivalence classes represented by them, and we write $m \simeq n$ for $m, n \in \mathcal{M}/X$ if there exists an isomorphism i of $\mathcal{C}$ with $mi = n$. Note that if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, then $\mathcal{M}/X$ becomes a preordered class with respect to the order defined by

$$m \leq n \iff \exists j : nj = m.$$

Furthermore, in this case $\mathrm{sub}(X)$ is a partially ordered class.

Now let $\mathcal{C}$ be a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category. For objects A and B of $\mathcal{C}$ let us form the category of relations from A to B as $\mathbf{Rel}(A, B) = \mathrm{sub}(A \times B)$. Note that $(\mathbf{Iso}, \mathbf{All})$ where $\mathbf{All}$ denotes the class of all morphisms and $\mathbf{Iso}$ denotes the class of isomorphisms of $\mathcal{C}$, is a factorization structure for every category $\mathcal{C}$. Thus $\mathbf{Span}(A, B)$ is a special case of a category of relations from A to B. We therefore adopt all the notation introduced in 3.1 for relations. As for spans we shall refer to objects of $\mathcal{M}/A \times B$ as *relations*, again meaning the equivalence class represented by them.

Finally observe that although our definition of relations and their composition follows [21] notationally, it in fact is slightly different since Pavlović does not work with equivalence classes but rather with objects of $\mathcal{M}/A \times B$.

3.3 Composition of relations

For objects A and B of $\mathcal{C}$ the $(\mathcal{E}, \mathcal{M})$-factorization system gives rise to a functor

$$\mathrm{im} : \mathbf{Span}(A, B) \to \mathbf{Rel}(A, B),$$

which assigns to an object of $\mathbf{Span}(A, B)$ represented by $s : S \to A \times B$ the object of $\mathbf{Rel}(A, B)$ obtained by $(\mathcal{E}, \mathcal{M})$-factorizing s. This is clearly well-defined and functorial. We denote by in the inclusion functor. In order to compose relations one uses the adjunction $\mathrm{im} \dashv \mathrm{in} : \mathbf{Rel}(A, B) \to \mathbf{Span}(A, B)$ as shown in the diagram

$$\begin{array}{ccc} \mathbf{Span}(A,B) & \xrightarrow{\sim} & \mathcal{C}/A\times B \\ \mathrm{im}\big\downarrow \dashv \big\uparrow \mathrm{in} & & \mathrm{im}_0\big\downarrow \dashv \big\uparrow \mathrm{in}_0 \\ \mathbf{Rel}(A,B) & \xrightarrow{\sim} & \mathcal{M}/A\times B\,. \end{array}$$

The functors im_0 and in_0 are obtained by choosing representatives of the equivalence classes in $\mathbf{Span}(A,B)$ and $\mathbf{Rel}(A,B)$ respectively.

The composite of two relations $r : A \mathrel{+\!\!\to} B$ and $s : B \mathrel{+\!\!\to} C$ is thus defined as follows:

$$s \circ r = \mathrm{im}(s \diamond r) \tag{6}$$

This means that first a pullback is formed as in diagram (5), and then the image of $\langle r_0p_0, s_1p_1\rangle$ is taken. Note that composition clearly defines a functor from $\mathbf{Rel}(B,C) \times \mathbf{Rel}(A,B)$ to $\mathbf{Rel}(A,C)$. Moreover, any 2-cell $\alpha : r \to r'$ in $\mathbf{Rel}(A,B)$ induces an arrow $a : \langle r_0p_0, s_1p_1\rangle \to \langle r'_0p'_0, s_1p'_1\rangle$ in $\mathbf{Span}(A,C)$, where $\langle p'_0, p'_1\rangle$ is obtained from r' and s in a diagram like (5). The 2-cell $s \circ \alpha : s \circ r \to s \circ r'$ is now the arrow $\mathrm{im}(a) : \mathrm{im}\langle r_0p_0, s_1, p_1\rangle \to \mathrm{im}\langle r'_0p'_0, s_1p'_1\rangle$ in $\mathbf{Rel}(A,B)$.

Given $\beta : s \to s'$, the 2-cell $\beta \circ r : s \circ r \to s' \circ r$ is obtained similarly. The constructions $s \circ (-)$ and $(-) \circ r$ are easily seen to be functorial. Furthermore α and β are natural in the sense that

$$(\beta \circ r')(s \circ \alpha) = (s' \circ \alpha)(\beta \circ r) \tag{7}$$

holds. This is taken to be the 2-cell $\beta \circ \alpha$.

From now on, we will no longer distinguish between im and im_0. For example, the notation $r \simeq \mathrm{im}(f)$, where $f \in \mathcal{C}/A \times B$ represents a span and $r \in \mathcal{M}/A \times B$ represents a relation, shall be used frequently.

Taking as its objects the objects of $\mathcal{C}$, as its hom-categories the categories $\mathbf{Rel}(A,B)$ and the horizontal composition as defined above, the only ingredients missing to make $\mathbf{Rel}(\mathcal{C})$ a bicategory would be associativity and the identity laws. Unfortunately it turns out that, without any restrictions on $(\mathcal{E},\mathcal{M})$, associativity fails, so that $\mathbf{Rel}(\mathcal{C})$ is in general not a bicategory. An example will be given in Section 3.4. However, once associativity and the identity laws are established, $\mathbf{Rel}(\mathcal{C})$ will even be a 2-category. Nevertheless, we shall refer to it as a bicategory of relations.

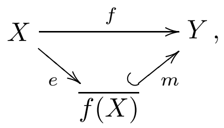
3.4 Associativity and identity relations

Composition of relations is in general not associative. We will first give an example for this (which is due to Klein [15]) and then derive sufficient conditions for relational composition to be associative, with identities given by

$$\iota_A := \mathrm{im}(\delta_A).$$

In the following example the opposite relation of a relation $r = \langle r_0, r_1\rangle$ shall be denoted by $r^o = \langle r_1, r_0\rangle$.

Example 3.2. Consider the category **Top** of topological spaces and continuous functions. Every morphism $f : X \to Y$ of **Top** factors



where e has dense range and m is a closed embedding. Taking $\mathcal{M}$ to be the class of closed embeddings and $\mathcal{E}$ the class of continuous maps with dense range, it is not difficult to check that **Top** is $(\mathcal{E}, \mathcal{M})$-structured.

Denote by Γf the relation $\mathrm{im}\langle 1, f\rangle$. Let $i : \mathbb{Q} \to \mathbb{R}$ be the usual embedding of the rationals into the real line. Clearly $i \in \mathcal{E}$. Further let $j : \mathbb{Q} \to \mathbb{R}$ be defined by $j(x) = ax$ for some fixed irrational number a. Note that we do not need to take images for any of the relations Γi, Γj and $\delta_{\mathbb{R}}$ since $\mathbb{R}$ is Hausdorff, which implies that graphs are closed. Now clearly $\Gamma i \circ (\Gamma i)^o \simeq \delta_{\mathbb{R}}$, and then $(\Gamma i \circ (\Gamma i)^o) \circ \Gamma j \simeq \Gamma j$. On the other hand we clearly get $(\Gamma i)^o \circ \Gamma j = \langle \mathsf{i}_{\mathbb{Q}}, \mathsf{i}_{\mathbb{Q}}\rangle : 0 \to \mathbb{Q} \times \mathbb{Q}$, where again taking images is not necessary since the empty set is closed in $\mathbb{Q} \times \mathbb{Q}$. But then $\Gamma i \circ ((\Gamma i)^o \circ \Gamma j) \simeq \langle \mathsf{i}_{\mathbb{Q}}, \mathsf{i}_{\mathbb{R}}\rangle : 0 \to \mathbb{Q} \times \mathbb{R}$ which is in $\mathcal{M}$. Recall that $\Gamma j = \langle 1, j\rangle$ is clearly not the empty relation from $\mathbb{Q}$ to $\mathbb{R}$. Therefore $(\Gamma i \circ (\Gamma i)^o) \circ \Gamma j \not\simeq \Gamma i \circ ((\Gamma i)^o \circ \Gamma j)$.

For the more restrictive setting of a *proper* $(\mathcal{E}, \mathcal{M})$-factorization system, i. e., with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ and $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, it has been shown by Klein in [15] that associativity of relational composition is equivalent to $\mathcal{E}$ being stable under all pullbacks. Note that in this setting the identities are given as in **Span**($\mathcal{C}$) by the diagonals $\delta_A : A \to A \times A$ since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ is equivalent to all diagonals being in $\mathcal{M}$ (see Proposition 2.4).

Pavlović in [21] claims that in the general case with no restrictions on $\mathcal{E}$ and $\mathcal{M}$ pullback stability of $\mathcal{E}$ is still necessary and sufficient for relational composition to be associative and the identities on A to be $\mathrm{im}(\delta_A)$. He only proves a very small part of this statement and does not prove necessity of the condition at all.

Jayewardene and Wyler analyse the situation much more thoroughly in [10]. They prove stability of $\mathcal{E}$ is still sufficient but that a weaker condition also suffices. The results of the next two subsections except 3.9 and 3.10 are due to [10].

3.4.1 Images and inverse images of relations

Definition 3.3. *Given an arrow* $f : A \to B$ *of* $\mathcal{C}$ *and relations* $r = \langle r_0, r_1\rangle : A \,+\!\!\!\to C$ *and* $s = \langle s_0, s_1\rangle : B \,+\!\!\!\to C$*, we define the* image functor $f^{\to} : \mathbf{Rel}(A, C) \to \mathbf{Rel}(B, C)$ *by*

$$f^{\to} r = \mathrm{im}\langle f r_0, r_1\rangle.$$

Further we define the inverse image functor $f^{\leftarrow} : \mathbf{Rel}(B, C) \to \mathbf{Rel}(A, C)$ *by*

$$f^{\leftarrow} s = \langle s_0', s_1 f'\rangle,$$

where s_0' and f' are obtained by forming a pullback as in

$$\begin{array}{ccccccc} & & & \bullet & & & \\ & & \swarrow^{s_0'} & & \searrow^{f'} & & \\ & A & & & & \bullet & \\ & & \searrow_{f} & & \swarrow_{s_0} & & \searrow^{s_1} \\ & & & B & & & & C\,. \end{array} \tag{8}$$

Note that (8) is a pullback if and only if

$$\begin{array}{ccc} \bullet & \xrightarrow{f'} & \bullet \\ {\scriptstyle \langle s_0', s_1 f' \rangle}\downarrow & & \downarrow{\scriptstyle s} \\ A \times C & \xrightarrow[f \times 1_C]{} & B \times C \end{array}$$

is a pullback. That means that we need not take images in the last definition since $\mathcal{M}$ is stable under pullback.

Let us now note some properties of the functors just defined.

Proposition 3.4. *(i) $f^{\rightarrow} \dashv f^{\leftarrow}$ for any $\mathcal{C}$-arrow f.*

(ii) If gf is defined in $\mathcal{C}$, then $(gf)^{\rightarrow} \simeq g^{\rightarrow} \circ f^{\rightarrow}$ and $(gf)^{\leftarrow} \simeq f^{\leftarrow} \circ g^{\leftarrow}$.

(iii) For relations $r = \langle r_0, r_1 \rangle : A \mapsto B$ and $s : B \mapsto C$ we have $s \circ r \simeq r_0^{\rightarrow} r_1^{\leftarrow} s$.

Proof. Items (ii) and (iii) are immediate consequences of the definitions. For (i) where $f : A \to B$ let $r = \langle r_0, r_1 \rangle : A \mapsto C$ and $s = \langle s_0, s_1 \rangle : B \mapsto C$. We must show that

$$X = \hom(\langle r_0, r_1 \rangle, f^{\leftarrow} \langle s_0, s_1 \rangle) \simeq \hom(f^{\rightarrow} \langle r_0, r_1 \rangle, \langle s_0, s_1 \rangle) = Y$$

is a natural isomorphism. To a given $k : r \to f^{\leftarrow} s$ assign $k \mapsto f'k : \langle f r_0, r_1 \rangle \to s$ of $\mathbf{Span}(B, C)$, where f' is obtained as in (8). This yields a natural isomorphism

$$X \simeq \hom(\langle f r_0, r_1 \rangle, s).$$

Then use the adjunction

$$\mathrm{im} \dashv \mathrm{in}$$

to get a natural isomorphism

$$\hom(\langle f r_0, r_1 \rangle, s) \simeq Y.$$

□

3.4.2 Legs and leg-stability

Definition 3.5. *For a relation $r = \langle r_0, r_1 \rangle : A \mapsto B$ we call r_0 and r_1 the* legs *of r. We say that $\mathcal{C}$ has* stable legs*, or that $\mathcal{C}$ is* leg-stable*, if $\mathcal{E}$ is stable under pullback along legs of relations.*

Observe that (8) shows legs to be pullback stable. Further note that since $\mathcal{C}$ is assumed to have products, legs are composites πm of a projection π of a binary product and an arrow $m \in \mathcal{M}$. Therefore the following result holds:

Proposition 3.6. $\mathcal{C}$ *is leg-stable if and only if*

(i) For $e_0, e_1 \in \mathcal{E}$*, the product* $e_0 \times e_1$ *is in* $\mathcal{E}$*.*

(ii) $\mathcal{E}$ *is stable under pullback along* $\mathcal{M}$*-arrows.*

The next result gives us sufficient conditions for composition of relations to be associative, with identities $\mathrm{im}(\delta_A)$.

Theorem 3.7. *For a finitely complete,* $(\mathcal{E}, \mathcal{M})$*-structured category* $\mathcal{C}$ *the following are equivalent:*

(i) $r \circ \mathrm{im}(a) \simeq \mathrm{im}(r \diamond a)$ *for all spans* a *and relations* r*.*

(ii) $\mathrm{im}(b) \circ r \simeq \mathrm{im}(b \diamond r)$ *for all spans* b *and relations* r*.*

(iii) $r \circ \mathrm{im}\langle a_0, a_1\rangle \simeq a_0^{\rightarrow} a_1^{\leftarrow} r$ *for all spans* $a = \langle a_0, a_1\rangle$ *and relations* r*.*

(iv) $e^{\rightarrow} e^{\leftarrow} r \simeq r$ *for all relations* r *and all* $e \in \mathcal{E}$*.*

If these conditions are satisfied, then composition of relations is associative with identities $\iota_A = \mathrm{im}(\delta_A) : A \nrightarrow A$*.*

Proof. Considering opposite relations it is clear that (i) and (ii) are equivalent. Using these we shall prove the statements at the end about relational composition. In (i) and (ii) choose a and b to be identity spans. Then

$$r \circ \mathrm{im}(\delta_A) \simeq r \simeq \mathrm{im}(\delta_B) \circ r \tag{9}$$

for any relation $r : A \nrightarrow B$. To see associativity of relational composition consider

$$t \circ (s \circ r) \simeq t \circ \mathrm{im}(s \diamond r) \simeq \mathrm{im}(t \diamond (s \diamond r)) \simeq \mathrm{im}((t \diamond s) \diamond r) \simeq \mathrm{im}(t \diamond s) \circ r \simeq (t \circ s) \circ r.$$

Now in (iii) let $a_1^{\leftarrow} r = \langle u_0, u_1\rangle$. Then $s \diamond \langle a_0, a_1\rangle \simeq \langle a_0 u_0, u_1\rangle$ and

$$\mathrm{im}(r \diamond \langle a_0, a_1\rangle) = \mathrm{im}\langle a_0 u_0, u_1\rangle = a_0^{\rightarrow} a_1^{\leftarrow} r.$$

Therefore if (i) is true, then the left-hand side is equal to $r \circ \mathrm{im}\langle a_0, a_1\rangle$. Conversely, if (iii) is true, then the right-hand side is equal to $r \circ \mathrm{im}\langle a_0, a_1\rangle$. This shows that (i) and (iii) are equivalent.

Finally, consider $e \in \mathcal{E}$ with codomain A. Then $\mathrm{im}\langle e, e\rangle = \mathrm{im}(\delta_A)$. So if (iii) holds then (9) implies (iv). Conversely, let $\langle a_0, a_1\rangle = \langle s_0, s_1\rangle e$ be an $(\mathcal{E}, \mathcal{M})$-factorization. Then

$$a_0^{\rightarrow} a_1^{\leftarrow} r \simeq s_0^{\rightarrow} e^{\rightarrow} e^{\leftarrow} s_1^{\leftarrow} r \simeq s_0^{\rightarrow} s_1^{\leftarrow} r \simeq r \circ \langle s_0, s_1\rangle \simeq r \circ im\langle a_0, a_1\rangle,$$

i. e., using (iv), (iii) is shown. □

Now let us investigate how leg-stability of $\mathcal{C}$ is connected to the last result.

Theorem 3.8. *If* $\mathcal{C}$ *is leg-stable, then*

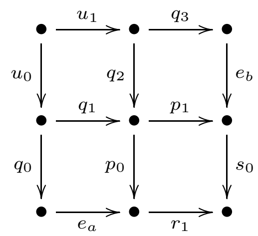
$$\mathrm{im}(\langle b_0, b_1\rangle \diamond \langle a_0, a_1\rangle) \simeq \mathrm{im}\langle b_0, b_1\rangle \circ \mathrm{im}\langle a_0, a_1\rangle \tag{10}$$

for all spans $a = \langle a_0, a_1\rangle$ *and* $b = \langle b_0, b_1\rangle$ *such that the composition is defined and* a_1 *or* b_0 *is a leg. Furthermore 3.7 (i)–(iv) are valid.*

Conversely if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ *and 3.7 (i)–(iv) are true, then* $\mathcal{C}$ *has stable legs.*

Proof. Conditions (i) and (ii) of 3.7 are clearly special cases of (10).

To prove (10) let $\langle a_0, a_1\rangle = \langle r_0, r_1\rangle e_a$ and $\langle b_0, b_1\rangle = \langle s_0, s_1\rangle e_b$ be $(\mathcal{E}, \mathcal{M})$-factorizations. Further let all squares in the diagram



(11)

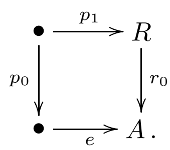
be pullbacks. Now observe that $a \diamond b = \langle r_0 e_a q_0 u_0, s_1 e_b q_3 u_1\rangle$. Recall that legs of relations are pullback stable. Hence, p_0 and p_1 are legs, and then q_1 and q_2 are in $\mathcal{E}$ by leg-stability of $\mathcal{C}$.

If $a_1 = r_1 e_a$ is a leg, then $p_1 q_1$ is a leg and therefore u_0 is in $\mathcal{E}$. But then

$$\mathrm{im}(b \diamond a) \simeq \mathrm{im}\langle r_0 p_0, s_1 p_1\rangle \simeq s \circ r \simeq \mathrm{im}(a) \circ \mathrm{im}(b).$$

An analogous argument applies if b_0 is a leg.

Conversely, let $r = \langle r_0, r_1\rangle : A \mapsto B$ be a relation, and let $e \in E$ have codomain A. Now form a pullback square in $\mathcal{C}$:



Thus we have

$$e^{\rightarrow} e^{\leftarrow} r = \mathrm{im}\langle e p_0, r_1 p_1\rangle = \mathrm{im}\langle r_0 p_1, r_1 p_1\rangle.$$

By 3.7 (iv), this is isomorphic to r. Hence, there is an $(\mathcal{E}, \mathcal{M})$-factorization $\langle r_0 p_1, r_1 p_1\rangle = \langle r_0, r_1\rangle e'$. Since we assume that $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ this implies $p_1 = e \in \mathcal{E}$. Therefore $\mathcal{C}$ has stable legs. □

Corollary 3.9. *For a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$ with $\mathcal{E}$ stable under pullback the image functor satisfies*

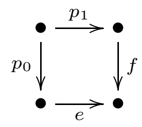
$$\mathrm{im}(b \diamond a) \simeq \mathrm{im}(b) \circ \mathrm{im}(a)$$

for spans a and b such that the composition is defined. In particular, the composition "$\circ$" of relations is associative, with identities given by $\iota_A : A \mapsto A$.

Proof. Considering diagram (11) we have, by pullback stability of $\mathcal{E}$, that u_0, u_1, q_1 and q_2 are in $\mathcal{E}$. As before this implies the result. □

Corollary 3.10. *If $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, then composition of relations is associative if and only if $\mathcal{E}$ is stable under pullback.*

Proof. One direction is 3.9. For the converse note that $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ implies that all arrows of the form $\langle 1, f\rangle$ lie in $\mathcal{M}$. Suppose



is a pullback square with $e \in \mathcal{E}$. It is easy to see that

$$r := \langle e, 1\rangle \circ (\langle 1, e\rangle \circ \langle f, 1\rangle) \simeq \mathrm{im}\langle ep_0, p_1\rangle \simeq \mathrm{im}\langle fp_1, p_1\rangle.$$

On the other hand $\langle e, 1\rangle \circ \langle 1, e\rangle \simeq \delta$, which implies that $r \simeq \langle f, 1\rangle$ by associativity. Hence, $\langle fp_1, p_1\rangle e' = \langle f, 1\rangle$ for some $e' \in \mathcal{E}$. Since $\langle f, 1\rangle$ is monic, $p_1 e' = 1$, whence p_1 lies in $\mathcal{E}$. □

The last result shows that if $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, then leg-stability of $\mathcal{C}$ is equivalent to pullback stability of $\mathcal{E}$. Moreover, observe that the results obtained essentially generalize those of Klein (cf. [15]).

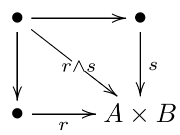
If $\mathcal{E}$ does not consist of epimorphisms, then it is possible that $\mathcal{E}$ is not stable under pullback but $\mathcal{C}$ is still leg-stable (see [10], 1.8.1 for an example). However, there remain two unsolved problems:

1. Can relational composition be associative, with identities ι_A, but 3.7 (i)–(iv) not valid?

2. Can 3.7 (i)–(iv) be valid if $\mathcal{C}$ does not have stable legs?

3.5 Structure of Rel($\mathcal{C}$)

The hom-categories $\mathbf{Rel}(A, B)$ of the bicategory of relations $\mathbf{Rel}(\mathcal{C})$ are finitely complete. Pullbacks are lifted from $\mathcal{C}$. The terminal object is $1_{A\times B} : A \times B \to A \times B$ which clearly is in $\mathcal{M}$. The product of $r : A \nrightarrow B$ and $s : A \nrightarrow B$ in $\mathbf{Rel}(A, B)$ is denoted by $r \wedge s$ and is given by forming the pullback

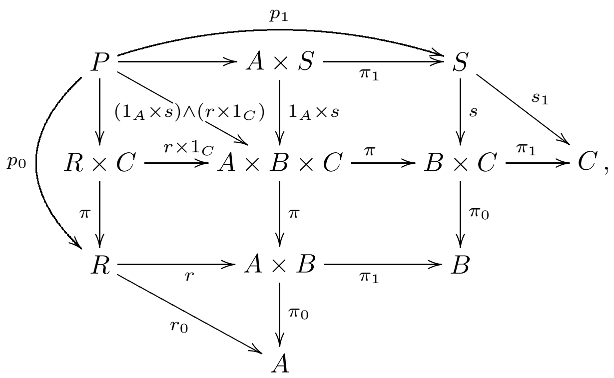


By the pullback stability of $\mathcal{M}$ (see Proposition 2.10) the arrow $r \wedge s$ is in $\mathcal{M}$.

Composition of relations can be presented using the local product just described and the global product of $\mathcal{C}$:

$$s \circ r = \mathrm{im}\,(\pi((1_A \times s) \wedge (r \times 1_C))) \tag{12}$$

where $\pi : A \times B \times C \to A \times C$ denotes a projection. To see this note that $(1_A \times s)$ and $(r \times 1_C)$ are in $\mathcal{M}$, recall diagram (5) and consider the following diagram:



where the 4 inner squares are pullbacks.

Every bicategory of relations is self-dual. Consider the global assignment

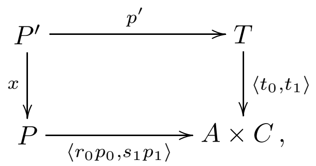
$$(-)^o : \mathbf{Rel}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Rel}(\mathcal{C})$$

that leaves the objects and 2-cells unchanged, while taking an $\mathcal{M}$-relation $r = \langle r_0, r_1 \rangle : R \to A \times B$ into the opposite relation $r^o = \langle r_1, r_0 \rangle : R \to B \times A$, which indeed is in $\mathcal{M}$ since $A \times B \simeq B \times A$. It defines a functor for each hom-category. Furthermore composition of 1-cells is preserved. So $(-)^o$ is a bifunctor.

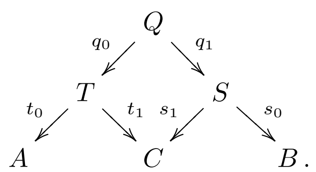
In any bicategory $\mathbf{Rel}(\mathcal{C})$, where $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured with $\mathcal{E}$ stable under pullback, Freyd's Modular Law (cf. [8]) holds true, in the sense that for any relations $r : A \nrightarrow B$, $s : B \nrightarrow C$ and $t : A \nrightarrow C$ there exist a 2-cell

$$s \circ r \wedge t \to s \circ (r \wedge s^o \circ t).$$

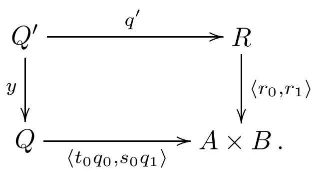
The relation on the left-hand side is obtained by factorizing the arrow $P' \to A \times C$ obtained by forming the pullback



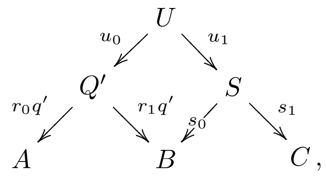
where $\langle r_0 p_0, s_1 p_1 \rangle$ is obtained as in diagram (5). Now let us consider $s^o \circ t$, which is obtained using



To get $r \wedge s^o \circ t$ we form the pullback



Now observe that $t_1 p' = s_1 p_1 x$. So there is a unique arrow $h : P' \to Q$ with $\langle q_0, q_1 \rangle h = \langle p', p_1 x \rangle$. Now $\langle t_0 q_0, s_0 q_1 \rangle h = \langle t_0 p', s_0 p_1 x \rangle = r p_0 x$. Hence, there is a unique arrow $h' : P' \to Q'$ with $yh' = h$ and $q' h' = p_0 x$. Finally, consider the diagram



which induces the relation $s \circ (r \wedge s^o \circ t)$. Then $r_1 q' h' = s_0 q_1 h = s_0 p_1 x$, whence there exists an arrow

$$\langle r_0 q' h', s_1 p_1 x \rangle = tp' \to \langle r_0 q' u_0, s_1 u_1 \rangle,$$

which gives us the desired 2-cell by the universal property of the $(\mathcal{E}, \mathcal{M})$-factorization system.

Finally, note that a bicategory of relations need not yield an *allegory* in the sense of Freyd and Scedrov (cf. [8]) if we want the local product to coincide with the "intersection" of relations (see Example A.3 in Appendix A).

3.6 Rel($\mathcal{C}$) as symmetric monoidal closed 2-category

It is well-known that **Rel**(**Set**) can be viewed as a symmetric monoidal closed 2-category with the tensor given by the cartesian product. This fact can be generalized. For every $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$ with $\mathcal{E}$ stable under pullback, **Rel**($\mathcal{C}$) is symmetric monoidal closed. For the definition of a symmetric monoidal closed category consult [13].

In **Rel**($\mathcal{C}$) the tensor

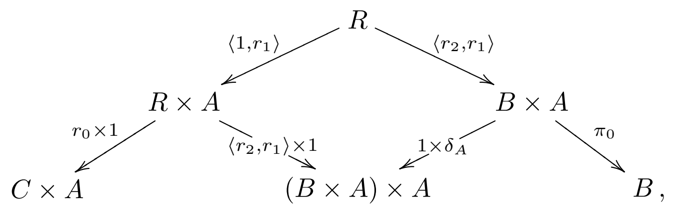
$$\otimes : \mathbf{Rel}(\mathcal{C}) \times \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{C})$$

is given by the binary product on $\mathcal{C}$, that means that $A \otimes B := A \times B$ on objects and $r \otimes s := \langle r_0 \times s_0, r_1 \times s_1 \rangle : A \otimes B \to A' \otimes B'$ for relations $r = \langle r_0, r_1 \rangle : A \mathrel{+\!\!\!\to} A'$ and $s : \langle s_0, s_1 \rangle : B \mathrel{+\!\!\!\to} B'$. Clearly $\otimes$ is 2-functorial since products commute with pullbacks in $\mathcal{C}$ and $\mathcal{E}$ and $\mathcal{M}$ are closed under products by stability. Note that the unit object is the terminal object 1 of $\mathcal{C}$. Associativity and symmetry of $\otimes$ as well as the necessary coherence conditions follow now easily from those of $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. As for closedness note the following result.

Proposition 3.11. *The functor* $- \otimes A : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{C})$ *is self-adjoint for every object* A *of* **Rel**($\mathcal{C}$).

Proof. The components of the counit $e_B : (B \times A) \times A \mathrel{+\!\!\!\to} B$ are given as $e_B := \mathrm{im}\langle 1_B \times \delta_A, \pi_0 \rangle$, where $\pi_0 : B \times A \to B$ is a product projection in $\mathcal{C}$. For any relation $r : \langle \langle r_0, r_1 \rangle, r_2 \rangle : C \times A \mathrel{+\!\!\!\to} B$ we define $\lambda(r) := \langle r_0, \langle r_2, r_1 \rangle \rangle : C \mathrel{+\!\!\!\to} B \times A$.

In order to see that $e \circ (\lambda(r) \otimes \iota_A) \simeq r$ consider the following diagram



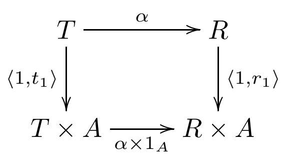
(13)

where the square is evidently a pullback. As for the uniqueness of $\lambda(r)$ suppose that $s = \langle s_0, \langle s_2, s_1 \rangle\rangle : C \nrightarrow B \times A$ is a relation with $e \circ (s \otimes 1_A) \simeq r$. Note that the left-hand side of this is formed as in (13) with all r_i replaced by s_i. Hence, $\langle\langle s_0, s_1\rangle, s_2\rangle \simeq r$, which means that $s \simeq \lambda(r)$ by associativity of $\times$ in $\mathcal{C}$. □

Clearly we cannot be satisfied with the universal property as shown in the last result in a 2-category. But unfortunately we have to impose a condition either on $\mathcal{M}$ or on $\mathcal{E}$ to obtain the universal property for 2-cells.

Corollary 3.12. *The 2-functor* $- \otimes A$ *is self-adjoint for every object* A*, if* $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ *or* $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$*.*

Proof. Suppose $t = \langle\langle t_0, t_1\rangle, t_2\rangle : C \times A \nrightarrow B$ is a relation and $\alpha : t \to r$ with r as above is a 2-cell. Then $\alpha : \lambda(t) \to \lambda(r)$ is a 2-cell with $e_B \circ (\alpha \otimes \iota_A) = \alpha$ because the 2-cell on the left-hand side of this equation can be formed by pulling back $\alpha \times 1_A$ along $\langle 1_R, r_1\rangle$. But clearly the square



(14)

is a pullback.

Uniqueness is clear for $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. If $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ and $\beta : \lambda(t) \to \lambda(r)$ is a 2-cell with $e_B \circ (\beta \otimes \delta_A) = \alpha$, then, since $r \otimes \delta_A$ is given by $\langle r_0 \times 1_A, \langle r_2, r_1\rangle \times 1_A\rangle : C \times A \nrightarrow (B \times A) \times A$ without taking an image, the composite $e_B \circ (\beta \otimes \delta_A)$ must be given by a pullback like (14) with $\alpha \times 1_A$ replaced by $\beta \times 1_A$. But then $\alpha = \beta$ is obvious. □

Note that the conditions on $\mathcal{M}$ and $\mathcal{E}$ respectively are only used to show the uniqueness of the 2-cell $\lambda(t) \to \lambda(r)$.

3.7 A calculus of relations using elements

Under certain circumstances it is possible to obtain a convenient calculus of relations using generalized elements. The convenience of this lies in the fact that relations may be treated almost as if they were relations between sets by referring to their elements.

This has for example been observed by Kelly in [14] for the case of a category $\mathcal{C}$ with a *stable proper* $(\mathcal{E}, \mathcal{M})$-factorizations system. Kelly's results however may be further generalized, by dropping the condition $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$.

But first let us clarify what is meant by an element or generalized element.

Definition 3.13. *Let $\mathcal{C}$ be an $(\mathcal{E},\mathcal{M})$-structured category and B an object of $\mathcal{C}$.*

*(i) An arrow $b : X \to B$ is called an X-*element *of the object B.*

(ii) If $m : M \to B$ is an $\mathcal{M}$-subobject of B we say an X-element b belongs *to m if there exist an arrow $x : X \to M$ such that $b = mx$.*

The next proposition is the key observation that allows us to develop a calculus of relations using elements.

Proposition 3.14. *Let $\mathcal{C}$ be an $(\mathcal{E},\mathcal{M})$-structured category with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. Then any $\mathcal{M}$-subobject is determined by the knowledge for all objects X of the X-elements that belong to it.*

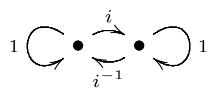
Proof. First observe that the elements of an $\mathcal{M}$-subobject do not depend on the representation of that subobject. Let $m : M \to B$ and $m' : M' \to B$ represent the same subobject, i. e., there is an isomorphism $i : M \to M'$ with $m'i = m$. Now if $b : X \to B$ belongs to m, that means that $mx = b$ for some arrow $x : X \to M$, then $m'ix = b$, whence b belongs to m'.

Note that every $\mathcal{M}$-subobject m belongs to itself. To complete the proof suppose that m and m' have the same X-elements for every object X. In particular m' belongs to m and vice versa. That means there exist arrows $i : M \to M'$ and $j : M' \to M$ such that $m'i = m$ and $mj = m'$. Using $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, it is easy to see that $m \simeq m'$, whence m and m' represent the same $\mathcal{M}$-subobject. □

The second part of the proof of the previous result does not in general remain true if the condition $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ is dropped, as the following example shows.

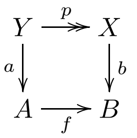
Example 3.15. Consider the category **CAT** of (small) categories and functors between them equipped with a factorizations system as follows. Let $\mathcal{E}$ be the class of functors that are bijective on objects and let $\mathcal{M}$ be the class of full and faithful functors. Then clearly every functor $F : \mathcal{A} \to \mathcal{B}$ factors through the category $\mathcal{C}$ which has the same objects as $\mathcal{A}$ and with $\mathcal{C}(A_1, A_2) := \mathcal{B}(FA_1, FA_2)$. Obviously F gives rise to functors $E : \mathcal{A} \to \mathcal{C}$ in $\mathcal{E}$ and $M : \mathcal{C} \to \mathcal{B}$ is $\mathcal{M}$ with $F = ME$. It is furthermore easy to check the universal property. Finally note that **CAT** is finitely complete and that $\mathcal{E}$ is stable under pullback. So **CAT** admits a calculus of relations. Observe however, that neither $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ nor $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ hold true.

To see that the second part of the previous proof fails, consider the functors $M : 2 \to 1$ and $\mathrm{Id}_1 : 1 \to 1$ where 1 denotes a discrete category with one object and 2 is the category given by



Clearly both functors are in $\mathcal{M}$, hence represent different $\mathcal{M}$-subobjects since obviously 2 is not isomorphic to 1. However, Id_1 and M have the same $\mathcal{X}$-elements. Clearly all $\mathcal{X}$-elements of Id_1 belong to M since 1 is a subcategory of 2. Moreover if $B : \mathcal{X} \to 1$ belongs to M, then B belongs to Id_1, too, since $B = \mathrm{Id}_1 B$.

Proposition 3.16. *Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback. Then an X-element b of B belongs to the image of an arrow $f : A \to B$ if and only if there is an arrow $p : Y \to X$ in $\mathcal{E}$ and an Y-element a of A such that the square*

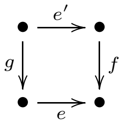


commutes.

Proof. Let $f = me$ be an $(\mathcal{E}, \mathcal{M})$-factorization of f. If b belongs to m, that means that $mi = b$ for some arrow i, then pulling back e along i yields the desired p and a.

Conversely, if we have a square $fa = bp$ with $p \in \mathcal{E}$, then the universal property of the factorization systems gives an arrows i with $mi = b$ so that b belongs to m. □

The previous result holds true, whenever $\mathcal{E}$ is stable under pullback. If $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, then the statement is even equivalent to stability of $\mathcal{E}$ under pullback. In fact, suppose that

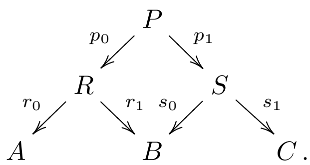


is a pullback square with e in $\mathcal{E}$, then f belongs to the image of e. So if we have p in $\mathcal{E}$ and an element a with $ea = fp$ we get a unique h with $gh = a$ and $e'h = p$, whence $e' \in \mathcal{E}$ by the dual of Proposition 2.4. For the rest of this section we assume that $\mathcal{E}$ is stable under pullback.

