

Equational Axiomatization of Algebras with Structure

Stefan Milius* and Henning Urbat**

Friedrich-Alexander-Universität Erlangen-Nürnberg

Abstract This paper proposes a new category theoretic account of equationally axiomatizable classes of algebras. Our approach is well-suited for the treatment of algebras equipped with additional computationally relevant structure, such as ordered algebras, continuous algebras, quantitative algebras, nominal algebras, or profinite algebras. Our main contributions are a generic HSP theorem and a sound and complete equational logic, which are shown to encompass numerous flavors of equational axiomizations studied in the literature.

1 Introduction

A key tool in the algebraic theory of data structures is their specification by operations (constructors) and equations that they ought to satisfy. Hence, the study of models of equational specifications has been of long standing interest both in mathematics and computer science. The seminal result in this field is Birkhoff's celebrated HSP theorem [8]. It states that a class of algebras over a signature Σ is a *variety* (i.e. closed under homomorphic images, subalgebras, and products) iff it is axiomatizable by equations $s = t$ between Σ -terms. Birkhoff also introduced a complete deduction system for reasoning about equations.

In algebraic approaches to the semantics of programming languages and computational effects, it is often natural to study algebras whose underlying sets are equipped with additional computationally relevant structure and whose operations preserve that structure. An important line of research thus concerns extensions of Birkhoff's theory of equational axiomatization beyond ordinary Σ -algebras. On the syntactic level, this requires to enrich Birkhoff's notion of an equation in ways that reflect the extra structure. Let us mention a few examples:

(1) *Ordered algebras* (given by a poset and monotone operations) and *continuous algebras* (given by a complete partial order and continuous operations) were identified by the ADJ group [15] as an important tool in denotational semantics. Subsequently, Bloom [9] and Adámek, Nelson, and Reiterman [2, 4] established ordered versions of the HSP theorem along with complete deduction systems. Here, the role of equations $s = t$ is taken over by inequations $s \leq t$.

* Supported by Deutsche Forschungsgemeinschaft (DFG) under project MI 717/5-1

** Supported by Deutsche Forschungsgemeinschaft (DFG) under project SCHR 1118/8-

(2) *Quantitative algebras* (given by an extended metric space and nonexpansive operations) naturally arise as semantic domains in the theory of probabilistic computation. In recent work, Mardare, Panangaden, and Plotkin [19, 20] presented an HSP theorem for quantitative algebras and a complete deduction system. In the quantitative setting, equations $s =_\varepsilon t$ are equipped with a non-negative real number ε , interpreted as “ s and t have distance at most ε ”.

(3) *Nominal algebras* (given by a nominal set and equivariant operations) are used in the theory of name binding [24] and have proven useful for characterizing logics for data languages [10, 12]. Varieties of nominal algebras were studied by Gabbay [14] and Kurz and Petrişan [17]. Here, the appropriate syntactic concept involves equations $s = t$ with constraints on the support of their variables.

(4) *Profinite algebras* (given by a profinite topological space and continuous operations) play a central role in the algebraic theory of formal languages [22]. They serve as a technical tool in the investigation of *pseudovarieties* (i.e. classes of finite algebras closed under homomorphic images, subalgebras, and finite products). As shown by Reiterman [25] and Eilenberg and Schützenberger [13], pseudovarieties can be axiomatized by *profinite equations* (formed over free profinite algebras) or, equivalently, by sequences of ordinary equations $(s_i = t_i)_{i < \omega}$, interpreted as “all but finitely many of the equations $s_i = t_i$ hold”.

The present paper proposes a general category theoretic framework that allows to study classes of algebras with extra structure in a systematic way. Our overall goal is to isolate the domain-specific part of any theory of equational axiomatization from its generic core. Our framework is parametric in the following data:

- a category \mathcal{A} with a factorization system $(\mathcal{E}, \mathcal{M})$;
- a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$;
- a class Λ of cardinal numbers;
- a class $\mathcal{X} \subseteq \mathcal{A}$ of objects.

Here, \mathcal{A} is the category of algebras under consideration (e.g. ordered algebras, quantitative algebras, nominal algebras). Varieties are formed within \mathcal{A}_0 , and the cardinal numbers in Λ determine the arities of products under which the varieties are closed. Thus, the choice $\mathcal{A}_0 = \text{finite algebras}$ and $\Lambda = \text{finite cardinals}$ corresponds to pseudovarieties, and $\mathcal{A}_0 = \mathcal{A}$ and $\Lambda = \text{all cardinals}$ to varieties. The crucial ingredient of our setting is the parameter \mathcal{X} , which is the class of objects over which equations are formed; thus, typically, \mathcal{X} is chosen to be some class of freely generated algebras in \mathcal{A} . Equations are modeled as \mathcal{E} -quotients $e: X \twoheadrightarrow E$ (more generally, filters of such quotients) with domain $X \in \mathcal{X}$.

The choice of \mathcal{X} reflects the desired expressivity of equations in a given setting. Furthermore, it determines the type of quotients under which equationally axiomatizable classes are closed. More precisely, in our general framework a *variety* is defined to be a subclass of \mathcal{A}_0 closed under $\mathcal{E}_{\mathcal{X}}$ -quotients, \mathcal{M} -subobjects, and Λ -products, where $\mathcal{E}_{\mathcal{X}}$ is a subclass of \mathcal{E} derived from \mathcal{X} . Due to its para-

metric nature, this concept of a variety is widely applicable and turns out to specialize to many interesting cases. The main result of our paper is the

General HSP Theorem. *A subclass of \mathcal{A}_0 forms a variety if and only if it is axiomatizable by equations.*

In addition, we introduce a generic deduction system for equations, based on two simple proof rules (see Section 4), and establish a

General Completeness Theorem. *The generic deduction system for equations is sound and complete.*

The above two theorems can be seen as the generic building blocks of the model theory of algebras with structure. They form the common core of numerous Birkhoff-type results and give rise to a systematic recipe for deriving concrete HSP and completeness theorems in settings such as (1)–(4). In fact, all that needs to be done is to translate our abstract notion of equation and equational deduction, which involves (filters of) quotients, into an appropriate syntactic concept. This is the domain-specific task to fulfill, and usually amounts to identifying an “exactness” property for the category \mathcal{A} . Subsequently, one can apply our general results to obtain HSP and completeness theorems for the type of algebras under consideration. Several instances of this approach are shown in Section 5. Proofs of all results and details for the examples can be found in the Appendix.

Related work. Generic approaches to universal algebra have a long tradition in category theory. They aim to replace syntactic notions like terms and equations by suitable categorical abstractions, most prominently Lawvere theories and monads [5,18]. Our present work draws much of its inspiration from the classical paper of Banaschewski and Herrlich [7] on HSP classes in $(\mathcal{E}, \mathcal{M})$ -structured categories. These authors were the first to model equations as quotients $e: X \rightarrow E$. However, their approach does not feature the parameter \mathcal{X} and assumes that equations are formed over \mathcal{E} -projective objects X . This limits the scope of their results to categories with enough projectives, a property that frequently fails in categories of algebras with structure (including continuous, quantitative or nominal algebras). The introduction of the parameter \mathcal{X} in our paper, along with the identification of the derived parameter $\mathcal{E}_{\mathcal{X}}$ as a key concept, is therefore a crucial step in order to gain a categorical understanding of such structures.

Equational logics on the level of abstraction of Banaschewski and Herrlich’s work were studied by Roşu [27,28] and Adámek, Hébert, and Sousa [1]. These authors work under assumptions on the category \mathcal{A} different from our framework, e.g. they require existence of pushouts. Hence, the proof rules and completeness results in *loc. cit.* are not directly comparable to our approach in Section 4.

In the present paper, we opted to model equations as filters of quotients rather than single quotients, which allows us to encompass several HSP theorems for finite algebras [13,23,25]. The first categorical generalization of such results was given by Adámek, Chen, Milius, and Urbat [11,30] who considered algebras for a monad \mathbb{T} on an algebraic category and modeled equations as filters of finite quotients of free \mathbb{T} -algebras (equivalently, as profinite quotients of free profinite

T-algebras). This idea was further generalized by Salamanca [29] to monads on concrete categories. However, again, this work only applies to categories with enough projectives, which excludes most of our present applications.

Acknowledgement. The authors would like to thank Thorsten Wißmann for insightful discussions on nominal sets.

2 Preliminaries

We start by recalling some notions from category theory. A *factorization system* $(\mathcal{E}, \mathcal{M})$ in a category \mathcal{A} consists of two classes \mathcal{E}, \mathcal{M} of morphisms in \mathcal{A} such that (1) both \mathcal{E} and \mathcal{M} contain all isomorphisms and are closed under composition, (2) every morphism f has a factorization $f = m \cdot e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and (3) the *diagonal fill-in* property holds: for every commutative square $g \cdot e = m \cdot f$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique d with $m \cdot d = g$ and $d \cdot e = f$. The morphisms m and e in (2) are unique up to isomorphism and are called the *image* and *coimage* of f , resp. The factorization system is *proper* if all morphisms in \mathcal{E} are epic and all morphisms in \mathcal{M} are monic. From now on, we will assume that \mathcal{A} is a category equipped with a proper factorization system $(\mathcal{E}, \mathcal{M})$. Quotients and subobjects in \mathcal{A} are taken with respect to \mathcal{E} and \mathcal{M} . That is, a *quotient* of an object X is represented by a morphism $e: X \twoheadrightarrow E$ in \mathcal{E} and a *subobject* by a morphism $m: M \rightarrowtail X$ in \mathcal{M} . The quotients of X are ordered by $e \leq e'$ iff e' factorizes through e , i.e. there exists a morphism h with $e' = h \cdot e$. Identifying quotients e and e' which are isomorphic (i.e. $e \leq e'$ and $e' \leq e$), this makes the quotients of X a partially ordered class. Given a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ we denote by $X \downarrow_{\mathcal{A}_0}$ the class of all quotients of X represented by \mathcal{E} -morphisms with codomain in \mathcal{A}_0 . The category \mathcal{A} is *\mathcal{E} -co-wellpowered* if for every object $X \in \mathcal{A}$ there is only a set of quotients with domain X . In particular, $X \downarrow_{\mathcal{A}_0}$ is then a *poset*. Finally, an object $X \in \mathcal{A}$ is called *projective* w.r.t. a morphism $e: A \rightarrow B$ if for every $h: X \rightarrow B$, there exists a morphism $g: X \rightarrow A$ with $h = e \cdot g$.

3 The Generalized Variety Theorem

In this section, we introduce our categorical notions of equation and variety, and derive the HSP theorem. For the rest of the paper, we fix the data mentioned in the introduction: a category \mathcal{A} with a proper factorization system $(\mathcal{E}, \mathcal{M})$, a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$, a class Λ of cardinal numbers, and a class $\mathcal{X} \subseteq \mathcal{A}$ of objects. An object of \mathcal{A} is called *\mathcal{X} -generated* if it is a quotient of some object in \mathcal{X} . A key role in the following development will be played by the subclass $\mathcal{E}_{\mathcal{X}} \subseteq \mathcal{E}$ defined by

$$\mathcal{E}_{\mathcal{X}} = \{ e \in \mathcal{E} : \text{every } X \in \mathcal{X} \text{ is projective w.r.t. } e \}.$$

Note that $\mathcal{X} \subseteq \mathcal{X}'$ implies $\mathcal{E}_{\mathcal{X}'} \subseteq \mathcal{E}_{\mathcal{X}}$. The choice of \mathcal{X} is a trade-off between “having enough equations” (that is, \mathcal{X} needs to be rich enough to make equations sufficiently expressive) and “having enough projectives” (that is, $\mathcal{E}_{\mathcal{X}}$ needs to generate \mathcal{A}_0 , as stated in (3) below).

Assumptions 3.1. Our data is required to satisfy the following properties:

- (1) \mathcal{A} has Λ -products, i.e. for every $\lambda \in \Lambda$ and every family $(A_i)_{i < \lambda}$ of objects in \mathcal{A} , the product $\prod_{i < \lambda} A_i$ exists.
- (2) \mathcal{A}_0 is closed under isomorphisms, Λ -products and \mathcal{X} -generated subobjects. The last statement means that for every subobject $m: A \rightarrow B$ in \mathcal{M} where $B \in \mathcal{A}_0$ and A is \mathcal{X} -generated, one has $A \in \mathcal{A}_0$.
- (3) Every object of \mathcal{A}_0 is an $\mathcal{E}_{\mathcal{X}}$ -quotient of some object of \mathcal{X} , that is, for every object $A \in \mathcal{A}_0$ there exists some $e: X \rightarrow A$ in $\mathcal{E}_{\mathcal{X}}$ with domain $X \in \mathcal{X}$.

Examples 3.2. Throughout this section, we will use the following three running examples to illustrate our concepts. For further applications, see Section 5.

(1) *Classical Σ -algebras.* The setting of Birkhoff's seminal work [8] in general algebra is that of algebras for a signature. Recall that a (*finitary*) *signature* is a set Σ of operation symbols each with a prescribed finite arity, and a Σ -*algebra* is a set A equipped with operations $\sigma: A^n \rightarrow A$ for each n -ary $\sigma \in \Sigma$. A *morphism* of Σ -algebras (or a Σ -*homomorphism*) is a map preserving all Σ -operations. The forgetful functor from the category $\mathbf{Alg}(\Sigma)$ of Σ -algebras and Σ -homomorphisms to \mathbf{Set} has a left adjoint assigning to each set X the *free Σ -algebra* $T_{\Sigma}X$, carried by the set of all Σ -terms in variables from X . To treat Birkhoff's results in our categorical setting, we choose the following parameters:

- $\mathcal{A} = \mathcal{A}_0 = \mathbf{Alg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, injective morphisms})$;
- $\Lambda =$ all cardinal numbers;
- $\mathcal{X} =$ all free Σ -algebras $T_{\Sigma}X$ with $X \in \mathbf{Set}$.

One easily verifies that $\mathcal{E}_{\mathcal{X}}$ consists of all surjective morphisms, that is, $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$.

(2) *Finite Σ -algebras.* Eilenberg and Schützenberger [13] considered classes of finite Σ -algebras, where Σ is assumed to be a signature with only finitely many operation symbols. In our framework, this amounts to choosing

- $\mathcal{A} = \mathbf{Alg}(\Sigma)$ and $\mathcal{A}_0 = \mathbf{Alg}_f(\Sigma)$, the full subcategory of finite Σ -algebras;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, injective morphisms})$;
- $\Lambda =$ all finite cardinal numbers;
- $\mathcal{X} =$ all free Σ -algebras $T_{\Sigma}X$ with $X \in \mathbf{Set}_f$.

As in (1), the class $\mathcal{E}_{\mathcal{X}}$ consists of all surjective morphisms.