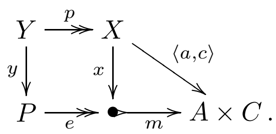
Now let us introduce some more notation. An element $\langle a, b\rangle : X \to A \times B$ belongs to a relation $r : R \to A \times B$ if and only if there is an $x : X \to R$ with $\langle a, b\rangle = rx$. In this case we shall write $b(r)a$.

Proposition 3.17. *Let $r : A \nrightarrow B$ and $s : B \nrightarrow C$ be relations and let $\langle a, c\rangle : X \to A \times C$ be an X-element. Then $c(s \circ r)a$ if and only if there exist an arrow $p : Y \to X$ in $\mathcal{E}$ and an Y-element b with $cp(s)b$ and $b(r)ap$.*

Proof. ($\Rightarrow$) The composite $s \circ r$ is obtained by forming a pullback as shown in the diagram

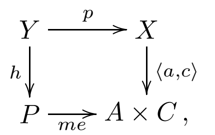


Let $me = \langle r_0p_0, s_1p_1\rangle$ be an $(\mathcal{E}, \mathcal{M})$-factorization. Now if $c(s \circ r)a$, then there is an arrow x with $mx = \langle a, c\rangle$. Form the pullback of e along x to obtain an arrow $p : Y \to X$ in $\mathcal{E}$ and an arrow $y : Y \to P$ as shown in the diagram



Define $b := r_1 p_0 y = s_0 p_1 y$. Then observe that $\langle ap, b\rangle = \langle r_0 p_0 y, r_1 p_0 y\rangle = r p_0 y$ and $\langle b, cp\rangle = \langle s_0 p_1 y, s_1 p_1 y\rangle = s p_1 y$. Hence, $cp(s)b$ and $b(r)ap$.

($\Leftarrow$) Conversely, suppose we have $cp(s)b$ and $b(r)ap$, which means that there are arrows $x : Y \to R$ and $y : Y \to S$ with $rx = \langle ap, b\rangle$ and $sy = \langle b, cp\rangle$. So $r_1 x = b = s_0 y$, whence there is a unique $h : Y \to P$ with $p_0 h = x$ and $p_1 h = y$. Thus we obtain a commutative square



where $p \in \mathcal{E}$. Thus, by Proposition 3.16, $\langle a, c\rangle$ belongs to m, i. e., $c(s \circ r)a$. $\square$

Note that the second part of the result still holds if we omit p. To be precise, $c(s)b$ and $b(r)a$ imply $c(s \circ r)a$ but in general not conversely. Also observe that $b(r)a$ implies $bx(r)ax$ for all arrows x such that the composition is defined and conversely $bp(r)ap$ implies $b(r)a$ if $p \in \mathcal{E}$ using the universal property of the factorization system.

Having established an "elementwise" calculus of relations one can now easily prove associativity of relation composition similar as in **Rel**(**Set**). This is a straightforward computation which may be done in 4 lines by the Reader.

To illustrate the strength of this calculus let us reprove Freyd's Modular Law, which needed quite a bit of diagram chasing, when we proved it in the general case (see 3.5). First note that for any relations x and y, $b(x \wedge y)a$ if and only if $b(x)a$ and $b(y)a$. Now let $r : A \leftrightarrow B$, $s : B \leftrightarrow C$ and $t : A \leftrightarrow C$ be relations. In order to show

$$s \circ r \wedge t \leq s \circ (r \wedge s^o \circ t),$$

it is sufficient to show that $b(s \circ r \wedge t)a$ implies $b(s \circ (r \wedge s^o \circ t))a$. But this can be checked easily enough. If $b(s \circ r \wedge t)a$, then $b(s \circ r)a$ and $b(t)a$. By Proposition 3.17, there is a $p \in \mathcal{E}$ and an element c such that $bp(s)c$ and $c(r)ap$. Clearly $bp(t)ap$, whence $c(s^o \circ t)ap$, which implies $c(r \wedge s^o \circ t)ap$. Finally, by Proposition 3.17, $b(s \circ (r \wedge s^o \circ t)a$.

4 Maps

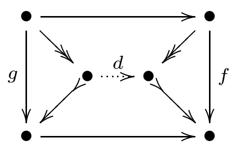
In Subsection 4.5 below the categorical notion of totality and single-valuedness of a relation will be introduced. One of the standard results about regular categories, i. e., categories that are (RegEpi, Mono) structured, is that every total and single-valued relation (also called a *map*) of $\mathbf{Rel}(\mathcal{C})$ corresponds to a unique arrow of $\mathcal{C}$. That means that every regular category is isomorphic to its category of maps:

$$\mathcal{C} \simeq \mathbf{Map}(\mathbf{Rel}(\mathcal{C})).$$

In [21] Pavlović gives a condition for an arbitrary $(\mathcal{E}, \mathcal{M})$-factorization system to allow an isomorphism like this. The condition is simply that $\mathcal{E}$ must consist of regular epis. Although easily stated, the proof of this result is quite involved. Here we present Pavlović's results in a slightly different order, which enables us to simplify some of his proofs. Moreover, it was possible to remove an error in the proof of the main result of [21] and its technical lemma. We give a proof of the main result of [22] for our setting. This result turns out to be useful for many subsequent results. Finally, note that the results of Section 4.7 do not appear elsewhere in this form.

For this section it is assumed that, for a given finitely complete $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$, the class $\mathcal{E}$ is stable under all pullbacks. Corollary 3.9 then allows us to form the bicategory $\mathbf{Rel}(\mathcal{C})$. Before we start let us just note a rather simple but very useful result. Its proof is immediate and therefore omitted.

Proposition 4.1. *Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback. Further let*



be a commutative square where the inner squares are formed by taking $(\mathcal{E}, \mathcal{M})$-factorizations of f and g and then using the universal property of the factorization to obtain d.

Then the outer square is a pullback if and only if the two inner ones are.

Now let us follow Pavlović and investigate maps in such a bicategory of relations.

4.1 Maps in arbitrary bicategories

Definition 4.2. (Lawvere) *A 1-cell in a bicategory is called a* map *if it has a right adjoint.*

That means for a map $r : A \nrightarrow B$ that there exists a 1-cell $r^* : B \nrightarrow A$ and 2-cells $\eta : \iota_A \to r^* \circ r$ and $\varepsilon : r \circ r^* \to \iota_B$ such that the adjunction equations

$$(\varepsilon \circ r)(r \circ \eta) = r, \quad (15)$$

$$(r^* \circ \varepsilon)(\eta \circ r^*) = r^* \quad (16)$$

hold.

Note that in every bicategory the maps form a subbicategory—they are closed under composition and contain the identities. Now let us prove a very useful lemma showing that two maps become isomorphic as soon as they are connected by a 2-cells commuting with the adjunction structure.

Lemma 4.3. *Let $\mathcal{B}$ be any bicategory. Let $r, s \in \mathcal{B}(A, B)$ be maps with right adjoints r^* and s^*, units η_r and η_s, and counits ε_r and ε_s. Further let $\alpha : r \to s$ and $\alpha^* : r^* \to s^*$ be 2-cells such that*

$$(\alpha^* \circ \alpha)\eta_r = \eta_s, \tag{17}$$

$$\varepsilon_s(\alpha^* \circ \alpha) = \varepsilon_r \tag{18}$$

are satisfied. Then α is an isomorphism so that r and s are isomorphic to each other. (Dually α^* is an isomorphism.)

Proof. First of all, let us show that α is split monic. Consider the diagram

$$\begin{array}{ccccc} r & \xrightarrow{r\circ\eta_r} & r \circ r^* \circ r & \xrightarrow{\varepsilon_r \circ r} & r \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\circ r^*\circ r} & & \uparrow{\scriptstyle\varepsilon_s\circ r} \\ s & \xrightarrow[s\circ\eta_r]{} & s \circ r^* \circ r & \xrightarrow[s\circ\alpha^*\circ r]{} & s \circ s^* \circ r \end{array} \tag{19}$$

The right-hand square of this commutes by condition (18), the left-hand square simply by the bicategorial naturality law ((7) on page 10). The adjunction $r \dashv r^*$ says that $(\varepsilon_r \circ r)(r \circ \eta_r) = 1_r$. Therefore

$$\tilde{\alpha} := (\varepsilon_s \circ r)(s \circ \alpha^* \circ r)(s \circ \eta_r)$$

is a left inverse of α. Since $s \dashv s^*$, a diagram dual to (19) yields a left inverse $\tilde{\alpha}^*$ of α^* defined by

$$\tilde{\alpha}^* := (r^* \circ \varepsilon_s)(r^* \circ \alpha \circ s^*)(\eta_r \circ s^*).$$

Then a third version of (19) obtained by switching r and s and replacing α by $\tilde{\alpha}$ and α^* by $\tilde{\alpha}^*$ yields a left inverse of $\tilde{\alpha}$. In this diagram the commutativity of the left-hand square will again just be naturality. In order to prove that the right-hand square is commutative, one has to show that

$$\varepsilon_r(\tilde{\alpha} \circ \tilde{\alpha}^*) = \varepsilon_s.$$

To see this start with $\varepsilon_r(\tilde{\alpha} \circ \tilde{\alpha}^*)$ and plug in the definitions of $\tilde{\alpha}$ and $\tilde{\alpha}^*$. Then use naturality, condition (17) and the adjunction equations (15) and (16), the former in the form $\eta_r = (r^* \circ \varepsilon_r \circ r)(\eta_r \circ \eta_r)$.

Finally, since $\tilde{\alpha}$ has both a left and right inverse, it must be an iso. Thus α is an iso, too. □

4.2 Induced relations

Before we further investigate maps in a bicategory of relations, let us divert for a while and introduce the notion of a graph of an arrow of $\mathcal{C}$ and list some properties that will simplify our investigation.

Firstly, some prerequisites. Recall that a *kernel* of an arrow $f : A \to B$ is a pair of arrows $\ker(f) = \langle k_0, k_1 \rangle : K \to A \times A$ obtained by pulling back f along itself. That means that $\ker(f)$ is a mono-subobject of $A \times A$. Then, of course, kernels of arrows with the same domain are partially ordered. Furthermore it is easy to see that

$$\ker(f) \leq \ker(g) \quad \text{if and only if} \quad (fx = fy \Rightarrow gx = gy)$$

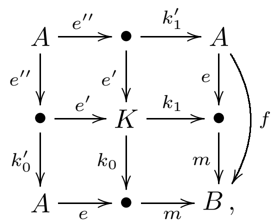
for all x, y.

Definition 4.4. *For any arrow $f \in \mathcal{C}(A, B)$, the relation $\Gamma f := \mathrm{im}\langle 1, f \rangle$ is called the* graph of f *or the relation* induced *by f.*

First note the following result, which will be needed in the proof of Lemma 4.6. It is a generalized version of 4.1(b) in [21].

Lemma 4.5. *If f is monomorphic in $\mathcal{C}$ and $f = me$ is an $(\mathcal{E}, \mathcal{M})$-factorization, where $\mathcal{E}$ is stable under pullback, then m is monomorphic, too.*

Proof. We shall show that the kernel pair of m consists of two equal isomorphisms. In order to see this let $f : A \to B$ be monic and consider the diagram



where all the squares are pullbacks. By stability of the factorization, k_0 lies in $\mathcal{M}$. Since $k_0' e'' = 1$, $k_0' \in \mathcal{E}$ by the dual of 2.3. But then, since $k_0 e' = e k_0'$, k_0 lies in $\mathcal{E}$ for the same reason. Hence, k_0 is an isomorphism. Being a kernel pair, k_0 and k_1 have a common right inverse. Thus, k_0 and k_1 must be equal. $\square$

The next lemma lists some important properties of graphs. Note that item (iv) is due to [10], that no proof of item (i) appears in [21] and that for (ii) we use the more general Lemma 4.5.

Lemma 4.6. *(i) For any $\mathcal{C}$-morphism f, Γf is a map.*

(ii) As an arrow of $\mathcal{C}$, a graph is always monic. In particular, every identity relation $\iota = \mathrm{im}\langle 1, 1 \rangle$ is monic.

(iii) Any relation $r = \langle r_0, r_1 \rangle : A \nrightarrow B$ can be written as the composite $r = \Gamma r_1 \circ (\Gamma r_0)^o$.

(iv) For any $\mathcal{C}$-morphism f and for relations r and s the following hold:

$$f^{\to} r = r \circ (\Gamma f)^o \quad \textit{and} \quad f^{\leftarrow} s = s \circ \Gamma f,$$

and dually

$$\Gamma f \circ r = (f^{\to} r^o)^o \quad \textit{and} \quad (\Gamma f)^o \circ s = (f^{\leftarrow} s^o)^o.$$

Proof. (i) We shall show that $(\Gamma f)^o$ is a right adjoint of Γf in $\mathbf{Span}(\mathcal{C})$, i. e. $\langle 1, f\rangle \dashv \langle f, 1\rangle$. Then the result follows from the fact that the image functor im is a homomorphism of bicategories.

Note that $\langle f, 1\rangle \diamond \langle 1, f\rangle = \langle k_0, k_1\rangle = ker(f) : K \to A \times A$. The common right inverse of k_0 and k_1 will be the unit $\eta : \delta_A \to (\Gamma f)^o \diamond \Gamma f$ of the adjunction. The counit $\varepsilon : \Gamma f \diamond (\Gamma f)^o \to \delta_B$ is simply $\varepsilon = f : \langle f, f\rangle \to \delta_B$ viewed as 2-cell.

Let us check the adjunction equation

$$(\varepsilon \diamond \Gamma f)(\Gamma f \diamond \eta) = \Gamma f.$$

It is easy to see that $\Gamma f \diamond (\Gamma f)^o \diamond \Gamma f = \langle k_0, f k_1\rangle$. Then we get that $\Gamma f \diamond \eta = \eta : \langle 1, f\rangle \to \langle k_0, f k_1\rangle$ and $\varepsilon \diamond \Gamma f = k_0 : \langle k_0, f k_1\rangle \to \langle 1, f\rangle$. But $k_0 \eta = 1_A$.

The second adjunction equation

$$((\Gamma f)^o \diamond \varepsilon)(\eta \diamond (\Gamma f)^o) = (\Gamma f)^o$$

holds by self-duality of $\mathbf{Span}(\mathcal{C})$ simply because the first holds.

(ii) For any arrow f of $\mathcal{C}$, $\langle 1, f\rangle$ is monic. Hence, $\Gamma f = \mathrm{im}\langle 1, f\rangle$ is monic by Lemma 4.5.

(iii) This is immediate knowing that $\mathrm{im}(b \diamond a) = \mathrm{im}(b) \circ \mathrm{im}(a)$ for all spans a, b.

(iv) This is again obvious from the definitions of $f^{\to}$, $f^{\leftarrow}$, and relation composition. □

Note that (ii) is not restricted to graphs. By Lemma 4.5, every span that is monomorphic in $\mathcal{C}$ induces a relation that is so.

4.3 Maps in the bicategory of relations

Observe that in light of Lemma 4.6(ii) every graph of an arrow can be the codomain of at most one 2-cell. This already greatly simplifies Lemma 4.3 in a bicategory of relations, namely conditions (17) and (18) are redundant.

Note that this result is not given in [21]. However, it simplifies the proofs of the important results 4.8 and 4.9 significantly. The tedious checking of conditions (17) and (18), which were omitted in [21], become obsolete, and the work to prove 4.9 will be nil. This is the payoff for having first proved Lemma 4.6 about graphs, especially item (ii).

Corollary 4.7. *Let $r, s : A \nrightarrow B$ be maps in a bicategory of relations with right adjoints r^* and s^*, units η_r and η_s and counits ε_r and ε_s. Further let $\alpha : r \to s$ and $\alpha^* : r^* \to s^*$ be 2-cells. Then α is an isomorphism.*

Proof. Proceed exactly as in the proof of 4.3 to produce left inverses of $\tilde{\alpha}$ and $\tilde{\alpha}^*$ of α and α^* respectively. Note that condition (18) holds now automatically since $\varepsilon_s(\alpha \circ \alpha^*)$ and ε_r have monic codomain ι_B.

For the last step of the proof one no longer needs a lengthy diagram chase to prove the right-hand side of the third version of (19) commutative. This time the required equation

$$\varepsilon_r(\tilde{\alpha} \circ \tilde{\alpha}^*) = \varepsilon_s$$

holds automatically. □

The next result shows that in a bicategory of relations a map r has a canonical right adjoint, namely its opposite relation r^o.

Proposition 4.8. *If $r : A \nrightarrow B$ is a map in the bicategory of relations $\mathbf{Rel}(\mathcal{C})$, its right adjoint is $r^o : B \nrightarrow A$.*

Proof. For a simpler notation let us denote a given right adjoint of r by $s : B \nrightarrow A$. Dualizing $r \dashv s$, we have $s^o \dashv r^o : B \nrightarrow A$. The plan is to construct 2-cells $\alpha : s^o \to r$ and $\alpha^* : r^o \to s$. Applying Corollary 4.7 we can then conclude that $r^o \simeq s$, which is the result.

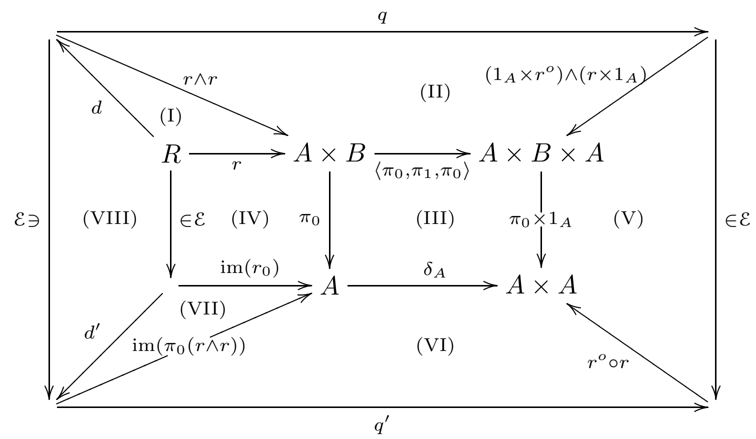
The 2-cell $\alpha : s^o \to r$ will be obtained by dualizing

$$\alpha^o := (r^o \circ \varepsilon)(\kappa \circ s) : s \to r^o \circ r \circ s \to r^o \qquad (20)$$

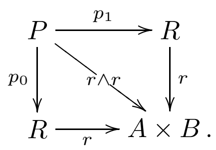
where ε is the counit of the adjunction $r \dashv s$. The 2-cell $\kappa : \iota_A \to r^o \circ r$ will be induced by an arrow $k : \delta_A \to r^o \circ r$ in $\mathbf{Span}(A, A)$. So let us construct k.

The following diagram in $\mathcal{C}$ summarizes the construction:

(21)



Now let us analyse how this comes about. The arrow d in (I) is induced using the universal property of the product $r \wedge r$ in $\mathbf{Rel}(A, B)$, which is given by the pullback



Note that d is simply the diagonal $\langle 1_r, 1_r \rangle$. The arrow q that makes (II) commutative is then obtained using the universal property of the pullback in $\mathcal{C}$ that forms the product $(1_A \times r^o) \wedge (r \times 1_A)$ in $\mathcal{C}/A \times B \times C$. Apply this property to $\langle p_0, \pi_0 r p_0 \rangle : P \to R \times A$ and $\langle \pi_0 r p_0, p_0 \rangle : P \to A \times R$ where p_0 is the left projection of $r \wedge r$ and π_0 the left projection of $A \times B$.

Square (III) is easily seen to be a pullback square. Squares (IV) and (V) are now obtained forming an $(\mathcal{E}, \mathcal{M})$-factorization of $\pi_0 r = r_0$ and

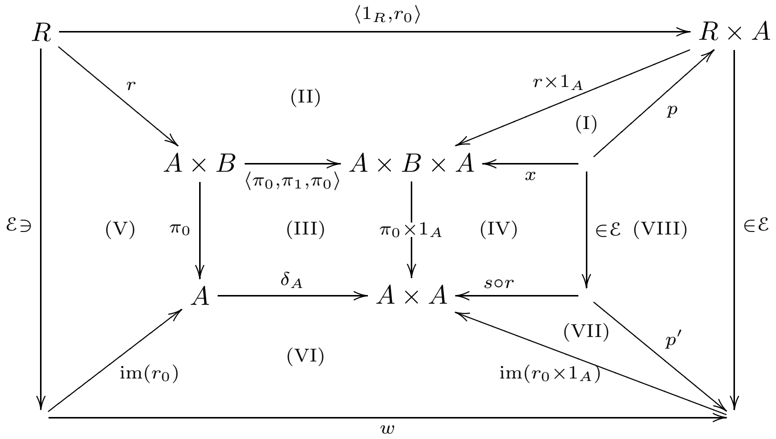
$$(\pi_0 \times 1_A) \cdot ((1_A \times r^o) \wedge (r \times 1_A)).$$

Observe that by (12) on page 15 the latter indeed yields $r^o \circ r$. Furthermore $(\mathcal{E}, \mathcal{M})$-factorize $\pi_0(r \wedge r)$ to form (VII) and (VIII). In order to obtain d' which actually separates them, use the diagonalization property of the factorization system. Finally use that property again to obtain q'.

Next we shall show that $\mathrm{im}(r_0)$ has a section j. Having done this it is possible to define

$$k := q'd'j : \delta_A \to r^o \circ r. \tag{22}$$

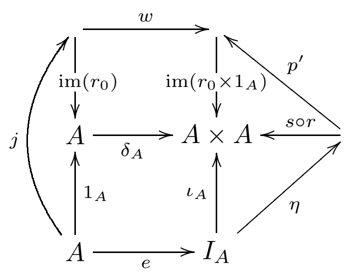
In order to construct j we shall first construct an arrow $p' : s \circ r \to \mathrm{im}(r_0 \times 1_A)$. The following diagram shows this construction:



(23)

Note that x denotes $(1_A \times s) \wedge (r \times 1_A)$. The arrow p in (I) is one of the projections of this product in $\mathcal{C}/A \times B \times A$. Square (II) can easily seen to be a pullback if extended to the right by the "left" product projections. Of course, (III) is still a pullback square, and (IV) and (V) are again $(\mathcal{E}, \mathcal{M})$-factorizations. Furthermore, $(\mathcal{E}, \mathcal{M})$-factorize $(\pi_0 \times 1_A)(r \times 1_A) = (r_0 \times 1_A)$ to get $\mathrm{im}(r_0 \times 1_A)$. The arrows w and p' are finally obtained by using the universal property of the factorization again. Observe that since (II) and (III) are pullbacks both squares glued together form a pullback, too. Since (VI) is obtained by $(\mathcal{E}, \mathcal{M})$-factorizing two opposite sides of this pullback it is a pullback by 4.1.

The universal property of this pullback now produces the desired section j of $\mathrm{im}(r_0)$. Just let $\delta_A = \iota_A e$ be an $(\mathcal{E}, \mathcal{M})$-factorization of the diagonal and consider the diagram



(24)

Recalling definition (22) we get $k : \delta_A \to r^o \circ r$, whence $\kappa : \iota_A \to r^o \circ r$ of (20).

The other 2-cell $\alpha^* : r^o \to s$ can be constructed similarly as

$$\alpha^* := (s \circ \varepsilon^o)(\chi \circ r^o) : r^o \to s \circ s^o \circ r^o \to s,$$

where $\chi : \iota_A \to s \circ s^o$ is derived from $\eta^o : \iota_A \to r^o \circ s^o$ just as $\kappa : \iota_A \to r^o \circ r$ was derived from $\eta : \iota_A \to s \circ r$. □

Corollary 4.9. *Let r, s be maps in a bicategory $\mathbf{Rel}(\mathcal{C})$ of relations. If there is a 2-cell $r \to s$, then $r \simeq s$.*

Proof. We have $r \dashv r^o$ and $s \dashv s^o$ by 4.8. Now every 2-cell $\alpha : r \to s$ induces a 2-cell $\alpha^o : r^o \to s^o$. Thus by 4.7, $r \simeq s$. □

Note that Proposition 4.8 significantly simplifies checking whether a relation r is a map. Firstly r^o is the only candidate for a right adjoint and secondly, one has to check only one of the adjunction equations since (15) and (16) become dual to each other:

$$(r^o \circ \varepsilon)(\eta \circ r^o) = ((\varepsilon \circ r)(r \circ \eta))^o.$$

The last result shows that maps in a bicategory of relations are absolutely rigid. The only 2-cells between them are isomorphisms. Therefore the maps of $\mathbf{Rel}(\mathcal{C})$ form even a category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. Its objects are the same as those of $\mathbf{Rel}(\mathcal{C})$, and its arrows are equivalence classes of isomorphic maps in $\mathbf{Rel}(\mathcal{C})$.

Proposition 4.8 also allows a characterization of equivalences (or equivalently, isomorphisms) of $\mathbf{Rel}(\mathcal{C})$ similar to 3.8 of [10], but generalized.

Proposition 4.10. *A relation r is an isomorphism (necessarily with inverse r^o), if and only if r and r^o are maps.*

Proof. ($\Rightarrow$) If s in an inverse of $r : A \leftrightarrow B$, then $\iota_A \simeq sr$ and $rs \simeq \iota_B$ show r to be a map with right adjoint s. Hence, by 4.8, $s \simeq r^o$. Moreover, r^o is a map.

($\Leftarrow$) Conversely, if r and r^o are maps, then $r^o \circ r$ and $r \circ r^o$ are maps, too. But there are 2-cells $\eta : \iota_A \to r^o \circ r$ and $\varepsilon : r \circ r^o \to \iota_B$. Hence, by 4.9, $\iota_A \simeq r^o \circ r$ and $r \circ r^o \simeq \iota_B$, whence r is an isomorphism with inverse r^o. □

4.4 Convergence and the graph functor

Corollary 4.9 and the remark following it give us a *functor*

$$\Gamma : \mathcal{C} \longrightarrow \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$$

which maps objects identically and which takes an arrow f to the equivalence class $\Gamma f = \mathrm{im}\langle 1, f\rangle$.

The following notions are again relative to an $(\mathcal{E}, \mathcal{M})$-factorization system.

Definition 4.11. *A map $r : A \leftrightarrow B$ is said to* converge *to an arrow $f : A \to B$ of $\mathcal{C}$ if $r \simeq \Gamma f$.*

An object B of $\mathcal{C}$ is separated *if a map to it can converge to at most one arrow; i. e., $\Gamma f \simeq \Gamma g$ implies $f = g$ for all $f, g \in \mathcal{C}(A, B)$.*

The next result will allow a characterization of the faithfulness of Γ in terms of a condition on the $(\mathcal{E}, \mathcal{M})$-factorization system. It is a combination of the results 4.6. and 4.9. of [21]. Item (ii) which is inherent in Pavlović's proofs is stated separately here because it will be helpful for many subsequent results.

Proposition 4.12. *The following are equivalent:*

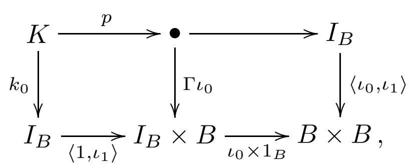
(i) *B is separated*

(ii) *For the identity morphism $\iota_B = \langle \iota_0, \iota_1 \rangle$, $\iota_0 = \iota_1$ holds.*

(iii) *The diagonal δ_B is in $\mathcal{M}$.*

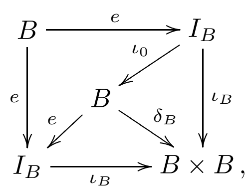
(iv) *For any pair $f, g \in \mathcal{C}(A, B)$ the existence of a 2-cell $\operatorname{im}\langle f, g\rangle \to \iota_B$ implies $f = g$.*

Proof. (i) $\Rightarrow$ (ii): Form the kernel pair of ι_B as shown in the diagram



where both squares are pullbacks. Being a component of $\ker(\iota_B)$, k_0 has a right inverse $h : I_B \to K$. Hence, in $\mathcal{C}/I_B \times B$, one has the morphism $ph : \langle 1, \iota_1\rangle \to \Gamma\iota_0$ and $\operatorname{im}(ph) : \Gamma\iota_1 \to \Gamma\iota_0$ by $(\mathcal{E}, \mathcal{M})$-factorizing. Applying Corollary 4.9 we have $\Gamma\iota_0 \simeq \Gamma\iota_1$. Now, since B is separated, $\iota_0 = \iota_1$.

(ii) $\Rightarrow$ (iii): Because $\delta_B = \iota_B e$ is an $(\mathcal{E}, \mathcal{M})$-factorization with some $e \in \mathcal{E}$, $1_B = \iota_0 e$ follows. Then consider the square



which clearly commutes. By the universal property of the factorization we must have $e\iota_0 = 1_{I_B}$. So e is an iso, whence $\iota_B \simeq \delta_B$, which means that $\delta_B \in \mathcal{M}$.

(iii) $\Rightarrow$ (i): Note that for every arrow $f : A \to B$, $\langle 1_A, f\rangle$ can be obtained by pulling back δ_B along $f \times 1_B$. So if δ_B is in $\mathcal{M}$, then $\langle 1_A, f\rangle$ is in $\mathcal{M}$, too. Therefore $\Gamma f = \langle 1, f\rangle$. But now $\langle 1, f\rangle \simeq \langle 1, g\rangle$ easily implies $f = g$. Hence B is separated.

(iii) $\Rightarrow$ (iv): If $\delta_B \in \mathcal{M}$, then $\iota_B \simeq \delta_B$. If further $\alpha : \operatorname{im}\langle f, g\rangle \to \delta_B$ is a 2-cell, then $\operatorname{im}\langle f, g\rangle = \langle \alpha, \alpha\rangle$. But then $f = \alpha e = g$ where $\langle f, g\rangle = \operatorname{im}\langle f, g\rangle e$ is an $(\mathcal{E}, \mathcal{M})$-factorization.

(iv) $\Rightarrow$ (ii): Take $\langle f, g\rangle = \langle \iota_0, \iota_1\rangle$ to conclude $\iota_0 = \iota_1$. □

The next result is in principle a corollary of Proposition 4.12. However, we shall give an alternative proof due to an earlier preprint of [10], which does not appear in the actual paper. Note that this proof does not use stability of $\mathcal{E}$ under pullback, as Proposition 4.12 does, but only the weaker condition of Theorem 3.7(iv).

Corollary 4.13. *For a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$, the following are equivalent:*

1. *the graph functor $\Gamma : \mathcal{C} \to \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is faithful,*

2. $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$.

Proof. (i) ⇒ (ii) Suppose $fe = ge$ for $e \in \mathcal{E}$. Then

$$e^{\leftarrow}(\Gamma f) \simeq \Gamma(fe) \simeq \Gamma(ge) \simeq e^{\leftarrow}(\Gamma g)$$

by 4.6(iv). Applying $e^{\rightarrow}$ to both sides we get $\Gamma f \simeq \Gamma g$ by 3.7(iv), whence $f = g$ by faithfulness of Γ. Thus e is epimorphic.

(ii) ⇒ (i) Let Γf and Γg be graphs. By 2.4, $\Gamma f \simeq \langle 1, f\rangle$ for all arrows f of $\mathcal{C}$. Hence, $\Gamma f \simeq \Gamma g$ implies $f = g$. □

4.5 Total and single-valued relations

The notion of a map as a relation with a right adjoint tries to capture the idea of total and single-valued relations being maps. Since the 2-cells in a bicategory of relations generalize inclusion of relations, the following definitions emerge.

Definition 4.14. *A relation* $r : A \nrightarrow B$ *is called*

(i) total *if there is a 2-cell* $\iota_A \to r^o \circ r$*,*

(ii) single-valued *if there is a 2-cell* $r \circ r^o \to \iota_B$*,*

(iii) injective *if* r^o *is single-valued,*

(iv) surjective *if* r^o *is total.*

Clearly every map is total and single-valued. The units and counits provide the needed 2-cells.

The following easy fact not given in [21] will again simplify the proof of the forthcoming result of [21], and it will also be useful later.

Lemma 4.15. *If* $r = \langle r_1, r_0\rangle : R \to A \times B$ *is a monomorphism, the following are equivalent:*

(i) r_0 *is monic,*

(ii) $\ker(r_0) \leq \ker(r_1)$

Proof. If r_0 is monic, the second statement clearly holds since $\ker(r_0) = \langle 1_R, 1_R\rangle$.

Conversely, suppose that $\ker(r_0) \leq \ker(r_1)$. But this is equivalent to saying

$$r_0 x = r_0 y \Rightarrow r_1 x = r_1 y$$

for all x, y. So since $\langle r_0, r_1\rangle$ is monic, $r_0 x = r_0 y$ implies $x = y$. □

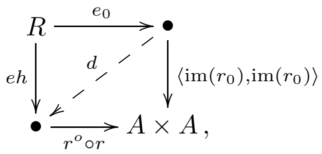
Proposition 4.16. *A relation* $r = \langle r_0, r_1\rangle : R \to A \times B$ *is*

(i) total if and only if $\mathrm{im}(r_0)$ *is a split epi,*

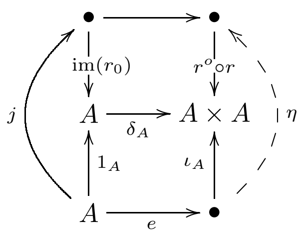
(ii) single-valued if $\ker(r_0) \leq \ker(r_1)$*; the converse holds true if and only if* B *is separated.*

Proof. (i) A 2-cell $\eta : \iota_A \to r^o \circ r$ induces a section of $\mathrm{im}(r_0)$ as shown in the proof of Proposition 4.8. Just exactly repeat the construction shown in diagrams (23) and (24) with s replaced by r^o.

Conversely, let j be a section of $\mathrm{im}(r_0)$. Note that the composite $r^o \circ r$ is given by $\mathrm{im}\langle r_0k_0, r_1k_1\rangle$, where $\langle k_0, k_1\rangle = ker(r_1)$. That means that there is a common right inverse h of k_0 and k_1. Then we get a commutative square



where e and e_0 are the coimages of $\langle r_0k_0, r_1k_1\rangle$ and r_0 respectively. The universal property of the $(\mathcal{E}, \mathcal{M})$-factorization system induces d so that the next diagram is commutative, and $\eta : \iota_A \to r^o \circ r$ is again induced by the universal property of the factorization system:



(ii) The statement $\ker(r_0) \le \ker(r_1)$ actually is equivalent to the identity

$$r_1k_0 = r_1k_1$$

for $\langle k_0, k_1\rangle = \ker(r_0)$. In other words, there is an arrow $r_1k_0 : \langle r_1k_0, r_1k_1\rangle \to \delta_B$. Applying the image functor yields

$$r \circ r^o = \mathrm{im}\langle r_1k_0, r_1k_1\rangle \to \mathrm{im}(\delta_B) = \iota_B.$$

Conversely, if there is a 2-cell $\mathrm{im}\langle r_1k_0, r_1k_1\rangle \to \iota_B$, then since B is separated we have $r_1k_0 = r_1k_1$ by Proposition 4.12.

Finally suppose that single-valuedness of a relation $r = \langle r_0, r_1\rangle$ implies that $\ker(r_0) \le \ker(r_1)$. In particular the identity relation $\iota_B = \langle \iota_0, \iota_1\rangle$ is single-valued. So $\ker(\iota_0) \le \ker(\iota_1)$ and since ι_B is monic, ι_0 is monic by Lemma 4.15. But ι_0 is also a split epi because $\iota_B = \mathrm{im}(\delta_B)$, which shows that $\iota_0 e = 1_B = \iota_1 e$ for some $e \in \mathcal{E}$. Therefore ι_0 is an iso; hence, $\iota_0 = \iota_1$. Thus B is separated by 4.12. □

The last result is a slightly strengthened version of Lemma 5.2 of [21]. Note that separatedness of B is not only sufficient but also necessary for the converse of item (ii) to hold.

Using ideas of the previous proof we can add another property of graphs to our list in Lemma 4.6.

Proposition 4.17. *For an arrow $f : A \to B$ of $\mathcal{C}$, the graph Γf is injective if f is monic. The converse holds true if A is separated.*

Proof. Our task is to find a 2-cell $(\Gamma f)^o \circ \Gamma f \to \iota_A$. Recall that $(\Gamma f)^o \circ \Gamma f$ can be obtained as $\mathrm{im}(\langle f, 1_A\rangle \diamond \langle 1_A, f\rangle) = \mathrm{im}(\ker(f))$. But if f is monic, then $\ker(f) = \delta_A$, whence $\mathrm{im}(\ker(f)) \simeq \iota_A$.

Conversely suppose that A is separated. Then $\delta_A \in \mathcal{M}$. So if there is a 2-cell $\eta : (\Gamma f)^o \circ \Gamma f = \mathrm{im}(\ker(f)) \to \delta_A$, we have $\ker(f) = \mathrm{im}(\ker(f))e = \delta_A \eta e$. So the kernel pair of f consists of two morphisms that are equal. Thus f is monic. □

The characterizations of Proposition 4.16 show that not all total and single-valued relations are maps. To see this consider the following result about spans taken from [5]. We omit the proof because it will be a corollary of Theorem 4.20.