(3) *Quantitative Σ -algebras.* In recent work, Mardare, Panangaden, and Plotkin [19,20] extended Birkhoff's theory to algebras endowed with a metric. Recall that an *extended metric space* is a set A with a map $d_A: A \times A \rightarrow [0, \infty]$ (assigning to any two points a possibly infinite distance), subject to the axioms (i) $d_A(a, b) = 0$ iff $a = b$, (ii) $d_A(a, b) = d_A(b, a)$, and (iii) $d_A(a, c) \leq d_A(a, b) + d_A(b, c)$ for all $a, b, c \in A$. A map $h: A \rightarrow B$ between extended metric spaces is *nonexpansive* if $d_B(h(a), h(a')) \leq d_A(a, a')$ for $a, a' \in A$. Let \mathbf{Met}_{∞} denote the category of extended metric spaces and nonexpansive maps. Fix a, not necessarily finitary,

signature Σ , that is, the arity of an operation symbol $\sigma \in \Sigma$ is any cardinal number. A *quantitative Σ -algebra* is a Σ -algebra A endowed with an extended metric d_A such that all Σ -operations $\sigma: A^n \rightarrow A$ are nonexpansive. Here, the product A^n is equipped with the sup-metric $d_{A^n}((a_i)_{i < n}, (b_i)_{i < n}) = \sup_{i < n} d_A(a_i, b_i)$. The forgetful functor from the category $\mathbf{QAlg}(\Sigma)$ of quantitative Σ -algebras and nonexpansive Σ -homomorphisms to \mathbf{Met}_∞ has a left adjoint assigning to each space X the free quantitative Σ -algebra $T_\Sigma X$. The latter is carried by the set of all Σ -terms (equivalently, well-founded Σ -trees) over X , with metric inherited from X as follows: if s and t are Σ -terms of the same shape, i.e. they differ only in the variables, their distance is the supremum of the distances of the variables in corresponding positions of s and t ; otherwise, it is ∞ .

We aim to derive the HSP theorem for quantitative algebras proved by Mardare et al. as an instance of our general results. The theorem is parametric in a regular cardinal number $c > 1$. In the following, an extended metric space is called *c-clustered* if it is a coproduct of spaces of size $< c$. Note that coproducts in \mathbf{Met}_∞ are formed on the level of underlying sets. Choose the parameters

- $\mathcal{A} = \mathcal{A}_0 = \mathbf{QAlg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M})$ given by morphisms carried by surjections and subspaces, resp.;
- $\Lambda =$ all cardinal numbers;
- $\mathcal{X} =$ all free algebras $T_\Sigma X$ with $X \in \mathbf{Met}_\infty$ a c -clustered space.

One can verify that a quotient $e: A \twoheadrightarrow B$ belongs to $\mathcal{E}_{\mathcal{X}}$ if and only if for each subset $B_0 \subseteq B$ of cardinality $< c$ there exists a subset $A_0 \subseteq A$ such that $e[A_0] = B_0$ and the restriction $e: A_0 \rightarrow B_0$ is isometric (that is, $d_B(e(a), e(a')) = d_A(a, a')$ for $a, a' \in A_0$). Following the terminology of Mardare et al., such a quotient is called *c-reflexive*. Note that for $c = 2$ every quotient is c -reflexive, so $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$. If c is infinite, $\mathcal{E}_{\mathcal{X}}$ is a proper subclass of \mathcal{E} .

Definition 3.3. An *equation over $X \in \mathcal{X}$* is a class $\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0$ that is

- (1) *Λ -codirected:* every subset $F \subseteq \mathcal{T}_X$ with $|F| \in \Lambda$ has a lower bound in F ;
- (2) *closed under $\mathcal{E}_{\mathcal{X}}$ -quotients:* for every $e: X \twoheadrightarrow E$ in \mathcal{T}_X and $q: E \twoheadrightarrow E'$ in $\mathcal{E}_{\mathcal{X}}$ with $E' \in \mathcal{A}_0$, one has $q \cdot e \in \mathcal{T}_X$.

An object $A \in \mathcal{A}$ *satisfies* the equation \mathcal{T}_X if every morphism $h: X \rightarrow A$ factorizes through some $e \in \mathcal{T}_X$. In this case, we write

$$A \models \mathcal{T}_X.$$

Remark 3.4. In many of our applications, one can simplify the above definition and replace classes of quotients by single quotients. Specifically, if \mathcal{A} is \mathcal{E} -co-wellpowered (so that every equation is a set, not a class) and $\Lambda =$ all cardinal numbers, then every equation $\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0$ contains a least element $e_X: X \rightarrow E_X$, viz. the lower bound of all elements in \mathcal{T}_X . Then an object A satisfies \mathcal{T}_X iff it satisfies e_X , in the sense that every morphism $h: X \rightarrow A$ factorizes through e_X . Therefore, in this case, one may equivalently define an equation to be a morphism $e_X: X \rightarrow E_X$ with $X \in \mathcal{X}$. This is the concept of equation investigated by Banaschewski and Herrlich [7].

Examples 3.5. In our running examples, we obtain the following concepts:

- (1) *Classical Σ -algebras.* By Remark 3.4, an equation corresponds to a quotient $e_X : T_\Sigma X \twoheadrightarrow E_X$ in $\mathbf{Alg}(\Sigma)$, where X is a set of variables.
- (2) *Finite Σ -algebras.* An equation \mathcal{T}_X over a finite set X is precisely a filter (i.e. a codirected and upwards closed subset) in the poset $T_\Sigma X \downarrow \mathbf{Alg}_f(\Sigma)$.
- (3) *Quantitative Σ -algebras.* By Remark 3.4, an equation can be presented as a quotient $e_X : T_\Sigma X \twoheadrightarrow E_X$ in $\mathbf{QAlg}(\Sigma)$, where X is a c -clustered space.

We shall demonstrate in Section 5 how to interpret the above abstract notions of equations, i.e. (filters of) quotients of free algebras, in terms of concrete syntax.

Definition 3.6. A *variety* is a full subcategory $\mathcal{V} \subseteq \mathcal{A}_0$ closed under \mathcal{E}_x -quotients, subobjects, and \wedge -products. More precisely,

- (1) for every \mathcal{E}_x -quotient $e : A \twoheadrightarrow B$ in \mathcal{A}_0 with $A \in \mathcal{V}$ one has $B \in \mathcal{V}$,
- (2) for every \mathcal{M} -morphism $m : A \rightarrow B$ in \mathcal{A}_0 with $B \in \mathcal{V}$ one has $A \in \mathcal{V}$, and
- (3) for every family of objects A_i ($i < \lambda$) in \mathcal{V} with $\lambda \in \Lambda$ one has $\prod_{i < \lambda} A_i \in \mathcal{V}$.

Examples 3.7. In our examples, we obtain the following notions of varieties:

- (1) *Classical Σ -algebras.* A *variety of Σ -algebras* is a class of Σ -algebras closed under quotient algebras, subalgebras, and products. This is Birkhoff's original concept [8].
- (2) *Finite Σ -algebras.* A *pseudovariety of Σ -algebras* is a class of finite Σ -algebras closed under quotient algebras, subalgebras, and finite products. This concept was studied by Eilenberg and Schützenberger [13].
- (3) *Quantitative Σ -algebras.* For any regular cardinal number $c > 1$, a *c -variety of quantitative Σ -algebras* is a class of quantitative Σ -algebras closed under c -reflexive quotients, subalgebras, and products. This notion of a variety was introduced by Mardare et al. [20].

Construction 3.8. Given a class \mathbb{E} of equations, put

$$\mathcal{V}(\mathbb{E}) = \{ A \in \mathcal{A}_0 : A \models \mathcal{T}_X \text{ for each } \mathcal{T}_X \in \mathbb{E} \}.$$

A subclass $\mathcal{V} \subseteq \mathcal{A}_0$ is called *equationally presentable* if $\mathcal{V} = \mathcal{V}(\mathbb{E})$ for some \mathbb{E} .

We aim to show that varieties coincide with the equationally presentable classes (see Theorem 3.16 below). The “easy” part of the correspondence is established by the following lemma, which is proved by a straightforward verification.

Lemma 3.9. *For every class \mathbb{E} of equations, $\mathcal{V}(\mathbb{E})$ is a variety.*

As a technical tool for establishing the general HSP theorem and the corresponding sound and complete equational logic, we introduce the following concept:

Definition 3.10. An *equational theory* is a family of equations

$$\mathcal{T} = (\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0)_{X \in \mathcal{X}}$$

with the following two properties (illustrated by the diagrams below):

(1) *Substitution invariance.* For every morphism $h: X \rightarrow Y$ with $X, Y \in \mathcal{X}$ and every $e_Y: Y \rightarrow E_Y$ in \mathcal{T}_Y , the coimage $e_X: X \rightarrow E_X$ of $e_Y \cdot h$ lies in \mathcal{T}_X .

(2) $\mathcal{E}_{\mathcal{X}}$ -*completeness.* For every $Y \in \mathcal{X}$ and every quotient $e: Y \rightarrow E_Y$ in \mathcal{T}_Y , there exists an $X \in \mathcal{X}$ and a quotient $e_X: X \rightarrow E_X$ in $\mathcal{T}_X \cap \mathcal{E}_{\mathcal{X}}$ with $E_X = E_Y$.

$$\begin{array}{ccc} X & \xrightarrow{\forall h} & Y \\ e_X \downarrow & & \downarrow \forall e_Y \\ E_X & \xrightarrow{\quad} & E_Y \end{array} \qquad \begin{array}{ccc} X & & Y \\ \exists e_X \downarrow & & \downarrow \forall e_Y \\ E_X & \equiv & E_Y \end{array}$$

Remark 3.11. In many settings, the slightly technical concept of an equational theory can be simplified. First, note that $\mathcal{E}_{\mathcal{X}}$ -completeness is trivially satisfied whenever $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$. If, additionally, every equation contains a least element (e.g. in the setting of Remark 3.4), an equational theory corresponds exactly to a family of quotients $(e_X: X \rightarrow E_X)_{X \in \mathcal{X}}$ such that $E_X \in \mathcal{A}_0$ for all $X \in \mathcal{X}$, and for every $h: X \rightarrow Y$ with $X, Y \in \mathcal{X}$ the morphism $e_Y \cdot h$ factorizes through e_X .

Example 3.12 (Classical Σ -algebras). Recall that a *congruence* on a Σ -algebra A is an equivalence relation $\equiv \subseteq A \times A$ that forms a subalgebra of $A \times A$. It is well-known that there is an isomorphism of complete lattices

$$\text{quotient algebras of } A \cong \text{congruences on } A \quad (3.1)$$

assigning to a quotient $e: A \rightarrow B$ its *kernel*, given by $a \equiv_e a'$ iff $e(a) = e(a')$. Consequently, in the setting of Example 3.2(1), an equational theory – presented as a family of single quotients as in Remark 3.11 – corresponds precisely to a family of congruences $(\equiv_X \subseteq T_{\Sigma}X \times T_{\Sigma}X)_{X \in \mathbf{Set}}$ closed under substitution, that is, for every $s, t \in T_{\Sigma}X$ and every morphism $h: T_{\Sigma}X \rightarrow T_{\Sigma}Y$ in $\mathbf{Alg}(\Sigma)$,

$$s \equiv_X t \quad \text{implies} \quad h(s) \equiv_Y h(t).$$

We saw in Lemma 3.9 that every class of equations, so in particular every equational theory \mathcal{T} , yields a variety $\mathcal{V}(\mathcal{T})$ consisting of all objects of \mathcal{A}_0 that satisfy every equation in \mathcal{T} . Conversely, to every variety one can associate an equational theory as follows:

Construction 3.13. Given a variety \mathcal{V} , form the family of equations

$$\mathcal{T}(\mathcal{V}) = (\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0)_{X \in \mathcal{X}},$$

where \mathcal{T}_X consists of all quotients $e_X: X \rightarrow E_X$ with codomain $E_X \in \mathcal{V}$.

Lemma 3.14. *For every variety \mathcal{V} , the family $\mathcal{T}(\mathcal{V})$ is an equational theory.*

We are ready to state the first main result of our paper, the HSP Theorem. Given two equations \mathcal{T}_X and \mathcal{T}'_X over $X \in \mathcal{X}$, we put $\mathcal{T}_X \leq \mathcal{T}'_X$ if every quotient in \mathcal{T}'_X factorizes through some quotient in \mathcal{T}_X . Theories form a poset with respect to the order $\mathcal{T} \leq \mathcal{T}'$ iff $\mathcal{T}_X \leq \mathcal{T}'_X$ for all $X \in \mathcal{X}$. Similarly, varieties form a poset (in fact, a complete lattice) ordered by inclusion.

Theorem 3.15 (HSP Theorem). *The complete lattices of equational theories and varieties are dually isomorphic. The isomorphism is given by*

$$\mathcal{V} \mapsto \mathcal{T}(\mathcal{V}) \quad \text{and} \quad \mathcal{T} \mapsto \mathcal{V}(\mathcal{T}).$$

One can recast the HSP Theorem into a more familiar form, using equations in lieu of equational theories:

Theorem 3.16 (HSP Theorem, equational version). *A class $\mathcal{V} \subseteq \mathcal{A}_0$ is equationally presentable if and only if it forms a variety.*

Proof. By Lemma 3.9, every equationally presentable class $\mathcal{V}(\mathbb{E})$ is a variety. Conversely, for every variety \mathcal{V} one has $\mathcal{V} = \mathcal{V}(\mathcal{T}(\mathcal{V}))$ by Theorem 3.15, so \mathcal{V} is presented by the equations $\mathbb{E} = \{ \mathcal{T}_X : X \in \mathcal{X} \}$ where $\mathcal{T} = \mathcal{T}(\mathcal{V})$. \square

4 Equational Logic

The correspondence between theories and varieties gives rise to the second main result of our paper, a generic sound and complete deduction system for reasoning about equations. The corresponding semantic concept is the following:

Definition 4.1. An equation $\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0$ *semantically entails* the equation $\mathcal{T}'_Y \subseteq Y \downarrow \mathcal{A}_0$ if every \mathcal{A}_0 -object satisfying \mathcal{T}_X also satisfies \mathcal{T}'_Y (that is, if $\mathcal{V}(\mathcal{T}_X) \subseteq \mathcal{V}(\mathcal{T}'_Y)$). In this case, we write $\mathcal{T}_X \models \mathcal{T}'_Y$.

The key to our proof system is a categorical formulation of term substitution:

Definition 4.2. Let $\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0$ be an equation over $X \in \mathcal{X}$. The *substitution closure* of \mathcal{T}_X is the smallest theory $\overline{\mathcal{T}} = (\overline{\mathcal{T}}_Y)_{Y \in \mathcal{X}}$ such that $\mathcal{T}_X \leq \overline{\mathcal{T}}_X$.

The substitution closure of an equation can be computed as follows:

Lemma 4.3. *For every equation $\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0$ one has $\overline{\mathcal{T}} = \mathcal{T}(\mathcal{V}(\mathcal{T}_X))$.*

The deduction system for semantic entailment consists of two proof rules:

- (Weakening) $\mathcal{T}_X \vdash \mathcal{T}'_X$ for all equations $\mathcal{T}'_X \leq \mathcal{T}_X$ over $X \in \mathcal{X}$.
- (Substitution) $\mathcal{T}_X \vdash \overline{\mathcal{T}}_Y$ for all equations \mathcal{T}_X over $X \in \mathcal{X}$ and all $Y \in \mathcal{X}$.

Given equations \mathcal{T}_X and \mathcal{T}'_Y over X and Y , respectively, we write $\mathcal{T}_X \vdash \mathcal{T}'_Y$ if \mathcal{T}'_Y arises from \mathcal{T}_X by a finite chain of applications of the above rules.

Theorem 4.4 (Completeness Theorem). *The deduction system for semantic entailment is sound and complete: for every pair of equations \mathcal{T}_X and \mathcal{T}'_Y ,*

$$\mathcal{T}_X \models \mathcal{T}'_Y \quad \text{iff} \quad \mathcal{T}_X \vdash \mathcal{T}'_Y.$$

5 Applications

In this section, we present some of the applications of our categorical results (see Appendix B for full details). Transferring the general HSP theorem of Section 3 into a concrete setting requires to perform the following four-step procedure:

Step 1. Instantiate the parameters \mathcal{A} , $(\mathcal{E}, \mathcal{M})$, \mathcal{A}_0 , A and \mathcal{X} of our categorical framework, and characterize the quotients in $\mathcal{E}_{\mathcal{X}}$.