Proposition 4.18. *A span $r : \langle r_0, r_1\rangle : A \nrightarrow B$ is a map in* **Span**$(\mathcal{C})$ *if and only if r_0 is an iso.*

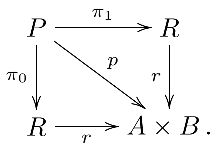
Proposition 4.16 shows that a span $r = \langle r_0, r_1\rangle$ is total and single-valued as soon as r_0 is a split epi and $\ker(r_0) \subseteq \ker(r_1)$. This is clearly not enough to make r_0 an iso. Take for example any non-monic split epi f. Then $\langle f, f\rangle$ is total and single-valued but not a map in **Span**$(\mathcal{C})$.

This means that in general maps cannot be reduced to total and single-valued relations. Totality and single-valuedness have to be suitably connected by the adjunction equations. However, a total and single-valued relation is a map as soon as it is monic as an arrow of $\mathcal{C}$. Moreover, it turns out that all maps are monic arrows of $\mathcal{C}$. This is essentially Theorem 5.3 of [22]. In this sequel of [21], Pavlović works in an even more general setting than that of $(\mathcal{E}, \mathcal{M})$-structured categories, namely in the context of regular fibrations. Here we give an easy proof of his result for relations relative to a factorization system.

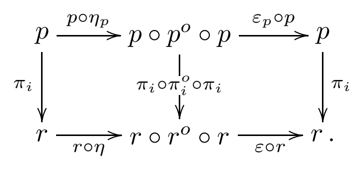
Theorem 4.19. *A relation r in* **Rel**$(\mathcal{C})$ *is a map if and only if it is total, single-valued, and monic as an arrow of $\mathcal{C}$.*

Proof. A total and single-valued relation that is monic is obviously a map since $(\varepsilon \circ r)(r \circ \eta) : r \to r$ must be the identity.

Conversely, every map $r : A \nrightarrow B$ is total and single-valued. We shall show that $r \wedge r$ is a map, too. The relation $r \wedge r$ is given as the arrow p in the pullback square



Furthermore, let $d : r \to r \wedge r$ be a diagonal in **Rel**(A, B). Then clearly, $\pi_i d = 1_r$ for $i = 0, 1$. Define 2-cells by $\eta_p := (d^o \circ d)\eta_r : \iota_A \to p^o \circ p$ and $\varepsilon_p := \varepsilon(\pi_0 \circ \pi_0^o) : p \circ p^o \to \iota_B$. Now consider the diagram



Note that since ι_B is monic in $\mathcal{C}$, and therefore $\varepsilon(\pi_0 \circ \pi_0^o) = \varepsilon(\pi_1 \circ \pi_1^o)$, both squares are commutative by definition for $i = 0, 1$. But the lower side of the whole rectangle is an identity since r is a map. Thus the upper side must be an identity, too, since π_0, π_1 are product projections in $\mathbf{Rel}(A, B)$. Hence, p is a map. But now, $\pi_0 : p \to r$ is a 2-cell between maps, whence an isomorphism by 4.9, which shows that $\pi_0 = \pi_1$, which implies that r is monic in $\mathcal{C}$. □

In a posetal bicategory maps correspond exactly to the total and single-valued relations because in a poset all arrows are monic. The hom-categories $\mathbf{Rel}(A, B)$ are posets precisely if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ (cf. [20], 3.7). This setting will be studied in Section 5.

The last theorem allows another characterization of maps. Note that this result strengthens the technical Lemma 5.4 of [21] but has a much easier proof.

Theorem 4.20. *Let $r = \langle r_0, r_1 \rangle : A \nrightarrow B$ be a relation. If $r_0 \in \mathcal{E}$ and $\ker(r_0) \leq \ker(r_1)$, then r is a map; the converse holds true if and only if B is separated.*

Proof. If $r_0 \in \mathcal{E}$ and $\ker(r_0) \leq \ker(r_1)$, then r is total and single-valued by Proposition 4.16. We shall show that r is monic as an arrow of $\mathcal{C}$. It is well-known that the kernel pair k_0, k_1 of r is given by pulling back $\ker(r_0)$ along $\ker(r_1)$, i.e., by forming the intersection of both kernels. But by hypothesis, $\langle k_0, k_1 \rangle = \ker(r) = \ker(r_0)$. Being the pullback of r along itself, k_0 and k_1 lie in $\mathcal{M}$. But both arrows lie in $\mathcal{E}$, too, since they also can be obtained by pulling back r_0 along itself. Hence k_0 and k_1 are isomorphisms. Being a kernel pair, k_0 and k_1 have a common right inverse, which implies that they must be equal. Thus r_0 and therefore r are monic.

Conversely, suppose that B is separated and r is a map. By Theorem 4.19, r is total, single-valued, and monic in $\mathcal{C}$. Applying Proposition 4.16 we see that if $r_0 = me$ is an $(\mathcal{E}, \mathcal{M})$-factorization, then m is split epic. Moreover, $\ker(r_0) \leq \ker(r_1)$ implies that r_0 must be monic since r is so. Hence, m is monic by Lemma 4.5. Thus m is an iso, which shows that r_0 lies in $\mathcal{E}$.

That separatedness of B is necessary for the converse to be true can be seen as in the proof for single-valuedness in Proposition 4.16. □

The last result together with Propositions 4.10 and 4.16 allows the following important Corollary, which does not appear in [21].

Corollary 4.21. *The following are equivalent:*

(i) $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$,

(ii) a relation is single-valued if and only if $\ker(r_0) \leq \ker(r_1)$,

(iii) a relation is injective if and only if $\ker(r_1) \leq \ker(r_0)$,

(iv) a relation $r = \langle r_0, r_1 \rangle$ is a map if and only if $r_0 \in \mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$,

(v) a relation is an isomorphism if and only if $r_0, r_1 \in \mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$.

4.6 Function comprehension

Here we discuss two more notions relative to a factorization system.

Definition 4.22. *An object B of $\mathcal{C}$ is said to be* Cauchy-complete *if a map to it converges to a unique arrow.*

The category $\mathcal{C}$ is function comprehensive *if all of its objects are Cauchy-complete.*

First of all, observe that every Cauchy-complete object is separated. Furthermore, the graph functor Γ is faithful if and only if all objects of $\mathcal{C}$ are separated.

Now let us characterize function comprehension in terms of a condition on the factorization system. The result will then also produce a condition for the desired isomorphism $\mathcal{C} \to \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ of categories.

Theorem 4.23. *For a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$ with $\mathcal{E}$ stable under pullback, the following are equivalent:*

(i) the graph functor Γ is an isomorphism of categories,

(ii) the category $\mathcal{C}$ is function comprehensive,

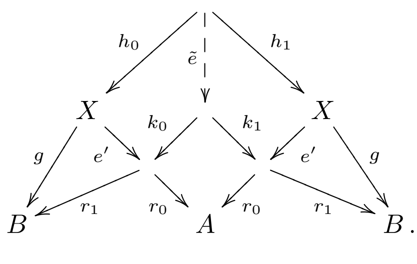
(iii) $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$.

Proof. (i) $\iff$ (ii): The first and second statements are equivalent, since, by definition, Γ maps objects identically, and Cauchy-completeness of *all* objects of $\mathcal{C}$ is equivalent to Γ being full and faithful.

(ii) $\Rightarrow$ (iii): Suppose that $\mathcal{C}$ is function comprehensive, i. e., all objects of $\mathcal{C}$ are Cauchy-complete, whence separated. But this implies that $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, or equivalently that all diagonal δ_A are in $\mathcal{M}$.

We must show that $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$. So let $e : X \to A$ be in $\mathcal{E}$, and let $g : X \to B$ be an arrow of $\mathcal{C}$ that equates the kernel $\langle h_0, h_1 \rangle$ of e in $\mathcal{C}$, i. e., $gh_0 = gh_1$. We have to construct a unique arrow f such that $g = fe$. Then e is a coequalizer of its kernel pair, whence a regular epi.

Now let $r = \langle r_0, r_1 \rangle = \mathrm{im}\langle e, g \rangle$. Clearly r_0 is in $\mathcal{E}$ since $r_0 e' = e$ for some e' in $\mathcal{E}$. We will now show that $\ker(r_0) \leq \ker(r_1)$. This together with $r_0 \in \mathcal{E}$ allows to apply Theorem 4.20 to conclude that r is a map. First we relate $\langle h_0, h_1 \rangle = \ker(e)$ and $\langle k_0, k_1 \rangle = \ker(r_0)$. To do this consider the diagram



Clearly both squares in the upper half are pullbacks; hence, by stability of $\mathcal{E}$ under pullback, $\tilde{e}$ is in $\mathcal{E}$. Therefore

$$\mathrm{im}\langle r_0 k_0, r_1 k_1 \rangle = \mathrm{im}\langle g h_0, g h_1 \rangle.$$

But since $gh_0 = gh_1$, there is a 2-cell $\mathrm{im}\langle gh_0, gh_1 \rangle \to \iota_B \simeq \delta_B$, whence $r_1 k_0 = r_1 k_1$. But this is equivalent to $\ker(r_0) \leq \ker(r_1)$. Thus r is a map. Using function comprehension, we conclude that there is a unique arrow f of $\mathcal{C}$ with $\langle 1, f \rangle \simeq r$. Recall that $\langle 1, f \rangle$ is in $\mathcal{M}$ since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. That means, we can choose $r = \langle 1, f \rangle$, which implies that $\langle e, g \rangle = \langle 1, f \rangle e$ is an $(\mathcal{E}, \mathcal{M})$-factorization that gives us r. Therefore we have a unique f with $g = fe$.

(iii) $\Rightarrow$ (ii): Suppose $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$. For every map $r : A \nrightarrow B$ we must find a unique arrow $f : A \to B$ with $r \simeq \Gamma f$. Note that by Corollary 4.13, all objects of $\mathcal{C}$ are separated. Furthermore, all diagonals as well as all arrows of the form $\langle 1, f\rangle$ are in $\mathcal{M}$ by Proposition 2.4. So graphs are of the form $\Gamma f = \langle 1, f\rangle$. It is therefore sufficient to find a $\mathcal{C}$ arrow $f : A \to B$ such that $r \simeq \langle 1_A, f\rangle$. Uniqueness is then easy since $\langle 1_A, g\rangle \simeq r \simeq \langle 1_A, f\rangle$ implies $f = g$.

The desired arrow f can be obtained easily. Since r is a map, r_0 must lie in $\mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$, which shows that r_0 is an iso by hypothesis. Hence, $r \simeq \langle 1, r_1 r_0^{-1}\rangle$, which completes the proof. □

Observe that Theorem 4.20 and Corollary 4.21 greatly simplify the proof of the last result as compared to the proof in [21].

Moreover, note that the proof of 4.23((ii) ⇒ (iii)) (Theorem 5.1 in [21]) and its technical Lemma 5.4 use the following wrong argument in [21]: If

$$\begin{array}{ccc} \bullet & \xrightarrow{e_1} & \bullet \\ {\scriptstyle e_2}\downarrow & & \downarrow{\scriptstyle m} \\ \bullet & \xrightarrow[m]{} & \bullet \end{array} \tag{25}$$

is a commutative square with $e_1, e_2 \in \mathcal{E}$ and $m \in \mathcal{M}$, then $e_1 = e_2$. This is of course true if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, but it is in general not true even if $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ as the following example shows.

Example 4.24. Consider the category $\mathbf{Top}_1$ of T_1 spaces and continuous maps between them, which is a full subcategory of **Top**. Note that $\mathbf{Top}_1$ is finitely complete since subspaces and direct products of T_1 spaces are T_1. Moreover, $\mathbf{Top}_1$ is equipped with the following $(\mathcal{E}, \mathcal{M})$-factorizations structure: $\mathcal{E}$ consists of *monotone quotient* mappings (a continuous mapping $f : X \to Y$ is monotone if all fibres $f^{-}(y)$ are connected) and $\mathcal{M}$ consists of so-called *light* mappings (a continuous mapping is light if all its fibres are totally disconnected). We leave the verification that this gives a $(\mathcal{E}, \mathcal{M})$-factorizations structure with $\mathcal{E}$ stable under pullback to the Reader. A reference for this is [7] Section 6.2. Note that $\mathcal{E} \subseteq \mathrm{Epi}(\mathbf{Top}_1)$ holds true.

Now consider the T_1 space $X = (-1, 0) \cup (0, 1) \subseteq \mathbb{R}$ equipped with the usual subspace topology. Let 1 and $2 = \{0, 1\}$ denote a one-point and a two-point discrete space. Then there are two obvious monotone quotients $e_1, e_2 : X \to 2$ with $e_1((-1, 0)) = e_2((0, 1)) = \{0\}$ and $e_1((0, 1)) = e_2((-1, 0)) = \{1\}$. But the unique continuous mapping $m : 2 \to 1$ is light, whence we get a commutative square $me_1 = me_2$ like (25) with $e_1 \neq e_2$.

4.7 Finite completeness of Map(Rel($\mathcal{C}$))

A reoccuring theme is the fact that for certain categories $\mathcal{C}$ the category $\mathbf{Rel}(\mathcal{C})$ of relations is isomorphic to the category $\mathbf{Rel}(\mathcal{B})$, where $\mathcal{B}$ is the category of maps in $\mathbf{Rel}(\mathcal{C})$, i. e.,

$$\mathbf{Rel}(\mathcal{C}) \simeq \mathbf{Rel}(\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))). \tag{26}$$

Kelly has shown this in [14] for categories $\mathcal{C}$ with a proper stable factorization system. He also showed that $\mathcal{B}$ is regular. Wyler and Jayewardene (cf. [10]) have generalized this to $(\mathcal{E}, \mathcal{M})$-structured categories, where $\mathcal{E}$ need not consist of epimorphisms but $\mathcal{M}$ consists of monomorphisms. They did not show that $\mathcal{B}$ is regular, which, however, is true since their result can be further generalized to tabular allegories in the sense of Freyd and Scedrov (cf. [8]).

The question arises whether this result is still valid in case $\mathcal{M}$ does not consist of monomorphisms. In this case, $\mathbf{Rel}(\mathcal{C})$ is not an allegory (see Example A.3 in Appendix A). The first step towards (26) is to show that $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is finitely complete. It turns out that the proof of this is essentially the same as for allegories with minor adjustments at some crucial points. We shall now show what kind of adjustments these are. Before going on the Reader will undoubtedly wish to consult [8] or Appendix A to familiarize oneself with the proof for allegories.

As earlier in this section, $\mathcal{C}$ is a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback. Our goal must be to prove results similar to those leading to Theorem A.20 in Section A.4.

But first let us note an important fact true in every bicategory.

Lemma 4.25. *Let $r : A \leftrightarrow B$ be a 1-cell in a bicategory $\mathcal{B}$. Then the following are equivalent:*

(i) r has a right adjoint s,

(ii) $r \circ - \dashv s \circ - : \mathcal{B}(X, B) \to \mathcal{B}(X, A)$,

(iii) $- \circ s \dashv - \circ r : \mathcal{B}(B, X) \to \mathcal{B}(A, X)$.

Proof. (i) ⇒ (ii) Suppose $r \dashv s$ with unit $\eta : \iota_A \to s \circ r$ and counit $\varepsilon : r \circ s \to \iota_B$. Let $F := r \circ -$ and $G := s \circ -$. Define $\alpha : \mathrm{Id} \to GF$ by $\alpha_x = \eta \circ x$ and $\beta : FG \to \mathrm{Id}$ by $\beta_x = \varepsilon \circ x$, which clearly are natural in x. Moreover, α and β are the unit and counit of the adjunction $F \dashv G$. That the adjunction equations

$$(G \circ \beta)(\alpha \circ G) = 1_G \qquad \text{and} \qquad (\beta \circ F)(F \circ \alpha) = 1_F$$

hold follows from the respective adjunction equations for η and ε.

(ii) ⇒ (i) If $F \dashv G$ with unit α and counit β, then the 2-cells α_{ι_A} and β_{ι_B} are readily checked to be the unit and counit of an adjunction $r \dashv s$.

(i) $\iff$ (iii) can be proved completely similar. □

The previous result has the following consequence in $\mathbf{Rel}(\mathcal{C})$. Recall that $\wedge$ denotes the local product.

Corollary 4.26. *1. If r and s are relations and f is a map, then*

$$(r \wedge s) \circ f \simeq r \circ f \wedge s \circ f.$$

2. For maps f and g, the relation $g^o \circ f$ is monic as an arrow in $\mathcal{C}$.

Proof. (i) The map f has a right adjoint, whence $- \circ f$ is a right adjoint, which preserves products.

(ii) A relation r is monic in $\mathcal{C}$ if and only if $r \wedge r \simeq r$. But

$$(g^o \circ f \wedge g^o \circ f) \simeq (g^o \wedge g^o) \circ f \simeq g^o \circ f,$$

since g^o is monic by Theorem 4.19. □

Recall from Section 3.5 that for any relations r, s and t there exists 2-cells

$$\begin{array}{rcll} r \circ s \wedge t & \to & s \circ (r \wedge s^o \circ t) & \quad \text{modular law,} \\ r \circ (s \wedge t) & \to & r \circ s \wedge r \circ t & \quad \text{semi-distributivity,} \end{array}$$

if the composites are defined.

Definition 4.27. *A relation r is called*

(i) reflexive *if there exists a 2-cell $\iota \to r$,*

(ii) coreflexive *if there exists a 2-cell $r \to \iota$.*

The domain *of r is the relation defined by* $\mathrm{dom}(r) := \iota \wedge r^o \circ r$.

Proposition 4.28. *A relation r is total if and only if its domain is reflexive.*

Proof. A 2-cell $\iota \to r^o \circ r$ induces a 2-cell $\iota \to \iota \wedge r^o \circ r$.

Conversely, if $\mathrm{dom}(r)$ is reflexive, then there is a 2-cell

$$\iota \longrightarrow \iota \wedge r^o \circ r \xrightarrow{\pi_1} r^o \circ r\,,$$

which shows r to be total. □

Proposition 4.29. *For any relations $r, s : A \nrightarrow B$ there exist 2-cells*

$$\mathrm{dom}(r \wedge s) \underset{m}{\overset{e}{\rightleftarrows}} \iota \wedge r^o \circ s\,.$$

These 2-cells satisfy $em = 1$ if r and s are maps.

Proof. The 2-cells m and s can be obtained exactly as in the proof of Lemma A.9 for allegories. The second part follows from the fact that that $\iota \wedge r^o \circ s$ is monic since $r^o \circ s$ is so by Corollary 4.26. □

Definition 4.30. *A relation r is called* tabular *if there are maps f, g with $r \simeq g \circ f^o$ and $f^o \circ f \wedge g^o \circ g \simeq \iota$. In this case f, g are said to* tabulate r.

Note that all relations $r = \langle r_0, r_1 \rangle$ that are monic as arrows in $\mathcal{C}$ are tabulated by $\Gamma r_0, \Gamma r_1$ since

$$(\Gamma r_0)^o \circ \Gamma r_0 \wedge (\Gamma r_1)^o \circ \Gamma r_1 \simeq \mathrm{im}(\ker(r_0) \cap \ker(r_1)) \simeq \mathrm{im}(\delta) \simeq \iota.$$

In particular, maps are tabular.

Proposition 4.31. *If $f^o \circ f \wedge g^o \circ g \simeq \iota$, then f, g is a monomorphic pair in* $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

Proof. As for allegories but using Corollary 4.26. □

Proposition 4.32. *Suppose f, g tabulate r. Then there exists a 2-cell $y \circ x^o \to r$ if and only if there is a unique map h such that $x \simeq f \circ h$ and $y \simeq g \circ h$.*

Proof. If $x \simeq f \circ h$ and $y \simeq g \circ h$, then $y \circ x^o \simeq g \circ h \circ h^o \circ f^o \to g \circ f^o \simeq r$, since h is single-valued.

Conversely, if there is a 2-cell $y \circ x^o \to g \circ f^o$, define

$$h := f^o \circ x \wedge g^o \circ y.$$

Then there is a 2-cell

$$\iota \to \iota \wedge y^o \circ y \circ x^o \circ x \to \iota \wedge y^o \circ g \circ f^o \circ x \to \mathrm{dom}(h),$$

whence h is total by 4.28.

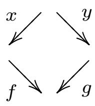
In order to see that h is monic as an arrow in $\mathcal{C}$, note that $f^o \circ x$ and $g^o \circ y$ are monic in $\mathcal{C}$ by Corollary 4.26.

Single-valuedness, the two equations and uniqueness of h follow now exactly as in Proposition A.16 for allegories. □

The next result can be proved exactly as for allegories.

Proposition 4.33. 1. *If r is coreflexive and tabular, then there is a unique monomorphic h in* $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ *such that $r = h \circ h^o$.*

2. *A square*

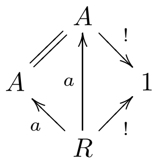


commutes in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ *if and only if there is a 2-cell*

$$y \circ x^o \to g^o \circ f.$$

Theorem 4.34. *The category $\mathcal{B} = \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is finitely complete.*

Proof. The *terminal object* of $\mathcal{B}$ is the same as the terminal object of $\mathcal{C}$. The diagram



shows that $\mathrm{im}\langle 1, !\rangle$ is the only relation $A \mathrel{+\!\!\to} 1$ in $\mathcal{B}$.

The *pullback* of a map f along a map g is given by a tabulation of $g^o \circ f$, which is a monic arrow in $\mathcal{C}$ by Corollary 4.26. In order to check that this really gives a pullback, proceed as in Theorem A.20 for allegories. □

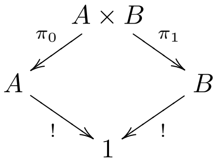
For the sake of completeness let us also point out how to obtain equalizers and binary products.

Remark 4.35. An *equalizer* of two parallel maps f and g is obtained as a tabulation of $\iota \wedge f^o \circ g$ as for allegories. However, note that this relation is not necessarily isomorphic to $\mathrm{dom}(f \wedge g)$.

The *product* of two objects A and B is the object $A \times B$ (their product in $\mathcal{C}$) with projections given by $\Gamma\pi_0$ and $\Gamma\pi_1$, where π_0 and π_1 are the respective projections in $\mathcal{C}$. For a pair $r : C \mathrel{+\!\!\to} A$, $s : \mathcal{C} \mathrel{+\!\!\to} B$ of relations the unique induced relation is given by

$$\langle r, s\rangle := (\Gamma\pi_0)^o \circ r \wedge (\Gamma\pi_1)^o \circ s,$$

because of Proposition 4.32 and the fact that $\Gamma\pi_0, \Gamma\pi_1$ tabulate $(\Gamma!)^o \circ \Gamma!$ since



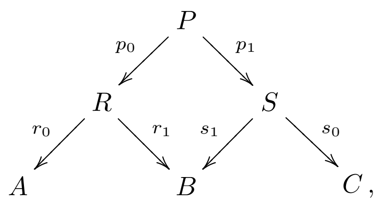
is a pullback square in $\mathcal{C}$.

Corollary 4.36. *The graph functor* Γ *preserves finite limits. In particular, the (non-full) subcategory* $\Gamma\mathcal{C}$ *of* $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ *formed by the graphs is finitely complete.*

Proof. Considering Theorem 4.34 we need only show that a pullback of graphs in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is also a pullback in $\Gamma\mathcal{C}$. Since tabulations consist of graphs the projections in a pullback square of two graphs Γf and Γg are graphs, too. Suppose that $\Gamma f \circ \Gamma x \simeq \Gamma g \circ \Gamma y$. By Proposition 4.33, there is a 2-cell

$$\alpha : \mathrm{im}\langle x, y\rangle \simeq \Gamma y \circ (\Gamma x)^o \to (\Gamma g)^o \circ \Gamma f.$$

Assume that $\Gamma f = \langle r_0, r_1\rangle$ and $\Gamma g = \langle s_0, s_1\rangle$. The tabulation $\Gamma t_0, \Gamma t_1$ of $(\Gamma g)^o \circ \Gamma f$, which gives the pullback projections in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$, is obtained using the diagram



and $(\mathcal{E}, \mathcal{M})$-factorizing such that $\langle t_0, t_1\rangle = \mathrm{im}\langle r_0 p_0, s_0 p_1\rangle$. Hence, $t_0 k = x$ and $t_1 k = y$, where $k = \alpha e$ for some $e \in \mathcal{E}$. Thus, there exist 2-cells

$$\begin{aligned} \Gamma k &\to (\Gamma t_0)^o \circ \Gamma t_0 \circ \Gamma k \simeq (\Gamma t_0)^o \circ \Gamma x, \\ \Gamma k &\to (\Gamma t_1)^o \circ \Gamma t_1 \circ \Gamma k \simeq (\Gamma t_1)^o \circ \Gamma y. \end{aligned}$$

But the canonical factorization h with $\Gamma x \simeq \Gamma t_0 \circ h$ and $\Gamma y \simeq \Gamma t_1 \circ h$ is given by the local product of the terms on the right-hand sides. Hence, there is a 2-cell $\Gamma k \to h$, which shows that the map h converges to the arrow k. □

5 Functional relations

Until now we investigated relations in categories in the most possible generality, without any further assumptions on the $(\mathcal{E}, \mathcal{M})$-factorization system with the exception of stability of $\mathcal{E}$ under certain pullbacks, which was needed to make compositions of relations associative.

The assumption considered in this section, namely that $\mathcal{M}$ consists of monomorphisms of $\mathcal{C}$, allows some more results. Some of the results have already been proved for the general case. The point in proving them again in this setting is that this can be done without using stability of $\mathcal{E}$ under pullback, which will allow us to prove the converse of Corollary 3.9 for $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, which means that stability of $\mathcal{E}$ under pullback is equivalent to im being a homomorphism of bicategories or, equivalently, that the equation

$$\mathrm{im}(b \diamond a) \simeq \mathrm{im}(b) \circ \mathrm{im}(a)$$

holds for all spans a and b such that the composition is defined.

In this section we present the results of [10], and some of [25] in Section 5.4. Note that the proofs may again differ from the original papers.

5.1 Maps as functional relations

With $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ the hom-categories $\mathbf{Rel}(A, B)$ become partially ordered. The notion of total and single-valued relations have already been defined. Let us unravel these for $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. A relation $r : A \leftrightarrow B$ is single-valued if $r \circ r^o \leq \iota_B$, total if $\iota_A \leq r^o \circ r$, injective if $r^o \circ r \leq \iota_A$, and surjective if $\iota_B \leq r \circ r^o$. Now let us add two more notions.

Definition 5.1. *A relation $r : A \leftrightarrow B$ is called* functional *if it is total and single-valued; r is said to be* bijective *if it is injective and surjective.*

Clearly a relation r is single-valued if and only if r^o is injective, and r is total if and only if r^o is surjective. The identity relation satisfies all four conditions since $\iota_A \simeq \iota_A^o$. Each of the classes defined by these notions can easily be seen to be closed under composition of relations.

The following results 5.2–5.9 are again due to [10]. However, the proof of the next proposition is different from the one given in [10].

Proposition 5.2. *A relation $r : A \leftrightarrow B$ is functional if and only if it is a map.*

Proof. This is an immediate consequence of Theorem 4.19. □

Corollary 5.3. *For any $\mathcal{C}$-arrow f the graph Γf is functional.*

Corollary 5.4. *A relation r is an isomorphism with inverse r^o if and only if it is functional and bijective.*

The next result strengthens Corollary 4.7 in case $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$.

Proposition 5.5. *If $r \leq s$ for a total relation r and a single-valued s, then $r \simeq s$.*

Proof. We have $s \leq s \circ r^o \circ r \leq s \circ s^o \circ r \leq r$, since $\iota \leq r^o \circ r$, $r^o \leq s^o$ and $s \circ s^o \leq \iota$. □

5.2 Total relations revisited

The following result is in principle a corollary of Proposition 4.16(1) in case $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$. However, we give a proof here that does not use stability of $\mathcal{E}$ under pullback.

Proposition 5.6. *For a relation $r = \langle r_0, r_1 \rangle : A \nrightarrow B$ the following are equivalent:*

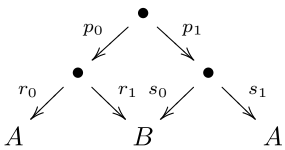
(i) r *is total,*

(ii) $\iota_A \leq s \circ r$ *for some relation* $s : B \nrightarrow A$,

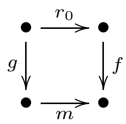
(iii) $r_0 \in \mathcal{E}$.

Proof. (i) ⇒ (ii) This is obvious using $s = r^o$.

(ii) ⇒ (iii) Form as usual the pullback



to compose s and r. Let $s \circ r = \langle t_0, t_1 \rangle$. Then $r_0 p_0$ factors as $t_0 e$ with $e \in \mathcal{E}$. By hypothesis there is an η such that $t_0 \eta = 1_A$. Now consider a commutative square



where m is in $\mathcal{M}$. Extend this using $r_0 p_0 = t_0 e$ to get $mgp_0 = ft_0 e$. The universal property of the $(\mathcal{E}, \mathcal{M})$-factorizations system implies the existence of an arrow d such that $md = ft_0$. Now $md\eta = ft_0\eta = f$, and therefore $mg = fr_0 = md\eta r_0$. Then, since m is monic, $g = d\eta r_0$ and, furthermore, $d\eta$ is uniquely determined. So r_0 has the diagonal property of the $(\mathcal{E}, \mathcal{M})$-system, and therefore $r_0 \in \mathcal{E}$.

(iii) ⇒ (i) We have $r^o \circ r = \text{im}\langle r_0 k_0, r_0 k_1 \rangle$ where $\langle k_0, k_1 \rangle = \ker(r_1)$. A common right inverse η of k_0 and k_1 provides a 2-cell $\langle r_0, r_0 \rangle \to r^o \diamond r$ in $\mathbf{Span}(A, A)$. Since $r_0 \in \mathcal{E}$ implies that $\text{im}\langle r_0, r_0 \rangle = \iota_A$, we get $\iota_A \leq r^o \circ r$, which completes the proof. □

Corollary 5.7. *For $f : A \to B$, the graph Γf is surjective if and only if $f \in \mathcal{E}$.*

Proof. Let $\Gamma f = \text{im}\langle 1_A, f \rangle = \langle r_0, r_1 \rangle$. Then $f = r_1 e$ for some $e \in \mathcal{E}$. The arrow f is in $\mathcal{E}$ if and only if r_1 is in $\mathcal{E}$. □

5.3 A characterization of stability of $\mathcal{E}$

To prove the next Theorem, we need the following facts true for every $\mathbf{Rel}(\mathcal{C})$.

Lemma 5.8.

(i) If $h : A \nrightarrow B$ is an injective map, then h is a section with right inverse h^o in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

(ii) If r_0, r_1 is a monomorphic pair in $\mathcal{C}$, then $\langle \Gamma r_0, \Gamma r_1 \rangle$ is injective, and therefore monomorphic in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

Proof. (i) Since h is a map, $h^o \circ h$ is monic as an arrow of $\mathcal{C}$. Hence, the existence of a 2-cell $h^o \circ h \to \iota_A$ is equivalent to $h^o \circ h \simeq \iota_A$ by totality of h.

(ii) Let $a := \Gamma r_0$, $b := \Gamma r_1$, $p_0 := \Gamma \pi_0$, and $p_1 := \Gamma \pi_1$, where π_0, π_1 are the appropriate product projections in $\mathcal{C}$. Recall that $\langle a, b \rangle = p_0^o \circ a \wedge p_1^o \circ b =: x$, by definition. Then, by semi-distributivity and single-valuedness of p_0 and p_1, there exists a 2-cell

$$x^o \circ x = (a^o \circ p_0^o \wedge b^o \circ p_1) \circ (p_0^o \circ a \wedge p_1^o \circ b) \to a^o \circ a \wedge b^o \circ b \simeq \iota,$$

where the last equation follows since r_0, r_1 is a monic pair in $\mathcal{C}$. □

Theorem 5.9. *Let $r = \langle r_0, r_1 \rangle$ be a relation. Then the following hold:*

(i) r is total if and only if Γr_0 is surjective,

(ii) r is single-valued if and only if Γr_0 is injective,

(iii) r is functional (a map) if and only if Γr_0 is bijective,

(iv) r is bijective if and only if Γr_1 is bijective.

Proof. We only need to show (ii), since (i) follows from 5.6 and 5.7, and (iii) and (iv) follow from (i) and (ii) or their dual respectively.

Suppose that Γr_0 is injective. Then there exists a 2-cell

$$r \circ r^o \simeq \Gamma r_1 \circ (\Gamma r_0)^o \circ \Gamma r_0 \circ (\Gamma r_1)^o \to \Gamma r_1 \circ (\Gamma r_1)^o \to \iota_B,$$

which shows that r is single-valued.

Conversely, consider the composite $r \circ r^o \simeq \mathrm{im}\langle r_1 k_0, r_1 k_1 \rangle$, where $\langle k_0, k_1 \rangle = \ker(r_0)$. If $\iota_B = \langle \iota_0, \iota_1 \rangle$, then existence of a 2-cell $\varepsilon : r \circ r^o \to \iota_B$ shows that there is an arrow t in $\mathcal{C}$ such that $r_1 k_0 = \iota_0 t$ and $r_1 k_1 = \iota_1 t$. Applying the graph functor, we see that

$$\Gamma(r_1 k_0) \simeq \Gamma(\iota_0 t) \simeq \Gamma(\iota_1 t) \simeq \Gamma(r_1 k_1),$$

using that $\Gamma \iota_0 \simeq \Gamma \iota_1$, which is true because

$$\Gamma \iota_0 \circ \Gamma e \simeq \Gamma(\iota_0 e) \simeq \Gamma 1 \simeq \Gamma(\iota_1 e) \simeq \Gamma \iota_1 \circ \Gamma e,$$

for some e in $\mathcal{E}$, which implies that Γe is split epic in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

Since clearly $\Gamma(r_0 k_0) \simeq \Gamma(r_0 k_1)$, we can write

$$\langle \Gamma r_0, \Gamma r_1 \rangle \circ \Gamma k_0 = \langle \Gamma r_0, \Gamma r_1 \rangle \circ \Gamma k_1.$$

But by Lemma 5.8, this implies that $\Gamma k_0 \simeq \Gamma k_1$, and therefore there exists a 2-cell

$$(\Gamma r_0)^o \circ \Gamma r_0 \simeq \mathrm{im}\langle k_0, k_1 \rangle \simeq \Gamma k_1 \circ (\Gamma k_0)^o \simeq \Gamma k_0 \circ (\Gamma k_0)^o \to \iota_B.$$

□

It seems unlikely that Theorem 5.9(ii) should hold if $\mathcal{M}$ does not consist of monomorphisms. However, it is not impossible that $\langle \Gamma r_0, \Gamma r_1 \rangle$ in Lemma 5.8 is monomorphic without r_0, r_1 being a monomorphic pair in $\mathcal{C}$.

We are now ready to prove the following important result.

Theorem 5.10. *For an* $(\mathcal{E}, \mathcal{M})$*-structured category* $\mathcal{C}$ *with* $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$*, the following are equivalent:*

(i) $\mathcal{E}$ *is stable under pullback,*

(ii) The equation

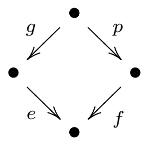
$$\mathrm{im}(b \diamond a) = \mathrm{im}(b) \circ \mathrm{im}(a)$$

holds for all spans a *and* b *such that the composition is defined.*

Proof. (i) $\Rightarrow$ (ii) This is exactly the statement of Corollary 3.9, which is also true without any condition on $\mathcal{M}$.

(ii) $\Rightarrow$ (i) It was proved in 3.8 that if (ii) holds, then $\mathcal{C}$ has regular legs. In particular, $\mathcal{E}$ is stable under pullbacks along product projections. In order to show stability of $\mathcal{E}$ under pullbacks along all arrows it suffices to show stability along monos since every arrow f of $\mathcal{C}$ factors as $f = \pi_1 \langle 1, f \rangle$.

Hence, let



be a pullback square with $e \in \mathcal{E}$ and f monic. By hypothesis we can write this pullback in the form $(\Gamma f)^o \circ \Gamma e \simeq \Gamma p \circ (\Gamma g)^o$. Furthermore it is easy to see that $(\Gamma m)^o \circ \Gamma m \simeq \iota_B$ for monic arrows $m : A \to B$ of $\mathcal{C}$. Now

$$\begin{aligned} \iota &\simeq (\Gamma f)^o \circ \Gamma f \simeq (\Gamma f)^o \circ \Gamma e \circ (\Gamma e)^o \circ (\Gamma f) \simeq \Gamma p \circ (\Gamma g)^o \circ \Gamma g \circ (\Gamma p)^o \\ &\simeq \Gamma p \circ (\Gamma p)^o, \end{aligned}$$

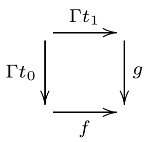
since f is monic, Γe is surjective and single-valued, and finally, g is monic. This equation tells us that Γp is surjective. Hence, by Corollary 5.7, p is in $\mathcal{E}$. □

Note that as a corollary of Corollary 3.10 the same result holds true with $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ instead of $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$.