Step 2. Establish an *exactness property* for the category \mathcal{A} , i.e. a correspondence between quotients $e: A \twoheadrightarrow B$ in \mathcal{A} and suitable relations between elements of A .

Step 3. Infer a suitable syntactic notion of equation, and prove it to be expressively equivalent to the categorical notion of equation given by Definition 3.3.

Step 4. Invoke Theorem 3.15 to deduce an HSP theorem.

The details of Steps 2 and 3 are application-specific, but typically straightforward. In each case, the bulk of the usual work required for establishing the HSP theorem is moved to our general categorical results and thus comes for free.

Similarly, to obtain a complete deduction system in a concrete application, it suffices to phrase the two proof rules of our generic equational logic in syntactic terms, using the correspondence of quotients and relations from Step 2; then Theorem 4.4 gives the completeness result.

5.1 Classical Σ -Algebras

The classical Birkhoff theorem emerges from our general results as follows.

Step 1. Choose the parameters of Example 3.2(1), and recall that $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$.

Step 2. The exactness property of $\mathbf{Alg}(\Sigma)$ is given by the correspondence (3.1).

Step 3. Recall from Example 3.5(1) that equations can be presented as single quotients $e: T_{\Sigma}X \twoheadrightarrow E_X$. The exactness property (3.1) leads to the following classical syntactic concept: a *term equation* over a set X of variables is a pair $(s, t) \in T_{\Sigma}X \times T_{\Sigma}X$, denoted as $s = t$. It is *satisfied* by a Σ -algebra A if for every map $h: X \rightarrow A$ we have $h^{\sharp}(s) = h^{\sharp}(t)$. Here, $h^{\sharp}: T_{\Sigma}X \rightarrow A$ denotes the unique extension of h to a Σ -homomorphism. Equations and term equations are expressively equivalent in the following sense:

(1) For every equation $e: T_{\Sigma}X \twoheadrightarrow E_X$, the kernel $\equiv_e \subseteq T_{\Sigma}X \times T_{\Sigma}X$ is a set of term equations equivalent to e , that is, a Σ -algebra satisfies the equation e iff it satisfies all term equations in \equiv_e . This follows immediately from (3.1).

(2) Conversely, given a term equation $(s, t) \in T_{\Sigma}X \times T_{\Sigma}X$, form the smallest congruence \equiv on $T_{\Sigma}X$ with $s \equiv t$ (viz. the intersection of all such congruences) and let $e: T_{\Sigma}X \twoheadrightarrow E_X$ be the corresponding quotient. Then a Σ -algebra satisfies $s = t$ iff it satisfies e . Again, this is a consequence of (3.1).

Step 4. From Theorem 3.16 and Example 3.7(1), we deduce the classical

Theorem 5.1 (Birkhoff [8]). *A class of Σ -algebras is a variety (i.e. closed under quotients, subalgebras, products) iff it is axiomatizable by term equations.*

Similarly, one can obtain Birkhoff's complete deduction system for term equations as an instance of Theorem 4.4; see Appendix B.1 for details.

5.2 Finite Σ -Algebras

Next, we derive Eilenberg and Schützenberger's equational characterization of pseudovarieties of algebras over a finite signature Σ using our four-step plan:

Step 1. Choose the parameters of Example 3.2(2), and recall that $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$.

Step 2. The exactness property of $\mathbf{Alg}(\Sigma)$ is given by (3.1).

Step 3. By Example 3.2(2), an equational theory is given by a family of filters $\mathcal{T}_n \subseteq T_{\Sigma n} \downarrow \mathbf{Alg}_f(\Sigma)$ ($n < \omega$). The corresponding syntactic concept involves sequences $(s_i = t_i)_{i < \omega}$ of term equations. We say that a finite Σ -algebra A *eventually satisfies* such a sequence if there exists $i_0 < \omega$ such that A satisfies all equations $s_i = t_i$ with $i \geq i_0$. Equational theories and sequences of term equations are expressively equivalent:

- (1) Let $\mathcal{T} = (\mathcal{T}_n)_{n < \omega}$ be a theory. Since Σ is a finite signature, for each finite quotient $e: T_{\Sigma n} \rightarrow E$ the kernel \equiv_e is a finitely generated congruence [13, Prop. 2]. Consequently, for each $n < \omega$ the algebra $T_{\Sigma n}$ has only countably many finite quotients. In particular, the codirected poset \mathcal{T}_n is countable, so it contains an ω^{op} -chain $e_0^n \geq e_1^n \geq e_2^n \geq \dots$ that is *cofinal*, i.e., each $e \in \mathcal{T}_n$ is above some e_i^n . The e_i^n can be chosen in such a way that, for each $m > n$ and $q: m \rightarrow n$, the morphism $e_i^n \cdot T_{\Sigma q}$ factorizes through e_i^m . For each $n < \omega$, choose a finite subset $W_n \subseteq T_{\Sigma n} \times T_{\Sigma n}$ generating the kernel of e_n^n . Let $(s_i = t_i)_{i < \omega}$ be a sequence of term equations where (s_i, t_i) ranges over $\bigcup_{n < \omega} W_n$. One can verify that a finite Σ -algebra lies in $\mathcal{V}(\mathcal{T})$ iff it eventually satisfies $(s_i = t_i)_{i < \omega}$.
- (2) Conversely, given a sequence of term equations $(s_i = t_i)_{i < \omega}$ with $(s_i, t_i) \in T_{\Sigma m_i} \times T_{\Sigma m_i}$, form the theory $\mathcal{T} = (\mathcal{T}_n)_{n < \omega}$ where \mathcal{T}_n consists of all finite quotients $e: T_{\Sigma n} \rightarrow E$ with the following property:

$$\exists i_0 < \omega : \forall i \geq i_0 : \forall (g: T_{\Sigma m_i} \rightarrow T_{\Sigma n}) : e \cdot g(s_i) = e \cdot g(t_i).$$

Then a finite Σ -algebra eventually satisfies $(s_i = t_i)_{i < \omega}$ iff it lies in $\mathcal{V}(\mathcal{T})$.

Step 4. The theory version of our HSP theorem (Theorem 3.16) now implies:

Theorem 5.2 (Eilenberg-Schützenberger [13]).

A class of finite Σ -algebras is a pseudovariety (i.e. closed under quotients, subalgebras, and finite products) iff it is axiomatizable by a sequence of term equations.

In an alternative characterization of pseudovarieties due to Reiterman [25], where the restriction to finite signatures Σ can be dropped, sequences of term equations are replaced by the topological concept of a *profinite equation*. This result can also be derived from our general HSP theorem, see Appendix B.4.

5.3 Quantitative Algebras

In this section, we derive an HSP theorem for quantitative algebras.

Step 1. Choose the parameters of Example 3.2(3). Recall that we work with fixed regular cardinal $c > 1$ and that $\mathcal{E}_{\mathcal{X}}$ consists of all c -reflexive quotients.

Step 2. To state the exactness property of $\mathbf{QAlg}(\Sigma)$, recall that an (*extended*) *pseudometric* on a set A is a map $p: A \times A \rightarrow [0, \infty]$ satisfying all axioms of an extended metric except possibly the implication $p(a, b) = 0 \Rightarrow a = b$. Given a quantitative Σ -algebra A , a pseudometric p on A is called a *congruence* if (i) $p(a, a') \leq d_A(a, a')$ for all $a, a' \in A$, and (ii) every Σ -operation $\sigma: A^n \rightarrow A$ ($\sigma \in \Sigma$) is nonexpansive w.r.t. p . Congruences are ordered by $p \leq q$ iff $p(a, a') \leq q(a, a')$ for all $a, a' \in A$. There is a dual isomorphism of complete lattices

$$\text{quotient algebras of } A \cong \text{congruences on } A \quad (5.1)$$

mapping $e: A \rightarrow B$ to the congruence p_e on A given by $p_e(a, b) = d_B(e(a), e(b))$.

Step 3. By Example 3.5(3), equations can be presented as single quotients $e: T_{\Sigma}X \rightarrow E$, where X is a c -clustered space. The exactness property (5.1) suggests to replace equations by the following syntactic concept. A *c -clustered equation* over the set X of variables is an expression

$$x_i =_{\varepsilon_i} y_i \ (i \in I) \vdash s =_{\varepsilon} t \quad (5.2)$$

where (i) I is a set, (ii) $x_i, y_i \in X$ for all $i \in I$, (iii) s and t are Σ -terms over X , (iv) $\varepsilon_i, \varepsilon \in [0, \infty]$, and (v) the equivalence relation on X generated by the pairs (x_i, y_i) ($i \in I$) has all equivalence classes of cardinality $< c$. In other words, the set of variables can be partitioned into subsets of size $< c$ such that only relations between variables in the same subset appear on the left-hand side of (5.2). A quantitative Σ -algebra A *satisfies* (5.2) if for every map $h: X \rightarrow A$ with $d_A(h(x_i), h(y_i)) \leq \varepsilon_i$ for all $i \in I$, one has $d_A(h^{\sharp}(s), h^{\sharp}(t)) \leq \varepsilon$. Here $h^{\sharp}: T_{\Sigma}X \rightarrow A$ denotes the unique Σ -homomorphism extending h .

Equations and c -clustered equations are expressively equivalent:

(1) Let X be a c -clustered space, i.e. $X = \coprod_{j \in J} X_j$ with $|X_j| < c$. Every equation $e: T_{\Sigma}X \rightarrow E$ induces a set of c -clustered equations over X given by

$$x =_{\varepsilon_{x,y}} y \ (j \in J, x, y \in X_j) \vdash s =_{\varepsilon_{s,t}} t \quad (s, t \in T_{\Sigma}X), \quad (5.3)$$

with $\varepsilon_{x,y} = d_X(x, y)$ and $\varepsilon_{s,t} = d_E(e(s), e(t))$. It is not difficult to show that e and (5.3) are equivalent: an algebra satisfies e iff it satisfies all equations (5.3).

(2) Conversely, to every c -clustered equation (5.2) over a set X of variables, we associate an equation in two steps:

- Let p the largest pseudometric on X with $p(x_i, y_i) \leq \varepsilon_i$ for all i (that is, the pointwise supremum of all such pseudometrics). Form the corresponding quotient $e_p: X \rightarrow X_p$, see (5.1). It is easy to see that X_p is c -clustered.

- Let q be the largest congruence on $T_\Sigma(X_p)$ with $q(T_\Sigma e_p(s), T_\Sigma e_p(t)) \leq \varepsilon$ (that is, the pointwise supremum of all such congruences). Form the corresponding quotient $e_q: T_\Sigma(X_p) \twoheadrightarrow E_q$.

A routine verification shows that (5.2) and e_q are expressively equivalent, i.e. satisfied by the same quantitative Σ -algebras.

Step 4. From Theorem 3.16 and Example 3.7(3), we deduce the following

Theorem 5.3 (Quantitative HSP Theorem). *A class of quantitative Σ -algebras is a c -variety (i.e. closed under c -reflexive quotients, subalgebras, and products) iff it is axiomatizable by c -clustered equations.*

The above theorem generalizes a recent result of Mardare, Panangaden, and Plotkin [20] who considered only signatures Σ with operations of finite or countably infinite arity and cardinal numbers $c \leq \aleph_1$. Theorem 5.3 holds without any restrictions on Σ and c . In addition to the quantitative HSP theorem, one can also derive the completeness of quantitative equational logic [19] from our general completeness theorem, see Appendix B.5.

5.4 Nominal Algebras

In this section, we derive an HSP theorem for algebras in the category **Nom** of nominal sets and equivariant maps; see Pitts [24] for the required terminology. We denote by \mathbb{A} the countably infinite set of atoms, by $\text{Perm}(\mathbb{A})$ the group of finite permutations of \mathbb{A} , and by $\text{supp}_X(x)$ the least support of an element x of a nominal set X . Recall that X is *strong* if, for all $x \in X$ and $\pi \in \text{Perm}(\mathbb{A})$,

$$[\forall a \in \text{supp}_X(x) : \pi(a) = a] \iff \pi \cdot x = x.$$

A *supported set* is a set X equipped with a map $\text{supp}_X: X \rightarrow \mathcal{P}_f(\mathbb{A})$. A *morphism* $f: X \rightarrow Y$ of supported sets is a function with $\text{supp}_Y(f(x)) \subseteq \text{supp}_X(x)$ for all $x \in X$. Every nominal set X is a supported set w.r.t. its least-support map supp_X . The following lemma, whose first part is a reformulation of [21, Prop. 5.10], gives a useful description of strong nominal sets in terms of supported sets.

Lemma 5.4. *The forgetful functor from **Nom** to **SuppSet** has a left adjoint $F: \mathbf{SuppSet} \rightarrow \mathbf{Nom}$. The nominal sets of the form FY ($Y \in \mathbf{SuppSet}$) are up to isomorphism exactly the strong nominal sets.*

Fix a finitary signature Σ . A *nominal Σ -algebra* is a Σ -algebra A carrying the structure of a nominal set such that all Σ -operations $\sigma: A^n \rightarrow A$ are equivariant. The forgetful functor from the category **NomAlg**(Σ) of nominal Σ -algebras and equivariant Σ -homomorphisms to **Nom** has a left adjoint assigning to each nominal set X the *free nominal Σ -algebra* $T_\Sigma X$, carried by the set of Σ -terms and with group action inherited from X . To derive a nominal HSP theorem from our general categorical results, we proceed as follows.

Step 1. Choose the parameters of our setting as follows:

- $\mathcal{A} = \mathcal{A}_0 = \mathbf{NomAlg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, injective morphisms})$;
- $\Lambda = \text{all cardinal numbers}$;
- $\mathcal{X} = \{T_\Sigma X : X \text{ is a strong nominal set}\}$.

One can show that a quotient $e: A \twoheadrightarrow B$ belongs to $\mathcal{E}_{\mathcal{X}}$ iff it is *support-reflecting*: for every $b \in B$ there exists $a \in A$ with $e(a) = b$ and $\text{supp}_A(a) = \text{supp}_B(b)$.

Step 2. A *nominal congruence* on a nominal Σ -algebra A is a Σ -algebra congruence $\equiv \subseteq A \times A$ that forms an equivariant subset of $A \times A$. In analogy to (3.1), there is an isomorphism of complete lattices

$$\text{quotient algebras of } A \cong \text{nominal congruences on } A. \quad (5.4)$$

Step 3. By Remark 3.4, an equation can be presented as a single quotient $e: T_\Sigma X \twoheadrightarrow E$, where X is a strong nominal set. Equations can be described by syntactic means as follows. A *nominal Σ -term* over a set Y of variables is an element of $T_\Sigma(\text{Perm}(\mathbb{A}) \times Y)$. Every map $h: Y \rightarrow A$ into a nominal Σ -algebra A extends to the Σ -homomorphism

$$\hat{h} = (T_\Sigma(\text{Perm}(\mathbb{A}) \times Y) \xrightarrow{T_\Sigma(\text{Perm}(\mathbb{A}) \times h)} T_\Sigma(\text{Perm}(\mathbb{A}) \times A) \xrightarrow{T_\Sigma(\dashv\dashv)} T_\Sigma A \xrightarrow{id^\sharp} A)$$

where id^\sharp is the unique Σ -homomorphism extending the identity map $id: A \rightarrow A$. A *nominal equation* over Y is an expression of the form

$$\text{supp}_Y \vdash s = t, \quad (5.5)$$

where $\text{supp}_Y: Y \rightarrow \mathcal{P}_f(\mathbb{A})$ is a function and s and t are nominal Σ -terms over Y . A nominal Σ -algebra A *satisfies* the equation $\text{supp}_Y \vdash s = t$ if for every map $h: Y \rightarrow A$ with $\text{supp}_A(h(y)) \subseteq \text{supp}_Y(y)$ for all $y \in Y$ one has $\hat{h}(s) = \hat{h}(t)$. Equations and nominal equations are expressively equivalent:

(1) Given an equation $e: T_\Sigma X \twoheadrightarrow E$ with X a strong nominal set, choose a supported set Y with $X = FY$, and denote by $\eta_Y: Y \rightarrow FY$ the universal map (see Lemma 5.4). Form the nominal equations over Y given by

$$\text{supp}_Y \vdash s = t \quad (s, t \in T_\Sigma(\text{Perm}(\mathbb{A}) \times Y) \text{ and } e \cdot T_\Sigma m(s) = e \cdot T_\Sigma m(t)) \quad (5.6)$$

where m is the composite $\text{Perm}(\mathbb{A}) \times Y \xrightarrow{\text{Perm}(\mathbb{A}) \times \eta_Y} \text{Perm}(\mathbb{A}) \times X \xrightarrow{\dashv\dashv} X$. It is not difficult to see that a nominal Σ -algebra satisfies e iff it satisfies (5.6).

(2) Conversely, given a nominal equation (5.5) over the set Y , let $X = FY$ and form the nominal congruence on $T_\Sigma X$ generated by the pair $(T_\Sigma m(s), T_\Sigma m(t))$, with m defined as above. Let $e: T_\Sigma X \twoheadrightarrow E$ be the corresponding quotient, see (5.4). One can show that a nominal Σ -algebra satisfies e iff it satisfies (5.5).

Step 4. We thus deduce the following result as an instance of Theorem 3.16:

Theorem 5.5 (Kurz and Petrişan [17]). *A class of nominal Σ -algebras is a variety (i.e. closed under support-reflecting quotients, subalgebras, and products) iff it is axiomatizable by nominal equations.*

For brevity and simplicity, in this section we restricted ourselves to algebras for a signature. Kurz and Petrişan proved a more general HSP theorem for algebras over an endofunctor on **Nom** with a suitable finitary presentation. This extra generality allows to incorporate, for instance, algebras for binding signatures.

5.5 Further Applications

Let us briefly mention some additional instances of our framework, all of which are given a detailed treatment in the Appendix.

Ordered algebras. Bloom [9] proved an HSP theorem for Σ -algebras in the category of posets: a class of such algebras is closed under homomorphic images, subalgebras, and products, iff it is axiomatizable by inequations $s \leq t$ between Σ -terms. This result can be derived much like the unordered case in Section 5.1.

Continuous algebras. A more intricate ordered version of Birkhoff’s theorem concerns *continuous algebras*, i.e. Σ -algebras with an ω -cpo structure on their underlying set and continuous Σ -operations. Adámek, Nelson, and Reiterman [4] proved that a class of continuous algebras is closed under homomorphic images, subalgebras, and products, iff it is axiomatizable by inequations between terms with formal suprema (e.g. $\sigma(x) \leq \bigvee_{i < \omega} c_i$). This result again emerges as an instance of our general HSP theorem. A somewhat curious feature of this application is that the appropriate factorization system $(\mathcal{E}, \mathcal{M})$ takes as \mathcal{E} the class of dense morphisms, i.e. morphisms of \mathcal{E} are not necessarily surjective. However, one has $\mathcal{E}_{\mathcal{X}} = \text{surjections}$, so homomorphic images are formed in the usual sense.

Abstract HSP theorems. Our results subsume several existing categorical generalizations of Birkhoff’s theorem. For instance, Theorem 3.15 yields Manes’ [18] correspondence between quotient monads $\mathbb{T} \rightarrow \mathbb{T}'$ and varieties of \mathbb{T} -algebras for any monad \mathbb{T} on **Set**. Similarly, Banaschewski and Herrlich’s [7] HSP theorem for objects in categories with enough projectives is a special case of Theorem 3.16.

6 Conclusions and Future Work

We have presented a categorical approach to the model theory of algebras with additional structure. Our framework applies to a broad range of different settings and greatly simplifies the derivation of HSP-type theorems and completeness results for equational deduction systems, as the generic part of such derivations now comes for free using our Theorems 3.15, 3.16 and 4.4. There remain a number of interesting directions and open questions for future work.

As shown in Section 5, the key to arrive at a syntactic notion of equation lies in identifying a correspondence between quotients and suitable relations, which we informally coined “exactness”. The similarity of these correspondences in our

applications suggests that there should be a (possibly enriched) notion of *exact category* that covers our examples; cf. Kurz and Velebil's [16] 2-categorical view of ordered algebras. This would allow to move more work to the generic theory.

Theorem 4.4 can be used to recover several known sound and complete equational logics, but it also applies to settings where no such logic is known, for instance, a logic of profinite equations (however, cf. recent work of Almeida and Klíma [6]). In each case, the challenge is to translate our two abstract proof rules into concrete syntax, which requires the identification of a syntactic equivalent of the two properties of an equational theory. While substitution invariance always translates into a syntactic substitution rule in a straightforward manner, $\mathcal{E}_{\mathcal{X}}$ -completeness does not appear to have an obvious syntactic counterpart. In most of the cases where a concrete equational logic is known, this issue is obfuscated by the fact that one has $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$, so $\mathcal{E}_{\mathcal{X}}$ -completeness becomes a trivial property. Finding a syntactic account of $\mathcal{E}_{\mathcal{X}}$ -completeness remains an open problem. One notable case where $\mathcal{E}_{\mathcal{X}} \neq \mathcal{E}$ is the one of nominal algebras. Gabbay's work [14] does provide an HSP theorem and a sound and complete equational logic in a setting slightly different from Section 5.4, and it should be interesting to see whether this can be obtained as an instance of our framework.

Finally, in previous work [30] we have introduced the notion of a *profinite theory* (a special case of the equational theories in the present paper) and shown how the dual concept can be used to derive Eilenberg-type correspondences between varieties of languages and pseudovarieties of finite algebras. Our present results pave the way to an extension of this method to new settings, such as nominal sets. Indeed, a simple modification of the parameters in Section 5.4 yields a new HSP theorem for *orbit-finite* nominal Σ -algebras. We expect that a dualization of this result in the spirit of *loc. cit.* leads to a correspondence between varieties of data languages and varieties of orbit-finite nominal monoids, an important step towards an algebraic theory of data languages.

References

1. Adámek, J., Hébert, M., Sousa, L.: A logic of injectivity. *J. Homotopy Relat. Struct.* 2(2), 13–47 (2007)
2. Adámek, J., Mekler, A.H., Nelson, E., Reiterman, J.: On the logic of continuous algebras. *Notre Dame J. Formal Logic* 29(3), 365–380 (1988)
3. Adámek, J., Herrlich, H., Strecker, G.E.: *Abstract and Concrete Categories - The Joy of Cats*. Dover Publications (2009)
4. Adámek, J., Nelson, E., Reiterman, J.: The Birkhoff variety theorem for continuous algebras. *Algebra Universalis* 20(3), 328–350 (1985)
5. Adámek, J., Rosický, J., Vitale, E.M.: *Algebraic Theories: A Categorical Introduction to General Algebra*. Cambridge Tracts in Mathematics, Cambridge University Press (2010)
6. Almeida, J., Klíma, O.: Towards a pseudoequational proof theory. arXiv preprint arXiv:1708.09681 (2017)
7. Banaschewski, B., Herrlich, H.: Subcategories defined by implications. *Houston J. Math.* 2(2), 149–171 (1976)

8. Birkhoff, G.: On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society* 10, 433—454 (1935)
9. Bloom, S.L.: Varieties of ordered algebras. *J. Comput. Syst. Sci.* 2(13), 200–212 (1976)
10. Bojańczyk, M.: Nominal monoids. *Theory of Computing Systems* 53(2), 194–222 (2013)
11. Chen, L.T., Adámek, J., Milius, S., Urbat, H.: Profinite monads, profinite equations and Reiterman’s theorem. In: Jacobs, B., Löding, C. (eds.) *Proc. FoSSaCS’16. Lecture Notes Comput. Sci.*, vol. 9634. Springer (2016)
12. Colcombet, T., Ley, C., Puppis, G.: Logics with rigidly guarded data tests. *Logical Methods in Computer Science* 11(3) (2015)
13. Eilenberg, S., Schützenberger, M.P.: On pseudovarieties. *Advances Math.* 10, 413–418 (1976)
14. Gabbay, M.J.: Nominal algebra and the HSP theorem. *Journal of Logic and Computation* 19, 341–367 (2009)
15. Goguen, J.A., Thatcher, J.W., Wagner, E.G., Wright, J.B.: Initial algebra semantics and continuous algebras. *J. ACM* 24(1), 68–95 (1977)
16. Kurz, A., Velebil, J.: Quasivarieties and varieties of ordered algebras: regularity and exactness. *Math. Structures Comput. Sci.* 27, 1153–1194 (2017)
17. Kurz, A., Petrisan, D.: On universal algebra over nominal sets. *Mathematical Structures in Computer Science* 20(2), 285–318 (2010)
18. Manes, E.G.: *Algebraic Theories*, Graduate Texts in Mathematics, vol. 26. Springer (1976)
19. Mardare, R., Panangaden, P., Plotkin, G.: Quantitative algebraic reasoning. In: *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*. pp. 700–709. *LICS ’16*, ACM (2016)
20. Mardare, R., Panangaden, P., Plotkin, G.: On the axiomatizability of quantitative algebras. In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20–23, 2017*. pp. 1–12. IEEE Computer Society (2017), <https://doi.org/10.1109/LICS.2017.8005102>
21. Milius, S., Schröder, L., Wißmann, T.: Regular behaviours with names. *Applied Categorical Structures* 24(5), 663–701 (2016)
22. Pin, J.É.: Profinite Methods in Automata Theory. In: Albers, S., Marion, J.Y. (eds.) *26th International Symposium on Theoretical Aspects of Computer Science STACS 2009*. pp. 31–50. IBFI Schloss Dagstuhl (2009)
23. Pin, J.É., Weil, P.: A Reiterman theorem for pseudovarieties of finite first-order structures. *Algebra Universalis* 35, 577–595 (1996)
24. Pitts, A.M.: *Nominal Sets: Names and Symmetry in Computer Science*. Cambridge University Press (2013)
25. Reiterman, J.: The Birkhoff theorem for finite algebras. *Algebra Universalis* 14(1), 1–10 (1982)
26. Ribes, L., Zalesskii, P.: *Profinite Groups*. Springer Berlin Heidelberg (2010)
27. Roşu, G.: Complete categorical equational deduction. In: Fribourg, L. (ed.) *Computer Science Logic*. pp. 528–538. Springer Berlin Heidelberg (2001)
28. Roşu, G.: Complete Categorical Deduction for Satisfaction as Injectivity, pp. 157–172. Springer Berlin Heidelberg (2006)
29. Salamánca, J.: Unveiling Eilenberg-type Correspondences: Birkhoff’s Theorem for (finite) Algebras + Duality (February 2017), <https://arxiv.org/abs/1702.02822>
30. Urbat, H., Adámek, J., Chen, L., Milius, S.: Eilenberg theorems for free. *CoRR* abs/1602.05831 (2017), <http://arxiv.org/abs/1602.05831>

Appendix

This appendix contains all omitted proofs, as well as a detailed treatment of the examples mentioned in the paper.

A Proofs

We first note some useful properties of the class $\mathcal{E}_{\mathcal{X}}$. Recall the following general properties of categories \mathcal{A} with a factorization system $(\mathcal{E}, \mathcal{M})$ [3, Prop. 14.6/14.9]:

- (1) The intersection $\mathcal{E} \cap \mathcal{M}$ consists precisely of the isomorphisms in \mathcal{A} .
- (2) The *cancellation law* holds: if p and q are composable morphisms with $p \in \mathcal{E}$ and $q \cdot p \in \mathcal{E}$, then $q \in \mathcal{E}$.

Lemma A.1. (1) *The class $\mathcal{E}_{\mathcal{X}}$ contains all isomorphisms and is closed under composition.*

(2) *Let $p : A \rightarrow B$ and $q : B \rightarrow C$ be morphisms in \mathcal{A} . If $p \in \mathcal{E}$ and $q \cdot p \in \mathcal{E}_{\mathcal{X}}$ then $q \in \mathcal{E}_{\mathcal{X}}$.*

Proof. (1) The first statement holds because \mathcal{E} contains all isomorphisms and, clearly, every object X is projective w.r.t. every isomorphism. For the second statement, let $p : A \rightarrow B$ and $q : B \rightarrow C$ be morphisms in $\mathcal{E}_{\mathcal{X}}$. Since \mathcal{E} is closed under composition, we have $q \cdot p \in \mathcal{E}$. Given $X \in \mathcal{X}$, we need to show that X is projective w.r.t. $q \cdot p$. This follows easily from the corresponding properties of p and q : for any morphism $h : X \rightarrow C$, we obtain $h' : X \rightarrow B$ with $q \cdot h' = h$ because $q \in \mathcal{E}_{\mathcal{X}}$, and then we obtain $h'' : X \rightarrow A$ with $p \cdot h'' = h'$ because $p \in \mathcal{E}_{\mathcal{X}}$. Thus $(q \cdot p) \cdot h'' = h$, which proves that $q \cdot p \in \mathcal{E}_{\mathcal{X}}$.

(2) Note first that $q \in \mathcal{E}$ by the cancellation law. Let $X \in \mathcal{X}$ and $h : X \rightarrow C$. Since $q \cdot p \in \mathcal{E}_{\mathcal{X}}$, we get a morphism $h' : X \rightarrow A$ with $h = (q \cdot p) \cdot h' = q \cdot (p \cdot h')$. This proves $q \in \mathcal{E}_{\mathcal{X}}$. □

Proof of Lemma 3.9

Since $\mathcal{V}(\mathbb{E}) = \bigcap_{\mathcal{T}_X \in \mathbb{E}} \mathcal{V}(\mathcal{T}_X)$ and intersections of varieties are varieties, it suffices to show that $\mathcal{V}(\mathcal{T}_X)$ is a variety for each equation \mathcal{T}_X over $X \in \mathcal{X}$.

(1) *Closure under $\mathcal{E}_{\mathcal{X}}$ -quotients.* Let $q : A \twoheadrightarrow B$ be an $\mathcal{E}_{\mathcal{X}}$ -quotient in \mathcal{A}_0 where $A \in \mathcal{V}(\mathcal{T}_X)$, and let $h : X \rightarrow B$. Since q lies in $\mathcal{E}_{\mathcal{X}}$, there exists $h' : X \rightarrow A$ with $h = q \cdot h'$. Then since $A \in \mathcal{V}(\mathcal{T}_X)$, the morphism h' factorizes through some $e \in \mathcal{T}_X$. Thus also h factorizes through e , see the commutative diagram below:

$$\begin{array}{ccccc}
 & & X & & \\
 & e \swarrow & \downarrow h' & \searrow h & \\
 E & \dashrightarrow & A & \twoheadrightarrow & B \\
 & & & q &
 \end{array}$$

This proves that $B \in \mathcal{V}(\mathcal{T}_X)$.