Next note the following results from [10].

Lemma 5.11. *The class of surjective maps is pullback stable in* $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

Proof. Recall that a pullback of two maps f, g is given by



where $\langle t_0, t_1 \rangle = g^o \circ f$. If f is surjective, then $\langle t_0, t_1 \rangle$ is surjective since g is total. Hence, t_1 lies in $\mathcal{E}$, which implies that Γt_1 is surjective. □

Of course, the proof just given uses stability of $\mathcal{E}$ under pullback, since this was the assumption in Section 4.7 to prove existence of finite limits in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. However, the same proof can be given for $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ without using stability of $\mathcal{E}$. The trick is that the existence of pullbacks in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ does not depend on this assumption. For the proof of this consult [10]. Hence, we obtain the following result.

Theorem 5.12. *For $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ the following are equivalent:*

(i) $\mathcal{E}$ is stable under pullbacks in $\mathcal{C}$,

(ii) all span compositions $s \diamond r$ satisfy

$$\mathrm{im}(s \diamond r) \simeq \mathrm{im}(s) \circ \mathrm{im}(r),$$

(iii) the graph functor $\Gamma : \mathcal{C} \to \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ preserves finite limits.

Proof. (i) $\iff$ (ii) is Theorem 5.10.

(i) $\Rightarrow$ (iii) is Corollary 4.36.

(iii) $\Rightarrow$ (i) Suppose

$$\begin{array}{ccc} \bullet & \xrightarrow{e'} & \bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[e]{} & \bullet \end{array}$$

is a pullback square in $\mathcal{C}$ with $e \in \mathcal{E}$. Then $\Gamma f \circ \Gamma e' \simeq \Gamma e \circ \Gamma g$ is a pullback square in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ with Γe surjective by Corollary 5.7. By Lemma 5.11, $\Gamma e'$ is surjective, whence e' lies in $\mathcal{E}$. $\square$

Finally note that one can now show that $\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Rel}(\mathbf{Map}(\mathbf{Rel}(\mathcal{C})))$ are isomorphic to each other as it is done in [10]. Those proofs shall not be given here. In Section 6 we shall rather present the result obtained by Kelly (cf. [14]), which is in a sense more elegant, though only true for categories with a proper stable $(\mathcal{E}, \mathcal{M})$-factorization system. In Appendix A we give a more general result about so-called tabular allegories, which will generalize both of the settings just mentioned.

5.4 Special relations for $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$

In this subsection we assume that the $(\mathcal{E}, \mathcal{M})$-factorization system of the category $\mathcal{C}$ is proper and stable, that means that $\mathcal{M}$ consists of monomorphisms and $\mathcal{E}$ consists of epimorphisms and is stable under pullbacks.

5.4.1 Total, single-valued and isomorphic relations

The following result except (viii) is again taken from [10].

Theorem 5.13. *For $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ the following are equivalent:*

(i) $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$,

(ii) a graph Γf is injective if and only if f is monic in $\mathcal{C}$,

(iii) a relation $r = \langle r_0, r_1 \rangle$ is single-valued if and only if r_0 is monic in $\mathcal{C}$,

(iv) *a relation $r = \langle r_0, r_1 \rangle$ is injective if and only if r_1 is monic in $\mathcal{C}$,*

(v) *a graph Γf is bijective if and only if f is monic and in $\mathcal{E}$,*

(vi) *a relation $r = \langle r_0, r_1 \rangle$ is functional (a map) if and only if r_0 is monic and in $\mathcal{E}$,*

(vii) *a relation $r = \langle r_0, r_1 \rangle$ is bijective if and only if r_1 is monic and in $\mathcal{E}$,*

(viii) *a relation $r = \langle r_0, r_1 \rangle$ is an isomorphism if and only if r_0 and r_1 are monic and in $\mathcal{E}$.*

Proof. We shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (vi) $\Rightarrow$ (viii) $\Rightarrow$ (i). Note that (iv) and (vii) are equivalent to (ii) and (iv) respectively by duality, and that (v) is just (ii) and 5.7.

(i) $\Rightarrow$ (ii) If $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, then $\langle 1, f \rangle$ lies in $\mathcal{M}$ for all $f : A \to B$. Furthermore, all objects of $\mathcal{C}$ are separated. Hence, Proposition 4.16 and Lemma 4.15 imply (ii).

(ii) $\Rightarrow$ (iii) is a consequence of Theorem 5.9.

(iii) $\Rightarrow$ (vi) Statement (vi) follows from (iii) and 5.6.

(vi) $\Rightarrow$ (viii) is obvious by Corollary 5.4.

(viii) $\Rightarrow$ (i) As for Theorem 4.20. □

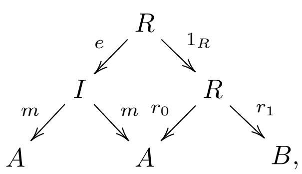
5.4.2 Sections and Monomorphisms

First of all note that, by self duality, characterizing sections and monomorphism of $\mathbf{Rel}(\mathcal{C})$ is the same as characterizing retractions and epimorphisms, respectively.

The results concerning sections and monomorphisms presented here are in principle due to Schröfel (cf. [25]) who does not work with $(\mathcal{E}, \mathcal{M})$-structured categories. However, the proofs carry over to a category with a proper stable $(\mathcal{E}, \mathcal{M})$-factorization system.

Theorem 5.14. *If $r : \langle r_0, r_1 \rangle : A \nrightarrow B$ is monomorphic in $\mathbf{Rel}(\mathcal{C})$, then r is total, or equivalently, $r_0 \in \mathcal{E}$.*

Proof. Take an $(\mathcal{E}, \mathcal{M})$-factorization $r_0 = me$. Note that $\langle m, m \rangle = \delta_A m$ is in $\mathcal{M}$. So if we compose r and $\langle m, m \rangle$ we get



whence $r \circ \langle m, m \rangle \simeq r \simeq r \circ \delta_A$. Now if r is monic, then $\langle m, m \rangle \simeq \delta_A$, and then $1_A = mi$ for some isomorphism i of $\mathcal{C}$. But this implies that m is an isomorphism, too. Hence, r_0 is in $\mathcal{E}$. □

This result together with Theorem 5.13(viii) shows that we have the following chain of inclusions:

$$\mathrm{Iso}(\mathbf{Rel}(\mathcal{C})) \subseteq [\mathcal{E}, \mathrm{Mono}(\mathcal{C})] \subseteq \mathrm{Sect}(\mathbf{Rel}(\mathcal{C})) \subseteq \mathrm{Mono}(\mathbf{Rel}(\mathcal{C})) \subseteq [\mathcal{E}, \mathcal{C}],$$

where $[X, Y] = \{r = \langle r_0, r_1 \rangle \mid r_0 \in X, r_1 \in Y\}$.

Of course, the converse of the last theorem does not hold true. Totality is not enough to make a relation monomorphic. As an example take the unique set map $2 \to 1$ from a two-element set to a one-element set, which is a total relation, but not monic.

The next example shows that left inverses of sections need not just be the opposite relation.

Example 5.15. Consider in **Rel**(**Set**) the objects $A = \{0, 1\}$ and $B = \{0, 1, 2\}$ and the relations

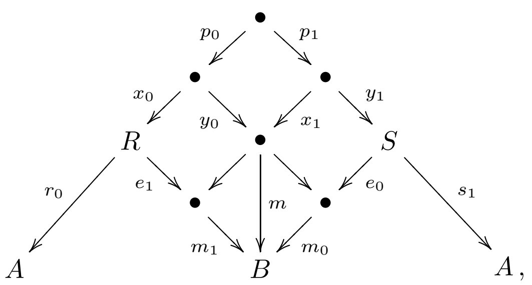
$$\begin{aligned} R &= \{(0,0), (0,1), (1,0), (1,2)\}, \\ S &= \{(1,0), (2,1)\}. \end{aligned}$$

Then certainly $S \circ R = \{(0,0), (1,1)\} = \delta_A$ holds; but $R^o \circ R \neq \delta_A$ since $(0,1) \in R^o \circ R$.

This leads to the following theorem.

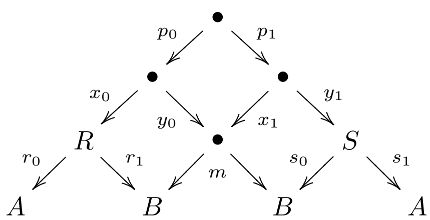
Theorem 5.16. *A relation $r = \langle r_0, r_1 \rangle : A \mapsto B$ is a section in* **Rel**($\mathcal{C}$) *if and only if there is an arrow $m \in \mathcal{M}$ such that for $u \simeq \langle m, m \rangle \circ r$, $u^o \circ u \simeq \delta_A$ holds.*

Proof. ($\Rightarrow$) Let s be a left inverse of r. Take $(\mathcal{E}, \mathcal{M})$-factorizations $r_1 = m_1 e_1$, and $s_0 = m_0 e_0$, and define m by the following diagram:



where all the squares are pullbacks. Let $u = \mathrm{im}\langle r_0 x_0, m y_0 \rangle$. Note that since m_0 and m_1 are monic, the lower squares in the diagram

(27)



are pullbacks, whence $\langle m, m \rangle \circ r = u$. Moreover, diagram (27) shows that $s \circ u \simeq s \circ \langle m, m \rangle \circ r \simeq s \circ r \simeq \delta_A$. By 5.6, this implies that $u^o \circ u \geq \delta_A$.

Now note that $s \circ r \simeq \mathrm{im}\langle r_0 x_0 p_0, s_1 y_1 p_1 \rangle \simeq \delta_A$. So $r_0 x_0 p_0 = e = s_1 y_1 p_1$ for some $e \in \mathcal{E}$. Clearly $m y_0 p_0 = s_0 y_1 p_1$, where p_0 is in $\mathcal{E}$ by stability of $\mathcal{E}$ under pullback, and $\langle r_0 x_0, m y_0 \rangle = uq$ for some $q \in \mathcal{E}$. Putting all this together, we conclude that

$$u q p_0 = \langle r_0 x_0, m y_0 \rangle p_0 = \langle s_1, s_0 \rangle y_1 p_1.$$

With the universal property of the factorization system this statement implies that $u \leq s^o$, or $u^o \leq s$, whence $u^o \circ u \leq s \circ r \simeq \delta_A$.

($\Leftarrow$) Conversely, suppose $u^o \circ u \simeq \delta_A$, where $u \simeq \langle m, m\rangle \circ r$ for some $m \in \mathcal{M}$. Then

$$\delta_A \simeq u^o \circ u \simeq r^o \circ \langle m, m\rangle \circ \langle m, m\rangle \circ r.$$

So r has a left inverse, and is therefore a section. □

If the relations of $\mathbf{Rel}(\mathcal{C})$ satisfy

$$r \simeq r \circ r^o \circ r, \tag{28}$$

then monomorphisms and sections of $\mathbf{Rel}(\mathcal{C})$ are the same. Moreover, there is a nice characterization for monomorphisms. Condition (28) is for example true for mono-relations over Abelian categories, for mono-relations over groups, and for all maps if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. Moreover, (28) is true, if r can be factorized as $r \simeq \langle e_1, m_1\rangle \circ \langle m_0, e_0\rangle$, where $m_i \in \mathcal{M}$ and $e_i \in \mathcal{E}$ (see [25] for details).

Theorem 5.17. *Let $r = \langle r_0, r_1\rangle$ be a relation satisfying (28). Then the following are equivalent:*

(i) r *is monic,*

(ii) $r^o \circ r \simeq \delta_A$,

(iii) $r_0 \in \mathcal{E}$ *and* $r_1 \in \mathrm{Mono}(\mathcal{C})$,

(iv) r *is a section.*

Proof. (i) $\Rightarrow$ (ii) follows immediately from (28).

(ii) $\Rightarrow$ (iii) is true by 5.6 and 5.13.

(iii) $\Rightarrow$ (iv) Note that $r^o \circ r \simeq \mathrm{im}\langle r_0, r_0\rangle \simeq \delta_A$, since $r_0 \in \mathcal{E}$.

(iv) $\Rightarrow$ (i) is obvious. □

Note that the same result without (iii) holds in the general case, where $\mathcal{M}$ can contain non-monomorphic and $\mathcal{E}$ non-epimorphic arrows.

6 Relations induced by maps

Relations have first been studied over regular categories. An obvious generalization of this was to study relations relative to a stable proper $(\mathcal{E}, \mathcal{M})$-factorization system (cf. [15]). But as Kelly (cf. [14]) observed, this generalization is somewhat illusory. He proved that for a finitely complete category $\mathcal{C}$ with stable proper factorization system the bicategory $\mathbf{Rel}(\mathcal{C})$ of relations is isomorphic to $\mathbf{Rel}(\mathcal{B})$, where $\mathcal{B}$ is a regular category, namely the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$.

Relations over regular categories have been axiomatized by Freyd (cf. [8]). Such a relational category is called an *allegory*. However, Freyd's axioms are general enough to cover all categories $\mathbf{Rel}(\mathcal{C})$, where $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, i. e., precisely the poset enriched bicategories of relations (cf. [20], 3.7). Moreover, Freyd gives an elegant proof that for a so-called tabular allegory $\mathcal{A}$,

$$\mathcal{A} \simeq \mathbf{Rel}(\mathbf{Map}(\mathcal{A})),$$

hence, generalizing Kelly's result. In this section we restrict ourselves to presenting Kelly's results. However, in Appendix A we outline the more general treatment using allegories.

Throughout this section we assume that $\mathcal{C}$ is a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ is stable under pullback. Note that unlike [14], we do not assume $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ a priori, since some of the results do not need this assumption. First let us investigate an important class of morphisms of $\mathcal{C}$.

6.1 The monic arrows of $\mathcal{E}$

This class of morphisms is defined by

$$\Sigma := \mathcal{E} \cap \mathrm{Mono}(\mathcal{C}).$$

The significance of Σ is clear immediately when we consider Theorem 4.20, namely a relation $r = \langle r_0, r_1 \rangle$ is a map precisely when r_0 lies in Σ, and r is an isomorphism of $\mathbf{Rel}(\mathcal{C})$ precisely when $r_0, r_1 \in \Sigma$. Next, let us look at some properties of Σ. Clearly Σ contains all isomorphisms of $\mathcal{C}$, is closed under composition, and stable under pullback. Recall that, by Proposition 2.4, $gf \in \mathcal{E}$ implies $g \in \mathcal{E}$, and that gf monomorphic implies that f is monomorphic. Note that for a monomorphic f, the $(\mathcal{E}, \mathcal{M})$-factorization $me = f$ has $e \in \Sigma$. Hence, Σ consists of isomorphisms alone if and only if $\mathrm{Mono}(\mathcal{C}) \subseteq \mathcal{M}$, which, in case $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, implies that $\mathcal{C}$ is regular (see Proposition 2.7 and the remark following it).

Proposition 6.1. *(i) $gf \in \mathcal{E}$ and g monomorphic $\Rightarrow f \in \mathcal{E}$; in particular $gf \in \Sigma$ and $g \in \Sigma \Rightarrow f \in \Sigma$,*

(ii) gf monomorphic and $f \in \mathcal{E} \Rightarrow g$ is monomorphic; in particular $gf \in \Sigma$ and $f \in \Sigma \Rightarrow g \in \Sigma$.

Proof. (i) Note that f is the pullback of gf along g, since g is monic, which makes the result obvious.

(ii) follows immediately from Lemma 2.6, since gf and f are monic. □

6.2 Maps as fractions

We shall show that $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is the category of fractions $\mathcal{C}[\Sigma^{-1}]$. References for the calculus of fractions are, for example, [9] and [3], Section 5.2.

Let us write $\mathcal{B}$ for the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$, which indeed is locally discrete as a sub-2-category of $\mathbf{Rel}(\mathcal{C})$ (see Corollary 4.9). By Theorem 4.20, we can write every relation r of $\mathcal{B}$ as $r \simeq \Gamma f \circ (\Gamma\alpha)^o$ for some arrow f of $\mathcal{C}$ and some α in Σ. (Here we follow Kelly's convention of denoting elements of Σ by Greek letters.) More conveniently, we will write $r = f\alpha^o$. Now recall that Theorem 4.20 implies that Σ consists precisely of those morphisms that are inverted by Γ. In fact, we get the following result.

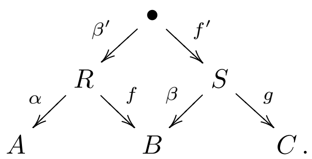
Proposition 6.2. *The class Σ admits a right calculus of fractions, and $\Gamma : \mathcal{C} \to \mathcal{B}$ is the projection of $\mathcal{C}$ to its category of fractions $\mathcal{C}[\Sigma^{-1}]$.*

Proof. One easily checks that the 4 conditions (see [3], 5.2.3) for allowing a right calculus of fractions are satisfied by Σ.

Now let $T : \mathcal{C} \to \mathcal{D}$ be a functor which inverts the elements of Σ. We have to find a unique functor $S : \mathcal{B} \to \mathcal{D}$ with $T = S\Gamma$. Uniqueness is clear, since S must necessarily be defined by

$$S(f\alpha^o) = Tf(T\alpha)^{-1}.$$

Then obviously $S\Gamma f = Tf$ and S clearly preserves identities. As for composites, consider two maps $\langle\alpha, f\rangle$ and $\langle\beta, g\rangle$ with their composite given by



Now clearly

$$\begin{aligned} S(\langle\beta, g\rangle \circ \langle\alpha, f\rangle) &= T(gf')(T(\alpha\beta'))^{-1} = TgTf'(T\beta')^{-1}(T\alpha)^{-1} \\ &= Tg(T\beta)^{-1}Tf(T\alpha)^{-1} = S(\langle\beta, g\rangle) \circ S(\langle\alpha, f\rangle), \end{aligned}$$

where the last but one equality follows from $f\beta' = \beta f'$. □

Recall from Theorem 4.34 and Corollary 4.36, that $\mathcal{B}$ has finite limits and Γ preserves these. Of course, this is also a corollary of the previous result because it is a general fact about categories of fractions (see [3], 5.2.5).

6.3 Regularity of Map(Rel($\mathcal{C}$))

Theorem 6.3. *Let $f\alpha^o$ be a morphism of $\mathcal{B}$. Then the following hold:*

(i) $f\alpha^o$ is monomorphic in $\mathcal{B}$ $\iff$ f is monomorphic in $\mathcal{C}$,

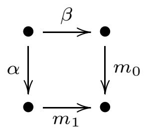
(ii) the subobjects in $\mathcal{B}$ of an object A may be identified with the $\hat{\mathcal{M}}$-subobjects of A in $\mathcal{C}$, where $\hat{\mathcal{M}} := \mathcal{M} \cap \mathrm{Mono}(\mathcal{C})$.

If $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, then additionally *(iii)* $f\alpha^o$ *is extremally epimorphic in* $\mathcal{B} \iff f \in \mathcal{E}$*,*

(iv) extremal epimorphisms are stable under pullback in $\mathcal{B}$*.*

Proof. (i) Since α^o is an iso in $\mathcal{B}$, $f\alpha^o$ is monic if and only if Γf is monic in $\mathcal{B}$. The pullback of Γf along itself in $\mathcal{B}$ can be obtained as a pullback in $\mathcal{C}$. Since Γ is faithful, it consists of two equal morphisms if and only if f is monic in $\mathcal{C}$.

(ii) By (i) and since $\hat{\mathcal{M}} \subseteq \mathrm{Mono}(\mathcal{C})$, the graph functor assigns to every $\hat{\mathcal{M}}$-subobject m in $\mathcal{C}$ a subobject Γm in $\mathcal{B}$. This assignment is obviously well-defined. It is surjective, for if $f\alpha^o$ represents a subobject in $\mathcal{B}$, $(\mathcal{E},\mathcal{M})$-factorize $f = me$, where e lies in Σ and m is monic by Lemma 4.5; hence, Γm represents the same subobject as $f\alpha^o$. Finally, if Γm_0 and Γm_1 represent the same subobject in $\mathcal{B}$, there is an isomorphism $i = \langle \alpha, \beta \rangle$ in $\mathcal{B}$ with $\Gamma m_1 = \Gamma m_0 \circ i$. Recall that, since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$, $\Gamma m_1 = \langle 1, m_1 \rangle$, and therefore $\Gamma m_0 \circ i = \mathrm{im}\langle \alpha, m_0\beta \rangle = \langle 1, m_1 \rangle$, whence we must have a commutative square

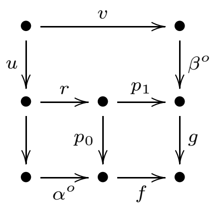


in $\mathcal{C}$. Since α and β are both in Σ, appealing to the universal property of the factorizations system yields $m_0 \simeq m_1$ as $\hat{\mathcal{M}}$-subobjects in $\mathcal{C}$.

(iii) Again $f\alpha^o$ is extremally epic in $\mathcal{B}$ if and only if Γf is so in $\mathcal{B}$. If f is in $\mathcal{E}$, then $\Gamma f \circ (\Gamma f)^o \simeq \iota$, which shows Γf to be split epic.

Conversely, if Γf is extremally epic in $\mathcal{B}$, $(\mathcal{E},\mathcal{M})$-factorize $f = me$, where m is monic, and therefore $\Gamma f \simeq \Gamma m \circ \Gamma e$, where Γm is monic in $\mathcal{B}$. By hypothesis, Γm is isomorphic, which implies that m lies in Σ and shows that f is in $\mathcal{E}$.

(iv) Let $f\alpha^o$ be an extremal epi in $\mathcal{B}$. Its pullback along an arrow $g\beta^o$ can be obtained as follows:



First pull back f along g in $\mathcal{C}$ to obtain p_0 and $p_1 \in \mathcal{E}$. The pullback $r = \langle r_0, r_1 \rangle$ of α^o along Γp_0 is an isomorphism in $\mathcal{B}$, whence $r_0, r_1 \in \Sigma$. The relation $\Gamma p_1 \circ r = \mathrm{im}\langle r_0, p_1 r_1 \rangle$ is extremally epic by (iii). If we finally pull back $\Gamma p_1 \circ r$ along β^o, we get $v = \Gamma\beta \circ \Gamma p_1 \circ r \circ u$ is an extremal epi in $\mathcal{B}$ since $\Gamma\beta$ and u are isos. □

Corollary 6.4. *If* $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$*, then* $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ *is a regular category with regular epimorphisms stable under pullback. The subobjects in* $\mathcal{B}$ *may be identified with the* $\mathcal{M}$*-subobjects in* $\mathcal{C}$*.*

6.4 The category of bicategories of relations

We define $\mathcal{K}$ to be the 2-category whose objects are finitely complete categories $\mathcal{C}$ with a stable $(\mathcal{E},\mathcal{M})$-factorization system. An arrow $F : \mathcal{C} \to \mathcal{C}'$ is a left-exact functor (i. e., a functor preserving finite limits) with $F\mathcal{E} \subseteq \mathcal{E}'$ and $F\mathcal{M} \subseteq \mathcal{M}'$. The 2-cells are given by natural transformations between such functors. Note that $\mathcal{K}$ has a full sub-2-category **Reg** given by the regular categories $\mathcal{C}$, where $\mathcal{M} = \mathrm{Mono}(\mathcal{C})$. An arrow between regular categories in $\mathcal{K}$ is just a left-exact functor preserving extremal epimorphisms.

Clearly any $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{K}$ induces a 2-functor $\mathbf{Rel}(F) : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{D})$ sending objects A to FA and relations $r = \langle r_0, r_1 \rangle : A \nrightarrow B$ to $\langle Fr_0, Fr_1 \rangle : FA \nrightarrow FB$. Note that this lies in $\mathcal{M}$, since $F\langle r_0, r_1 \rangle : FR \to F(A \times B)$ lies in $\mathcal{M}'$ and F preserves binary products. A 2-cell $\alpha : r \to s$ is sent to $F\alpha : \mathbf{Rel}(F)(r) \to \mathbf{Rel}(F)(s)$. It is easy to check that $\mathbf{Rel}(F)$ is indeed a 2-functor.

Now we are ready to form the 2-category of 2-categories of relations, $\mathcal{RK}$, whose objects are the 2-categories $\mathbf{Rel}(\mathcal{C})$ and whose arrows are the 2-functors $F : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{D})$ which commute with $(-)^o$, that means that $(Fr)^o = Fr^o$ for any relation r. The 2-cells are the natural transformations between such functors. In fact, $\mathcal{RK}$ is even a 3-category (see [3], 7.3.2), but we will not use that structure here.

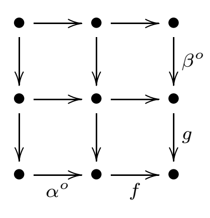
Note that since every 2-functor preserves adjoint situations we get a functor $\mathbf{Map}(F) : \mathbf{Map}(\mathbf{Rel}(\mathcal{C})) \to \mathbf{Map}(\mathbf{Rel}(\mathcal{D}))$ by restricting F to maps. This assignment certainly gives rise to a 2-functor $\mathbf{Map} : \mathcal{RK} \to \mathbf{CAT}$. Unfortunately it is not at all clear whether $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ lies in $\mathcal{K}$ in general. However, for categories $\mathcal{C}$ with a proper stable factorization system this is true, as we have seen in Corollary 6.4.

6.5 Comparison of Rel($\mathcal{C}$) and Rel(Map(Rel($\mathcal{C}$)))

Let $\mathcal{K}_p$ be the full sub-2-category of $\mathcal{K}$ with $\mathcal{K}$ as in Section 6.4 consisting of those $\mathcal{C}$ with proper and stable $(\mathcal{E}, \mathcal{M})$-factorizations system. Then **Reg** is a reflective sub-2-category of $\mathcal{K}_p$.

Proposition 6.5. *The 2-functor* $\Gamma : \mathcal{C} \to \mathcal{B}$ *is the reflection of the 2-category* $\mathcal{K}_p$ *into* **Reg**.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be an arrow of $\mathcal{K}_p$, where $\mathcal{D}$ is regular. Being left-exact, F preserves monos. Moreover, $F\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{D})$, whence F inverts the elements of Σ. By 6.2, $F = S\Gamma$ for a unique $S : \mathcal{B} \to \mathcal{D}$. Since by 6.4, $\Gamma\mathcal{M} = \mathrm{Mono}(\mathcal{B})$ and $\Gamma\mathcal{E} = \mathrm{RegEpi}(\mathcal{B})$, we must have $S[\mathrm{Mono}(\mathcal{B})] \subseteq \mathrm{Mono}(\mathcal{D})$ and $S[\mathrm{RegEpi}(\mathcal{B})] \subseteq \mathrm{RegEpi}(\mathcal{D})$. Furthermore, S is left-exact. Since $F = S\Gamma$, it preserves terminal objects. Recall that pullbacks in $\mathcal{B}$ are formed as follows:



The pullback of f along g may be formed in $\mathcal{C}$ and must be preserved by S since $S\Gamma = F$ and Γ preserve pullbacks. The other three squares are pullbacks of isos, which must be preserved by every functor, which shows that S is left-exact.

That the universal property of Γ also holds for 2-cells is true for any category of fractions (see [9], Ch. I). □

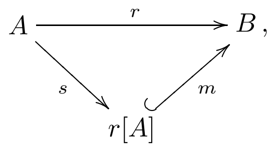
Theorem 6.6. *For* $\mathcal{B} = \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$*, the 2-functor* $\mathbf{Rel}(\Gamma) : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{B})$ *is an isomorphism of 2-categories.*

Proof. Clearly $\mathbf{Rel}(\Gamma)$ is the identity on objects. Moreover, it is an isomorphisms on the hom-categories, since by Corollary 6.4, the subobjects of $A \times B$ in $\mathcal{B}$ can be identified with the $\mathcal{M}$-subobjects of $A \times B$ in $\mathcal{C}$. □

Observe that the last result also shows that $\mathbf{Rel} : \mathbf{Reg} \to \mathcal{RK}_p$ is an equivalence of categories with equivalence inverse $\mathbf{Map} : \mathcal{RK}_p \to \mathbf{Reg}$, since $\mathcal{C} \simeq \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ holds for regular $\mathcal{C}$ (see Theorem 4.23). If we consider finitely complete categories $\mathcal{C}$ with a stable proper $(\mathcal{E}, \mathcal{M})$-factorization system, then we get an adjunction $\mathbf{Rel} \dashv \mathbf{Map} : \mathcal{RK}_p \to \mathcal{K}_p$. Theorem 6.6 tells us, that the counit of this adjunction is an isomorphism. The unit is given by $\Gamma : \mathcal{C} \to \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. We will not present proofs for this here. These results can be obtained easily with the theory of allegories (see [8] or Appendix A).

7 Factorization systems in $\mathbf{Rel}(\mathcal{C})$

Every relation $r \subseteq A \times B$ between sets has a canonical image-factorization when considered as a multivalued function. This factorization is given by



where $r[A] = \{b \in B \mid \exists a \in A.(a,b) \in r\}$ is the image of A under r. So every set-relation factorizes as $m \circ s$, where s is a surjective relation and m an injective mapping. That this can be generalized has already been observed by Jayewardene and Wyler in [10] in the case $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. However, they do not show that this actually yields an $(\mathcal{E}, \mathcal{M})$-factorization system for $\mathbf{Rel}(\mathcal{C})$. We shall show this, and that even without the assumption that $\mathcal{M}$ consists of monomorphisms, a weaker (2-categorial) universal property still holds. This discussion on factorization systems in the 2-category $\mathbf{Rel}(\mathcal{C})$ is a new contribution. Let us first turn our attention to the general case.

7.1 Factorization in the general case

Throughout this subsection we assume that $\mathcal{C}$ is an $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback. The first result about the existence of a canonical factorization for every relation is due to Wyler and Jayewardene ([10]) in the case of $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. However, their proof carries over without further adjustment to the general case.

Proposition 7.1. *Every relation r factors as $r \simeq \Gamma m \circ s$, where $m \in \mathcal{M}$ and where $s = \langle s_0, s_1 \rangle$ with $s_1 \in \mathcal{E}$.*

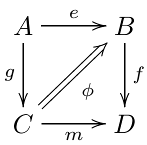
Proof. Let $r = \langle r_0, r_1 \rangle$ and $(\mathcal{E}, \mathcal{M})$-factorize $r_1 = me$. We have $r = (1 \times m)\langle r_0, e\rangle$ with r and $(1 \times m)$ in $\mathcal{M}$. So $s = \langle r_0, e \rangle$ is an $\mathcal{M}$-relation as desired. Finally, 4.6(iv) tells us that $r \simeq \Gamma m \circ s$. $\square$

Observe now that this canonical factorization gives rise to two classes of relations in $\mathbf{Rel}(\mathcal{C})$, as follows:

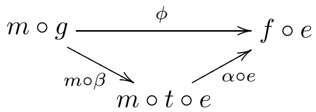
$$\begin{aligned} \bar{\mathcal{E}} &= \{\langle s_0, s_1 \rangle \in \mathcal{M} \mid s_1 \in \mathcal{E}\} \\ \bar{\mathcal{M}} &= \{\mathrm{im}\langle 1, m \rangle \mid m \in \mathcal{M}\}. \end{aligned}$$

It is easy to see that both classes are closed under horizontal composition. Now it is time to define what shall be meant by an $(\mathcal{E}, \mathcal{M})$-factorization system in a 2-category. This has been done recently by Kasangian and Vitale in [11] for a 2-category with invertible 2-cells. However, in the 2-category $\mathbf{Rel}(\mathcal{C})$, 2-cells are not necessarily invertible. So the definition of [11] needs some adjustment here.

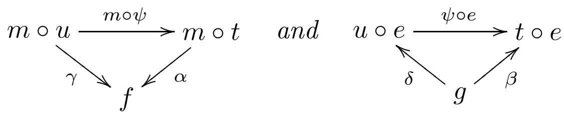
Definition 7.2. *Let $\mathcal{A}$ be a 2-category. Given 1-cells $e : A \,+\!\!\!\rightarrow B$ and $m : C \,+\!\!\!\rightarrow D$, we say e has the* fill-in property *with respect to m, in symbols $e \perp m$, if for* *each pair of 1-cells $f : B \nrightarrow D$ and $g : A \nrightarrow C$ and for each 2-cell $\phi : m \circ g \to f \circ e$*



there exists a 1-cell $t : B \nrightarrow C$ and 2-cells $\alpha : m \circ t \to f$ and $\beta : g \to t \circ e$ such that the diagram



commutes (we say (α, t, β) is a fill-in *for (f, ϕ, g)); moreover, if (γ, u, δ) is another fill-in for (f, ϕ, g), then there exists a unique 2-cell $\psi : u \to t$ such that*



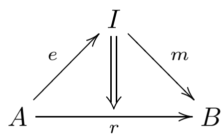
are commutative.

Definition 7.3. *A factorization system in a 2-category $\mathcal{A}$ is given by two classes $\mathcal{E}$ and $\mathcal{M}$ of 1-cells in $\mathcal{A}$ such that*

(i) *$\mathcal{E}$ and $\mathcal{M}$ contain all identities and are closed under composition,*

(ii) *$\mathcal{E}$ and $\mathcal{M}$ are stable under isomorphic 2-cells,*

(iii) *for each 1-cell $r : A \nrightarrow B$ of $\mathcal{A}$ there exist $e \in \mathcal{E}$, $m \in \mathcal{M}$ and an isomorphic 2-cell such that*



commutes,

(iv) *for each $e \in \mathcal{E}$ and for each $m \in \mathcal{M}$, we have $e \perp m$.*

Before we turn back to $\mathbf{Rel}(\mathcal{C})$ let us quickly prove a general fact about factorization structures as defined in 7.3.

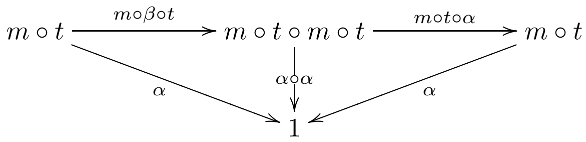
Proposition 7.4. *Let $\mathcal{A}$ be a 2-category with classes $\mathcal{E}$ and $\mathcal{M}$ of morphisms that satisfy (iv) in Definition 7.3. Then*

$$\mathcal{E} \cap \mathcal{M} \subseteq \mathbf{Map}(\mathcal{A}).$$

Proof. Let $m \in \mathcal{E} \cap \mathcal{M}$. The fill-in property (iv) gives a fill-in (α, t, β) of $(1, 1_m, 1)$ with

$$(\alpha \circ m)(m \circ \beta) = 1_m. \tag{29}$$

Now consider the 2-cell $(t \circ \alpha)(\beta \circ t) : t \to t$. It is easy to see that



is commutative. The right triangle clearly commutes. To see that the left-hand triangle commutes, note that $\alpha \circ \alpha = \alpha(\alpha \circ m \circ t)$ and then use (29). Similarly it can be shown that $\beta = (t \circ \alpha \circ m)(\beta \circ t \circ m)\beta$. But by uniqueness we must have $(t \circ \alpha)(\beta \circ t) = 1_t$. Hence, t is a right adjoint of m, i. e., m is a map. □

Now we shall show that $\mathbf{Rel}(\mathcal{C})$ is $(\bar{\mathcal{E}}, \bar{\mathcal{M}})$-structured in the sense of Definition 7.3. Conditions (i) and (ii) clearly hold true, and condition (iii) has been shown in Proposition 7.1. So let us prove condition (iv).

Proposition 7.5. *If $e \in \bar{\mathcal{E}}$ and $m \in \bar{\mathcal{M}}$, then $e \perp m$.*

Proof. Let $e : A \nrightarrow B$ a relation in $\bar{\mathcal{E}}$, $m : C \nrightarrow D$ be a relation in $\bar{\mathcal{M}}$, and let $f : B \nrightarrow D$, $g : A \nrightarrow C$ be relations so that there exists a 2-cell $\phi : m \circ g \to f \circ e$. Recall that the graph m is a map. So we have $m \dashv m^o$ with unit η and counit ε. Define t, α and β by

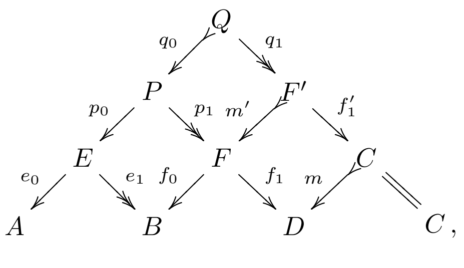
$$\begin{aligned} t &:= m^o \circ f, \\ \alpha &:= \varepsilon \circ f : m \circ m^o \circ f \to f, \\ \beta &:= (m^o \circ \phi)(\eta \circ g) : g \to m^o \circ f \circ e. \end{aligned}$$

It is now a straightforward task to check $\phi = (\alpha \circ e)(m \circ \beta)$ using only naturality (see (7) on page 10) and the adjunction equation

$$(\varepsilon \circ m)(m \circ \eta) = 1_m. \tag{30}$$

Now given any other fill-in (γ, u, δ) of (f, ϕ, g), we define $\psi := (m^o \circ \gamma)(\eta \circ u) : u \to m^o \circ f$. Again it is easy to check that $\alpha(m \circ \psi) = \gamma$ and $\beta = (\psi \circ e)\delta$ using only naturality and the adjunction equation.

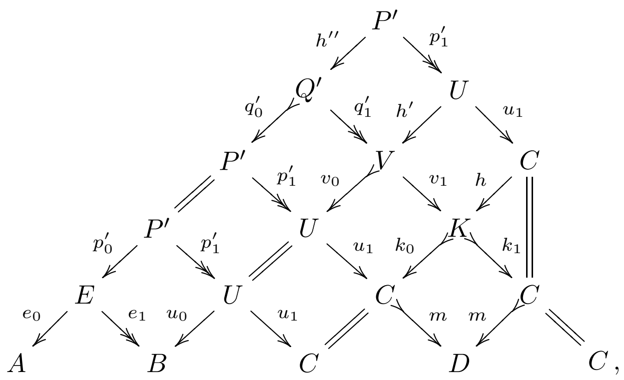
Finally we need to show that ψ is unique. To do this we must analyse how $(m^o \circ \gamma)$ and $(\eta \circ u)$ come about. First observe that the composite $m^o \circ f \circ e$ is obtained using the diagram



(31)

where, by (8), $\langle f_0 m', f_1' \rangle$ is in $\mathcal{M}$. Now consider the composite $m^o \circ m \circ u$ which we extend backwards by e, i. e., we obtain the diagram

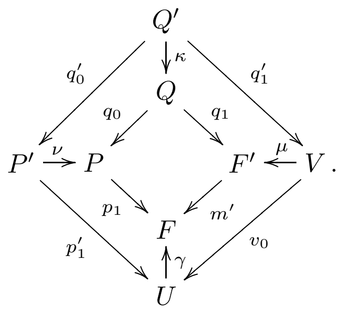
(32)



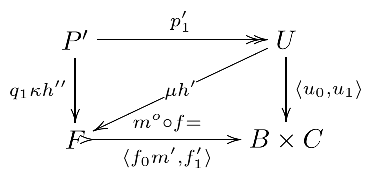
where all squares are pullbacks. Note that the common left-inverse h of k_0 and k_1 induces η, whence h' induces $\eta \circ u$. Furthermore, observe that h' and h'' are left-inverses of v_0 and q_0' respectively, and that $\langle u_0 v_0, k_1 v_1 \rangle$ lies in $\mathcal{M}$.

Now the pullback of $\gamma : U \to F$ along $m' : F' \to F$ induces the 2-cell $\mu = m^o \circ \gamma : V \to F'$. Pulling μ back along q_1 we get an arrow κ that induces $m^o \circ \gamma \circ e : m^o \circ m \circ u \circ e \to m^o \circ f \circ e$. Note that κ can also be obtained by first pulling back γ along p_1 to obtain $\nu : P \to P'$ which induces $\gamma \circ e : m \circ u \circ e \to f \circ e$, and then pulling back ν along q_0. Hence, we have the following diagram formed by pullbacks inscribed into diagram (32):

(33)



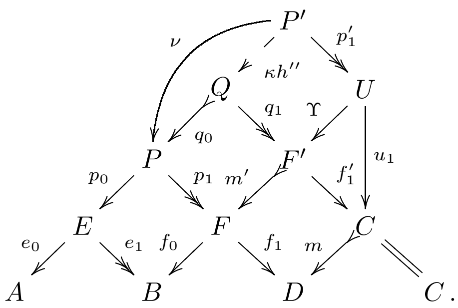
Clearly $\psi : u \to m^o \circ f$ is (not only induced but) given by $\mu h' : U \to F'$. It is now straightforward to check that the square



commutes.

Now suppose we are given a 2-cell $\Upsilon : u \to m^o \circ f$ with $\alpha(m \circ \Upsilon) = \gamma$ and $\beta = (\Upsilon \circ e)\delta$. Obviously $(m^o \circ f)\Upsilon = \langle u_0, u_1 \rangle$. So we need to check only that

$\Upsilon p_1' = q_1\kappa h''$. To do this consider



Note that $\alpha : m \circ m^o \circ f \to f$ is given by $m' : \langle f_0 m', m f_1' \rangle \to \langle f_0, f_1 \rangle$. Moreover, the 2-cell $m \circ \Upsilon$ is given by the arrow Υ since m lies in $\mathcal{M}$. Hence, $\alpha(m \circ \Upsilon) = \gamma$ translates into $m'\Upsilon = \gamma$ as arrows in $\mathcal{C}$. Therefore the pullback of $m'\Upsilon$ along p_1 must be ν. Clearly $m(u_1 p_1') = (f_1 p_1)\nu$. Using diagrams (32) and (33), it is easy to see that the arrow $P' \to Q$ induced by the universal property of the pullback must be $\kappa h''$. Therefore $q_1 \kappa h'' = \Upsilon p_1'$. Hence, the universal property of the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathcal{C}$ implies $\Upsilon = \psi$, which completes the proof. □

Remark 7.6. (i) I feel that condition (i) in Definition 7.3 is somewhat unnatural and should rather be:

(i') $\mathcal{E}$ and $\mathcal{M}$ contain all equivalences and are closed under composition with equivalences.

But for our example **Rel**($\mathcal{C}$) it is not at all clear that this holds since we do not have a characterization of equivalences, which are in fact the isomorphisms of **Rel**($\mathcal{C}$), in terms of $\mathcal{E}$ and $\mathcal{M}$ in general. However, for $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ there is the characterization of Theorem 4.20. Hence, $\text{Iso}(\mathcal{C}) \subseteq \bar{\mathcal{E}}$ clearly holds true. But for $\bar{\mathcal{M}}$ this is not even true if $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$.

Example 7.7. Consider the category **Top** of topological spaces and continuous maps equipped with the following $(\mathcal{E}, \mathcal{M})$-factorization system. The class $\mathcal{E}$ consists of surjective continuous mappings and the class $\mathcal{M}$ of subspace inclusions. Note that $\mathcal{E} = \text{Epi}(\mathbf{Top})$. Clearly the mapping $e : 2_d \to 2_i$, where 2_d is the discrete and 2_i is an indiscrete two-point space, is continuous, and therefore in $\mathcal{E} \cap \text{Mono}(\mathbf{Top})$. So clearly the relation $\langle e, 1 \rangle$ is an isomorphism in **Rel**(**Top**). But it cannot be isomorphic to any graph in $\mathbf{Rel}(\mathbf{Top})(2_i, 2_d)$ since e does not have a continuous inverse.

It seems as if the class $\bar{\mathcal{M}}$ is not yet large enough. On the other hand, to prove that ψ in (iv) of Definition 7.3 is unique we need to be able to exploit the special structure of $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$.

(ii) Note that except for the uniqueness of ψ the proof of the previous result only uses the fact that $m \in \bar{\mathcal{M}}$ is a map. So if we do not insist on the uniqueness of ψ we might define

$$\begin{aligned} \hat{\mathcal{E}} &= \{\text{all relations}\} \\ \hat{\mathcal{M}} &= \mathbf{Map}(\mathbf{Rel}(\mathcal{C})) \end{aligned}$$

to get the weak version of a factorization system for $\mathbf{Rel}(\mathcal{C})$ as defined in Definition 7.3. Moreover, the classes $\hat{\mathcal{E}}$ and $\hat{\mathcal{M}}$ satisfy (i') above. Further observe that we were not forced to use the dual of the adjunction equation (30). So if uniqueness of ψ is left aside, we might even blow up $\bar{M}$ to the class of relations $r : A \nrightarrow B$ for which there is a 1-cell $s : B \nrightarrow A$ and 2-cells $\eta : \iota_A \to s \circ r$ and $\varepsilon : r \circ s \to \iota_B$ with $(\varepsilon \circ r)(r \circ \eta) = 1_r$.

7.2 A functorial approach

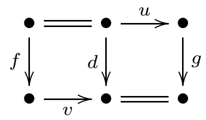
It is somewhat unsatisfactory that the classes $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$ are not closed under composition with equivalences. It would therefore be very convenient to change the view on which should be the important data determining a factorization system. The 2-category $\mathbf{Rel}(\mathcal{C})$ provides a canonical factorization for each relation as seen in Proposition 7.1. However, our definition of the classes $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$ relies more or less on an educated guess. Our goal should be to have a description of a factorization system that automatically induces the classes $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$ just from the fact that there exists a canonical factorization of each relation.

Fortunately such a view on factorization structures exists and is well-known for $(\mathcal{E}, \mathcal{M})$-factorization systems on ordinary categories. A reference for this is [16]. We recall the basic facts from there.

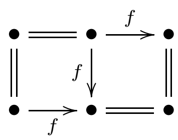
Definition 7.8. *A functor $F : \mathcal{C}^2 \to \mathcal{C}$ where 2 is the category $\{\bullet \to \bullet\}$ is called a weak factorization system on $\mathcal{C}$ if there is a natural isomorphism $FE \to \mathrm{Id}_{\mathcal{C}}$, where E is the embedding of $\mathcal{C}$ into $\mathcal{C}^2$ with $E(A) = 1_A$ on objects and $E(f) = (f, f) : 1_A \to 1_B$ for morphisms $f : A \to B$ of $\mathcal{C}$.*

Without loss of generality one may assume $FE = \mathrm{Id}_{\mathcal{C}}$. Note that this requires that identities are factorized into identities.

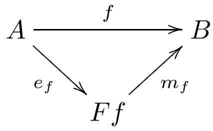
The free factorization system on $\mathcal{C}^2$, which gives for $(u, v) : f \to g$ the factorization



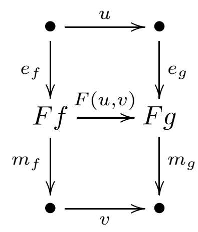
with $d = gu = vf$, induces a factorization of arrows of $\mathcal{C}$ as follows. For every arrow f there is the generic decomposition



in $\mathcal{C}^2$. Applying F, we obtain



with $e_f := F(1, f)$ and $m_f := F(f, 1)$. Note that e_f and m_f are isomorphisms if f is so. It is easy to see that for any $(u, v) : f \to g$ in $\mathcal{C}^2$ the diagram



(34)

is commutative. Putting

$$\mathcal{E}_F = \{h \mid m_h \text{ iso}\} \quad \text{and} \quad \mathcal{M}_F = \{h \mid e_h \text{ iso}\},$$

we can ask whether the following properties hold for F:

(i) $e_f \in \mathcal{E}_F$ for all arrows f,

(ii) $m_f \in \mathcal{M}_F$ for all arrows f,

(iii) $F(u, v)$ is uniquely determined by f, g, u and v and the commutativity of (34).

It can easily be checked that weak factorization systems satisfying these conditions provide an equivalent description of $(\mathcal{E}, \mathcal{M})$-factorization systems. Moreover, condition (iii) is redundant so that we obtain the following result.

Theorem 7.9. *The $(\mathcal{E}, \mathcal{M})$-factorization systems are equivalently described by the weak factorization systems that satisfy (i) and (ii).*

The proof of this is given in [16] together with another abstract and purely 2-categorial description of factorization systems.

Naturally the question arises what happens if we replace $\mathcal{C}$ by a 2-category $\mathcal{B}$, for example $\mathbf{Rel}(\mathcal{C})$, and F by a lax functor (or pseudo functor). For the definitions of both notions see [3]. In short, a *lax functor* F between 2-categories $\mathcal{A}$ and $\mathcal{B}$ is given by a family of functors

$$F_{AB} : \mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$$

such that identity 1-cells and composition of 1-cells are preserved up to coherent 2-cells. If all these 2-cells are isomorphisms, then F is called a *pseudo functor*.

Let us now investigate what kind of functor the canonical factorization of Proposition 7.1 induces in $\mathbf{Rel}(\mathcal{C})$. From now on we regard $2 = \{\bullet \to \bullet\}$ as a 2-category with only identity 2-cells and the embedding E as a 2-functor. Clearly $F : \mathbf{Rel}(\mathcal{C})^2 \to \mathbf{Rel}(\mathcal{C})$ should be defined with diagram (34) in mind. Two possible choices for F immediately come about. For $(\alpha, \beta) : (u, v) \to (x, y) : f \to g$ in $\mathbf{Rel}(\mathcal{C})^2$ define

$$\begin{array}{rclcrcl} F_0(u, v) & := & \bar{m}_g^o \circ v \circ \bar{m}_f, & \quad & F_0(\alpha, \beta) & := & \bar{m}_g^o \circ \beta \circ \bar{m}_f \\ F_1(u, v) & := & \bar{e}_g \circ u \circ \bar{e}_f^o, & \quad & F_1(\alpha, \beta) & := & \bar{e}_g \circ \alpha \circ \bar{e}_f^o, \end{array}$$

where $f \simeq \bar{m}_f \circ \bar{e}_f$ and $g \simeq \bar{m}_g \circ \bar{e}_g$ are the canonical factorizations as given in Proposition 7.1. The object Ff is in both cases given by $(\mathcal{E}, \mathcal{M})$-factorizing the right leg f_1 of $f = \langle f_0, f_1 \rangle$. It turns out that only one of these definitions is reasonable.

Proposition 7.10. *The assignment $F = F_0$ is a lax functor $\mathbf{Rel}(\mathcal{C})^2 \to \mathbf{Rel}(\mathcal{C})$ with $FE = \mathrm{Id}_{\mathbf{Rel}(\mathcal{C})}$. Moreover, F preserves identities if and only if for all relations r, $\bar{m}_r$ is injective.*

Proof. First let us check that $FE = \mathrm{Id}_{\mathbf{Rel}(\mathcal{C})}$. If $f : A \nrightarrow B$ is a relation, then Ef is given by the square

$$\begin{array}{ccc} A & \overset{f}{\nrightarrow} & B \\ {\scriptstyle \iota_A}\downarrow & & \downarrow{\scriptstyle \iota_B} \\ A & \underset{f}{\nrightarrow} & B. \end{array}$$

But clearly the identities are factorized into identities since $\iota_0, \iota_1 \in \mathcal{E}$ for $\iota = \langle \iota_0, \iota_1 \rangle$. Hence,

$$F(f, f) = \iota_B^o \circ f \circ \iota_A \simeq f.$$

Let us abbreviate $\mathbf{Rel}(\mathcal{C})$ by $\mathcal{B}$. Since composition is functorial, F is functorial on $\mathcal{B}^2(f, g)$ for all relations f and g. We write F_{fg} for the restriction of F to this hom-category.

For all relations f, g and h we get a natural transformation

$$\gamma_{f,g,h} : \circ \cdot (F_{fg} \times F_{gh}) \to F_{fh} \cdot \circ_2,$$

where $\circ_2 : \mathcal{B}^2(f, g) \times \mathcal{B}^2(g, h) \to \mathcal{B}^2(f, h)$ and $\circ : \mathcal{B}(Ff, Fg) \times \mathcal{B}(Fg, Fh) \to \mathcal{B}(Ff, Fh)$ are composition functors. Its component at $((u, v), (x, y))$ is given by

$$\bar{m}_h^o \circ y \circ \bar{m}_g \circ \bar{m}_g^o \circ v \circ \bar{m}_f \xrightarrow{\bar{m}_h^o \circ y \circ \varepsilon_{m_g} \circ v \circ \bar{m}_g^o} \bar{m}_h^o \circ y \circ v \circ \bar{m}_f,$$

where ε_{m_g} is the counit of the adjunction $\bar{m}_g \dashv \bar{m}_g^o$.

Furthermore for all relations f there is a natural transformation

$$\delta_f : u_{Ff} \to F_{ff} \cdot u_f,$$

where $u_f : 1 \to \mathcal{B}(Ff, Ff)$ and $u_{Ff} : 1 \to \mathcal{B}^2(f, f)$ are the functors that give the identities. The only component of δ_f is the unit η_f of $\bar{m}_f \dashv \bar{m}_f^o$.

Checking the coherence axioms is now a straightforward task involving only naturality, as well as both of the adjunction equations, in order to show the unit axioms.

Finally, check that for all relations $r : A \nrightarrow B$, $F(1_r) = \bar{m}_r^o \circ \bar{m}_r$. Hence, F preserves identities if and only if $\bar{m}_r^o \circ \bar{m}_r \simeq \iota$, which is equivalent to injectivity because $\bar{m}_r$ is a map, and therefore monic as an arrow of $\mathcal{C}$. □

Note that for F_1 the unit axioms of the coherence part do not hold true since $\bar{e}_f$ is in general not a map. The components of the transformations $\gamma_{f,g,h}$ and δ_f are induced by units and counits of adjunctions. These are in general no isomorphisms. Thus, one cannot expect F to be a pseudo functor in general.

The fact that F is a lax functor naturally leads to the notion of a weak lax factorization system of a 2-category.

Definition 7.11. *Let $\mathcal{B}$ be a 2-category. A lax (pseudo) functor $F : \mathcal{B}^2 \to \mathcal{B}$ with $FE = \mathrm{Id}_{\mathcal{B}}$ is called a* weak lax (pseudo) *factorization system of $\mathcal{B}$.*

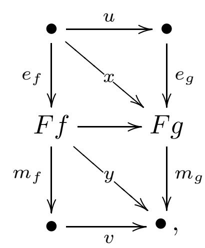
For every arrow f of $\mathcal{B}$ we put

$$\begin{aligned} e_f &:= F(1,f), \\ m_f &:= F(f,1). \end{aligned}$$

Then the coherence transformation $\gamma_{(1,f),(f,1)}$ yields a 2-cell

$$m_f \circ e_f = F(1,f) \circ F(f,1) \to F(f,f) = f.$$

Moreover, for every arrow $(u,v) : f \to g$ in $\mathcal{B}^2$ the lax (pseudo) functoriality of F provides coherent 2-cells that fill the diagram



(35)

where $x = F(u, gu) = F(u, vf)$ and $y = F(vf, v) = F(gu, v)$. If F is a pseudo functor, then the 2-cell $m_f \circ e_f \to f$ and the 2-cells in (35) are isomorphisms.

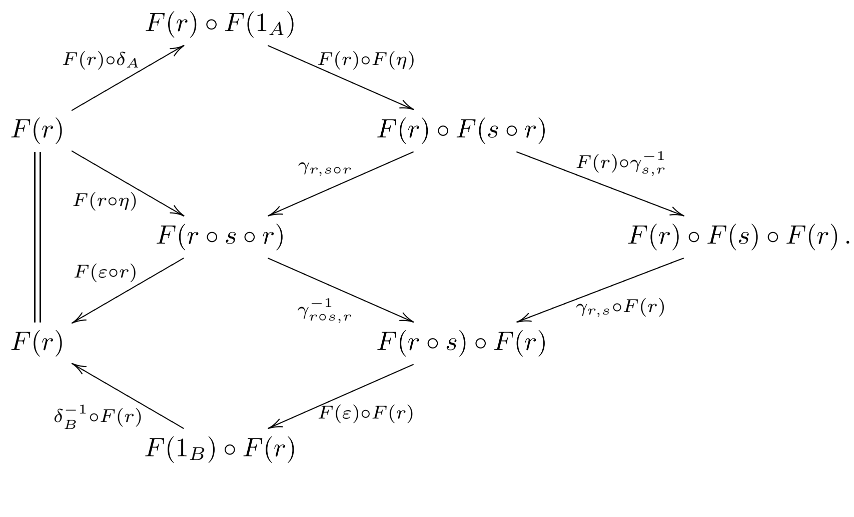
The question arises what the classes $\mathcal{E}_F$ and $\mathcal{M}_F$ should be in this setting. For a pseudo functor F this is not too difficult to answer.

Proposition 7.12. *Let $F : \mathcal{B} \to \mathcal{B}$ be a pseudo functor between 2-categories. Then F preserves adjoint situations.*

Proof. Let $r \dashv s : B \leftrightarrow A$ be a pair of adjoint 1-cells with unit η and counit ε. We shall show that $F(r) \dashv F(s)$ in $\mathcal{B}'$. The unit and counit of this adjunction are given by

$$\begin{aligned} \eta' :&\quad 1_{FA} \xrightarrow{\delta_A} F1_A \xrightarrow{F\eta} F(s \circ r) \xrightarrow{\gamma_{s,r}^{-1}} F(s) \circ F(r)\,, \\ \varepsilon' :&\quad F(r) \circ F(s) \xrightarrow[\gamma_{r,s}]{} F(s \circ r) \xrightarrow[F\varepsilon]{} F1_B \xrightarrow[\delta_B^{-1}]{} 1_{FB}\,, \end{aligned}$$

where δ and γ are components of the coherence transformations. To check the adjunction equation $(F(r) \circ \eta')(\varepsilon' \circ F(r)) = F(r)$ consider the following diagram of 2-cells in $\mathcal{B}'$



By the unit axioms for F, $\delta_B^{-1} \circ F(r) = \gamma_{1_B,r}$ and $F(r) \circ \delta_A = \gamma_{r,1_A}^{-1}$. Note that the left-hand triangle commutes since F is functorial on hom-categories. The right-hand square commutes because of the coherence axioms for F, and the upper and lower square by naturality of γ. The second adjunction equation can be shown analogously. $\square$

This also shows that a pseudo functor preserves equivalences because if η and ε are isomorphisms, then η' and ε' are so, too. Hence, a weak pseudo factorization system F on $\mathcal{B}$ factorizes equivalences into equivalences. Putting

$$\begin{array}{rcl} \bar{\mathcal{E}}_F & = & \{h \mid m_h \text{ equivalence}\}, \\ \bar{\mathcal{M}}_F & = & \{h \mid e_h \text{ equivalence}\}, \end{array}$$

we obtain two classes of 1-cells with $\mathrm{Eq}(\mathcal{B}) \subseteq \bar{\mathcal{M}}_F \cap \bar{\mathcal{E}}_F$, where $\mathrm{Eq}(\mathcal{B})$ denotes the class of equivalences of $\mathcal{B}$. Moreover, it can quite easily be seen that $\bar{M}_F$ and $\bar{\mathcal{E}}_F$ are closed under composition with equivalences. More precisely, we obtain the following result.

Proposition 7.13. *If $m : B \nrightarrow C$ lies in $\bar{\mathcal{M}}_F$ and $i : A \nrightarrow B$ is an equivalence in $\mathcal{B}$, then $m \circ i \in \bar{\mathcal{M}}_F$; dually, if $e : C \nrightarrow A$ lies in $\bar{\mathcal{E}}_F$, then $i \circ e \in \bar{\mathcal{E}}_F$.*

Proof. Note that up to isomorphism $(1, m) : 1_B \to m$ can be written as

$$\begin{array}{ccccccc} B & \xrightarrow{i^{-1}} & A & = & A & \xrightarrow{i} & B \\ \| & & \| & & \downarrow{\scriptstyle m\circ i} & & \downarrow{\scriptstyle m} \\ B & \xrightarrow[i^{-1}]{} & A & \xrightarrow[m\circ i]{} & C & = & C \end{array}$$

in $\mathcal{B}^2$, where i^{-1} is the equivalence inverse of i. Since $e_m = F(1, m)$, $F(i^{-1}, i^{-1})$ and $F(i, 1_C)$ must be equivalences in $\mathcal{B}$, $F(1, m \circ i) = e_{m\circ i}$ must be so, too. Hence, $m \circ i \in \bar{\mathcal{M}}_F$. The proof for $e \in \bar{\mathcal{E}}_F$ is similar. $\square$

Note that in order to prove the last result we had to evoke the so-called 2 out of 3 property for the class $\mathrm{Eq}(\mathcal{B})$. A class $\mathcal{F}$ of morphisms in a category is said to satisfy the 2 out of 3 property if the following holds true. For any morphisms f, g and h with $gf = h$ and with any two of these morphisms lying in $\mathcal{F}$, the third morphisms lies in $\mathcal{F}$, too.

As for ordinary categories we may ask whether the following hold:

($\bar{\text{i}}$) $e_f \in \bar{\mathcal{E}}_F$ for all 1-cells f of $\mathcal{B}$,

($\overline{\text{ii}}$) $m_f \in \bar{\mathcal{M}}_F$ for all 1-cells f of $\mathcal{B}$,

($\overline{\text{iii}}$) If $\alpha : F(u, v) \circ e_f \to x$ and $\beta : m_g \circ F(u, v) \to y$ are the isomorphic 2-cells of (35), i. e., $(\alpha, F(u, v), \beta)$ is a fill-in, and if (γ, t, δ) is another such fill-in, then there exist a unique isomorphic 2-cell $\phi : t \to F(u, v)$ with

$$\alpha(\phi \circ e_f) = \gamma \quad \text{and} \quad \beta(m_g \circ \phi) = \delta. \tag{36}$$

Moreover, it seems quite reasonable to ask whether condition ($\overline{\text{iii}}$) is redundant again. Indeed, existence of the 2-cell ϕ of ($\overline{\text{iii}}$) can easily be seen.

Proposition 7.14. *If F is a weak pseudo factorization system with $(\bar{i})$ and $(\bar{\bar{ii}})$, then condition $(\overline{\overline{\overline{iii}}})$ holds true with the possible exception of the uniqueness requirement for ϕ.*

Proof. Suppose a fill-in (γ, t, δ) for (35) with isomorphic 2-cells γ and δ has been given. The isomorphic 2-cells α and β of (35) yield an isomorphism in $\mathcal{B}^2(e_f, m_g)$ as follows:

$$(1, m_g) \circ (t, t) \circ (e_f, 1) \xrightarrow{(\gamma,\delta)} (x, y) \xrightarrow{(\alpha^{-1},\beta^{-1})} (1, m_g) \circ (z, z) \circ (e_f, 1),$$

where z stands for $F(u, v)$. Applying the pseudofunctor F we get

$$e_{m_g} \circ t \circ m_{e_f} \simeq e_{m_g} \circ F(u, v) \circ m_{e_f}.$$

By $(\bar{\text{i}})$ and $(\bar{\bar{\text{ii}}})$, e_{m_g} and m_{e_f} are equivalences. Hence, $t \simeq F(u, v)$. □

In order to show uniqueness of ϕ in $(\overline{\overline{\overline{\text{iii}}}})$ one would have to analyze the situation much more thoroughly, e. g., similarly as it has been done in [16] for the setting of ordinary categories. For the time being we leave this as an open problem and make the following definition.

Definition 7.15. *A* pseudo factorization system *of a 2-category $\mathcal{B}$ is a weak pseudo factorization system $F : \mathcal{B}^2 \to \mathcal{B}$ that satisfies conditions $(\bar{i})$–$(\overline{\overline{\overline{iii}}})$.*

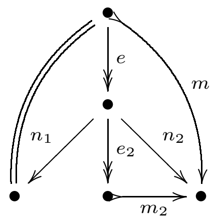
It should not be too difficult to see whether this notion is equivalent to the notion of $(\mathcal{E}, \mathcal{M})$-factorization system in a 2-category as defined in [11]. The proof of this should just be a 2-categorial version of the proof for the respective result for ordinary categories. At the moment we leave this as an open problem for further study.

However, if we further relax our setting and assume that F is just a lax functor, we run into difficulties. Unfortunately a lax functor does not preserve equivalences. But in order to define $\bar{\mathcal{E}}_F$ and $\bar{\mathcal{M}}_F$ reasonably, i. e., such that a result like Proposition 7.14 holds true, it was important to use the 2 out of 3 property of Eq($\mathcal{B}$). In other words, if we think of equivalences as being the pseudo isomorphisms, we are looking for a decent class of *lax isomorphisms*, i. e., a class of 1-cells that is preserved by each lax functor and satisfies the 2 out of 3 property. It does not seem obvious at all what that class should be. The obvious generalization of Eq($\mathcal{B}$) for a lax functor F yields a rather weird class that is not even closed under composition. If F preserves at least identities up to isomorphism, then we obtain a class that is closed under composition and 2-cells. But unfortunately the required cancellation laws do not hold true. On the other hand, for our example $\mathbf{Rel}(\mathcal{C})$ with the canonical factorization of Proposition 7.1 the lax functor $F : \mathbf{Rel}(\mathcal{C})^2 \to \mathbf{Rel}(\mathcal{C})$ preserves identities if and only if for all relations r, $\bar{m}_r$ is injective. At least in the setting with $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ this is equivalent to $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$. However, the last condition is sufficient to make F a functor as we shall see in the next subsection. Hence, at this point it does not seem to make much sense to investigate the weak lax factorization systems much further.

Denote by $(\overline{\overline{\overline{\text{iii}}}}\text{'})$ the condition obtained from $(\overline{\overline{\overline{\text{iii}}}})$ by dropping the assumption that the 2-cells are isomorphic. Let us check whether $(\bar{\text{i}})$, $(\bar{\bar{\text{ii}}})$ and $(\overline{\overline{\overline{\text{iii}}}}\text{'})$ hold true for $\mathcal{B} = \mathbf{Rel}(\mathcal{C})$. Let $r : A \nrightarrow B$ be a relation with canonical factorization $\bar{m}_r \circ \bar{e}_r \simeq r$. Note that

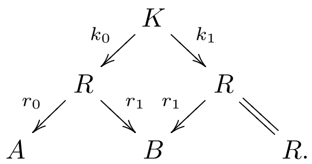
$$\begin{aligned} e_r &= F(1, r) \simeq \bar{m}_r \circ r, \\ m_r &= F(r, 1) \simeq \bar{m}_r. \end{aligned}$$

Surprisingly, $e_r \neq \bar{e}_r$. However, if $\bar{m}_r^o \circ \bar{m}_r \simeq \iota_{Ff}$, then $e_r \simeq \bar{e}_r$, and moreover, $(\bar{\text{i}})$ and $(\overline{\text{ii}})$ hold for r. For $(\bar{\text{i}})$ note that if $e_r = \langle e_0, e_1 \rangle$ with $e_1 \in \mathcal{E}$, then m_{e_r} is an identity. For $(\overline{\text{ii}})$ consider the diagram



and note that for $m_r = \Gamma m$, $e_{m_r} = \mathrm{im}\langle n_1, e_2 \rangle \simeq \mathrm{im}\langle n_1 e, e_2 e \rangle \simeq \mathrm{im}\langle 1, e_2 e \rangle$. But $e_2 e$ is an isomorphism in $\mathcal{C}$. Thus e_{m_r} is an isomorphism in $\mathbf{Rel}(\mathcal{C})$.

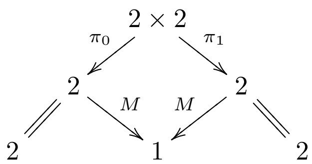
In general, neither $(\bar{\text{i}})$ nor $(\overline{\text{ii}})$ are true. For $(\bar{\text{i}})$ let $\mathcal{B} = \mathbf{Span}(\mathcal{C})$ and note that for a span $r = \langle r_0, r_1 \rangle : A \nrightarrow B$, m_{e_r} is given by Γk_1, where k_1 can be obtained by pulling back r_1 along itself as shown in the diagram



Note that Γk_1 is an equivalence (i.e., an isomorphism) if and only if k_1 is an isomorphism in $\mathcal{C}$ if and only if r_1 is monomorphic, which is not true in general for every span r.

For $(\overline{\text{ii}})$ note that $e_{m_r} \simeq m_r^o \circ m_r$, which is an isomorphism if and only if it is an identity. But $(\Gamma m)^o \circ \Gamma m \simeq \iota$ does not hold true in general if $m \in \mathcal{M}$.

Example 7.16. Consider the category **CAT** with the factorization structure as in Example 3.15 on page 19. Recall that $M : 2 \to 1$ lies in $\mathcal{M}$. But $T := (\Gamma M)^o \circ \Gamma M \not\simeq \iota_2$. In order to see this note that the kernel of M is given as in diagram



since 1 is a terminal object of **CAT**. Hence, when we $(\mathcal{E}, \mathcal{M})$-factorize $\langle \pi_0, \pi_1 \rangle$ we obtain a category with 4 objects whereas factorizing δ_2 yields a category with 2 objects. This also shows that ΓM is not injective. Indeed, any 2-cell $T \to \iota_2$ must be isomorphic, since T is monic as arrow of **CAT**.

What about ($\overline{\text{iii}}$') then? For any given fill-in (γ, t, δ) of (35) it is easy to show the existence of a 2-cell $\phi : t \to F(u, v)$. This 2-cell is given by

$$\phi := (m_g^o \circ \delta)(\eta_{m_g} \circ t).$$

Checking the two identities (36), where

$$\begin{array}{ll} \alpha: & m_g^o \circ v \circ m_f \circ m_f^o \circ f \xrightarrow{m_g^o \circ v \circ \varepsilon_{m_f} \circ f} m_g^o \circ v \circ f\,, \\ \beta: & m_g \circ m_g^o \circ v \circ m_f \xrightarrow{\varepsilon_{m_g} \circ v \circ m_f} v \circ m_f \end{array}$$

is again straightforward using naturality and the adjunction equations. Despite the striking resemblance of this with the proof of Proposition 7.5, the uniqueness of ϕ does not seem to hold true since, unlike the situation in Proposition 7.5, there is no arrow in $\mathcal{E}$ which could be used to form a square to which the universal property of the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathcal{C}$ could be applied to show that ϕ is the unique diagonal.

7.3 Factorization systems for $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$

In this subsection $\mathcal{C}$ is still an $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback. But now we also assume $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. The following proposition is the result obtained by Wyler and Jayewardene in [10].

Proposition 7.17. *Every relation r factors $r = \Gamma m \circ s$, where s is surjective and $m \in \mathcal{M}$, hence Γm injective.*

Furthermore s is total if and only if r is total, and s is single-valued if and only if r is single-valued.

Proof. The first part is just Proposition 7.1, where s is surjective by 5.6 for s^o.

Moreover, by 5.6, r is total if and only if s is total. We have $s = (\Gamma m)^o \circ r$ since m is monic, and therefore $(\Gamma m)^o$ is single-valued. So s is single-valued if and only if r is single-valued, since the composite of single-valued relations is single-valued, too. □

Theorem 7.18. *For any commutative square*

$$\begin{array}{ccc} A & \overset{e}{\nrightarrow} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \underset{m}{\nrightarrow} & D \end{array}$$

in $\mathbf{Rel}(\mathcal{C})$ with $e \in \bar{\mathcal{E}}$ and $m \in \bar{\mathcal{M}}$ there is a unique relation $t : B \nrightarrow C$ such that $m \circ t \simeq f$ and $t \circ e \simeq g$.

Proof. Note that $m \in \bar{\mathcal{M}}$ means that $m = \mathrm{im}\langle 1, m_0 \rangle$ for some $m_0 \in \mathcal{M}$. So clearly $m^o \circ m \simeq \iota_C$. Now define $t := m^o \circ f$. Then clearly

$$g \simeq m^o \circ m \circ g \simeq m^o \circ f \circ e \simeq t \circ e,$$

and

$$m \circ t \simeq m \circ m^o \circ f \leq f,$$

since m is single-valued.

To prove that $f \leq m \circ t$ we use the calculus of 3.7 to mimic the proof for set-relations. Suppose $d(f)b$. Since $e = \langle e_0, e_1 \rangle$ is surjective (or by pulling back b along e_1) there exists an $p \in \mathcal{E}$ and an element a of A with $bp(e)a$. Clearly $dp(f)bp$, whence $dp(f \circ e)a$, and $dp(m \circ g)a$. By Proposition 3.16, there is an $q \in \mathcal{E}$ and an element c of C with $dpq(m)c$ and $c(g)aq$. Obviously, $c(m^o)dpq$ and $dpq(f)bpq$, whence $c(m^o \circ f)bpq$, and $dpq(m)c$ as before. Thus $dpq(m \circ m^o \circ f)bpq$, which implies that $d(m \circ t)b$ by the universal property of the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathcal{C}$.

Uniqueness of t follows easily because $m \circ t' \simeq f$ implies

$$t' \simeq m^o \circ m \circ t' \simeq m^o \circ f.$$

□

Of course, this proof can also be done by diagram chasing as in Proposition 7.5. Though rather lengthy, it is quite instructive to do so because one sees more clearly how and where the assumption $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ is used, and what can be said without it. But this shall not be pursued here. Let us just point out that for example the 2-cell $m \circ t \to f$ need not be an isomorphism anymore but still can be shown to be split epic.

We shall now show that $\mathbf{Rel}(\mathcal{C})$ is $(\bar{\mathcal{M}}_F, \bar{\mathcal{E}}_F)$-structured if $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$.

Theorem 7.19. *If $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, then the lax functor $F : \mathbf{Rel}(\mathcal{C})^2 \to \mathbf{Rel}(\mathcal{C})$ of Proposition 7.10 is a functor.*

Proof. For $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ we have $m_r^o \circ m_r \leq \iota$ for all relations r since m_r is a graphs of an arrow of $\mathcal{M}$. By 7.10, F preserves identities.