(2) *Closure under subobjects.* Let $m : A \rightarrow B$ be a subobject in \mathcal{A}_0 where $B \in \mathcal{V}(\mathcal{T}_X)$, and let $h : X \rightarrow A$. Then $m \cdot h$ factorizes through some $e \in \mathcal{T}_X$ since $B \models \mathcal{T}_X$. This implies that h factorizes through e using diagonal fill-in:

$$\begin{array}{ccc} X & \xrightarrow{e} & E \\ h \downarrow & \nearrow & \downarrow \\ A & \xrightarrow{m} & B \end{array}$$

Therefore, $A \in \mathcal{V}(\mathcal{T}_X)$.

(3) *Closure under Λ -products.* Let A_i ($i < \lambda$) be a family of objects in $\mathcal{V}(\mathcal{T}_X)$, where $\lambda \in A$. We denote by $p_i : \prod_{i < \lambda} A_i \rightarrow A_i$ the product projections. First note that $\prod_{i < \lambda} A_i$ lies in \mathcal{A}_0 by Assumption 3.1(2). Let $h : X \rightarrow \prod_{i < \lambda} A_i$. Since $A_i \in \mathcal{V}(\mathcal{T}_X)$, there exists for every $i < \lambda$ some $e_i : X \rightarrow E_i$ in \mathcal{T}_X and $k_i : E_i \rightarrow A_i$ with $k_i \cdot e_i = p_i \cdot h$. Since \mathcal{T}_X is Λ -codirected, we may choose e_i independently of i , that is, we obtain one $e : X \rightarrow E$ in \mathcal{T}_X through which all $p_i \cdot h$ factorize. Then h factorizes through e via $\langle k_i \rangle$, as shown by the commutative diagram below:

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow h & & \\ E & \xleftarrow{\langle k_i \rangle} & \prod_i A_i & \xrightarrow{p_i} & A_i \\ & \searrow e & \nearrow & & \\ & & X & & \end{array}$$

k_i

This proves that $\prod_i A_i \in \mathcal{V}(\mathcal{T}_X)$. □

Lemma A.2. *Let \mathcal{T} be an equational theory. An object $A \in \mathcal{A}_0$ belongs to $\mathcal{V}(\mathcal{T})$ if and only if, for some $Y \in \mathcal{X}$, the equation \mathcal{T}_Y contains a quotient with codomain A .*

Proof. For the “if” direction, suppose that \mathcal{T}_Y contains the quotient $e_Y : Y \rightarrow A$. By $\mathcal{E}_{\mathcal{X}}$ -completeness of \mathcal{T} , we may assume that $e_Y \in \mathcal{E}_{\mathcal{X}}$. Let $h : X \rightarrow A$ with $X \in \mathcal{X}$. Since $e_Y \in \mathcal{E}_{\mathcal{X}}$, there exists a morphism $g : X \rightarrow Y$ with $e_Y \cdot g = h$. By substitution invariance, the coimage e_X of $e_Y \cdot g$ lies in \mathcal{T}_X . Then h factorizes through e_X , as shown by the commutative diagram below.

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ e_X \downarrow & \searrow h & \downarrow e_Y \\ E_X & \xrightarrow{\quad} & A \end{array}$$

This proves that $A \in \mathcal{V}(\mathcal{T})$.

For the “only if” direction, let $A \in \mathcal{V}(\mathcal{T})$. By Assumption 3.1(3), we can express A as an $\mathcal{E}_{\mathcal{X}}$ -quotient $e : Y \rightarrow A$ of some $Y \in \mathcal{X}$. Since $A \in \mathcal{V}(\mathcal{T})$, we know that A satisfies \mathcal{T}_Y , i.e. there exists $e_Y : Y \rightarrow E_Y$ in \mathcal{T}_Y and a morphism

$\bar{e} : E_Y \twoheadrightarrow A$ with $\bar{e} \cdot e_Y = e$. By Lemma A.1(2), we have $\bar{e} \in \mathcal{E}_X$, and thus $e \in \mathcal{T}_Y$ because \mathcal{T}_Y is closed under \mathcal{E}_X -quotients. This proves that \mathcal{T}_Y contains a quotient with codomain A . \square

Proof of Lemma 3.14

Let $\mathcal{T}(\mathcal{V}) = (\mathcal{T}_X)_{X \in \mathcal{X}}$. We first prove that \mathcal{T}_X is an equation for each $X \in \mathcal{X}$.

The closure of \mathcal{T}_X under \mathcal{E}_X -quotients follows immediately from the fact that \mathcal{V} is closed under \mathcal{E}_X -quotients.

To show that \mathcal{T}_X is Λ -codirected, let $e_i : X \twoheadrightarrow A_i$ ($i < \lambda$) be a family of quotients in \mathcal{T}_X with $\lambda \in \Lambda$. Form the \mathcal{E}/\mathcal{M} -factorization of $\langle e_i \rangle : X \rightarrow \prod_{i < \lambda} A_i$:

$$\begin{array}{ccc}
 & X & \\
 e \swarrow & \downarrow \langle e_i \rangle & \searrow e_i \\
 A \xrightarrow{m} & \prod_{i < \lambda} A_i & \xrightarrow{p_i} A_i
 \end{array}$$

By Assumption 3.1(2), A lies in \mathcal{A}_0 and, since \mathcal{V} is closed under subobjects and Λ -products, one has $A \in \mathcal{V}$. Thus $e \in \mathcal{T}_X$ and e is an upper bound of the e_i 's.

In order to prove substitution invariance for $\mathcal{T}(\mathcal{V})$, suppose that $e_Y \in \mathcal{T}_Y$ and $h : X \rightarrow Y$ are given, and take the \mathcal{E}/\mathcal{M} -factorization $e_Y \cdot h = \bar{h} \cdot e_X$ of $e_Y \cdot h$:

$$\begin{array}{ccc}
 X & \xrightarrow{h} & Y \\
 e_X \downarrow & & \downarrow e_Y \\
 E_X & \xrightarrow{\bar{h}} & E_Y
 \end{array}$$

Then $E_X \in \mathcal{A}_0$ because $E_Y \in \mathcal{A}_0$ and \mathcal{A}_0 is closed under \mathcal{X} -generated subobjects. Moreover, since $E_Y \in \mathcal{V}$ and \mathcal{V} is closed under subobjects in \mathcal{A}_0 , we get $E_X \in \mathcal{V}$. This shows that $e_X \in \mathcal{T}_X$ by definition of \mathcal{T}_X . Thus, $\mathcal{T}(\mathcal{V})$ is substitution invariant.

For \mathcal{E}_X -completeness of $\mathcal{T}(\mathcal{V})$, let $Y \in \mathcal{X}$ and $e_Y : Y \twoheadrightarrow E$ in \mathcal{T}_Y . By definition, this means that $E \in \mathcal{A}$. By Assumption 3.1(3), there exists an \mathcal{E}_X -quotient $e_X : X \twoheadrightarrow E$ for some $X \in \mathcal{X}$. Then $e_X \in \mathcal{T}_X$ by definition of $\mathcal{T}(\mathcal{V})$. Thus, $\mathcal{T}(\mathcal{V})$ is \mathcal{E}_X -complete. \square

Lemma A.3. *For every variety \mathcal{V} , we have $\mathcal{V} = \mathcal{V}(\mathcal{T}(\mathcal{V}))$.*

Proof. Let $\mathcal{T}(\mathcal{V}) = (\mathcal{T}_X)_{X \in \mathcal{X}}$.

To prove \subseteq , let $A \in \mathcal{V}$. By Assumption 3.1(3), there exists a quotient $e : X \twoheadrightarrow A$ with $X \in \mathcal{X}$. Thus $e \in \mathcal{T}_X$ by the definition of \mathcal{T}_X , and therefore $A \in \mathcal{V}(\mathcal{T}(\mathcal{V}))$ by Lemma A.2.

For \supseteq , let $A \in \mathcal{V}(\mathcal{T}(\mathcal{V}))$. By Lemma A.2, for some $X \in \mathcal{X}$, \mathcal{T}_X contains a quotient $e : X \twoheadrightarrow A$ with codomain A . Thus $A \in \mathcal{V}$ by definition of \mathcal{T}_X . \square

Lemma A.4. *For every equational theory \mathcal{T} , we have $\mathcal{T} = \mathcal{T}(\mathcal{V}(\mathcal{T}))$.*

Proof. Let $\mathcal{T} = (\mathcal{T}_X)_{X \in \mathcal{X}}$ and $\mathcal{T}(\mathcal{V}(\mathcal{T})) = (\mathcal{T}'_X)_{X \in \mathcal{X}}$. We need to prove $\mathcal{T}_X = \mathcal{T}'_X$ for all $X \in \mathcal{X}$.

For \subseteq , let $e : X \twoheadrightarrow A$ in \mathcal{T}_X . Then $A \in \mathcal{V}(\mathcal{T})$ by Lemma A.2, and thus $e \in \mathcal{T}'_X$ by the definition of $\mathcal{T}(\mathcal{V}(\mathcal{T}))$.

For \supseteq , let $e : X \twoheadrightarrow A$ in \mathcal{T}'_X . Then $A \in \mathcal{V}(\mathcal{T})$ by the definition of $\mathcal{T}(\mathcal{V}(\mathcal{T}))$. Thus, by Lemma A.2, there exists some $Y \in \mathcal{X}$ and $e_Y : Y \twoheadrightarrow A$ in \mathcal{T}_Y . Since the theory \mathcal{T} is $\mathcal{E}_{\mathcal{X}}$ -complete, we may assume that $e_Y \in \mathcal{E}_{\mathcal{X}}$. Since X is projective w.r.t. e_Y , there is a morphism $h : X \rightarrow Y$ with $e_Y \cdot h = e$. Let e_X and \bar{h} be the \mathcal{E}/\mathcal{M} -factorization of $e_Y \cdot h$. By substitution invariance, e_X lies in \mathcal{T}_X :

$$\begin{array}{ccc}
 X & \xrightarrow{h} & Y \\
 e_X \downarrow & \searrow e & \downarrow e_Y \\
 E_X & \xrightarrow{\bar{h}} & A
 \end{array}$$

Since $e = \bar{h} \cdot e_X$, the cancellation law implies that \bar{h} lies in \mathcal{E} . Since it also lies in \mathcal{M} , we have that \bar{h} is an isomorphism. Thus e_X and e represent the same quotient of X , which implies $e \in \mathcal{T}_X$. \square

Proof of Theorem 3.15

By Lemma A.3 and Lemma A.4, the two maps $\mathcal{V} \mapsto \mathcal{T}(\mathcal{V})$ and $\mathcal{T} \mapsto \mathcal{V}(\mathcal{T})$ are mutually inverse bijections. It only remains to show that they are antitone.

(1) Suppose that $\mathcal{V} \subseteq \mathcal{V}'$ are varieties, and let $e : X \twoheadrightarrow A$ be a quotient in $[\mathcal{T}(\mathcal{V})]_X$. Then $A \in \mathcal{V}$ by definition of $\mathcal{T}(\mathcal{V})$, and thus $A \in \mathcal{V}'$, i.e. the quotient e also lies in $[\mathcal{T}(\mathcal{V}')]_X$. This shows $\mathcal{T}' \leq \mathcal{T}$.

(2) Suppose that $\mathcal{T} \leq \mathcal{T}'$ are theories, and let $A' \in \mathcal{V}(\mathcal{T}')$. Then, by Lemma A.2, there exists $X \in \mathcal{X}$ and a quotient $e' : X \twoheadrightarrow A'$ in \mathcal{T}'_X with codomain A' . By $\mathcal{E}_{\mathcal{X}}$ -completeness of \mathcal{T}'_X , we may assume that $e' \in \mathcal{E}_{\mathcal{X}}$. Since $\mathcal{T} \leq \mathcal{T}'$, the quotient e' factorizes through some quotient $e : X \twoheadrightarrow A$ in \mathcal{T}_X , i.e. $e' = q \cdot e$ for some $q : A \twoheadrightarrow A'$. Since $e' \in \mathcal{E}_{\mathcal{X}}$ we have $q \in \mathcal{E}_{\mathcal{X}}$ by Lemma A.1(2). Moreover, $A \in \mathcal{V}(\mathcal{T})$ by Lemma A.2, and thus $A' \in \mathcal{V}(\mathcal{T})$ because $\mathcal{V}(\mathcal{T})$ is closed under $\mathcal{E}_{\mathcal{X}}$ -quotients. This shows $\mathcal{V}(\mathcal{T}') \subseteq \mathcal{V}(\mathcal{T})$. \square

Proof of Lemma 4.3

Let $\mathcal{S} = \mathcal{T}(\mathcal{V}(\mathcal{T}_X))$.

(1) One has $\mathcal{T}_X \leq \mathcal{S}_X$. Indeed, suppose that $e : X \twoheadrightarrow E$ is a quotient in \mathcal{S}_X . Then, by definition of $\mathcal{T}(-)$, one has $E \in \mathcal{V}(\mathcal{T}_X)$, i.e. $E \models \mathcal{T}_X$. Thus $e : X \twoheadrightarrow E$ factorizes through some $e' \in \mathcal{T}_X$, which proves $\mathcal{T}_X \leq \mathcal{S}_X$.

(2) Now suppose that \mathcal{S}' is any theory with $\mathcal{T}_X \leq \mathcal{S}'_X$. We need to show $\mathcal{S} \leq \mathcal{S}'$. Since $\mathcal{S} = \mathcal{T}(\mathcal{V}(\mathcal{T}_X))$, this is equivalent to showing that $\mathcal{V}(\mathcal{S}') \subseteq \mathcal{V}(\mathcal{T}_X)$ by Theorem 3.15. Thus let $A \in \mathcal{V}(\mathcal{S}')$, and let $h : X \rightarrow A$. Since $A \models \mathcal{S}'_X$, the morphism h factorizes through some $e' \in \mathcal{S}'_X$. Since $\mathcal{T}_X \leq \mathcal{S}'_X$, the quotient e' factorizes through some $e \in \mathcal{T}_X$. Thus h factorizes through e , which shows that $A \models \mathcal{T}_X$, i.e. $A \in \mathcal{V}(\mathcal{T}_X)$. \square

Proof of Theorem 4.4

Soundness. The soundness of (Weakening) easily follows from the definitions of semantic entailment and satisfaction of equations. For the soundness of (Substitution), let $\mathcal{T}_X \subseteq X \downarrow \mathcal{A}_0$ be an equation and $\overline{\mathcal{T}}$ its substitution closure. We need to prove that $\mathcal{T}_X \models \overline{\mathcal{T}}_Y$ for all $Y \in \mathcal{X}$, equivalently, $\mathcal{V}(\mathcal{T}_X) \subseteq \mathcal{V}(\overline{\mathcal{T}})$. In fact, this holds even with equality:

$$\begin{aligned} \mathcal{V}(\mathcal{T}_X) &= \mathcal{V}(\mathcal{T}(\mathcal{V}(\mathcal{T}_X))) && \text{by Theorem 3.15} \\ &= \mathcal{V}(\overline{\mathcal{T}}) && \text{by Lemma 4.3.} \end{aligned}$$

Completeness. Suppose that \mathcal{T}_X and \mathcal{T}'_Y are equations over X and Y , respectively, and denote by $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}'}$ their substitution closures. Suppose that $\mathcal{T}_X \models \mathcal{T}'_Y$. Then $\mathcal{T}'_Y \leq \overline{\mathcal{T}}_Y$ because

$$\begin{aligned} \mathcal{T}'_Y &\leq \overline{\mathcal{T}'_Y} && \text{by def. of } \overline{\mathcal{T}'} \\ &= [\mathcal{T}(\mathcal{V}(\mathcal{T}'_Y))]_Y && \text{by Lemma 4.3} \\ &\leq [\mathcal{T}(\mathcal{V}(\mathcal{T}_X))]_Y && \text{see below} \\ &= \overline{\mathcal{T}}_Y && \text{by Lemma 4.3} \end{aligned}$$

In the penultimate step, we use that $\mathcal{V}(\mathcal{T}_X) \subseteq \mathcal{V}(\mathcal{T}'_Y)$ by assumption and that the map $\mathcal{T}(-)$ is antitone. Thus we obtain the proof

$$\mathcal{T}_X \vdash \overline{\mathcal{T}}_Y \vdash \mathcal{T}'_Y$$

where step first step uses (Substitution) and the second one uses (Weakening).

B Details for the Examples of Section 5

In this section, we provide full details for all the applications mentioned in the paper. Let us start with two general remarks:

Remark B.1. To characterize $\mathcal{E}_{\mathcal{X}}$ in a category \mathcal{A} of algebras with structure, it suffices to look at the category of underlying structures. Indeed, suppose that

- (1) the category \mathcal{A} is part of an adjoint situation $F \dashv U : \mathcal{A} \rightarrow \mathcal{B}$;
- (2) there is a subclass $\mathcal{X}' \subseteq \mathcal{B}$ such that $\mathcal{X} = \{FX' : X' \in \mathcal{X}'\}$;
- (3) there is a class \mathcal{E}' of morphisms in \mathcal{B} such that $\mathcal{E} = \{e \in \mathcal{A} : Ue \in \mathcal{E}'\}$.

Let $\mathcal{E}'_{\mathcal{X}'}$ be the class of all $e' \in \mathcal{E}'$ such that every $X' \in \mathcal{X}'$ is projective w.r.t. e' . Then

$$\mathcal{E}_{\mathcal{X}} = \{e \in \mathcal{E} : Ue \in \mathcal{E}'_{\mathcal{X}'}\}.$$

Indeed, for all $e \in \mathcal{E}$, one has

$$\begin{aligned} e \in \mathcal{E}_{\mathcal{X}} &\Leftrightarrow \forall X \in \mathcal{X} : \mathcal{A}(X, e) \text{ is surjective} \\ &\Leftrightarrow \forall X' \in \mathcal{X}' : \mathcal{A}(FX', e) \text{ is surjective} \\ &\Leftrightarrow \forall X' \in \mathcal{X}' : \mathcal{B}(X', Ue) \text{ is surjective} \\ &\Leftrightarrow Ue \in \mathcal{E}'_{\mathcal{X}'}. \end{aligned}$$

Remark B.2. In the situation of Remark 3.11, our equational logic can be stated in terms of single quotients $e_X : X \twoheadrightarrow E_X$ in lieu of sets \mathcal{T}_X of them. More precisely, given a quotient $e : X \twoheadrightarrow E$ with $X \in \mathcal{X}$ and $E \in \mathcal{A}_0$, its *substitution closure* is the smallest substitution invariant family $(\bar{e}_X : X \twoheadrightarrow E_X)_{X \in \mathcal{X}}$ with $e \leq \bar{e}_X$, where families are ordered componentwise by the order of quotients in $X \downarrow \mathcal{A}_0$. Then the two rules of our deduction system are given by

Weakening: $e_X \vdash e'_X$ for all $e'_X \leq e_X$ in $X \downarrow \mathcal{A}_0$.

Substitution: $e_X \vdash \bar{e}_Y$ for every component \bar{e}_Y of the substitution closure of e .

B.1 Birkhoff's Equational Logic

In Section 5.1 we derived Birkhoff's HSP theorem from our general HSP theorem. In this section, we demonstrate that the completeness of Birkhoff's equational deduction system follows from our general completeness result (Theorem 4.4). A set Γ of term equations *semantically entails* the term equation $s = t$ (notation: $\Gamma \models s = t$) if every Σ -algebra that satisfies all equations in Γ also satisfies $s = t$. Birkhoff's proof system consists of the following rules, where s, t, u, s_i, t_i are Σ -terms over an arbitrary set X of variables, $\sigma \in \Sigma$ is an n -ary operation symbol, and $h : T_\Sigma X \rightarrow T_\Sigma Y$ a Σ -homomorphism:

(Refl) $\vdash t = t$

(Sym) $s = t \vdash t = s$

(Trans) $s = t, t = u \vdash s = u$

(Cong) $s_i = t_i (i = 1, \dots, n) \vdash \sigma(s_1, \dots, s_n) = \sigma(t_1, \dots, t_n)$

(Subst) $s = t \vdash h(s) = h(t)$

We write $\Gamma \vdash s = t$ if there exists a proof of $s = t$ from the axioms in Γ using the above rules. Observe that

(1) A set $\Gamma \subseteq T_\Sigma X \times T_\Sigma X$ is a congruence iff it is closed under (Refl), (Sym), (Trans), and (Cong).

(2) A family of sets $(\Gamma_X \subseteq T_\Sigma X \times T_\Sigma X)_{X \in \mathbf{Set}}$ corresponds to an equational theory (cf. Example 3.12) iff it is closed under (Refl), (Sym), (Trans), (Cong), and (Subst).

Theorem B.3 (Birkhoff [8]). $\Gamma \models s = t$ implies $\Gamma \vdash s = t$.

Proof. We derive this statement from Theorem 4.4. Choose a set X of variables such that $\Gamma \subseteq T_\Sigma X \times T_\Sigma X$ and $(s, t) \in T_\Sigma X \times T_\Sigma X$, and suppose that $\Gamma \models s = t$. Let $e : T_\Sigma X \twoheadrightarrow E_X$ and $e' : T_\Sigma X \twoheadrightarrow E'_X$ be the quotients corresponding to the congruences generated by Γ and (s, t) , respectively. Then $e \models e'$, so by Theorem 4.4 (cf. also Remark B.2), there exists a proof

$$e = e_0 \vdash e_1 \vdash \dots \vdash e_n = e'$$

in our abstract calculus for some $e_i : T_\Sigma X_i \twoheadrightarrow E_i$. Denote by Γ_i the kernel of e_i . We show that for every $i = 0, \dots, n$ and $(s', t') \in \Gamma_i$ one has $\Gamma \vdash s' = t'$; this

then implies $\Gamma \vdash s = t$ by putting $i = n$ and $(s', t') = (s, t)$. The proof is by induction on i .

For $i = 0$, we have that Γ_0 is the congruence on $T_\Sigma X$ generated by Γ , so Γ_0 is the closure of Γ under the rules (Refl), (Sym), (Trans), (Cong). Thus, every pair $(s', t') \in \Gamma_0$ can be proved from Γ using these four rules.

Now suppose that $0 < i < n$. If the step $e_i \vdash e_{i+1}$ is an application of the weakening rule, the statement follows trivially by induction because then $\Gamma_{i+1} \subseteq \Gamma_i$. Thus suppose that $e_i \vdash e_{i+1}$ uses the substitution rule. Identifying equational theories with families of congruences, see Example 3.12, the substitution closure of e_i is the family $\overline{\Gamma}_i = (\equiv_Y \subseteq T_\Sigma Y \times T_\Sigma Y)_{Y \in \mathbf{Set}}$ obtained by closing Γ_i under the rules (Refl), (Sym), (Trans), (Cong), (Subst). Thus Γ_{i+1} is equal to $\equiv_{X_{i+1}}$. Therefore, every pair $(s', t') \in \Gamma_{i+1}$ can be proved from Γ_i using (Refl), (Sym), (Trans), (Cong), (Subst). By induction, it follows that $\Gamma \vdash s' = t'$. \square

B.2 Ordered Algebras

In this section, we show that Bloom's variety theorem for ordered algebras [9] emerges as a special case of our general HSP theorem. Given a finitary signature Σ , an *ordered Σ -algebra* is a Σ -algebra A in the category of posets; that is, A endowed with a partial order on its underlying set such that all Σ -operations $\sigma: A^n \rightarrow A$ are monotone. The category $\mathbf{Alg}_{\leq}(\Sigma)$ of ordered Σ -algebras and monotone Σ -homomorphisms has a factorization system given by surjective morphism and order-embeddings, respectively. Here, a morphism $h: A \rightarrow B$ is called an *order-embedding* if $a \leq a' \Leftrightarrow h(a) \leq h(a')$ for all $a, a' \in A$. The forgetful functor from $\mathbf{Alg}_{\leq}(\Sigma)$ to \mathbf{Set} has a left adjoint mapping to each set X the term algebra $T_\Sigma X$, discretely ordered.

Step 1. To treat ordered algebras in our setting, we choose

- (1) $\mathcal{A} = \mathcal{A}_0 = \mathbf{Alg}_{\leq}(\Sigma)$;
- (2) $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, order-embeddings})$;
- (3) $\Lambda = \text{all cardinal numbers}$;
- (4) $\mathcal{X} = \text{all free algebras } T_\Sigma X \text{ with } X \in \mathbf{Set}$.

Lemma B.4. *The class $\mathcal{E}_{\mathcal{X}}$ consists of all surjective morphisms, i.e. $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$.*

Proof. Apply Remark B.1 to the adjunction $\mathbf{Alg}_{\leq}(\Sigma) \xrightleftharpoons[\mathcal{U}]{\mathcal{F}}$ \mathbf{Set} with $\mathcal{X}' = \mathbf{Set}$ and $\mathcal{E}' = \text{surjections}$. Since every surjection in \mathbf{Set} splits (i.e. has a left inverse), that class $\mathcal{E}'_{\mathcal{X}'}$ consists precisely of the surjective maps. \square

Let us check that our Assumptions 3.1 are satisfied. For (1), just note that products in $\mathbf{Alg}_{\leq}(\Sigma)$ are formed on the level of underlying sets (with partial order and Σ -structure taken pointwise). (2) is trivial since $\mathcal{A} = \mathcal{A}_0$. For (3), let $A \in \mathbf{Alg}_{\leq}(\Sigma)$ and choose a surjective map $e: X \twoheadrightarrow A$ for some set X . Then the unique extension $e^\sharp: T_\Sigma X \twoheadrightarrow A$ to a morphism in $\mathbf{Alg}_{\leq}(\Sigma)$ is surjective, i.e. $T_\Sigma X \in \mathcal{X}$ and $e^\sharp \in \mathcal{E}_{\mathcal{X}}$.

Step 2. Given an ordered algebra A , a preorder \preceq on A is called *stable* if it refines the order of A (i.e. $a \leq_A a'$ implies $a \preceq a'$) and every Σ -operation $\sigma: A^n \rightarrow A$ ($\sigma \in \Sigma$) is monotone with respect to \preceq . It is well-known and easy to prove that there is an isomorphism of complete lattices

$$\text{quotients algebras of } A \cong \text{stable preorders on } A \quad (\text{B.1})$$

assigning to $e: A \twoheadrightarrow B$ the stable preorder given by $a \preceq_e a'$ iff $e(a) \leq_B e(a')$.

Step 3. The exactness property (B.1) suggests that one may replace equations $e: T_\Sigma X \twoheadrightarrow E$ by the following syntactic concept: a *term inequation* over the set X of variables is a pair $(s, t) \in T_\Sigma X \times T_\Sigma X$, denoted as $s \leq t$. It is *satisfied* by an algebra $A \in \mathbf{Alg}_\leq(\Sigma)$ if for every morphism $h: T_\Sigma X \rightarrow A$ one has $h(s) \leq_A h(t)$. Equations $e: T_\Sigma X \twoheadrightarrow E$ and term inequations are expressively equivalent in the following sense:

(1) For every equation $e: T_\Sigma X \twoheadrightarrow E$, the corresponding preorder $\preceq_e \subseteq T_\Sigma X \times T_\Sigma X$ is a set of term inequations equivalent to e , that is, an algebra $A \in \mathbf{Alg}_\leq(\Sigma)$ satisfies e iff it satisfies all term inequations given by the pairs in \preceq_e . This follows immediately from (B.1).

(2) Conversely, given a term inequation $(s, t) \in T_\Sigma X \times T_\Sigma X$, form the smallest stable preorder \preceq on $T_\Sigma X$ with $s \preceq t$ (viz. the intersection of all such preorders) and let $e: T_\Sigma X \twoheadrightarrow E$ be the corresponding quotient. Then, by (B.1) again, an algebra $A \in \mathbf{Alg}_\leq(\Sigma)$ satisfies $s \leq t$ iff it satisfies e .

Step 4. We therefore deduce from Theorem 3.16:

Theorem B.5 (Bloom [9]). *A class of ordered Σ -algebras is a variety (i.e. closed under quotient algebras, subalgebras, and products) iff it is axiomatizable by term inequations.*

B.3 Eilenberg-Schützenberger Theorem

In this section, we derive Eilenberg and Schützenberger's HSP theorem [13] for finite algebras. Fix a finitary signature Σ containing only finitely many operation symbols.

Step 1. To treat finite algebras in our setting, choose the parameters

- $\mathcal{A} = \mathbf{Alg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms}, \text{injective morphisms})$;
- $\mathcal{A}_0 = \mathbf{Alg}_f(\Sigma)$, the full subcategory of finite Σ -algebras;
- $\Lambda =$ all finite cardinal numbers;
- $\mathcal{X} =$ all free Σ -algebras $T_\Sigma X$ with $X \in \mathbf{Set}_f$.

As in Section 5.1, we have $\mathcal{E}_{\mathcal{X}} = \mathcal{E} =$ surjective morphisms because surjections in \mathbf{Set} split. Clearly, all our Assumptions 3.1 are satisfied.

Step 2. The exactness property of $\mathbf{Alg}(\Sigma)$ has already been stated in (3.1).

Step 3. In the present setting, an equational theory is given by a family $\mathcal{T} = (\mathcal{T}_n)_{n < \omega}$, where each $\mathcal{T}_n \subseteq T_{\Sigma n} \downarrow \mathbf{Alg}_f(\Sigma)$ is a filter (i.e. a codirected and upwards closed set) in the poset of finite quotient algebras of $T_{\Sigma n}$.

Remark B.6. Note that since $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$, substitution invariance (see Definition 3.10) has the following equivalent statement: for every $e: T_{\Sigma n} \rightarrow E$ in \mathcal{T}_n and every Σ -homomorphism $h: T_{\Sigma m} \rightarrow T_{\Sigma n}$, $h \cdot e$ factorizes through some $e': T_{\Sigma m} \rightarrow E'$ in \mathcal{T}_m . This is easy to see using the upwards closedness of \mathcal{T}_m .