As for composition note that for any arrow $(u, v) : f \to g$ in $\mathbf{Rel}(\mathcal{C})^2$, $F(u, v)$ is the unique arrow that makes (35) commutative, by Theorem 7.18. Hence, F must be a functor. □

Corollary 7.20. *The category $\mathbf{Rel}(\mathcal{C})$ is $(\bar{\mathcal{E}}_F, \bar{\mathcal{M}}_F)$-structured.*

Note that in $\mathbf{Rel}(\mathcal{C})$ conditions (i) and (ii) are equivalent to $(\bar{\mathrm{i}})$ and $(\bar{\mathrm{ii}})$ respectively. Also recall that $\bar{\mathcal{E}}$ and $\bar{\mathcal{M}}$ as defined in Section 7.1 are closed under composition, contain the identities and are contained in $\bar{\mathcal{E}}_F$ and $\bar{\mathcal{M}}_F$ respectively.

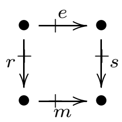
Finally, note that Example 3.15 above shows that the proof of Theorem 7.18 does not work without the assumption that $\mathcal{M}$ consists of monomorphisms. Indeed, consider the square

$$\begin{array}{ccc} 2 & \xrightarrow{\iota_2} & 2 \\ {\scriptstyle \iota_2}\downarrow & & \downarrow{\scriptstyle \Gamma M} \\ 2 & \xrightarrow[\Gamma M]{} & 1 \end{array}$$

in $\mathbf{Rel}(\mathbf{CAT})$ with $\iota_2 \in \bar{\mathcal{E}}$ and $\Gamma M \in \bar{\mathcal{M}}$. This certainly has ι_2 as its unique diagonal. But according to Theorem 7.18 the canonical choice for the diagonal should be $(\Gamma M)^o \circ \Gamma M$, which, however, is not an identity.

To complete this section let us note Wyler's and Jayewardene's result about diagonals for commutative squares in $\mathbf{Rel}(\mathcal{C})$ from [10].

Theorem 7.21. *For a commutative square*



of relations with e *surjective and single-valued and* m *injective and total, the following are equivalent and determine the relation* t*:*

$$r \simeq t \circ e, \qquad t \simeq r \circ e^o, \qquad s \simeq m \circ t \qquad t \simeq m^o \circ s.$$

Moreover r *and* t *are single-valued if* s *is single-valued, and* s *and* t *are total if* r *is total.*

Proof. If $r \simeq t \circ e$, then $e \circ e^o \simeq \iota$ implies $t \simeq r \circ e^o$ and further

$$m \circ t \simeq m \circ r \circ e^o \simeq s \circ e \circ e^o \simeq s.$$

Similarly, $s \simeq m \circ t$ implies $t \simeq m^o \circ s$ and $r \simeq t \circ e$ using $m^o \circ m \simeq \iota$.

The last part of the result is now immediate from the four statements and the hypothesis using the fact that composites of total and single-valued relations are total and single-valued, respectively. □

8 Some limits in **Rel**($\mathcal{C}$)

It is well-known that **Rel**(**Set**) has finite (co)products. The generalization of this to **Rel**($\mathcal{C}$) has apparently not been studied anywhere before. In this section we shall show how to obtain finite coproducts in the ordinary as well as in the 2-category **Rel**($\mathcal{C}$) of relations if $\mathcal{C}$ is an extensive category. Note that by self duality of **Rel**($\mathcal{C}$) we also will have described finite products. First let us recall some facts about extensive categories (see [6] for the proofs).

8.1 Extensive categories

Definition 8.1. *A category $\mathcal{C}$ with coproducts of pairs of objects is* extensive *if and only if it has pullbacks along injections of coproducts, and if every commutative diagram*

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{x_1} & X_1 + X_2 & \xleftarrow{x_2} & X_2 \end{array}$$

comprises a pair of pullback squares in $\mathcal{C}$ if and only if the top row is a coproduct diagram in $\mathcal{C}$.

Definition 8.2. *Let $\mathcal{C}$ be a category. If $(X_i \to X \mid i \in I)$ is a cocone under a diagram with vertices X_i, we say X is a* universal colimit *of the diagram provided that, for each morphism $Y \to X$, the cocone $(Y \times_X X_i \to Y \mid i \in I)$ is a colimit for the "pulled-back" diagram with vertices $Y \times_X X_i$.*

Note that a universal initial object is an initial object 0 with the following property: For any object A, if $A \to 0$ is an arrow then A is also an initial object; equivalently, any arrow $A \to 0$ is an isomorphism. A universal initial object is often referred to as a *strict* initial object.

Definition 8.3. *Let $\mathcal{C}$ be a category with coproducts and pullbacks. A coproduct $X = \coprod_{i \in I} X_i$ is said to be* disjoint *if*

(i) for every $j \in I$ the coproduct injection $X_i \xrightarrow{i_j} X$ is monic,

(ii) for each pair $j, k \in I$ with $j \neq k$, the pullback of the two injections i_j, i_k is an initial object of $\mathcal{C}$.

Definition 8.4. *A category with binary products and coproducts is said to be* distributive *if the canonical arrow*

$$(A \times C) + (B \times C) \xrightarrow{\text{dist}} (A + B) \times C$$

is an isomorphism for any objects A, B and C.

Proposition 8.5. *(i) In a category with universal binary coproducts, initial objects are strict.*

(ii) In an extensive category, binary coproducts are universal.

(iii) A category with finite coproducts and pullbacks along their injections is extensive if and only if the coproducts are disjoint and universal.

(iv) An extensive category with binary products is distributive.

8.2 2-limits and bilimits

The definitions given below are taken from [3]. For the sake of brevity we omit the definitions of 2-functor and 2-natural transformation which can be found in [3]. Roughly speaking, a 2-functor is a functor $F : \mathcal{B} \to \mathcal{C}$ between 2-categories that preserves the additional structure, i. e., when restricted to the hom-categories we obtain functors $F_{AB} : \mathcal{B}(A, B) \to \mathcal{C}(A, B)$ such that certain coherence axioms are satisfied.

Before giving the definitions let us first introduce some more notation. Let $\mathcal{D}$ and $\mathcal{C}$ be categories with $\mathcal{D}$ small. For every object B of $\mathcal{C}$ there is the constant 2-functor

$$\Delta_B : \mathcal{D} \to \mathcal{C}$$

with $\Delta_B(A) := B$ for any object A of $\mathcal{D}$, $\Delta_B(f) := 1_B$ for any 1-cell f of $\mathcal{D}$, and $\Delta_B(\alpha) := \mathrm{id}_{1_B}$ for any 2-cell α of $\mathcal{D}$. Given a 2-functor $F : \mathcal{D} \to \mathcal{C}$ and an object B of $\mathcal{C}$ we write 2-$\mathbf{cone}(B, F)$ for the category whose objects are 2-natural transformations $\Delta_B \to F$ (the "2-cones on F with vertex B") and whose morphisms are the modifications between them (see [3] for the definition of modifications).

Definition 8.6. *The 2-limit of a 2-functor F, if it exists, is a pair (L, π), where L is an object of $\mathcal{C}$ and π is an object of 2-$\mathbf{cone}(L, F)$ such that the functor*

$$\mathcal{C}(B, L) \to 2\text{-}\mathbf{cone}(B, F)$$

of composition with π becomes an isomorphism of categories for each object B of $\mathcal{C}$.

Explicitly, (L, π) is a limit with the following additional properties: for any 2-cell $\alpha : f \to g : X \to Y$ of $\mathcal{D}$ we have $F\alpha \circ \mathrm{id}_{\pi_X} = \mathrm{id}_{\pi_Y}$, and whenever we have a cone (B, σ) with this property there is a unique 1-cell $b : B \to L$ with $\pi_A \circ b = \sigma_A$ for each object A of $\mathcal{D}$. Furthermore, given a natural transformation $\tau : \Delta_B \to F$ with all these properties, so that a unique 1-cell $c : B \to L$ is induced, and given a family of 2-cells $\Sigma_A : \sigma_A \to \tau_A$ with $F\alpha \circ \Sigma_X = \Sigma_Y$ for any 2-cell α of $\mathcal{D}$, then there is a unique 2-cell $\beta : b \to c$ such that $\Sigma_A = \mathrm{id}_{\pi_A} \circ \beta$.

Since 2-limits are special limits, we immediately get the following result.

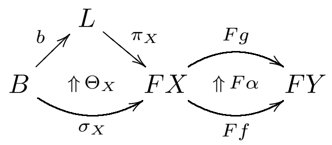
Proposition 8.7. *If (L, π) and (M, μ) are 2-limits of the same 2-functor $F : \mathcal{D} \to \mathcal{C}$, then there exists an isomorphism $b : L \to M$ with $\mu_A \circ b = \pi_A$ for each object A of $\mathcal{D}$.*

Definition 8.8. *The bilimit of a 2-functor $F : \mathcal{D} \to \mathcal{C}$, if it exists, is a pair (L, π) where L is an object of $\mathcal{C}$ and $\pi : \Delta_L \to F$ is a 2-natural transformation such that the functor*

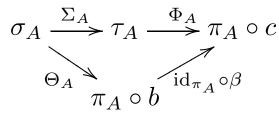
$$\mathcal{C}(B, L) \to 2\text{-}\mathbf{cone}(B, F)$$

of composition with π is an equivalence of categories for every object B of $\mathcal{C}$.

Explicitly, (L, π) is a cone with $F\alpha \circ \mathrm{id}_{\pi_X} = \mathrm{id}_{\pi_Y}$ for any 2-cell $\alpha : f \to g : X \to Y$ of $\mathcal{D}$. Moreover, if (B, σ) is a cone with the same properties, then there is a 1-cell $b : B \to L$ and a family Θ of isomorphic 2-cells such that the diagram



commutes, i. e. $F\alpha \circ \Theta_X = \Theta_Y$, and the following additional property holds. Given another cone (B, τ) with a family of isomorphic 2-cells $\Phi_A : \tau_A \to \pi_A \circ c$, where c is obtained as before b, $F\alpha \circ \Phi_X = \Phi_Y$, and given a family of 2-cells $\Sigma_A : \sigma_A \to \tau_A$ with $F\alpha \circ \Sigma_X = \Sigma_Y$, then there exists a unique 2-cell $\beta : b \to c$ such that the diagram



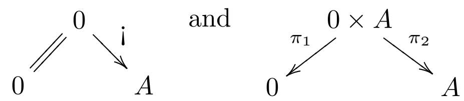
is commutative for any object A of $\mathcal{D}$.

Unfortunately, two bilimits of a 2-functor are not in general isomorphic to each other. However, there is the following result. Its proof can be found in [3].

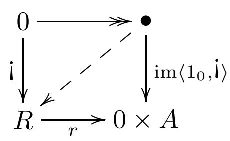
Proposition 8.9. *Two bilimits of a 2-functor are weakly equivalent, i. e., if (L, π) and (M, σ) are bilimits of a 2-functor, then there exist 1-cells $i : L \to M$ and $j : M \to L$ and isomorphic 2-cells $1_L \simeq j \circ i$ and $1_M \simeq i \circ j$.*

8.3 Initial and terminal objects in Rel($\mathcal{C}$)

It turns out that if $\mathcal{C}$ has a strict initial object 0, then 0 is also initial in the 2-category **Rel**($\mathcal{C}$). But first let us observe what can be said about **Rel**$(0, A)$ without assuming strictness of the initial object 0 of $\mathcal{C}$. Clearly the hom-categories **Span**$(0, A)$ have initial and terminal objects given by



respectively, where π_i are product projections. Note that $\langle \pi_1, \pi_2 \rangle = 1_{0 \times A}$ always belongs to $\mathcal{M}$, but for a non-strict 0, $\langle 1_0, \mathsf{i} \rangle$ does not belong to $\mathcal{M}$ in general. However, $\mathrm{im}\langle 1_0, \mathsf{i} \rangle$ is initial in **Rel**$(0, A)$. To see this let $r : 0 \leftrightarrow A$ be a relation. The universal property of the factorization system yields a 2-cell $\alpha : \mathrm{im}\langle 1_0, \mathsf{i} \rangle \to r$ such that the diagram



is commutative. Since the upper left triangle commutes for any 2-cell $\mathrm{im}\langle 1_0, \mathsf{i} \rangle \to r$, α must be the unique such 2-cell. Note that this also shows that 0 is initial in **Map**(**Rel**($\mathcal{C}$)).

Now we add strictness to the initial object of $\mathcal{C}$.

Proposition 8.10. *Let $\mathcal{C}$ be $(\mathcal{E}, \mathcal{M})$-structured category with universal, that means strict, initial object* 0*. Then* 0 *is also initial in the 2-category* $\mathbf{Rel}(\mathcal{C})$*.*

Proof. First observe that if $me = \langle 1_0, \mathsf{i} \rangle$ is an $(\mathcal{E}, \mathcal{M})$-factorization with $\mathrm{cod}(e) = E$, then strictness of 0 implies that E is initial in $\mathcal{C}$. Thus e is an isomorphism, which implies that $\langle 1_0, \mathsf{i} \rangle$ is in $\mathcal{M}$.

Suppose $r : R \to 0 \times A$ is a relation. Then R must be an initial object of $\mathcal{C}$. So $\mathrm{im}\langle 1_0, \mathsf{i} \rangle$ is the only 1-cell in $\mathbf{Rel}(0, A)$. Furthermore, note that the unique 1-cell $0 \nrightarrow A$ is given by a graph. Hence, $\mathbf{Rel}(0, A)$ has only the identity 2-cell. Therefore, 0 is initial in $\mathbf{Rel}(\mathcal{C})$. □

Because $\mathbf{Rel}(\mathcal{C})$ is self dual, 0 is also a terminal object of $\mathbf{Rel}(\mathcal{C})$, whence it is a zero object.

8.4 Products and Coproducts in $\mathbf{Rel}(\mathcal{C})$

Again let us begin by formulating what can be said without assuming extensivity of $\mathcal{C}$. If $\mathcal{C}$ is not extensive a weaker construction than that of a (co)product is still possible. We will call this a pre-product.

Definition 8.11. *In a bicategory a* pre-product *of two* 0*-cells* A *and* B *is given by an object* P *together with two* 1*-cells* $\pi_0 : P \nrightarrow A$ *and* $\pi_1 : P \nrightarrow B$ *satisfying the following property: for any* 1*-cells* $r : C \nrightarrow A$ *and* $s : C \nrightarrow B$ *there is a* 1*-cell* $p : C \nrightarrow P$ *such that there exist* 2*-cells* $\alpha : r \to \pi_0 \circ p$ *and* $\beta : s \to \pi_1 \circ p$*, and* p *is the "least" such* 1*-cell, in the sense that for any* 1*-cell* $h : C \nrightarrow P$ *and* 2*-cells* $\gamma : r \to \pi_0 \circ h$ *and* $\delta : s \to \pi_1 \circ h$ *there is a* 2*-cell* $\eta : p \to h$ *with* $(\pi_0 \circ \eta)\alpha = \gamma$ *and* $(\pi_1 \circ \eta)\beta = \delta$*.*

Note that in a bicategory of relations self duality implies that the dual notion (i. e., the notion with all the 1-cells reversed) coincides with this. This shall be called a *pre-coproduct*.

Proposition 8.12. *Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$-structured category with binary coproducts, and let* A *and* B *be objects of* $\mathcal{C}$*. Then*

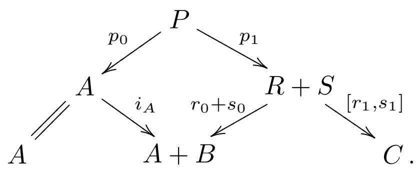
$$\begin{array}{ccccccc} & A & & & & B & \\ /\!/ & & \searrow^{i_A} & & {}^{i_B}\swarrow & & \backslash\!\backslash \\ A & & \xrightarrow[i_0]{\quad\nmid\quad} & A+B & \xleftarrow[i_1]{\quad\nmid\quad} & & B\,, \end{array}$$

where $A + B$ *denotes the coproduct of* A *and* B *in* $\mathcal{C}$ *with inclusions* i_A *and* i_B*, forms a pre-coproduct of* A *and* B *in the 2-category* $\mathbf{Rel}(\mathcal{C})$ *with the injections given by* $\mathrm{im}\langle 1_A, i_A \rangle$ *and* $\mathrm{im}\langle 1_B, i_B \rangle$*.*

Proof. Let $r = \langle r_0, r_1 \rangle : A \nrightarrow C$ and $s = \langle s_0, s_1 \rangle : B \nrightarrow C$ be relations. We must construct a relation $[r, s] : A + B \nrightarrow C$ such that there exist 2-cells $r \to [r, s] \circ i_0$ and $s \to [r, s] \circ i_1$. The relation $[r, s]$ is the image of the span

$$\begin{array}{ccc} & R+S & \\ {}^{r_0+s_0}\swarrow & & \searrow^{[r_1,s_1]} \\ A+B & & C\,. \end{array} \tag{37}$$

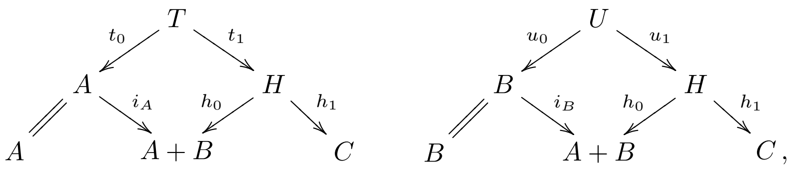
Consider the composite $[r, s] \circ i_0$ given by a pullback



(38)

Since $i_A r_0 = (r_0 + s_0) i_R$ we obtain a unique arrow $\alpha_0 : R \to P$ with $\langle p_0, p_1 \rangle \alpha_0 = r$, and therefore a 2-cell $\alpha : r \to [r, s] \circ i_0$. A 2-cell $\beta : s \to [r, s] \circ i_1$ can be obtained similarly.

Now suppose there is a relation $h = \langle h_0, h_1 \rangle : A + B \,+\!\!\!\to C$, with 2-cells $r \to h \circ i_0$ and $s \to h \circ i_1$. That means that, if we consider the composites $h \circ i_0$ and $h \circ i_1$ as shown in the diagrams

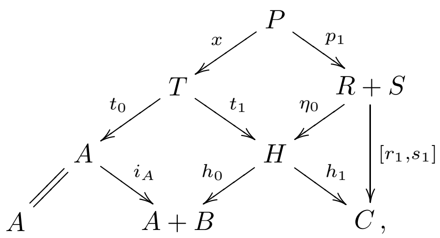


there are 2-cells $\gamma : r \to \langle t_0, h_1 t_1 \rangle$ and $\delta : s \to \langle u_0, h_1 u_1 \rangle$. Note that the codomains of γ and δ are automatically in $\mathcal{M}$ (see (8) on page 12).

It is now straightforward to check that $\eta_0 := [t_1, u_1](\gamma + \delta)$ yields an arrow $\langle r_0 + s_0, [r_1, s_1] \rangle \to \langle h_0, h_1 \rangle$ in $\mathcal{C}/(A+B) \times C$, whence induces the desired 2-cell $\eta : [r, s] \to h$. We have to check that

$$(i_0 \circ \eta)\alpha = \gamma \quad \text{and} \quad (i_1 \circ \eta)\beta = \delta. \tag{39}$$

The 2-cell $i_0 \circ \eta$ is constructed using the diagram



where the squares are pullbacks. Note that since $h_0 \eta_0 = r_0 + s_0$, we can choose the pullback of η_0 along t_1 so that $t_0 x = p_0$. The 2-cell $i_0 \circ \eta$ is now induced by the arrow x. To show (39) it is sufficient to show that $x \alpha_0 = \gamma$, where α_0 is the arrow obtained from diagram (38). But this is quickly done extending by the monic $\langle t_0, t_1 \rangle$:

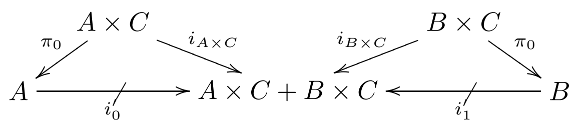
$$\langle t_0, t_1 \rangle x \alpha_0 = \langle p_0, \eta_0 p_1 \rangle \alpha_0 = \langle r_0, [t_1, u_1](\gamma + \delta) i_R \rangle = \langle r_0, t_1 \gamma \rangle = \langle t_0, t_1 \rangle \gamma.$$

□

For given relations $r : A \,+\!\!\!\to C$ and $s : B \,+\!\!\!\to C$ let $t = \langle \pi_{A+B}, \pi_C \rangle : A + B \,+\!\!\!\to C$ be the terminal object of $\mathbf{Rel}(A + B, C)$. It is easy to verify that there exist

2-cells $r \to t \circ i_0$ and $s \to t \circ i_1$. So if we consider the full subcategory $\mathcal{S}$ of $\mathbf{Rel}(A + B, C)$ formed by the relations u for which 2-cells $r \to u \circ i_0$ and $s \to u \circ i_1$ exist, Proposition 8.12 and the last remark show that $\mathcal{S}$ has terminal and weakly initial objects[2], given by $\langle \pi_{A+B}, \pi_C \rangle$ and $[r, s]$ respectively.

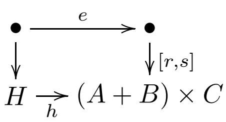
Note that if we fix the object C, the object $(A \times C) + (B \times C)$ together with the relations given by



will have the same property as $A + B$ together with i_0, i_1. The relation $[r, s]$ is now given by $\mathrm{im}\langle r + s, [r_1, s_1] \rangle$. The proof then completely follows the previous one and is therefore left to the Reader.

Remark 8.13. Observe that, provided $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$, all 2-cells in the construction will be uniquely determined. Moreover the 1-cell $[r, s]$ will be unique, for suppose there is a 1-cell h with the same property. Then $[r, s] \leq h$ and $h \leq [r, s]$, whence $[r, s] \simeq h$.

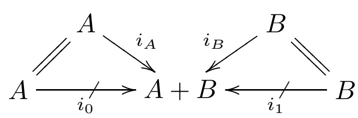
However, in the general case one cannot expect that η is uniquely determined. In order to show that η is equal to another 2-cell ϵ with the properties of Definition 8.11, we would have to use the universal property of the factorization system, i. e., placing both η and ϵ on the diagonal of a commutative square like



with $e \in \mathcal{E}$. But unfortunately our construction lacks the necessary $\mathcal{E}$-arrow with codomain $R + S$ to produce such a square.

If we now add extensivity to the category $\mathcal{C}$, the same construction as before yields a (co)product, and even a 2-(co)product in $\mathbf{Rel}(\mathcal{C})$ as we shall see later, provided that $\mathcal{M}$ is closed under binary coproducts in $\mathcal{C}$.

Proposition 8.14. *Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$-structured extensive category and let A and B be objects of $\mathcal{C}$. Then*

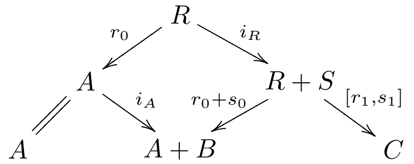


is a coproduct diagram in the ordinary category $\mathbf{Rel}(\mathcal{C})$.

Proof. Let $r = \langle r_0, r_1 \rangle : A \mathrel{+\!\!\to} C$ and $s = \langle s_0, s_1 \rangle : B \mathrel{+\!\!\to} C$ relations. We must construct a unique relation $[r, s] : A + B \mathrel{+\!\!\to} C$ such that $[r, s] \circ i_0 \simeq r$ and $[r, s] \circ i_1 \simeq s$.

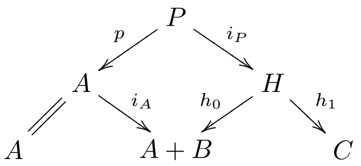
[2] An object I of a category is *weakly initial* if for any other object A of that category there is an arrow $I \to A$.

The relation $[r,s]$ is given by $\operatorname{im}\langle r_0 + s_0, [r_1, s_1]\rangle$ again (see diagram (37)). Observe that $[r,s] \circ i_0 \simeq r$ because by extensivity of $\mathcal{C}$, the square in the middle of the diagram

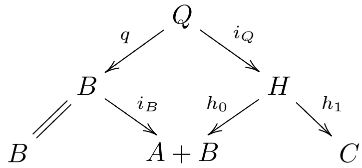


is a pullback. That $[r,s] \circ i_1 \simeq s$ holds true can be shown analogously.

Finally let us show that $[r,s]$ is unique. To prove it let $h = \langle h_0, h_1\rangle : H \to (A+B) \times C$ be a relation for which $h \circ i_0 \simeq r$ and $h \circ i_1 \simeq s$ hold. That means that the composites

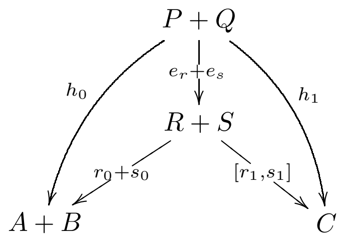


and



represent the same relation as r and s respectively. Therefore there are $\mathcal{E}$-arrows $e_r : P \to R$ and $e_s : Q \to S$ such that $\langle r_0, r_1\rangle e_r$ and $\langle s_0, s_1\rangle e_s$ are $(\mathcal{E}, \mathcal{M})$-factorizations of $\langle p, h_1 i_P\rangle$ and $\langle q, h_1 i_Q\rangle$ respectively.

By extensivity of $\mathcal{C}$ we have $H = P + Q$ and $h_0 = p + q$. Observe now that the following diagram commutes:



This proves $h \simeq [r,s]$ since, by Corollary 2.9, $e_r + e_s \in \mathcal{E}$. □

Finally, note again that, since $\mathbf{Rel}(\mathcal{C})$ is self dual, we also have characterized products in $\mathbf{Rel}(\mathcal{C})$. Note that the existence of general (co)products in $\mathbf{Rel}(\mathcal{C})$ follows from the existence of universal and disjoint general coproducts in $\mathcal{C}$ similarly.

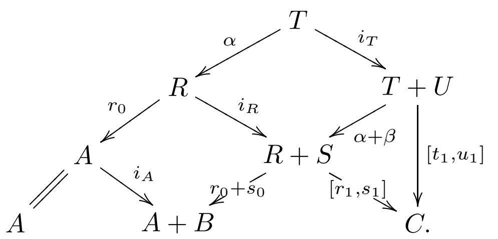
8.5 2-products in $\mathbf{Rel}(\mathcal{C})$

We will now show that the (co)product constructed in the previous section is even a 2-co(op)product in the sense of Definition 8.6. Note that "co" here means that the direction of the 2-cells in the definition is reversed and "op" means that the direction of the 1-cells is reversed. Before we prove the additional property let us unravel Definition 8.6 for the case of a binary product in $\mathbf{Rel}(\mathcal{C})$. A 2-co(op)product of two objects A and B in $\mathbf{Rel}(\mathcal{C})$ is a coproduct in the usual sense, i. e. an object $A + B$ together with injections $i_0 : A \nrightarrow A + B$ and $i_1 : B \nrightarrow A + B$ such that for any pair $r : A \nrightarrow C$, $s : B \nrightarrow C$ of relations there exists a unique relation $[r,s] : A + B \mathrel{+\!\!\to} C$ with $[r,s] \circ i_0 \simeq r$ and $[r,s] \circ i_1 \simeq s$. Moreover, the following additional property must be satisfied. Given a second pair of relations $t : A \mathrel{+\!\!\to} C$ and $u : B \mathrel{+\!\!\to} C$, and 2-cells $\alpha : t \to r$ and $\beta : u \to s$, then there exists a unique 2-cell $\gamma : [t,u] \to [r,s]$ with $\gamma \circ i_0 = \alpha$ and $\gamma \circ i_1 = \beta$.

Proposition 8.15. *If $\mathcal{C}$ is an $(\mathcal{E},\mathcal{M})$-structured extensive category and $\mathcal{M}$ is closed under binary coproducts, then 2-co(op)products exist in the 2-category* $\mathbf{Rel}(\mathcal{C})$ *and are given as in Proposition 8.14.*

Proof. The proof of Proposition 8.14 shows that we have a universal cocone. Hence, we must only show the additional property. First note that the hypothesis that $\mathcal{M}$ is closed under coproducts implies that $\langle r_0+s_0, [r_1,s_1]\rangle = \mathrm{dist}(r+s)$, where dist is the canonical arrow of Definition 8.4, is in $\mathcal{M}$ since $\mathcal{C}$ is extensive with binary products, whence distributive.

Now, given r, s, t, u, α and β as above, consider the diagram



(40)

By extensivity, the upper right square is a pullback. So $\alpha+\beta$ is a 2-cell with $(\alpha+\beta)\circ i_0 = \alpha$. Similarly $(\alpha+\beta)\circ i_1 = \beta$. We have to show that $\alpha+\beta$ is unique. But if we put any 2-cell $\gamma : [t,u] \to [r,s]$ in the place of $\alpha+\beta$ in diagram (40), extensivity of $\mathcal{C}$ implies that $\gamma = \gamma_0 + \gamma_1$ and furthermore, that $\gamma_0 = \gamma \circ i_0 = \alpha$ and $\gamma_1 = \gamma \circ i_1 = \beta$. ☐

8.6 Rel($\mathcal{C}$) is not finitely complete

We have seen that under certain conditions on $\mathcal{C}$, $\mathbf{Rel}(\mathcal{C})$ has finite (co)products. One might ask about (co)equalizers, i. e., whether or not $\mathbf{Rel}(\mathcal{C})$ is finitely (co)complete. However, this is not even true in $\mathbf{Rel}(\mathbf{Set})$ as the following example that has been pointed out by Koslowski (cf. [17]) shows.

Example 8.16. On the set of rational numbers consider the relations $< : \mathbb{Q} \mathrel{+\!\!\to} \mathbb{Q}$ and $\delta_{\mathbb{Q}}$. Assume that the equalizer $e : E \mathrel{+\!\!\to} \mathbb{Q}$ of $<$ and $\delta_{\mathbb{Q}}$ exists. Note that $<$ is idempotent, whence $< \circ < = \delta_{\mathbb{Q}} \circ <$, which implies that there exists a unique relation $h : \mathbb{Q} \mathrel{+\!\!\to} E$ with $< = e \circ h$. But now

$$e \circ h \circ e = < \circ\, e = \delta_{\mathbb{Q}} \circ e = e \circ \delta_E,$$

and therefore $h \circ e = \delta_E$ since e is monomorphic. That implies that for all $x \in E$ there exists a $q \in \mathbb{Q}$ with $(x,q) \in e$ and $(q,x) \in h$, whence $(q,q) \in <$. But $q < q$ is a contradiction.

8.7 Application to maps

One can easily see that the graph functor preserves coproducts. The injections in Proposition 8.12 are graphs. Moreover, the relation $[\Gamma f, \Gamma g] : A + B \looparrowright C$ induced by two graphs Γf and Γg is given by

$$[\Gamma f, \Gamma g] \simeq \mathrm{im}\langle 1 + 1, [f, g]\rangle,$$

and therefore a graph. Note that no extensivity is needed to make $A + B$ a coproduct in $\Gamma\mathcal{C}$ since graphs are equal as soon as they are connected by a 2-cell.

The question about coproducts in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ now arises naturally. The answer is that they exist; but the coproduct in $\mathcal{C}$ needs to be extensive once more. Before giving the proof, we note the following fact.

Lemma 8.17. *In any extensive category pullbacks commute with coproducts.*

Proof. Suppose $f_i p_i = g_i p_i$ for $i = 0, 1$ are two pullback squares. We shall show that

$$\begin{array}{ccc} P_0 + P_1 & \xrightarrow{q_0+q_1} & B_0 + B_1 \\ {\scriptstyle p_0+p_1}\downarrow & & \downarrow{\scriptstyle g_0+g_1} \\ A_0 + A_1 & \xrightarrow[f_0+f_1]{} & C_0 + C_1 \end{array}$$

is a pullback square, too. The square is clearly commutative. Suppose then that for $x : X \to A_0 + A_1$ and $y : X \to B_0 + B_1$, $(f_0 + f_1)x = (g_0 + g_1)y$ holds. By extensivity, we must have that $X = X_0 + X_1$, $x = x_0 + x_1$ and $y = y_0 + y_1$ with $x_i : X_i \to A_i$ and $y_i : X_i \to B_i$, for $i = 0, 1$. Since the coproduct in an extensive category is disjoint, i. e., the injections are monic, one readily checks that $f_i x_i = g_i y_i$ for $i = 0, 1$, when pulling back along the injections. The unique induced arrow $h_i : X_i \to P_i$ yields the desired unique arrow $h = h_0 + h_1$ with $(p_0 + p_1)h = x$ and $(q_0 + q_1)h = y$. ☐

Theorem 8.18. *Let $\mathcal{C}$ be an $(\mathcal{E}, \mathcal{M})$-structured extensive category. Then coproducts exist in* $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ *and are given as in Proposition 8.14.*

Proof. We need only check whether for given maps $r : A \looparrowright C$ and $s : B \looparrowright C$ the induced relation $[r, s] : A + B \looparrowright C$ is a map, too. To see that $[r, s]$ is single-valued, consider the composite $[r, s] \circ [r, s]^o$ obtained by taking $\ker(r_0 + s_0)$ and then $(\mathcal{E}, \mathcal{M})$-factorizing the appropriate span. By Lemma 8.17,

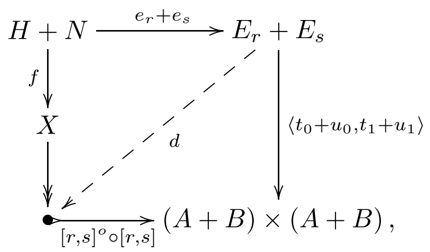
$$\ker(r_0 + s_0) = \langle k_0 + m_0, k_1 + m_1\rangle,$$

where $\langle k_0, k_1\rangle = \ker(r_0)$ and $\langle m_0, m_1\rangle = \ker(s_0)$. Using the single-valuedness of r and s we obtain an arrow $\varepsilon_r : \langle r_1 k_0, r_1 k_1\rangle \to \iota_C$ and $\varepsilon_s : \langle s_1 m_0, s_1 m_1\rangle \to \iota_C$. Thus $[\varepsilon_r, \varepsilon_s]$ induces a 2-cell $[r, s] \circ [r, s]^o \to \iota_C$.

Now consider the composite $[r, s]^o \circ [r, s]$ formed by taking

$$\langle x_0, x_1\rangle = \ker([r_1, s_1]),$$

and then $(\mathcal{E}, \mathcal{M})$-factorizing the resulting span. If $\langle h_0, h_1\rangle = \ker(r_1)$ and $\langle n_0, n_1\rangle = \ker(s_1)$, then clearly $[r_1, s_1](h_0 + n_0) = [r_1, s_1](h_1 + n_1)$. Hence, there exists an arrow $f : \langle h_0 + n_0, h_1 + n_1\rangle \to \langle x_0, x_1\rangle$. Let $r^o \circ r = \langle t_0, t_1\rangle$ and $s^o \circ s = \langle u_0, u_1\rangle$. Then we get a commutative square



where e_r, e_s are some arrows in $\mathcal{E}$. By the universal property of the factorization system there is a diagonal arrow d. Now by totality of r and s, there are arrows $\eta_r : \delta_A \to \langle t_0, t_1\rangle$ and $\eta_s : \delta_A \to \langle u_0, u_1\rangle$. It is easy to see that $d(\eta_r + \eta_s) : \delta_A \to [r, s]^o \circ [r, s]$ induces the 2-cell that shows $[r, s]$ to be total.

To complete the proof we shall show that $[r, s]$ is monic in $\mathcal{C}$. By Lemma 4.5, it is sufficient to show that the span

$$m := \langle r_0 + s_0, [r_1, s_1]\rangle$$

is monic in $\mathcal{C}$. Suppose that $mf = mg$ for a parallel pair $f, g : X \to R + S$ of arrows. By extensivity, $X = X_0 + X_1$, $f = f_0 + f_1$ and $g = g_0 + g_1$ with obvious domains and codomains. Using the same trick as in Lemma 8.17, i. e., pulling back along injections and using that injections are monic, we conclude that $rf_0 = rg_0$ and $sf_1 = sg_1$. Thus $f_i = g_i$ for $i = 0, 1$, and therefore $f = g$, since the maps r and s are monic in $\mathcal{C}$. □

We have seen that **Rel**($\mathcal{C}$) cannot be finitely (co)complete. For the category **Map**(**Rel**($\mathcal{C}$)) the situation is different. We already know that this category is finitely complete. Moreover, if $\mathcal{C}$ is extensive it has coproducts as just shown. The question whether these coproducts are extensive as in $\mathcal{C}$ arises immediately. Furthermore, it should be interesting to see what conditions must be imposed on $\mathcal{C}$ to force the existence of coequalizers or pushouts in **Map**(**Rel**($\mathcal{C}$)).