The syntactic concept corresponding to equational theories involves sequences $(s_i = t_i)_{i < \omega}$ of term equations, where $(s_i, t_i) \in T_{\Sigma m_i} \times T_{\Sigma m_i}$ for some $m_i < \omega$. A finite Σ -algebra A *eventually satisfies* $(s_i = t_i)_{i < \omega}$ if there exists $i_0 < \omega$ such that A satisfies the equations $s_i = t_i$ for all $i \geq i_0$. Equational theories and sequences of term equations are expressively equivalent in the following sense:

Lemma B.7. (1) *For each equational theory \mathcal{T} , there exists a sequence $(s_i = t_i)_{i < \omega}$ of term equations such that, for all finite Σ -algebras A ,*

$$A \in \mathcal{V}(\mathcal{T}) \quad \text{iff} \quad A \text{ eventually satisfies } (s_i = t_i)_{i < \omega}. \quad (\text{B.2})$$

(2) *For each sequence $(s_i = t_i)_{i < \omega}$ of term equations, there exists an equational theory \mathcal{T} such that, for all finite Σ -algebras A , (B.2) holds.*

The proof rests on an observation on congruences (see lemma below) that crucially relies on the finiteness of the signature Σ . In the following, a congruence $\equiv \subseteq \times A \times A$ on a Σ -algebra A is called *finite* if the corresponding quotient algebra A/\equiv , see (3.1), is finite. It is called *finitely generated* if there exists a finite subset $W \subseteq \equiv$ such that \equiv is the least congruence on A containing W .

Lemma B.8 ([13], Proposition 2). *Let Σ be a finite signature and $n < \omega$. Then every finite congruence on $T_{\Sigma n}$ is finitely generated.*

Proof (Lemma B.7).

(1) Let \mathcal{T} be an equational theory. Since Σ is finite, $T_{\Sigma n}$ is countable for each $n < \omega$. Hence, there are only countably many finitely generated congruences on $T_{\Sigma n}$, whence only countably many finite quotients, by Lemma B.8. In particular, \mathcal{T}_n is a countable co-directed poset and thus contains an ω^{op} -chain $e_0^n \geq e_1^n \geq e_2^n \geq \dots$ that is *cofinal*, which means that for every element $e \in \mathcal{T}_n$ there exists $i < \omega$ with $e \geq e_i^n$. The e_i^n can be chosen in a way that, for each $i, n < \omega$ and each map $q: n + 1 \rightarrow n$, the morphism $e_i^n \cdot T_{\Sigma} q$ factorizes through e_i^{n+1} :

$$\begin{array}{ccc} T_{\Sigma}(n+1) & \xrightarrow{T_{\Sigma} q} & T_{\Sigma} n \\ e_i^{n+1} \downarrow & & \downarrow e_i^n \\ E_i^{n+1} & \dashrightarrow & E_i^n \end{array} \quad (\text{B.3})$$

To see this, suppose inductively that this property already holds for all $i < \omega$ and $n' < n$. Since \mathcal{T} is a theory, each $e_i^n \cdot T_{\Sigma} q$ factorizes through some $e \in \mathcal{T}_{n+1}$.

Since there are only finitely many maps $q: n+1 \rightarrow n$ and \mathcal{T}_{n+1} is codirected, we may choose e independently of q . The quotient e lies above some element of the cofinal chain $e_0^{n+1} \geq e_1^{n+1} \geq e_2^{n+1} \geq \dots$. Replacing this chain by a suitable subchain, we can ensure that $e \geq e_i^{n+1}$. Then (B.3) holds.

Iterating (B.3) shows that for all $i, m, n < \omega$ with $n < m$ and all $q: m \rightarrow n$, the morphism $e_i^n \cdot T_\Sigma q$ factorizes through e_i^m , see the diagram below:

$$\begin{array}{ccc}
 T_\Sigma m & \xrightarrow{T_\Sigma q} & T_\Sigma n \\
 e_i^m \downarrow & & \downarrow e_i^n \\
 E_i^m & \dashrightarrow & E_i^n
 \end{array} \tag{B.4}$$

For each $n < \omega$, the kernel of e_n^n has a finite set W_n of generators by Lemma B.8. Let $(s_i = t_i)_{i < \omega}$ be a sequence of terms equations where (s_i, t_i) ranges over all elements in the countable set $\bigcup_{n < \omega} W_n$. We claim that, for each finite Σ -algebra A , the equivalence (B.2) holds.

(\Rightarrow) Suppose that $A \in \mathcal{V}(\mathcal{T})$. Choose a surjective map $h: n \rightarrow A$ with $n < \omega$. Then $h^\sharp: T_\Sigma n \rightarrow A$ factorizes through some e_i^n , and by (B.4) (replacing n by a larger number if necessary), we may assume that h^\sharp factorizes through e_n^n . We claim that A satisfies all equations $s_i = t_i$ with $(s_i, t_i) \in \bigcup_{m > n} W_m$. To see this, suppose that $(s_i, t_i) \in W_m$ for some $m > n$, and let $k: m \rightarrow A$. By projectivity of m in **Set**, we may choose $q: m \rightarrow n$ with $h \cdot q = k$, which implies $h^\sharp \cdot T_\Sigma q = k^\sharp$. Moreover, we have that $e_n^n \cdot T_\Sigma q$ factorizes through e_m^m by (B.4), thus also through e_m^m because $e_m^m \leq e_n^n$. In other words, we obtain the following commutative diagram, which shows that k^\sharp factorizes through e_m^m .

$$\begin{array}{ccc}
 & & k^\sharp \\
 & \curvearrowright & \\
 T_\Sigma m & \xrightarrow{T_\Sigma q} & T_\Sigma n \\
 e_m^m \downarrow & & \downarrow e_n^n \quad \searrow h^\sharp \\
 E_m^m & \dashrightarrow & E_n^n \dashrightarrow A
 \end{array}$$

Since $e_m^m(s_i) = e_m^m(t_i)$, it follows that $k^\sharp(s_i) = k^\sharp(t_i)$. Thus, A satisfies $s_i = t_i$.

(\Leftarrow) Suppose that A eventually satisfies the term equations $(s_i = t_i)_{i < \omega}$. Then, for some $n < \omega$, the algebra A satisfies all equations $s_i = t_i$ with $(s_i, t_i) \in \bigcup_{m > n} W_m$. To show that $A \in \mathcal{V}(\mathcal{T})$, let $m < \omega$ and $h: m \rightarrow A$. We need to prove that h^\sharp factorizes through some $e \in \mathcal{T}_m$.

- (a) If $m > n$, then h^\sharp merges all pairs in W_m . Since the kernel of e_m^m is generated by W_m , this implies that h factorizes through e_m^m .
- (b) If $m = 0$ and $T_\Sigma 0 = 0$ (i.e., the signature Σ contains no constant symbol), then the only quotient in \mathcal{T}_0 is the empty quotient $e: T_\Sigma 0 \rightarrow 0$, through which h trivially factorizes.

- (c) It remains to consider the case where $m \leq n$ and $T_\Sigma m \neq 0$. Then there exist morphisms $q: T_\Sigma(n+1) \rightarrow T_\Sigma m$ and $j: T_\Sigma m \rightarrow T_\Sigma(n+1)$ with $q \cdot j = id$. Indeed: (i) if $m = 0$, then $T_\Sigma m$ is the initial algebra. Choose j to be unique initial morphism, and q to be an arbitrary morphism, which exists because $T_\Sigma m \neq 0$. Then $q \cdot j = id$ by initiality; (ii) If $m > 0$, choose $q': n+1 \rightarrow m$ and $j': m \rightarrow n+1$ with $q' \cdot j' = id$. Then $j = T_\Sigma j'$ and $q = T_\Sigma q'$ satisfy $q \cdot j = id$.

Since \mathcal{T} is a theory, we know that $e_{n+1}^{n+1} \cdot j$ factorizes through some $e \in \mathcal{T}_m$, say $e_{n+1}^{n+1} \cdot j = k \cdot e$. Moreover, by (a) above, the morphism $h^\# \cdot q$ factorizes as $h^\# \cdot q = g \cdot e_{n+1}^{n+1}$ for some g .

$$\begin{array}{ccccc}
 T_\Sigma(n+1) & \xrightleftharpoons[q]{j} & T_\Sigma m & & \\
 \downarrow e_{n+1}^{n+1} & & \downarrow e & \searrow h^\# & \\
 E_{n+1}^{n+1} & \xleftarrow{k} & E & & A \\
 & \searrow g & & &
 \end{array}$$

It follows that

$$h^\# = h^\# \cdot q \cdot j = g \cdot e_{n+1}^{n+1} \cdot j = g \cdot k \cdot e, .$$

so $h^\#$ factorizes through $e \in \mathcal{T}_m$, as required.

- (2) Let $(s_i = t_i)_{i < \omega}$ be a sequence of term equations, where $(s_i, t_i) \subseteq T_\Sigma m_i \times T_\Sigma m_i$. For each $n < \omega$, form the set $\mathcal{T}_n \subseteq \mathbf{Alg}(\Sigma) \downarrow \mathbf{Alg}_f(\Sigma)$ of all finite quotients $e: T_\Sigma n \rightarrow E$ with the following property:

$$\exists i_0 < \omega : \forall i \geq i_0 : \forall (g: T_\Sigma m_i \rightarrow T_\Sigma n) : e \cdot g(s_i) = e \cdot g(t_i). \quad (\text{B.5})$$

We first show that $\mathcal{T} = (\mathcal{T}_n)_{n < \omega}$ is an equational theory. To see this, note first that \mathcal{T}_n is a filter: upward closure is obvious, and for codirectedness observe that given $e: T_\Sigma n \rightarrow E$ and $e': T_\Sigma n \rightarrow E'$ in \mathcal{T}_n , the subdirect product (i.e. the coimage of the map $\langle e, e' \rangle: T_\Sigma n \rightarrow E \times E'$) clearly lies in \mathcal{T}_n . To show that \mathcal{T} is substitution-invariant, let $e \in \mathcal{T}_n$ and $h: T_\Sigma m \rightarrow T_\Sigma n$. Factorize $e \cdot h = m \cdot \bar{e}$ with \bar{e} surjective and m injective. Since $e \in \mathcal{T}_n$, there exists $i_0 < \omega$ as in (B.5). Then, for every $i \geq i_0$ and $g: T_\Sigma m_i \rightarrow T_\Sigma m$ we have $e \cdot h \cdot g(s_i) = e \cdot h \cdot g(t_i)$. This implies $m \cdot \bar{e} \cdot g(s_i) = m \cdot \bar{e} \cdot g(t_i)$, so $\bar{e} \cdot g(s_i) = \bar{e} \cdot g(t_i)$ because m is injective. This shows that $\bar{e} \in \mathcal{T}_m$, i.e. \mathcal{T} is substitution-invariant. $\mathcal{E}_{\mathcal{X}}$ -completeness is trivial because $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$ (see Remark 3.11).

We claim that a finite Σ -algebra A lies in $\mathcal{V}(\mathcal{T})$ iff it eventually satisfies $(s_i = t_i)_{i < \omega}$.

(\Rightarrow) Let $A \in \mathcal{V}(\mathcal{T})$. Choose a surjective morphism $e: T_\Sigma n \rightarrow A$ for some $n < \omega$. Then e factorizes through some element of \mathcal{T}_n , which implies $e \in \mathcal{T}_n$ because this set is upwards closed. Thus, there exists $i_0 < \omega$ as in (B.5). We claim that A satisfies all the equations $s_i = t_i$ with $i \geq i_0$. Indeed, let $h: T_\Sigma m_i \rightarrow A$. By

projectivity of $T_\Sigma m_i$, there exists $g: T_\Sigma m_i \rightarrow T_\Sigma n$ with $h = e \cdot g$. By (B.5) we have $e \cdot g(s_i) = e \cdot g(t_i)$ and thus $h(s_i) = h(t_i)$. Thus A satisfies $s_i = t_i$ for $i \geq i_0$.

(\Leftarrow) Suppose that A eventually satisfies $(s_i = t_i)_{i < \omega}$; say, it satisfies $s_i = t_i$ for all $i \geq i_0$. To show that $A \in \mathcal{V}(\mathcal{T})$, let $n < \omega$ and $h: T_\Sigma n \rightarrow A$. For all $i \geq i_0$ and $g: T_\Sigma m_i \rightarrow T_\Sigma n$ we have $h \cdot g(s_i) = h \cdot g(t_i)$ because A satisfies $s_i = t_i$. Letting e denote the coimage of h , this implies $e \cdot g(s_i) = e \cdot g(t_i)$ for all $i \geq i_0$, and thus $e \in \mathcal{T}_n$ by definition of \mathcal{T}_n . We have thus shown that h factorizes through $e \in \mathcal{T}_n$, which proves that $A \in \mathcal{V}(\mathcal{T})$. \square

Step 4. From the theory version of our HSP theorem (Theorem 3.15) and the previous lemma, we conclude:

Theorem B.9 (Eilenberg-Schützenberger [13]).

A class of finite Σ -algebras is closed under finite products, subalgebras and quotients if and only if it is axiomatizable by a sequence of term equations.

Our above derivation of this theorem is overall not shorter than the original proof of Eilenberg and Schützenberger, and also rests on their Lemma B.8. However, the present approach has the advantage of explicitly relating the syntactic concept of a sequence of term equations to the order-theoretic concept of an equational theory, which is missing in the original paper.

B.4 Reiterman's Theorem and Pin & Weil's Theorem

Reiterman [25] proved another HSP theorem for finite Σ -algebras, in which one uses profinite equations rather than sequences of equations as in Eilenberg and Schützenberger's result (see Section B.3). In contrast to the latter, Reiterman's theorem applies to algebras over arbitrary finitary signatures Σ , not only signatures with finitely many operations. In this section, we show how to derive this theorem from our general results. We omit some of the details because Reiterman's theorem has already been treated categorically in previous work [11].

A *topological Σ -algebra* is a Σ -algebra A with a topology on its underlying set such that all Σ -operations $\sigma: A^n \rightarrow A$ are continuous. A *profinite Σ -algebra* is a topological Σ -algebra that can be expressed as a limit of finite algebras with discrete topology. We write $\mathbf{ProAlg}(\Sigma)$ for the category of profinite Σ -algebras and continuous Σ -homomorphisms. The category $\mathbf{Alg}_f(\Sigma)$ of finite Σ -algebras forms a full subcategory of $\mathbf{ProAlg}(\Sigma)$ by identifying finite Σ -algebras with profinite Σ -algebras with discrete topology. The forgetful functor from $\mathbf{ProAlg}(\Sigma)$ to \mathbf{Set} has a left adjoint assigning to each set X the *free profinite Σ -algebra* $\widehat{T}_\Sigma X$. The latter can be computed as the limit of all finite quotient algebras of $T_\Sigma X$, i.e. the limit of the diagram

$$D: T_\Sigma X \downarrow \mathbf{Alg}_f(\Sigma) \rightarrow \mathbf{ProAlg}(\Sigma), \quad (e: T_\Sigma \twoheadrightarrow A) \rightarrow A.$$

To deduce Reiterman's theorem from our HSP theorem, we proceed as follows.

Step 1. Choose the parameters

- $\mathcal{A} = \mathbf{ProAlg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, injective morphisms})$;
- $\mathcal{A}_0 = \mathbf{Alg}_f(\Sigma)$;
- $\Lambda =$ all finite cardinal numbers;
- $\mathcal{X} =$ all finitely generated free profinite algebras $\widehat{T}_\Sigma X$ ($X \in \mathbf{Set}_f$).

The class $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$ consists of all surjective morphisms. This follows from Remark B.1 applied to $\mathbf{ProAlg}(\Sigma) \xrightarrow{\leftarrow \top} \mathbf{Set}$, $\mathcal{X}' = \mathbf{Set}_f$ and $\mathcal{E}' =$ surjections.