9 Colimits of ω-chains in $\mathbf{Rel}(\mathcal{C})$

Let $\mathcal{A}$ be an arbitrary category. For a given endofunctor $F : \mathcal{A} \to \mathcal{A}$, a pair (A, a), where $a : FA \to A$ is an arrow of $\mathcal{A}$, is called an F-*algebra*. In theoretical Computer Science initial F-algebras are used as models for recursively defined data types (cf. [19]). Recently, Bird and de Moor (cf. [2]) have used initial algebras in allegories for the derivation of programs for formally specified optimization problems. However, they assume the existence of so-called power objects to get initial algebras. At least for regular categories $\mathcal{C}$, $\mathbf{Rel}(\mathcal{C})$ has power objects if and only if $\mathcal{C}$ is a topos (cf. [8]). But this seems to be a quite restrictive assumption. In general, initial F-algebras can be constructed as follows. If $\mathcal{A}$ has initial objects and colimits of ω-chains, and if F is cocontinuous, that means F preserves these colimits, an initial algebra can be obtained by an iterative construction, namely by taking the colimit of the ω-chain

$$0 \longrightarrow F0 \longrightarrow \cdots \longrightarrow F^n0 \longrightarrow \cdots .$$

One can ask whether such a construction is possible if $\mathcal{A}$ is replaced by $\mathbf{Rel}(\mathcal{C})$ for a suitable category $\mathcal{C}$. We have seen that initial objects exist in $\mathbf{Rel}(\mathcal{C})$ if they exist in $\mathcal{C}$ and are strict there. The question of finding initial algebras therefore essentially boils down to finding colimits of ω-chains.

The existence of these colimits in $\mathbf{Rel}(\mathcal{C})$ has not been studied before, and unfortunately it turns out that even in a very reasonable category such as $\mathbf{Rel}(\mathbf{Set})$ these do not exist in general. However, the construction which mimics the appropriate construction in $\mathbf{Set}$ fails, at least for total relations, only by one little part. It seems natural to ask whether this canonical construction can be generalized to all $(\mathcal{E}, \mathcal{M})$-structured $\mathcal{C}$, or what the conditions to make it possible might be. But before we turn to this question let us present a counterexample for $\mathbf{Rel}(\mathbf{Set})$.

9.1 A counterexample

We shall show that even in $\mathbf{Rel}(\mathbf{Set})$, the category of binary relations between sets, colimits of ω-chains do not exist in general.

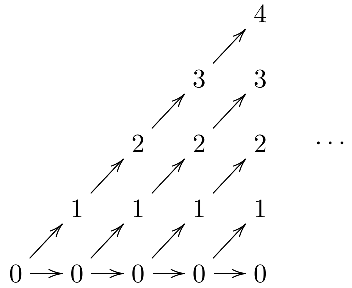
In order to see this, consider the family $\{A_i \mid i \in \mathbb{N} \setminus \{0\}\}$ of sets, where

$$A_i = \{0, \ldots, i-1\},$$

together with the relations $a_{i,i+1} : A_i \nrightarrow A_{i+1}$, defined by

$$a_{i,i+1} = \{(j, j+1) \mid 0 \leq j \leq i-1\} \cup \{(0,0)\}.$$

These sets and relations form an ω-chain in $\mathbf{Rel}(\mathbf{Set})$ as shown by the following figure:



Observe that the pair $(D, \{d_i\}_{i\in\mathbb{N}})$, where $D = \{0,1\}$, and

$$d_i = \{(0,0)\} \cup \{(j,1) \mid 0 \leq j \leq i-1\},$$

forms a cocone on that chain.

Now assume that the colimit $(C, \{c_i\}_{i\in\mathbb{N}})$ of the ω-chain exists. Obviously, C is not the empty set since otherwise $c_i = \emptyset$ for all i. Moreover, the unique relation $C \nrightarrow D$ would have to be empty, too, since $C = \emptyset$ is an initial object in **Rel**(**Set**). Thus, $d_i = \emptyset \circ \emptyset = \emptyset$, which is not true. Furthermore, observe that the relations c_i have to be total since the relations d_i are total.

Next we will consider chains of elements in our given ω-chain. By a *chain* (of elements) we mean a sequence $\{x_i \mid i \in \mathbb{N}\}$ such that $x_i \in A_i$ and $(x_i, x_{i+1}) \in a_{i,i+1}$ for all $i \geq 1$. An example of this is $(0,0,0,\ldots)$, which will be called the *0-chain*.

Note that there must be a point x in $\mathcal{C}$ such that all elements in the 0-chain are in relation with x but none of the other elements of the A_i are. In order to see this, observe that if any element of the 0-chain is in relation with an $x \in C$, then by commutativity of the cone $(C, \{c_i\})$ all the other elements of that chain also are. Moreover, since $(C, \{c_i\})$ is a colimit of the given ω-chain there is a unique relation $h : C \nrightarrow D$ such that $d_i = hc_i$ for all i. Furthermore, $(0,0) \in d_i$ for all i, which implies that there is an $x \in C$ such that $(0,x) \in c_i$ and $(x,0) \in h$ for all i. Now suppose that there is a set A_i and an element $n \in A_i$ with $0 < n \leq i-1$ such that $(n,x) \in c_i$. Observe that $(n,0) \notin d_i$. But we have $(n,x) \in c_i$ and $(x,0) \in h$, which implies $(n,0) \in h \circ c_i = d_i$, a contradiction.

Thus, taking into account the totality of the relations c_i, we proved that C has at least two elements, one of which is the element x such that $(0,x) \in c_i$ but $(n,x) \notin c_i$ for all i and $n \in A_i$ for $0 < n \leq i-1$.

Consider the following two relations with domain C and codomain D:

$$\begin{aligned} h_1 &= \{(x,0)\} \cup \{(y,1) \mid y \in C, y \neq x\} \\ h_2 &= h_1 \cup \{(x,1)\}. \end{aligned}$$

It can easily be checked that $d_i = h_1 c_i$ and $d_i = h_2 c_i$ hold for any i. We shall show this for h_1. So suppose that $(j,0) \in d_i$. This is the case precisely if $j = 0$. Hence, $(j,x) \in c_i$ and $(x,0) \in h_1$, which means $(j,0) \in h_1 \circ c_i$. Conversely, if $(j,0) \in h_1 \circ c_i$, then we must have $(j,x) \in c_i$ and $(x,0) \in h_1$, which implies $j = 0$ since no other element of A_i is in relation c_i with $x \in C$.

Finally suppose that $(j,1) \in d_i$. Since c_i is total there exists a $y \in C$ such that $(j,y) \in c_i$. If $j \neq 0$, then $y \neq x$, and $(y,1) \in h_1$, which shows that $(j,1) \in h_1 \circ c_i$. If $j = 0$, then we have $(j,1) \in a_{i,i+1}$, whence there is an element $y \in C$ with $x \neq y$ such that $(1,y) \in c_{i+1}$. So $(0,y) \in c_{i+1} \circ a_{i,i+1} = c_i$. Since $y \neq x$, we have $(y,1) \in h_1$, which implies that $(j,1) \in h_1 \circ c_i$. On the other hand, $(j,1) \in h_1 \circ c_i$ trivially implies that $(j,1) \in d_i$ just because the latter holds for all j.

Showing that $d_i = h_2 \circ c_i$ for all i is a very similar computation, which is left to the Reader. Since h_1 and h_2 are clearly non-equal, C cannot be a colimit of the given chain. However, it turns out that, at least for total relations there is always a largest factorization $h : C \nrightarrow D$.

Note that this example also shows that bicolimits of ω-chains cannot exits. Indeed, every equivalence of categories

$$\mathbf{Rel}(\mathbf{Set})(C,D) \to \mathrm{Cocone}(\mathcal{A}, D),$$

where $\mathcal{A} : \mathbb{N} \to \mathbf{Rel}(\mathbf{Set})$ is the obvious diagram, must be an isomorphism because both categories are posets.

9.2 Lax adjoint limits of ω-chains

In the preceding section we saw that colimits of ω-chains in $\mathbf{Rel}(\mathcal{C})$ do not exist in general. However, a good portion of the canonical construction in **Set** carries over to $\mathbf{Rel}(\mathbf{Set})$. Moreover, this can be generalized to the case of an $(\mathcal{E}, \mathcal{M})$-structured category $\mathcal{C}$ with certain additional conditions imposed on it. The "almost" universal cocone for an ω-chain in $\mathbf{Rel}(\mathcal{C})$ satisfies the conditions of the next definition. To have a handy name we will call this notion a *lax adjoint limit*.

Definition 9.1. *Let $D : \mathcal{D} \to \mathcal{C}$ be a 2-functor between 2-categories. A pair (L, ℓ) where L is an object of $\mathcal{C}$ and $\ell : \Delta L \to D$ is a lax natural transformation is called a* lax adjoint limit *of D if for any other such pair (M, m) where m is a lax natural transformation there is a 1-cell $h : M \to L$ and a modification η such that*

$$\begin{array}{ccc} \Delta M & \xrightarrow{m} & D\,. \\ {\scriptstyle \Delta h}\downarrow & \Downarrow \eta \quad \nearrow_{\ell} & \\ \Delta L & & \end{array}$$

Moreover, given any other 1-cell $k : M \to L$ and modification $\mu : m \to \ell \cdot \Delta k$ there is a unique 2-cell $\alpha : h \to k$ such that

$$\begin{array}{ccc} m & \xrightarrow{\eta} & \ell \cdot \Delta h \\ & {\scriptstyle \mu}\searrow & \downarrow {\scriptstyle \ell \cdot \Delta\alpha} \\ & & \ell \cdot \Delta k \end{array}$$

is commutative.

Observe that this can be stated more compactly by saying that the functor

$$\ell \circ (-) : \mathcal{C}(M, L) \to \text{Lax-Cone}(M, D)$$

of composition with ℓ is a right adjoint. Note that, in fact, we are interested in the dual notion of this, that means the notion with all cells reversed. Hence, we discuss lax adjoint cooplimits, which means that the functor F of composition with the canonical cocone is a left adjoint. It turns out that in $\mathbf{Rel}(\mathcal{C})$ the canonical cocone is even strict. Moreover, under a certain assumption on $\mathcal{C}$, the counit of the adjunction will be an isomorphism if we restrict the codomain of F to strict cocones; in other words, every cocone factors through the canonical cocone. Despite the fact that this additional assumption is true in **Set** for total relations, it is a somewhat strange one. But this will be discussed later.

Let us now analyze what a lax adjoint cooplimit of an ω-chain in the 2-category $\mathbf{Rel}(\mathcal{C})$ really is. Throughout the rest of this section we shall assume that $\mathcal{C}$ is a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback and $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$.

Suppose that an ω-chain, that means a functor $\mathcal{A} : \mathbb{N} \to \mathbf{Rel}(\mathcal{C})$, is given. Then a cocone $(A, (a_i)_{i \in \mathbb{N}})$ will be constructed such that for any lax cocone

$(B, (b_i)_{i \in \mathbb{N}})$, i. e. $b_i \geq b_j \circ A_{i,j}$ for all $j \geq i$, there exists a relation $h : A \nrightarrow B$ with $h \circ a_i \leq b_i$ for all i. Moreover, if $k : A \nrightarrow B$ is another factorization with $k \circ a_i \leq b_i$, then $k \leq h$. The uniqueness of the last 2-cell and the fact that the given families of 2-cells form modifications are true automatically since **Rel**($\mathcal{C}$) is a partial-order-enriched category. However, note that it is not for the sake of this convenience that we assume $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$. The construction seems to fail much earlier without this assumption. But here are the sufficient conditions.

Theorem 9.2. *Let* $\mathcal{C}$ *be a category as above. Then* **Rel**($\mathcal{C}$) *has lax adjoint cooplimits of* ω*-chains whenever the following conditions are satisfied:*

(i) $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$*,*

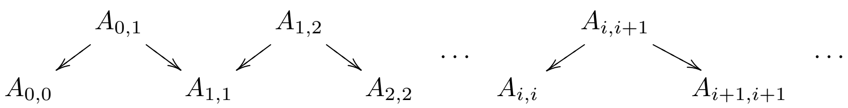
(ii) $\mathcal{C}$ *has universal colimits of* ω*-chains,*

(iii) pullbacks commute with colimits of ω*-chains in* $\mathcal{C}$*.*

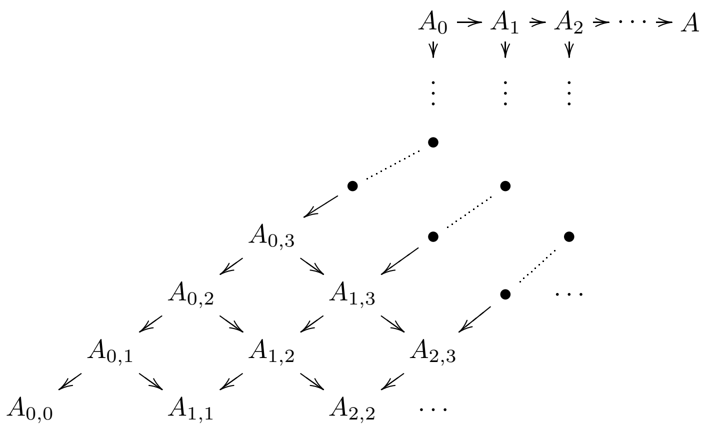
Proof. The construction is given in the next subsections. □

9.2.1 Construction of the canonical cocone

Suppose that an ω-chain in **Rel**($\mathcal{C}$), i. e., a zig-zag



in $\mathcal{C}$, has been given. The following diagram outlines the construction of the canonical cone for that ω-chain:

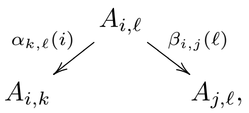


(41)

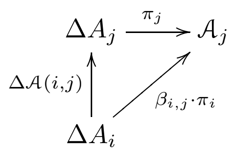
Start by forming the pullbacks $A_{i,i+2}$ of $A_{i,i+1}$ and $A_{i+1,i+2}$ for all $i \in \mathbb{N}$, and then iterate this process to obtain objects $A_{i,j}$ for all $i, j \in \mathbb{N}$. Next take the limits (A_i, π_i) of the cochains

$$A_{i,i} \longleftarrow A_{i,i+1} \longleftarrow \cdots \longleftarrow A_{i,j} \longleftarrow \cdots \qquad (42)$$

with projections given by natural transformations $\pi_i : \Delta A_i \to \mathcal{A}_i$, where the $\mathcal{A}_i$ are the obvious diagram functors. Furthermore, denote by $\mathcal{A}_{i,j}$ the diagram given by the tail of the cochain (42) starting in $A_{i,j}$. Note that A_i is the limit of $\mathcal{A}_{i,j}$, because the inclusion of a tail is a final functor. By abuse of notation we will denote the projections in this case with $\pi_i : \Delta A_i \to \mathcal{A}_{i,j}$ for all j, too, since their components are given by the components $\pi_i(n)$ of π_i. Observe that we use a non-standard notation for components of a natural transformation to avoid double indexation. Now note that the arrows pointing down-right in (41) give natural transformations $\beta_{i,j} : \mathcal{A}_{i,j} \to \mathcal{A}_j$ for all $j \geq i \in \mathbb{N}$. We denote the arrows pointing down-left in (41) by $\alpha_{i,j}(k) : A_{k,j} \to A_{k,i}$. Hence, we have



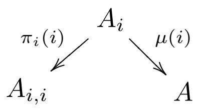
where ℓ is not necessarily equal to k. Clearly $(A_i, \beta_{i,j} \cdot \pi_i)$ is a cone on $\mathcal{A}_j$ so that there exist unique arrows $\mathcal{A}(i,j) : A_i \to A_j$ with



Thus we obtain an ω-chain

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_i \longrightarrow \cdots$$

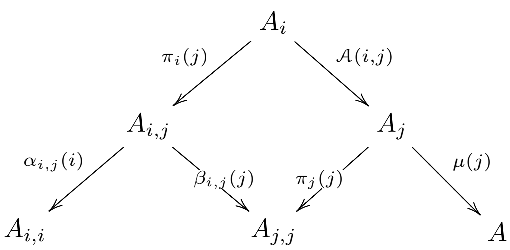
whose colimit (A, μ) is formed in $\mathcal{C}$, where $\mu : \mathcal{A} \to \Delta A$ for the obvious diagram functor $\mathcal{A}$. The claim is now that the images of the spans given by



for $i \in \mathbb{N}$ form the desired cocone for the given chain in $\mathbf{Rel}(\mathcal{C})$.

9.2.2 Commutativity

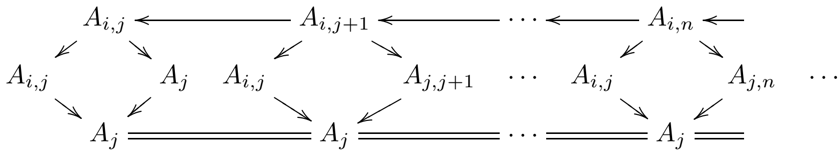
To show that we really have constructed a cocone we must show that the square in the diagram



(43)

is a pullback for all $j \geq i$. Considering (41), the square clearly commutes. To show the universal property we use the fact that limits commute with limits.

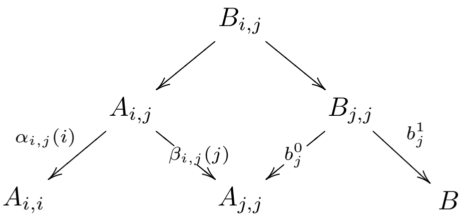
Consider the following four cochains all of whose objects at index n form a pullback:



Note that the objects on the left side and on the bottom do not change. If the limits of the four cochains are formed first, we obtain the square in diagram (43), which shows that it is a pullback square.

9.2.3 The universal property

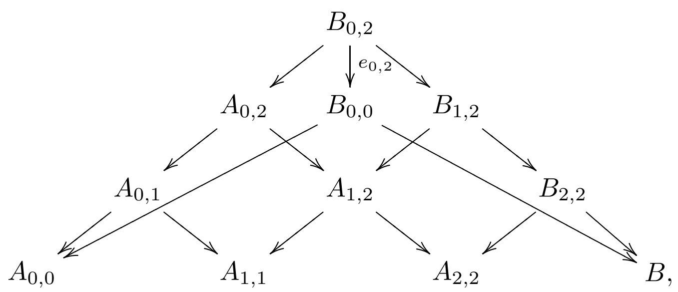
Denote by $a_i : A_{i,i} \nrightarrow A$ the components of the cocone constructed in the last subsection. We shall show that (A, a) satisfies the universal property of a lax adjoint cooplimit. So suppose that (B, b) is a lax cocone, i. e. $b_i \geq b_j \circ A_{i,j}$ for all $i \geq j \in \mathbb{N}$, where $b_i : B_{i,i} \to A_{i,i} \times A$, and for a moment $A_{i,j}$ denotes the relation $A_{i,i} \nrightarrow A_{j,j}$. That means, if we form the pullbacks



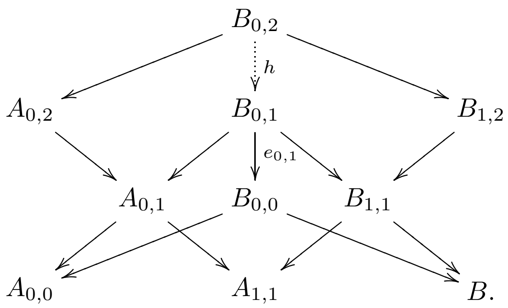
(44)

where $b_j = \langle b^0_j, b^1_j \rangle$, then there is a 2-cell from this span to the relation b_i given by an arrow $e_{i,j}$. Note that $b_i \simeq b_j \circ A_{i,j}$ if and only if $e_{i,j}$ lies in $\mathcal{E}$.

The next step is to construct connecting arrows between the $B_{i,j}$'s to get cochains similar to the ones given by the $A_{i,j}$'s. These arrows arise by using the universal property of the pullbacks. We shall explicitly construct the arrow $B_{02} \to B_{01}$. Consider the composite $b_2 \circ A_{02}$ formed as in diagram



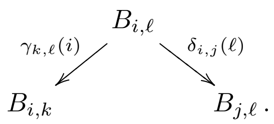
where all squares are pullbacks. Now use the universal property of the pullback in the following diagram:



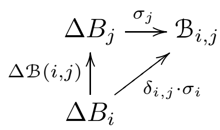
An arrow $h : B_{0,2} \to B_{0,1}$ is induced. One readily checks that $e_{0,1}h = e_{0,2}$ using that b_0 as an $\mathcal{M}$-relation is a monomorphic arrow of $\mathcal{C}$. Iterating this construction, we obtain cochains

$$B_{i,i} \longleftarrow B_{i,i+1} \longleftarrow \cdots \longleftarrow B_{i,j} \longleftarrow \cdots$$

for all $i \in \mathbb{N}$ as before. Form the limits (B_i, σ_i) where $\sigma_i : \Delta B_i \to \mathcal{B}_i$ for the obvious diagrams $\mathcal{B}_i$. Again, the diagrams obtained by taking tails are denoted by $\mathcal{B}_{i,j}$. For all $j \leq i$, their limits are the B_i with projections given by σ_i. Similar to the $\alpha_{i,j}$ and $\beta_{i,j}$ for the $A_{i,j}$ there are natural transformations $\delta_{i,j} : \mathcal{B}_{i,j} \to \mathcal{B}_j$ and arrows $\gamma_{i,j}$ as shown in the diagram



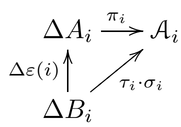
Now clearly $(B_i, \delta_{i,j} \cdot \sigma_i)$ is a cone on $\mathcal{B}_j$ so that there exist unique arrows $\mathcal{B}(i,j) : B_i \to B_j$ with



commutative. Hence, we obtain an ω-chain

$$B_0 \longrightarrow B_1 \longrightarrow \cdots \longrightarrow B_i \longrightarrow \cdots$$

whose diagram functor is called $\mathcal{B}$. Now note that, for a fixed $i \in \mathbb{N}$, the arrows $B_{i,j} \to A_{i,j}$, that are the pullback projections obtained in diagram (44), are the components of a natural transformation $\tau_i : \mathcal{B}_i \to \mathcal{A}_i$. Therefore $(B_i, \tau_i \cdot \sigma_i)$ is a cone on $\mathcal{A}_i$. This implies the existence of unique arrows $\varepsilon(i) : B_i \to A_i$ with



for all $i \in \mathbb{N}$, which clearly yield a natural transformation $\varepsilon : \mathcal{B} \to \mathcal{A}$. Moreover, the following statement holds true.

Lemma 9.3. *The square*

$$\begin{array}{ccc} B_i & \xrightarrow{\mathcal{B}(i,j)} & B_j \\ {\scriptstyle \varepsilon(i)}\downarrow & & \downarrow{\scriptstyle \varepsilon(j)} \\ A_i & \xrightarrow[\mathcal{A}(i,j)]{} & A_j \end{array}$$

is a pullback for all $j \geq i \in \mathbb{N}$.

Proof. Again use the limit interchange rule considering the following cochain of pullback squares

$$\begin{array}{ccccccccc} & B_{i,j} & & \longleftarrow & & B_{i,j+1} & & & \\ \swarrow & & \searrow & & \swarrow & & \searrow & & \\ A_{i,j} & & B_{j,j} & & A_{i,j+1} & & B_{j,j+1} & & \cdots \\ \searrow & & \swarrow & & \searrow & & \swarrow & & \\ & A_{j,j} & & \longleftarrow & & A_{j,j+1} & & & \end{array}$$

which "converges" to the desired square. □

To complete the construction of the canonical factorization $h : A \leftrightarrow B$ form the colimit (H, ν) where $\nu : \mathcal{B} \to \Delta H$. Clearly $(A, \mu \cdot \varepsilon)$ and (B, η), where the components of η are given by $\eta(i) := b_i^2 \cdot \sigma_i(i)$ form cocones on $\mathcal{B}$. Therefore there are unique arrows

$$\begin{array}{ccc} & H & \\ {\scriptstyle h_0}\swarrow & & \searrow{\scriptstyle h_1} \\ A & & B \end{array}$$

with

$$\begin{array}{ccc} \Delta H & \xleftarrow{\nu} & \mathcal{B} \\ {\scriptstyle \Delta h_0}\downarrow & \swarrow{\scriptstyle \mu\cdot\varepsilon} & \\ \Delta A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Delta H & \xleftarrow{\nu} & \mathcal{B}\,. \\ {\scriptstyle \Delta h_1}\downarrow & \swarrow{\scriptstyle \eta} & \\ \Delta B & & \end{array}$$

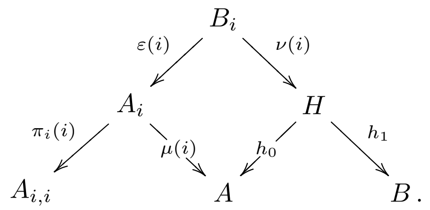
The claim is that $\mathrm{im}\langle h_0, h_1 \rangle$ is the canonical factorization $h : A \leftrightarrow B$. To prove it we must first show that

$$b_i \geq h \circ a_i$$

for all $i \in \mathbb{N}$. Here we use condition (iii) of Theorem 9.2, namely that colimits of ω-chains commute with pullbacks in $\mathcal{C}$. For $i \in \mathbb{N}$ consider the following four ω-chains, which together form pullback squares at each index, by Lemma 9.3

$$\begin{array}{ccc} & B_1 & \\ \swarrow & & \| \\ A_1 & & B_1 \\ \| & & \swarrow \\ & A_1 & \end{array} \quad \cdots \quad \begin{array}{ccc} & B_i & \\ \swarrow & & \| \\ A_i & & B_i \\ \| & & \swarrow \\ & A_i & \end{array} \quad \cdots \quad \begin{array}{ccc} & B_i & \\ \swarrow & & \searrow \\ A_i & & B_j\,. \\ \searrow & & \swarrow \\ & A_j & \end{array}$$

Note that from index i on upward the left and upper objects do not change any more. Taking colimits, we see that the composite $h \circ a_i$ is given by the pullback square in the following diagram



But clearly this span factorizes through $b_i : A_{i,i} \nrightarrow B$, since

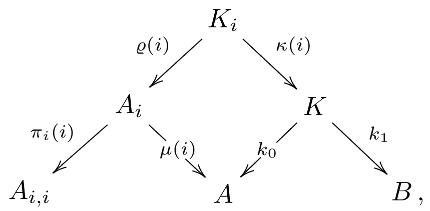
$$b_i\sigma_i(i) = \langle \pi_i(i)\varepsilon(i), h_1\nu(i)\rangle. \quad (45)$$

It is now left to show that for any other factorization $k : A \nrightarrow B$ with $b_i \geq k \circ a_i$ there exists a 2-cell $k \to h$.

9.2.4 The weak uniqueness

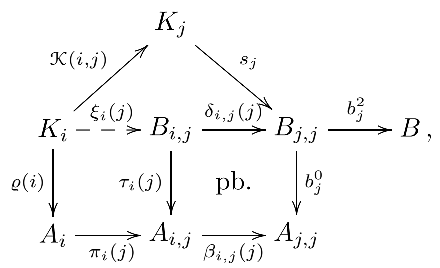
Suppose we are given a relation $k : A \nrightarrow B$ such that $b_i \geq k \circ a_i$ for all $i \in \mathbb{N}$. We shall construct a 2-cell $k \to h$. First let us consider the composites $k \circ a_i$, which are formed as in the following diagram:

(46)



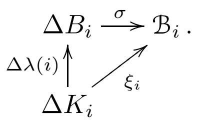
where the square is a pullback and, furthermore, the composite span factorizes through b_i, that means that $b_i s_i = \langle \pi_i(i)\varrho(i), k_1\kappa(i)\rangle$ for some arrow s_i. Now it is time to use condition (ii) of Theorem 9.2. Hence, by universality of the colimit (A, μ), we know that (K, κ), where the components of $\kappa : \mathcal{K} \to \Delta K$ are given as in (46) for an obvious functor $\mathcal{K}$, is a colimit of the ω-chain given by the objects K_i. Moreover, the $\varrho(i)$ yield a natural transformation $\varrho : \mathcal{K} \to \mathcal{A}$ with $\mu \cdot \varrho = \Delta k_0 \cdot \kappa$. We shall construct an arrow $s : K \to H$. This must obviously be done by using the universal property of the colimit (K, κ). Hence, we need a natural transformation $\lambda : \mathcal{K} \to \mathcal{B}$. In order to get its components, we can use the universal property of the limits (B_i, σ_i) (recall that $\mathcal{B}(i) = B_i$). So all amounts to constructing cones $\xi_i : \Delta K_i \to \mathcal{B}_i$. Its components can be obtained using the following diagram

(47)

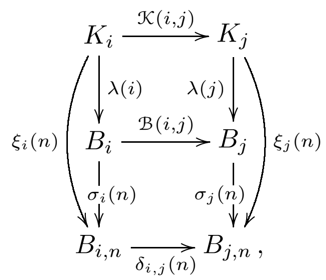


where i is fixed and $j \geq i$. Proving the naturality of ξ_i is now a straightforward chase through an admittedly rather huge diagram using only that the pullback projections in (47) are jointly monomorphic and that b_i is so, too. For the sake of brevity this task must be left to the Reader. Note that here the fact that b_i is monic, i. e., that $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ is used heavily.

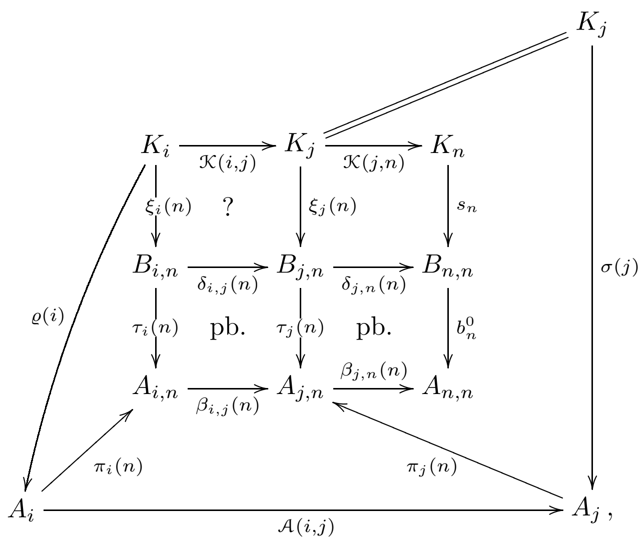
The universal property of the limit (B_i, σ_i) now induces unique factorizations $\lambda(i) : K_i \to B_i$ with



We need to show that the λ_i are components of the desired natural transformation λ. To do this we consider the diagram

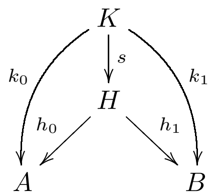


where $n \geq j \geq i$. Note that the diagram is known to be commutative except for the outer and the upper inner square. If we can show that the outer square commutes, then the fact that $\sigma_j : \Delta B_j \to \mathcal{B}_j$ is a monic family shows the naturality of λ. But this can be done by a quick chase through the diagram



using that the projections of the lower-right pullback form a monomorphic pair.

It is now time to construct an arrow $s : K \to H$, which will induce the desired 2-cell $k \to h$. But s can be obtained easily enough. We just have to evoke the universal property of the colimit (K, κ) to obtain a factorization of the cocone $(H, \nu \cdot \lambda)$ through κ; that is an arrow s such that $\nu \cdot \lambda = \Delta s \cdot k$. To complete the proof we must show that we indeed have a 2-cell, that means that the diagram



commutes.

One readily sees that $\pi_i \cdot \varrho = \pi_i \cdot \varepsilon \cdot \lambda$, which implies that $\varrho = \varepsilon \cdot \lambda$. Now

$$\Delta(h_0 s)\kappa = \Delta h_0 \Delta s \cdot \kappa = \Delta h_0 \cdot \nu\lambda = \mu\varepsilon\lambda = \mu\varrho = \Delta k_0 \cdot \kappa,$$

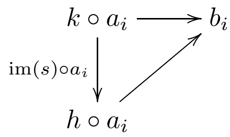
which shows that $h_0 s = k_0$, by uniqueness. Moreover,

$$\Delta(h_1 s)\kappa = \Delta h \cdot \eta\lambda = \Delta k_1 \cdot \kappa,$$

where the last step follows if we unfold the definition of η:

$$(\eta\lambda)(i) = b_i^1 \sigma_i(i)\lambda(i) = b_i^1 \xi_i(i) = b_i^1 s_i = k_1 \kappa(i).$$

Hence, by uniqueness, $h_1 s = k_1$. Finally, note that we need not prove anything towards the uniqueness of the 2-cell $im(s) : k \to h$ and that



commutes for all $i \in \mathbb{N}$ automatically, since $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$.

9.3 Consequences and Open Problems

As promised in Section 9.2 we shall now give a condition that forces the couniversal arrows of the functor

$$(-) \circ a : \mathbf{Rel}(\mathcal{C})(A, B) \to \mathrm{Cocone}(\mathcal{D}, B),$$

for a given ω-chain $\mathcal{D} : \mathbb{N} \to \mathbf{Rel}(\mathcal{C})$, to be isomorphisms. Note that the objects of the codomain of this functor are now strict cocones. Unfortunately, even in $\mathbf{Rel}(\mathbf{Set})$ the class of morphisms must be restricted because the condition does not hold true in general. But here it is:

> For all objects C of $\mathcal{C}$ the full subcategory $\mathcal{E}/C$ of the slice $\mathcal{C}/C$ is closed under limits of ω-cochains. (48)

More explicitly, if $\mathcal{A} : \mathbb{N} \to \mathcal{C}$ is an ω-cochain with $\mathcal{A}(i, 0) \in \mathcal{E}$ for all $i \in \mathbb{N}$, and if (L, ℓ) is the limit of $\mathcal{A}$, then $\ell(0)$ lies in $\mathcal{E}$, too.

The task is now to check whether condition (48) implies that $b_i \simeq h \circ a_i$. Recall that if (B, b) is a strict cocone, then the arrows $e_{i,j} : B_{i,j} \to B_{i,i}$ (see page 83) lie in $\mathcal{E}$. Hence, for the cochains $\mathcal{B}_i$, satisfying (48), the projection $\sigma_i(i)$

lies in $\mathcal{E}$. But we have seen that these projections induce the 2-cells $h \circ a_i \to b_i$ (see equation (45) on page 86). Thus $b_i \simeq h \circ a_i$.

Next let us discuss condition (48) in **Rel**(**Set**). As pointed out earlier, it is not true in general. Here is a counterexample.

Example 9.4. Consider the following cochain in **Set**:

$$\{0\} \xleftarrow{\;!\;} \mathbb{N} \xleftarrow{\text{succ}} \mathbb{N} \xleftarrow{\text{succ}} \mathbb{N} \xleftarrow{\text{succ}} \cdots$$

Its limit must be given by $\emptyset$, for suppose (L, ℓ) is any cone, and there exists an $x \in L$. Let $n := \ell_1(x)$. Then $\ell_{n+1}(x) = 0$, and there cannot be an element in $\mathbb{N}$ with $\ell_{n+2}(x) + 1 = 0 = \ell_{n+1}(x)$. Hence, (L, ℓ) is not a cone, a contradiction.

Clearly the projection $¡ : \emptyset \to \{0\}$ is not surjective. However, the map $\mathbb{N} \to \{0\}$ is surjective, whence (48) does not hold in this example.

The situation is different if we only consider those cochains that arise when the relations $A_{i,j} : A_{i,i} \mathrel{+\!\!\!\rightarrow} A_{j,j}$ are total. One readily checks that in **Rel**(**Set**),

$$A_i = \{(x_i, x_{i+1}, \ldots, x_j, \ldots) \mid x_j \in A_{j,j}, (x_j, x_{j+1}) \in A_{j,j+1} \forall j \geq i\}. \qquad (49)$$

Hence, clearly the projections $\sigma_i(i) : A_i \to A_{i,i}$ are surjective, if all $A_{j,j+1}$ are total. This leads to the following questions:

- Can the fact that (48) holds for total relations be generalized to a category $\mathcal{C}$ as in Theorem 9.2?
- What additional conditions must be imposed on $\mathcal{C}$ such that (48) holds?

Now let us turn to another open problem. A quick look at diagram (41) on page 81 shows that if the spans $A_{i,i+1}$ are graphs, i. e. given as $\langle 1, f_{i,i+1} \rangle$ for arrows $f_{i,i+1} : A_{i,i} \to A_{i+1,i+1}$ of $\mathcal{C}$, then A is just the colimit of the chain induced by the f's. Hence, the graph functor maps colimits of ω-chains to lax adjoint cooplimits. So the lax adjoint cooplimit of ω-chains becomes a colimit when we restrict the arrows in **Rel**($\mathcal{C}$) to graphs. It seems reasonable to ask:

- Does a lax adjoint cooplimit give a colimit of ω-chains **Map**(**Rel**($\mathcal{C}$))?

There is really not so much missing to answer this affirmatively. Recall that two maps are equal as soon as there exists a 2-cell between them (Corollary 4.9). Thus, one needs only to check whether the injections $a_i : A_{i,i} \mathrel{+\!\!\!\rightarrow} A$ and the canonical factorization $h : A \mathrel{+\!\!\!\rightarrow} B$ are maps, if all the $A_{i,j}$ and $b_i : A_{i,i} \mathrel{+\!\!\!\rightarrow} B$ are so. Of course, this is obvious if all the maps in **Rel**($\mathcal{C}$) are given by graphs (equivalently $\mathcal{E} \subseteq \mathrm{RegEpi}(\mathcal{C})$, see Theorem 4.23). Unfortunately, in general this seems not to be obvious at all.

However, the question can be answered positively if $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. The answer does not come completely for free, though.

Theorem 9.5. *If $\mathcal{C}$ is a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E} \subseteq$ $\mathrm{Epi}(\mathcal{C})$ stable under pullback, and if condition (ii) and (iii) of Theorem 9.2 hold in $\mathcal{C}$, then the construction of Theorem 9.2 yields colimits of ω-chains in* **Map**(**Rel**($\mathcal{C}$)) *if and only if*

for all objects C of $\mathcal{C}$ the full subcategory Σ/C of the slice $\mathcal{C}/C$, where $\Sigma = \mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$ is closed under limits of ω-cochains. (50)

Proof. First note that the condition $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$ is not needed here since maps are already monic in $\mathcal{C}$ by Theorem 4.19. Furthermore, observe that, by Proposition 6.1, condition (50) can be restated more explicitly as follows. For all ω-cochains $\mathcal{D} : \mathbb{N} \to \mathcal{C}$ with $\mathcal{D}(j,i) \in \Sigma = \mathcal{E} \cap \mathrm{Mono}(\mathcal{C})$ for all $j \geq i$, the projections of its limit are in Σ, too.

Suppose condition (50) holds. Assume that the relations $A_{i,i+1} : A_{i,i} \nrightarrow A_{i+1,i+1}$ are maps. Equivalently, $\alpha_{i,i+1}(i) : A_{i,i+1} \to A_{i,i}$ lies in Σ. But Σ is a pullback stable class. Hence, all $\alpha_{i,j}(k) : A_{k,j} \to A_{k,i}$, $j \geq i \geq k$ are in Σ, too. Applying condition (50), we see that $\pi_i(i) : A_i \to A_{i,i}$ lies in Σ for all i, whence $a_i : A_{i,i} \nrightarrow A$ is a map (recall that if $\langle s_0, s_1 \rangle$ is a span with $s_0 \in \Sigma$, then $r_0 \in \Sigma$ for $\langle r_0, r_1 \rangle = \mathrm{im}\langle s_0, s_1 \rangle$, by 6.1).

Now suppose that a second cocone (B, b) on the ω-chain in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is given. Clearly the natural transformations $\tau_i : \mathcal{B}_i \to \mathcal{A}_i$ are in Σ componentwise. By 6.1, all $\gamma_{i,j}(k) : B_{k,j} \to B_{k,i}$, $j \geq i \geq k$, lie in Σ. Hence, σ_i must lie in Σ componentwise as before π_i. This implies that $\varepsilon : \mathcal{B} \to \mathcal{A}$ lies in Σ componentwise, by 6.1 again, since $\pi_i \cdot \Delta\varepsilon(i) = \tau_i \cdot \sigma_i$. Since $\mathcal{E}$ commutes with all colimits, $h_0 : H \to A$ must therefore lie in $\mathcal{E}$, too. In order to see that h_0 is monic, we use condition (iii) of Theorem 9.2, recalling that an arrow is monic in $\mathcal{C}$ if and only if its kernel pair, which is given by the pullback of that arrow along itself, is a diagonal in $\mathcal{C}$.

Finally, to see that condition (50) is necessary, suppose that $\mathcal{D} : \mathbb{N} \to \mathcal{C}$ is an ω-cochain with $\mathcal{D}(j,i) \in \Sigma$ for all $j \geq i \in \mathbb{N}$. This gives an ω-chain $\langle \mathcal{D}(i+1,i), 1 \rangle$ in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. We know that its colimit injections are formed by taking the limit (A, π) of $\mathcal{D}$ (and its tails). Note that these injections obtained in $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ are of the form $\langle \pi(i), 1 \rangle$; taking images is not necessary since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. Therefore, all $\pi(i)$ must lie in Σ. $\square$

Note that if Σ (as a full subcategory of $\mathcal{C}^2$) is closed under limits of ω-cochains, then condition (50) in the previous result holds true.

Moreover, $\mathrm{Mono}(\mathcal{C})$ (as full subcategory of $\mathcal{C}^2$) is always closed under all limits. Hence, if $\mathcal{E}$ is closed under limits of ω-cochains, then condition (50) holds true. The converse of this, however, need not to be true necessarily. Also note that the earlier condition (48) implies condition (50).

Observe that condition (50) is much more decent than condition (48). For example, (50) is true in every regular category, since the class Σ consists of isomorphisms there, which implies that it is closed under all limits and colimits. Other (non-regular) examples are **Top** with the usual $(\mathrm{Epi}, \mathrm{RegMono})$-factorization, $\mathbf{Top}_1$ with the factorization system of Example 4.24, even **CAT** with $\mathcal{E}$ and $\mathcal{M}$ as in Example 3.15, which does not satisfy $\mathcal{E} \subseteq \mathrm{Epi}(\mathbf{CAT})$, though.

This leaves the question, whether a result like Theorem 9.5 can be obtained for an $(\mathcal{E}, \mathcal{M})$-structured category with non-epimorphic arrows in $\mathcal{E}$.

Finally, recall that every functor $F : \mathcal{C} \to \mathcal{C}'$ in the 2-category $\mathcal{K}$ as defined in Section 6.4 gives rise to a 2-functor $\mathbf{Rel}(F) : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{C}')$. If $\mathcal{C}$ and $\mathcal{C}'$ both satisfy the conditions of Theorem 9.2, and therefore admit lax adjoint cooplimits of ω-chains, then clearly $\mathbf{Rel}(F)$ preserves these if F is ω-cocontinuous and preserves limits of ω-cochains in $\mathcal{C}$. Moreover, Theorem 9.5 shows that the restriction of $\mathbf{Rel}(F)$ to maps is ω-cocontinuous in case $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$ and $\mathcal{E}' \subseteq \mathrm{Epi}(\mathcal{C}')$.

A Allegories

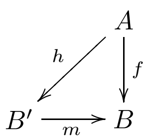
In this appendix we shall provide enough allegory theory to prove the results of Section 6 for every allegory satisfying certain conditions. This will also show that the results of Section 6 are true for any $\mathbf{Rel}(\mathcal{C})$, where $\mathcal{C}$ is finitely complete $(\mathcal{E},\mathcal{M})$-structured with $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$. Moreover, virtually all the results of [10], especially those of Section 5 and 6 in there, can also be extracted from the results about allegories given here. The moral of this is that although allegories are an attempt to axiomatize relations over regular categories, all bicategories of monic relations are covered by the axioms as well. Hence, the structure of $\mathbf{Rel}(\mathcal{C})$ is not very different regardless of $\mathcal{C}$ being regular, a proper stable $(\mathcal{E},\mathcal{M})$-structured category or even without $\mathcal{E}$ consisting of epimorphisms. However, the connection to the category $\mathcal{C}$ becomes more difficult for the more general cases. For example, a regular category $\mathcal{C}$ is isomorphic to the category $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. In a finitely complete $\mathcal{C}$ with a stable proper $(\mathcal{E},\mathcal{M})$-factorization system, this is not true unless $\mathcal{E} \subseteq \text{RegEpi}(\mathcal{C})$. But $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ is the category of fractions of a canonical class of morphisms characterizing the maps. If we further generalize and drop the assumption that $\mathcal{E}$ consists of epimorphisms, then the graph functor Γ is not faithful any more. Moreover, we lose the nice characterization of maps in terms of properties on just one leg of a relation.

The only cases not covered by allegories are the cases of general $(\mathcal{E},\mathcal{M})$-structured $\mathcal{C}$ with non-monic arrows in $\mathcal{M}$.

In principle we present here pp. 195–204 of [8], but fill in some details and skip the examples given there.

A.1 Preliminaries and Terminology

Definition A.1. *A subobject represented by* $m : B' \to B$ allows *an arrow* $f : A \to B$ *if there exists an arrow* $h : A \to B'$ *such that*



commutes.

The image *of f, if it exists, is the smallest subobject that allows f. A category* has images *if every morphism of it has an image.*

A morphism $c : A \to B$ is called a cover *if its image is entire, that means that it can be represented by an identity.*

Having introduced these notions, we may define a regular category to be a finitely complete category with images in which covers are stable under pullback. Note that in any category the notion of cover is precisely that of an extremal epimorphism. Thus stability of covers under pullback implies that an arrow is a cover if and only if it is a regular epimorphism (see Proposition 2.7). Hence, the definition given here of regular category really coincides with the usual one of a finitely complete $(\mathcal{E}, \text{Mono})$-structured category with $\mathcal{E}$ stable under pullback.

Finally, an object in a category is called *subterminal* (or a *subterminator*) if for all objects A there is at most one arrow $A \to S$. A terminal object is also called *terminator*.

A.2 Basic definitions

Definition A.2. *An* allegory *is a category $\mathcal{A}$ in which $\mathcal{A}(A, B)$ is a meet-semilattice for any objects A and B. Meets are denoted by $R \cap S$ for $R, S \in \mathcal{A}(A, B)$. Moreover, for any objects A and B, there is a monotone operation $(-)^o : \mathcal{A}(A, B) \to \mathcal{A}(B, A)$ (sometimes called* involution*) such that the following axioms are true:*

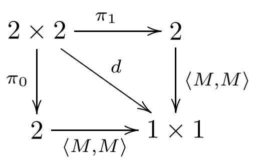
$$\begin{aligned} R^{oo} &= R, \\ (RS)^o &= S^o R^o, \\ R(S \cap T) &\subset RS \cap RT \qquad \textit{semi-distributivity}, \\ RS \cap T &\subset R(S \cap R^o T) \qquad \textit{modular-law}, \end{aligned}$$

where $R \subset S$ if and only if $R \cap S = R$, and composition in $\mathcal{A}$ is denoted by juxtaposition.

Throughout this section assume that $\mathcal{C}$ is a finitely complete $(\mathcal{E}, \mathcal{M})$-structured category with $\mathcal{E}$ stable under pullback and $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$. Observe that $\mathbf{Rel}(\mathcal{C})$ is an allegory, but not so if we drop the last assumption about $\mathcal{C}$.

Example A.3. *Recall* $\mathbf{Rel}(\mathbf{CAT})$ *and the functor $M : 2 \to 1$ from Example 3.15 on page 19.*

Now define a relation $r := \mathrm{im}\langle M, M\rangle$ in $\mathbf{Rel}(\mathbf{CAT})$*. The relation $r \wedge r$ is given by taking the image of d in*



Clearly the images of d and of $\langle M, M\rangle$ cannot be isomorphic, for when factorizing, we get categories with 2 objects and 4 objects respectively. So $r \wedge r \not\simeq r$, whence $\mathbf{Rel}(\mathbf{CAT})$ *is not an allegory.*

Now note the following properties of every allegory.

Proposition A.4. *(i) $RS \cap T \subset R(S \cap R^o S) \iff RS \cap T \subset (R \cap TS^o)S$,*

(ii) $R \subset RR^oR$,

(iii) $1^o = 1$.

Proof. (i) Apply $(-)^o$ to either inequality.
(ii) $R \subset 1R \cap R \subset (1 \cap RR^o)R \subset RR^oR$.
(iii) $1 = 1^{oo} = (11^o)^o = 1^{oo}1^o = 11^o = 1^o$. □

A.3 Special Morphisms

Definition A.5. *An endomorphism R in an allegory is called*

(i) reflexive *if $1 \subset R$,*

(ii) symmetric *if $R^o \subset R$,*

(iii) transitive *if* $RR \subset R$*,*

(iv) an equivalence relation *if* R *is reflexive, symmetric and transitive,*

(v) coreflexive *if* $R \subset 1$*.*

Note that $R^o \subset R$, if and only if $R \subset R^o$ if and only if $R = R^o$ since $(-)^o$ is monotone. Note that for regular categories the coreflexive relations on an object correspond to the subobjects of that object. For $\mathbf{Rel}(\mathcal{C})$ this is true if $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{C})$. However, without this condition the legs of identity relations are not in general equal, which makes this kind of statement not at all obvious. Here are some properties of the special relations just defined.

Proposition A.6. *Let* R *be an endomorphism in an allegory.*

(i) If R *is reflexive and transitive, then it is idempotent.*

(ii) If R *is symmetric and transitive, then it is idempotent.*

(iii) If R *is coreflexive, then it is symmetric and idempotent.*

(iv) For coreflexive morphisms A*,* B*,* $AB = A \cap B$*.*

Proof. (i) $R \subset RR$ using reflexivity, whence $R = RR$ by transitivity.

(ii) $R \subset RR^oR \subset R^3 \subset R^2$, and therefore $R = RR$.

(iii) $R \subset RR^oR \subset 1R^o1 = R^o$ and $RR \subset 1R = R$. Then by (ii), R is idempotent.

(iv) Clearly $AB \subset A1 = A$ and $AB \subset B$, and then $AB \subset A \cap B$. On the other hand, $A \cap B \subset A \subset 1$, and therefore $A \cap B \subset (A \cap B)^2 \subset AB$. □

Definition A.7. *The* domain *of a morphism* R*,* $\mathrm{dom}(R)$*, is defined by*

$$\mathrm{dom}(R) := 1 \cap R^oR.$$

Note that we deviate from Freyd's notation here by writing composition as before in this thesis not as in [8].

Proposition A.8. *For morphisms* $R : \alpha \to \beta$ *and* $S : \gamma \to \delta$ *and for* $A \subset 1_\alpha$ *the following hold:*

(i) $\mathrm{dom}(R) \subset A \iff R \subset RA$*,*

(ii) $\mathrm{dom}(SR) \subset \mathrm{dom}(R)$*.*

Proof. (i) If $1 \cap R^oR \subset A$, then $R = R1 \cap R \subset R(1 \cap R^oR) \subset RA$.

Conversely, if $R \subset RA$, then $1 \cap R^oR \subset 1 \cap R^oRA \subset (A^o \cap R^oR)A \subset A^oA \subset A$ since A is symmetric idempotent.

(ii) Notice that $SR = S(R \cap R) \subset SR(1 \cap R^oR) = SR\,\mathrm{dom}(R)$, which by (i) suffices to show the result. □

Lemma A.9. $\mathrm{dom}(R \cap S) = 1 \cap R^oS$*.*

Proof. We know that $\mathrm{dom}(RS) = 1 \cap (R \cap S)^o(R \cap S) \subset 1 \cap R^oS$ by semi-distributivity. Moreover, $1 \cap R^oS \subset 1 \cap (1 \cap (1 \cap R^oS) \subset 1 \cap (1 \cap (S^o \cap R^o)S) \subset 1 \cap (S^o \cap R^o)((S \cap R) \cap S) = \mathrm{dom}(S \cap R)$, where the second and the third step uses the modularity law. □

Definition A.10. *A relation R is called*

(i) total *if* $\mathrm{dom}(R) = 1$*; equivalently if* $1 \subset R^o R$*,*

(ii) single-valued *if* $R^o R \subset 1$*,*

(iii) a map *if it is total and single-valued.*

Note that for $\mathbf{Rel}(\mathcal{C})$ these notions coincide with the notions of totality, single-valuedness, and map as defined earlier.

Proposition A.11. *(i) If R and S are total (single-valued, maps), then SR is total (single-valued, a map), too.*

(ii) If SR is total, then R is total.

Proof. (i) $1 \subset R^o R \subset R^o S^o SR = (SR)^o (SR)$ from which SR is total. Similarly for single-valuedness, which implies that the statement holds for maps, too.

(ii) $1 \subset \mathrm{dom}(SR) \subset \mathrm{dom}(R) \subset 1$. □

For an allegory $\mathcal{A}$, $\mathbf{Map}(\mathcal{A})$ denotes the subcategory of maps. Morphisms therein will be denoted by lowercase letters.

Proposition A.12. *(i) The partial order restricted to $\mathbf{Map}(\mathcal{A})$ is discrete, i. e., $f \subset g$ implies $f = g$.*

(ii) A relation R is an isomorphism if and only if R and R^o are maps; furthermore in this case $R^{-1} = R^o$.

Proof. (i) $f \subset g$ implies $f^o \subset g^o$, whence $g1 \subset g f^o f \subset g g^o f \subset 1 f = f$.

(ii) If R and R^o are maps, then $1 \subset R^o R \subset 1$ and $1 \subset R R^o \subset 1$.

Conversely, $1 = R^{-1} R$ implies that R is total by A.11. Similarly, R^{-1} is total. Moreover, R is single-valued because

$$RR^o \subset (R^{-1})^o R^{-1} R R^o = (R^{-1})^o R^o = (R^{-1} R)^o = 1.$$

Similarly R^{-1} is single-valued, and finally, $R^{-1} \subset R^o R R^{-1} = R^o$ and $R^o \subset R^o (R^{-1})^o R^{-1} = R^{-1}$. □

Proposition A.13. *If R is single-valued, then $(S \cap T)R = SR \cap TR$.*

Proof. $SR \cap TR \subset (S \cap TRR^o)R \subset (S \cap T)R$, and by semi-distributivity $(S \cap T)R \subset SR \cap TR$. □

In particular this last result holds for maps.

A.4 Tabular Allegories

Definition A.14. *A pair of maps f, g* tabulates *a morphism R if $R = gf^o$ and $f^of \cap g^og = 1$. In this case R is called* tabular. *An allegory $\mathcal{A}$ is* tabular *if all its morphisms are.*

Note that $\mathbf{Rel}(\mathcal{C})$ is tabular. Every relation $r = \langle r_0, r_1 \rangle$ can be written as $\Gamma r_1 \circ (\Gamma r_0)^o$, where r_0, r_1 is a monomorphic pair in $\mathcal{C}$. It is well-known that a pair f, g is monic in a category with pullbacks if and only if $\ker(f) \cap \ker(g) = \delta$. But the last expression can be written as $(\Gamma f)^o(\Gamma f) \cap (\Gamma g)^o\Gamma g \simeq \iota$ in $\mathbf{Rel}(\mathcal{C})$. However, it is not at all obvious that f, g must be a monic pair in $\mathcal{C}$ if their graphs satisfy this condition.

Proposition A.15. *If $f^of \cap g^og = 1$ in an allegory $\mathcal{A}$, then f, g is a monomorphic pair in $\mathbf{Map}(\mathcal{A})$.*

Proof. If $fx = fy$ and $gx = gy$, then $x = (f^of \cap g^og)x = f^ofx \cap g^ogx = f^ofy \cap g^ogy = y$. □

Proposition A.16. *If f, g tabulates R, then $yx^o \subset R$ if and only if there exists a unique map h with $x = fh$ and $y = gh$.*

Proof. If $x = fh$ and $y = gh$, then $yx^o = gh(fh)^o = ghh^of^o \subset gf^o = R$.

Conversely, if $yx^o \subset gf^o$, define $h = f^ox \cap g^oy$. By Lemma A.9,

$$1 \subset 1 \cap (y^oy)(x^ox) \subset 1 \cap y^ogf^ox = 1 \cap (g^oy)^o(f^ox) = \mathrm{dom}(h) \subset 1,$$

which shows that h is total. Single-valuedness can be seen from

$$hh^o = (f^ox \cap g^oy)(f^ox \cap g^oy)^o \subset f^oxx^of \cap g^oyy^og \subset f^of \cap g^og = 1.$$

Moreover, $fh \subset ff^ox \subset x$, whence $x = fh$. Similarly $gh = y$. Uniqueness of h follows, since f, g is a monic pair in $\mathbf{Map}(\mathcal{A})$. □

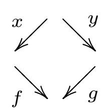
This also shows that tabulations are unique up to unique isomorphism. More precisely:

Corollary A.17. *If f, g and f', g' both tabulate R, then there exists a unique isomorphism u in $\mathbf{Map}(\mathcal{A})$ such that $f' = fu$ and $g' = gu$.*

Proposition A.18. *If R is coreflexive and tabular in $\mathcal{A}$, then there exists a monomorphic h in $\mathbf{Map}(\mathcal{A})$ such that $R = hh^o$.*

Proof. If $R \subset 1$, and f, g is a tabulation of R, then $g \subset gf^of = Rf \subset f$, which shows that $f = g$. So clearly $g^og = 1$, whence g is split monic. □

Proposition A.19. *A square*

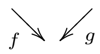


commutes in $\mathbf{Map}(\mathcal{A})$ if and only if $yx^o \subset g^of$.

Proof. $fx = gy$ implies that $yx^o \subset yx^o f^o f = yy^o g^o f \subset g^o f$. Conversely, $yx^o \subset g^o f$ implies that $gy \subset gyx^o x \subset gg^o fx \subset fx$. By A.12, $gy = fx$. □

Theorem A.20. *Let $\mathcal{A}$ be a tabular allegory. Then* $\mathbf{Map}(\mathcal{A})$ *has pullbacks, equalizers, images and covers are stable under pullback.*

Proof. The pullback of the map f along a map g is given by a tabulation of



that is the relation $g^o f$. Suppose that p, q tabulate $g^o f$. By A.19, $fp = gq$. If $fx = gy$, then $yx^o \subset qp^o$; hence, there is a unique map h with $x = ph$ and $y = qh$ by A.16.

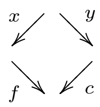
An equalizer of two parallel maps f and g is given by a tabulation of $\mathrm{dom}(f \cap g)$. Since $\mathrm{dom}(f \cap g)$ is coreflexive there exists a monic map h with $\mathrm{dom}(f \cap g) = hh^o$. By A.9, $hh^o \subset g^o f$, and therefore by A.19, $gh = fh$. Suppose $gx = fx$. Then $xx^o \subset g^o f$, and, by single-valuedness, $xx^o \subset 1 \cap g^o f = hh^o$; hence, there is a unique map y with $x = yh$ by A.16.

The image of a map f is given by a tabulation of $\mathrm{dom}(f^o) = 1 \cap ff^o$ by a monic map h with $h^o h = 1$. But by single-valuedness of f, $\mathrm{dom}(f^o) = ff^o = hh^o$. By A.16, there is a map x with $f = hx$. (Note that x is not necessarily an iso since ff^o need not be a tabulation.) Thus we have shown that h allows f. We must still show that h represents the smallest such subobject. So suppose we are given a subobject m that allows f, i. e., $f = mx$ for some map x. Then

$$hh^o = ff^o = mxx^o m^o \subset mm^o.$$

The map m is monic. Hence $m^o m = 1$, which means that m tabulates mm^o. By A.16, there is a map y such that $h = my$, showing that $h \leq m$ as subobjects in $\mathbf{Map}(\mathcal{A})$.

If g is a cover in $\mathbf{Map}(\mathcal{A})$, its image can be represented by an identity, whence $gg^o = 11^o = 1$, which holds if and only if $1 \subset gg^o$, or equivalently, if g^o is total. Finally, we need to show that covers are stable under pullback. If



is a pullback square, where c is a cover, then $yx^o = c^o f$. Since c^o is total, $c^o f$ is so, and therefore, by A.11, x^o is total or equivalently x is a cover. □

The last theorem almost shows that $\mathbf{Map}(\mathcal{A})$ is a regular category. There is only one little ingredient missing, namely terminal objects. However, recall that despite the missing 1, which implies that no binary products are available in $\mathbf{Map}(\mathcal{A})$, one may still define (with some extra technical effort) $\mathbf{Rel}(\mathbf{Map}(\mathcal{A}))$, which leads us to the following result.

Theorem A.21. *If $\mathcal{A}$ is a tabular allegory, then* $\mathcal{A} \simeq \mathbf{Rel}(\mathbf{Map}(\mathcal{A}))$.

Proof. This isomorphism of categories is given by assigning to each relation R, the span formed by its tabulation f, g. This assignment clearly is identity on objects. Moreover, for any monic pair f, g of maps we have a relation $R = gf^o$ since tabulations are unique, which shows bijectivity on the hom-lattices. □

This result is a very crucial one. It shows that every tabular allegory is a category of relations over a regular category, namely the subcategory formed by its maps. It is the reason why the structure of $\mathbf{Rel}(\mathcal{C})$ is always the same regardless of $\mathcal{C}$ being regular or just $(\mathcal{E}, \mathcal{M})$-structured with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$.

Now let us deal with terminal objects in $\mathbf{Map}(\mathcal{A})$.

A.5 Unitary Allegories

Definition A.22. *An object π in an allegory $\mathcal{A}$ is said to be a* partial unit *if 1_π is maximal in $\mathcal{A}(\pi, \pi)$. If, in addition, for any object λ there is a total relation in $\mathcal{A}(\lambda, \pi)$, then π is said to be a* unit.

Note that for regular categories $\mathcal{C}$ (partial) units coincide with (sub)terminators, because $r = \langle r_0, r_1 \rangle : S \leftrightarrow S$ is a relation, where S is a subterminator of $\mathcal{C}$ if and only if $r_0 = r_1$ if and only if $r \leq \delta_S$. If $\mathcal{C}$ has a terminal object, then clearly $\Gamma !_A$ is a total relation, and δ_1 is the maximum of $\mathbf{Rel}(1, 1)$. Conversely, if U is a unit, then for all objects A there is a total $r : A \leftrightarrow U$, and $rr^o \leq \delta_U$ since U is partial unit, whence r is a map. Thus r is given as a graph Γf of some arrow $f : A \to U$ which must be unique by Theorem 4.23. Hence, U is a terminator.

Note that for non-regular $\mathcal{C}$ (but still with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{C})$), (sub)terminators still give (partial) units, but the converse is not so obvious.

For an object α of an allegory let $\mathrm{Cor}(\alpha)$ be the meet-semilattice of coreflexive relations. Recall that for a poset X an *ideal* is a subset $I \subseteq X$ such that $x \in I$ and $y \leq x$ implies that $y \in I$.

Proposition A.23. *If π is a partial unit of an allegory $\mathcal{A}$, then*

$$\mathrm{dom} : \mathcal{A}(\alpha, \pi) \to \mathcal{A}(\alpha, \alpha)$$

is an order isomorphism onto an ideal of $\mathrm{Cor}(\alpha)$.

Proof. The mapping dom is obviously order preserving. Suppose that $\mathrm{dom}(R) \subset \mathrm{dom}(S)$. By A.8, $R \subset R\,\mathrm{dom}(S) \subset RS^oS \subset 1S$ since 1 is the maximum of $\mathcal{A}(\pi, \pi)$. Hence, dom is order reflecting. Therefore $\mathrm{dom}(S) = \mathrm{dom}(R)$ implies that $S = R$, which shows that dom is injective.

To see that dom is surjective and its image is an ideal, let $A \subset \mathrm{dom}R$. Then A is coreflexive and

$$\mathrm{dom}(RA) \subset \mathrm{dom}(A) \subset A^oA = A,$$

since firstly, using A.8, $R \subset R\mathrm{dom}(A) \subset R$ and secondly, A is symmetric idempotent. Finally,

$$A \subset AA \subset A^o\mathrm{dom}(R)A \subset A^o \cap A^oR^oRA \subset \mathrm{dom}(RA).$$

□

Proposition A.24. *(i) If λ is a unit in $\mathcal{A}$, then* $\operatorname{dom} : \mathcal{A}(\alpha, \lambda) \to \operatorname{Cor}(\alpha)$ *is an isomorphism of semilattices.*

(ii) The unique total morphism $p_\alpha : \alpha \to \lambda$ is single-valued, whence a map.

(iii) A unit λ is a terminal object in $\mathbf{Map}(\mathcal{A})$.

(iv) For any two objects α and β of an allegory, $(p_\beta)^o p_\alpha$ is the maximum in $\mathcal{A}(\alpha, \beta)$.

Proof. (i) Given a coreflexive relation A in $\mathcal{A}(\alpha, \alpha)$, take a total $p_\alpha : \alpha \to \lambda$, which exists since λ is a unit. We know that $A \subset 1$ and $A \subset A^o A \subset A^o p_\alpha^o p_\alpha A$, which implies that $A \subset \operatorname{dom}(p_\alpha A)$. Moreover, $\operatorname{dom}(p_\alpha A) \subset A$ holds by A.8, since $p_\alpha A \subset p_\alpha AA = p_\alpha A$. Hence, dom is surjective, and therefore an order iso by A.23.

(ii) We shall show that p_α is uniquely determined and single-valued. If $p_\alpha, p'_\alpha : \alpha \to \lambda$ are total, then as in (i), $\operatorname{dom}(p_\alpha A) = A = \operatorname{dom}(p'_\alpha A)$ for all $A \subset 1$. In particular, this holds for $A = 1$, and then $p_\alpha = p'_\alpha$ by injectivity of dom. Clearly $p_\alpha p_\alpha^o \subset 1_\lambda$ since the latter is the maximum of $\mathcal{A}(\lambda, \lambda)$.

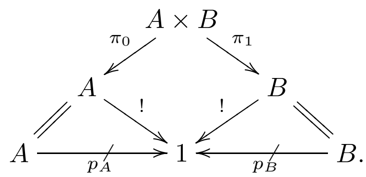
(iii) The existence and uniqueness of p_α in (ii) immediately imply that λ is terminal in $\mathbf{Map}(\mathcal{A})$.

(iv) Note that p_α is the maximum of $\mathcal{A}(\alpha, \lambda)$ since

$$\operatorname{dom}(A) = 1 \cap A^o A \subset 1 = 1 \cap 1 \subset 1 \cap p_\alpha^o p_\alpha = \operatorname{dom}(p_\alpha),$$

and dom is order reflecting. Hence, for all $R : \alpha \to \beta$, $p_\beta R \subset p_\alpha$, and therefore $R \subset p_\beta^o p_\beta R \subset p_\beta^o p_\alpha$, which shows the desired maximality. $\square$

Note that this result shows that for a unitary, tabular allegory $\mathcal{A}$, $\mathbf{Map}(\mathcal{A})$ is finitely complete. In $\mathbf{Rel}(\mathcal{C})$, $p_B^o p_A$ is tabulated by product projections as shown in the diagram



A.6 Functors between allegories

Definition A.25. *A (unitary)* representation *of allegories is a functor between allegories preserving (units), $(-)^o$ and intersection.*

Note that since composition is preserved, every representation of allegories preserves maps and tabulations. So given a representation $T : \mathcal{A} \to \mathcal{B}$ of tabular allegories, we get a functor $\mathbf{Map}(T) : \mathbf{Map}(\mathcal{A}) \to \mathbf{Map}(\mathcal{B})$. The functor $\mathbf{Map}(T)$ clearly preserves pullbacks, equalizers, covers and, furthermore, terminal objects if $\mathcal{A}$ and $\mathcal{B}$ are unitary.

Therefore, being just restriction on functors, the assignment $\mathcal{A} \mapsto \mathbf{Map}(\mathcal{A})$ yields a 2-functor

$$\mathbf{Map} : \mathbf{Al} \to \mathbf{Reg}$$

from the 2-category of unitary tabular allegories and unitary representations to the sub-2-category $\mathbf{Reg}$ of $\mathcal{K}$ of regular categories as defined in Section 6.

Proposition A.26. *Let $\mathcal{A}$ and $\mathcal{B}$ be tabular allegories. Then a representation $T : \mathcal{A} \to \mathcal{B}$ is faithful if and only if* $\mathbf{Map}(T)$ *is faithful and conservative.*

Proof. ($\Rightarrow$) If TR is an isomorphism in $\mathcal{B}$, its inverse must be $(TR)^o$, and then $T1 = 1 = (TR)(TR)^o = T(RR^o)$ and similarly, $T1 = T(R^oR)$. By faithfulness of T, R is an isomorphism in $\mathcal{A}$. Hence, T is faithful and conservative showing that the restriction $\mathbf{Map}(T)$ is so, too.

($\Leftarrow$) Suppose that $TR = TS$. Let f, g and h, k be tabulations of R and S respectively. Since tabulations are preserved by T there must be a unique isomorphic map u in $\mathcal{B}$ such that $Tf = (Th)u$ and $Tg = (Tk)u$. The same holds if T is replaced by $T' := \mathbf{Map}(T)$. Recall from the proof of A.16 that $u = (Tf)^oTh \cap (Tg)^oTk = T(f^oh \cap g^ok) =: T'(v)$, where v is a map, and therefore an isomorphism in $\mathcal{A}$ since T' is conservative. By faithfulness, $f = hv$ and $g = hv$, which, by A.16, implies the result. □

Note that only the second part of the proof actually uses tabularity of $\mathcal{A}$ and $\mathcal{B}$.

Now consider the full sub-2-category $\mathcal{K}_{\text{Mono}}$ of the 2-category $\mathcal{K}$ (as defined in Section 6 on page 50) consisting of those finitely complete $\mathcal{C}$ with stable $(\mathcal{E}, \mathcal{M})$-factorization system, where $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$. Observe that for any arrow $T : \mathcal{C} \to \mathcal{C}'$ of $\mathcal{K}_{\text{Mono}}$, $\mathbf{Rel}(T)$ is a unitary representation of the unitary tabular allegories $\mathbf{Rel}(\mathcal{C})$ and $\mathbf{Rel}(\mathcal{C}')$. In fact, we have found a left-adjoint for the map-functor.

Theorem A.27. $\mathbf{Rel} \dashv \mathbf{Map} : \mathbf{Al} \to \mathcal{K}_{\text{Mono}}$.

Proof. We shall show that the components of the unit of this adjunction are given by the graph functors

$$\Gamma : \mathcal{C} \longrightarrow \mathbf{Map}(\mathbf{Rel}(\mathcal{C})) .$$

Suppose $T : \mathcal{C} \to \mathbf{Map}(\mathcal{B})$ is an arrow in $\mathcal{K}_{\text{Mono}}$, where $\mathcal{B}$ is an object of $\mathbf{Al}$. On $r = \langle r_0, r_1 \rangle$ we define $S : \mathbf{Rel}(\mathcal{C}) \to \mathcal{B}$ by

$$S(r) := Tr_1 \circ (Tr_0)^o.$$

This clearly defines a unitary representation of allegories since it can be written as $S = I^{-1} \cdot \mathbf{Rel}(T)$, where $I : \mathcal{B} \to \mathbf{Rel}(\mathbf{Map}(\mathcal{B}))$ is the isomorphism of A.21. Now we have $(\mathbf{Map}(S) \cdot \Gamma)(f) = Tf_1 \circ (Tf_0)^o$ for $f : A \to B$ in $\mathcal{C}$, where $\langle f_0, f_1 \rangle e = \langle 1, f \rangle$ is an $(\mathcal{E}, \mathcal{M})$-factorization. Note that Tf is tabulated by itself and 1_{TA} in $\mathcal{B}$. Therefore

$$T(f) = Tf \circ (T1_A)^o = Tf_1 \circ Te \circ (Te)^o \circ (Tf_0)^o \subset Tf_1 \circ (Tf_0)^o,$$

since Te must be a cover in $\mathbf{Map}(\mathcal{B})$. Hence, $T = \mathbf{Map}(S) \cdot \Gamma$ by A.12.

As for uniqueness suppose that $S' : \mathbf{Rel}(\mathcal{C}) \to \mathcal{B}$ is an arrow in $\mathbf{Al}$ with $\mathbf{Map}(S') \cdot \Gamma = T$. Then

$$S'(r) = S'(\Gamma r_1 \circ (\Gamma r_0)^o) = S'\Gamma r_1 \circ (S'\Gamma r_0)^o = Tr_1 \circ (Tr_0)^o = S(r).$$

□

Note that the counit of the adjunction is an isomorphism since

$$\mathcal{A} \xrightarrow{\sim} \mathbf{Rel}(\mathbf{Map}(\mathcal{A}))$$

for any unitary tabular allegory $\mathcal{A}$.

This result also shows that **Reg**, the full sub-2-category of regular categories in $\mathcal{K}$ is a reflective subcategory of $\mathcal{K}_{\text{Mono}}$. Finally, we note the following result.

Theorem A.28. *The functor* $\mathbf{Rel} : \mathbf{Reg} \to \mathbf{Al}$ *is an equivalence of categories.*

Proof. Theorem A.21 shows **Rel** to be essentially surjective. It remains to show that it is fully faithful. But every unitary representation $T : \mathbf{Rel}(\mathcal{C}) \to \mathbf{Rel}(\mathcal{C}')$, where $\mathcal{C}$ and $\mathcal{C}'$ are regular, induces an arrow

$$\mathcal{C} \simeq \mathbf{Map}(\mathbf{Rel}(\mathcal{C})) \xrightarrow{\mathbf{Map}(T)} \mathbf{Map}(\mathbf{Rel}(\mathcal{C}')) \simeq \mathcal{C}'$$

in **Reg**. Moreover, if $\mathbf{Rel}(T) = \mathbf{Rel}(T')$ for functors $T, T' : \mathcal{C} \to \mathcal{C}'$, then in particular $\mathbf{Rel}(T)$ and $\mathbf{Rel}(T')$ agree on maps, and therefore on $\mathcal{C} \simeq \mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$. Hence, $T = T'$, which completes the proof. □

Note that the equivalence inverse is, of course, the functor **Map**.

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