Our Assumptions 3.1 are satisfied: for (1), note that finite products of finite (and thus discrete) profinite Σ -algebras are computed in \mathbf{Set} . (2) is clear. For (3), let A be a finite Σ -algebra and choose a surjective map $e: X \rightarrow A$ for some finite set X . Then the unique extension $\widehat{e}: \widehat{T}_\Sigma X \rightarrow A$ is surjective, i.e. $\widehat{T}_\Sigma X \in \mathcal{X}$ and $\widehat{e} \in \mathcal{E}_{\mathcal{X}}$.

Step 2. Given a profinite Σ -algebra A , a *profinite congruence* on A is a Σ -algebra congruence $\equiv \subseteq A \times A$ such that the quotient algebra A/\equiv , equipped with the quotient topology, is profinite. In analogy to (3.1), there is an isomorphism of complete lattices

$$\text{profinite quotient algebras of } A \cong \text{profinite congruences on } A \quad (\text{B.6})$$

mapping a profinite quotient $e: A \rightarrow B$ to its kernel $\equiv_e \subseteq A \times A$. To see this, one just needs to show that given profinite congruences $\equiv \subseteq \equiv'$ on A , one has $e \leq e'$ for the corresponding quotients $e: A \rightarrow A/\equiv$ and $e': A \rightarrow A/\equiv'$, i.e. e' factorizes through e in $\mathbf{ProAlg}(\Sigma)$. But this follows immediately from the fact that the codomain A/\equiv of e carries the quotient topology, i.e., every function h with $e' = h \cdot e$ is continuous.

Step 3. In the present setting, an equation over a finite set X of variables is given by a filter $\mathcal{T}_X \subseteq \widehat{T}_\Sigma X \downarrow \mathbf{Alg}_f(\Sigma)$ in the poset of finite quotient algebras of $\widehat{T}_\Sigma X$. One can view \mathcal{T}_X as a diagram of finite algebras in $\mathbf{ProAlg}(\Sigma)$ and take its limit cone $\pi_q: P_X \rightarrow A$ (where $q: \widehat{T}_\Sigma X \rightarrow A$ ranges over \mathcal{T}_X). Its universal property gives a unique morphism $e_X: \widehat{T}_\Sigma X \rightarrow P_X$ with $\pi_q \cdot e = q$ for all $q \in \mathcal{T}_X$. By standard properties of inverse limits of topological spaces, the map e is surjective [26, Corollary 1.1.6]. Then a finite Σ -algebra A satisfies the equation \mathcal{T}_X iff every $h: \widehat{T}_\Sigma X \rightarrow A$ factorizes through e_X . We have thus shown that every equation $\mathcal{T}_{\mathcal{X}}$ can be presented as a single quotient e_X .

A *profinite equation* over a finite set X of variables is a pair $(s, t) \in \widehat{T}_\Sigma X \times \widehat{T}_\Sigma X$, denoted as $s = t$. It is *satisfied* by a finite Σ -algebra A if for every map $h: X \rightarrow A$ we have $h^\sharp(s) = h^\sharp(t)$. Here, $h^\sharp: \widehat{T}_\Sigma X \rightarrow A$ denotes the unique extension of h to a morphism in $\mathbf{ProAlg}(\Sigma)$, using the universal property of the free profinite algebra $\widehat{T}_\Sigma X$.

Equations are expressively equivalent to profinite equations:

- (1) For every equation expressed as a profinite quotient $e: \widehat{T}_\Sigma X \rightarrow E$, the corresponding profinite congruence $\equiv_e \subseteq \widehat{T}_\Sigma X \times \widehat{T}_\Sigma X$ is a set of profinite equations equivalent to e , that is, a Σ -algebra A satisfies e iff it satisfies all term inequations in \equiv_e . This follows immediately from the exactness property (B.6).

(2) Conversely, given a profinite equation $(s, t) \in \widehat{T}_\Sigma X \times \widehat{T}_\Sigma X$, form the smallest profinite congruence \equiv on $\widehat{T}_\Sigma X$ with $s \equiv t$ (viz. the intersection of all such congruences) and let $e: \widehat{T}_\Sigma X \rightarrow E$ be the corresponding quotient. Then a profinite Σ -algebra A satisfies $s = t$ iff it satisfies e . This is once again a consequence of the exactness property (B.6).

Step 4. From Theorem 3.16, we deduce:

Theorem B.10 (Reiterman [25]). *A class of finite Σ -algebras is a pseudovariety (i.e. closed under quotients, subalgebras and finite products) iff it is axiomatizable by profinite equations.*

As for Birkhoff's classical HSP theorem, there is an ordered version of this result. An *ordered profinite Σ -algebra* is a profinite Σ -algebra carrying an additional partial order such that all operations are continuous and monotone. Morphisms are monotone continuous Σ -homomorphisms. Accordingly, take the parameters

- $\mathcal{A} = \mathbf{ProAlg}_{\leq}(\Sigma)$ (ordered profinite Σ -algebras);
- $\mathcal{A}_0 = \mathbf{Alg}_{\leq, f}(\Sigma)$ (finite ordered Σ -algebras);
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, order-embeddings})$;
- $\mathcal{X} =$ all finitely generated free ordered profinite algebras $\widehat{T}_\Sigma X$ ($X \in \mathbf{Set}_f$);
- $\Lambda =$ all finite cardinals.

In analogy to the above unordered case, replacing profinite equations $s = t$ by profinite inequations $s \leq t$, we obtain

Theorem B.11 (Pin and Weil [23]). *A class of finite ordered Σ -algebras is closed under quotients, subalgebras and finite products iff it can be presented by profinite inequations.*

B.5 Quantitative Algebras

In this section, we derive an HSP theorem for quantitative algebras as an instance of our general results. Recall that an *extended metric space* is a set A with a map $d_A: A \times A \rightarrow [0, \infty]$ (assigning to any two points a possibly infinite distance), subject to the axioms (i) $d_A(a, b) = 0$ iff $a = b$, (ii) $d_A(a, b) = d_A(b, a)$ and (iii) $d_A(a, c) \leq d_A(a, b) + d_A(b, c)$ for all $a, b, c \in A$. A map $h: A \rightarrow B$ between extended metric spaces is *nonexpansive* if $d_B(h(a), h(a')) \leq d_A(a, a')$ for $a, a' \in A$. Let \mathbf{Met}_∞ denote the category of extended metric spaces and nonexpansive maps. Note that products $\prod_{i \in I} A_i$ in \mathbf{Met}_∞ are given by cartesian products with the sup metric $d((a_i)_{i \in I}, (b_i)_{i \in I}) = \sup_{i \in I} d_{A_i}(a_i, b_i)$, and coproducts $\coprod_{i \in I} A_i$ by disjoint unions, where points in distinct components have distance ∞ .

Fix a, not necessarily finitary, signature Σ , that is, the arity of an operation symbol $\sigma \in \Sigma$ is any cardinal number. A *quantitative Σ -algebra* is a Σ -algebra A endowed with an extended metric d_A such that all Σ -operations $\sigma: A^n \rightarrow A$ are nonexpansive. The forgetful functor from the category $\mathbf{QAlg}(\Sigma)$ of quantitative Σ -algebras and nonexpansive Σ -homomorphisms to \mathbf{Met}_∞ has a left adjoint

assigning to each space X the free quantitative Σ -algebra $T_\Sigma X$. The latter is carried by the set of all Σ -terms (equivalently, well-founded Σ -trees) over X , with metric inherited from X as follows: if s and t are Σ -terms of the same shape, i.e. they differ only in the variables, their distance is the supremum of the distances of the variables in corresponding positions of s and t ; otherwise, it is ∞ .

The HSP theorem for quantitative algebras is parametric in a regular cardinal number $c > 1$. In the following, an extended metric space is called *c-clustered* if it is a coproduct of spaces of cardinality $< c$.

Step 1. Choose the parameters of our setting as

- $\mathcal{A} = \mathcal{A}_0 = \mathbf{QAlg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M})$ is given by morphisms carried by surjections and subspaces, resp.;
- $\Lambda =$ all cardinal numbers;
- $\mathcal{X} =$ all free algebras $T_\Sigma X$ with $X \in \mathbf{Met}_\infty$ a c -clustered space.

Let us characterize the class $\mathcal{E}_{\mathcal{X}}$:

Lemma B.12. *A quotient $e: A \twoheadrightarrow B$ belongs to $\mathcal{E}_{\mathcal{X}}$ if and only if for every subset $B_0 \subseteq B$ of size $< c$ there exists a subset of $A_0 \subseteq A$ such that $e[A_0] = B_0$ and the restriction $e: A_0 \rightarrow B_0$ is isometric.*

Following the terminology of Mardare et al. [20], we call a quotient with the property stated in the lemma *c-reflexive*. Note that every quotient is 2-reflexive.

Proof. By Remark B.1 applied to the adjunction $\mathbf{QAlg}(\Sigma) \xrightleftharpoons[\top]{\perp} \mathbf{Met}_\infty$ with $\mathcal{X}' = c$ -clustered spaces and $\mathcal{E}' =$ surjective nonexpansive maps, the statement of the lemma can be reduced to the case where the signature Σ is empty, that is, we can assume that $\mathcal{A} = \mathbf{Met}_\infty$ and $\mathcal{X} = c$ -clustered spaces.

Note that \mathcal{X} is the closure of the class $\mathcal{X}_c = \{X \in \mathbf{Met}_\infty : |X| < c\}$ under coproducts. Since a coproduct is projective w.r.t. some morphism e iff all of the coproduct components are, one has $\mathcal{E}_{\mathcal{X}} = \mathcal{E}_{\mathcal{X}_c}$. Therefore, it suffices to show that, for every $e: A \twoheadrightarrow B$ in \mathbf{Met}_∞ ,

$$e \in \mathcal{E}_{\mathcal{X}_c} \iff e \text{ is } c\text{-reflexive.}$$

For the “ \Rightarrow ” direction, suppose that $e \in \mathcal{E}_{\mathcal{X}_c}$, and let $m: B_0 \hookrightarrow B$ be a subspace of size $< c$. Then $B_0 \in \mathcal{X}_c$ and thus there exists $g: B_0 \rightarrow A$ with $e \cdot g = m$. Let $A_0 = g[B_0]$. It follows that $e[A_0] = B_0$, and for every pair of elements $g(b), g(b') \in A_0$ one has

$$d_B(e(g(b)), e(g(b'))) = d_B(m(b), m(b')) = d_B(b, b'),$$

i.e. $e: A_0 \rightarrow B_0$ is isometric. Thus e is c -reflexive.

For the “ \Leftarrow ” direction, suppose that e is c -reflexive and let $h: X \rightarrow B$ be a nonexpansive map with $X \in \mathcal{X}_c$, i.e. $|X| < c$. Then $h[X] \subseteq B$ has cardinality $< c$, so there exists a subset $A_0 \subseteq A$ such that $e[A_0] = h[X]$ and $e: A_0 \rightarrow$

$h[X]$ is isometric. For every $x \in X$, let $g(x)$ be the unique element of A_0 with $h(x) = e(g(x))$. This defines a function $g: X \rightarrow A$ with $e \cdot g = h$. Moreover, g is nonexpansive: for all $x, y \in X$ we have

$$d_A(g(x), g(y)) = d_B(e(g(x)), e(g(y))) = d_B(h(x), h(y)) \leq d_X(x, y).$$

This proves $e \in \mathcal{E}_{\mathcal{X}_c}$.

Remark B.13. It follows that our Assumptions 3.1 are satisfied. For (1), just observe that products in $\mathbf{QAlg}(\Sigma)$ are formed on the level of underlying metric spaces. (2) is trivial. For (3), we need to show that every algebra $A \in \mathbf{QAlg}(\Sigma)$ is a c -reflexive quotient of some algebra in \mathcal{X} . To this end, consider the family $m_i: A_i \rightarrow A$ ($i \in I$) of all subspaces of A of size $< c$. Then the map $[m_i]: \prod_{i \in I} A_i \rightarrow A$ in \mathbf{Met}_∞ is c -reflexive, as is its unique extension $T_\Sigma(\prod_{i \in I} A_i) \rightarrow A$ to a morphism of $\mathbf{QAlg}(\Sigma)$. Moreover, $T_\Sigma(\prod_{i \in I} A_i) \in \mathcal{X}$, which proves (3).

Step 2. Next, we establish the required exactness property for quantitative algebras. Recall that an (*extended*) *pseudometric* on a set A is a map $p: A \times A \rightarrow [0, \infty]$ satisfying all axioms of a metric except possibly the implication $p(a, b) \Rightarrow a = b$; that is, two distinct points may have distance 0 with respect to p . Given a quantitative Σ -algebra A , a pseudometric p on A is a *congruence* if

- (1) $p(a, a') \leq d_A(a, a')$ for all $a, a' \in A$, and
- (2) every Σ -operation $\sigma: A^n \rightarrow A$ ($\sigma \in \Sigma$) is nonexpansive with respect to p , that is, for each n -ary operation symbol $\sigma \in \Sigma$ and $a_i, b_i \in A$ one has

$$p(\sigma((a_i)_{i < n}), \sigma((b_i)_{i < n})) \leq \sup_{i < n} p(a_i, b_i).$$

Congruences are ordered by $p \leq q$ iff $p(a, a') \leq q(a, a')$ for all $a, a' \in A$.

Lemma B.14. *For each quantitative Σ -algebra A , there is a dual isomorphism of complete lattices*

$$\text{quotients of } A \cong \text{congruences on } A.$$

Proof. Every quotient $e: A \rightarrow B$ in $\mathbf{QAlg}(\Sigma)$ defines a congruence p_e on A given by $p_e(a, a') = d_B(e(a), e(a'))$ for $a, a' \in A$. Conversely, let p be a congruence on A . Then the equivalence relation \equiv_p on A given by $a \equiv_p a'$ iff $p(a, a') = 0$ is a Σ -algebra congruence. This yields the quotient $e_p: A \rightarrow A_p$, where A_p is the Σ -algebra A/\equiv_p equipped with the metric $d_{A_p}([a], [a']) = p(a, a')$ for $a, a' \in A$.

The two maps $e \mapsto p_e$ and $p \mapsto e_p$ are clearly antitone and mutually inverse. \square

Remark B.15. (1) Given $A \in \mathbf{QAlg}(\Sigma)$ and a family of triples $(a_j, b_j, \varepsilon_j)$ ($j \in J$) with $a_j, b_j \in A$ and $\varepsilon_j \in [0, \infty]$, there is a largest congruence p on A with $p(a_j, b_j) \leq \varepsilon_j$ for all j , viz. the pointwise supremum of all such congruences. We call p the *congruence generated by the relations* $a_j =_{\varepsilon_j} b_j$. If A is just a set (viewed as a discrete algebra over the empty signature) we call p the *pseudometric generated by the relations* $a_j =_{\varepsilon_j} b_j$.

(2) As an immediate consequence of the above lemma, we obtain the *homomorphism theorem* for quantitative algebras: given any two morphisms $e: A \rightarrow B$ and $f: A \rightarrow C$ in $\mathbf{QAlg}(\Sigma)$ with e surjective, then f factorizes through e if and only if $p_f \leq p_e$, that is, $d_C(f(a), f(a')) \leq d_B(e(a), e(a'))$ for all $a, a' \in A$.

Note that if the congruence p_e is generated by the relations $a_j =_{\varepsilon_j} b_j$ ($j \in J$) then it suffices to verify that $d_C(f(a_j), f(b_j)) \leq \varepsilon_j$ for all j .

Step 3. By Remark 3.4, in the current setting an equation can be presented as a single quotient $e_X: T_\Sigma X \rightarrow E$ with X a c -clustered space. The corresponding syntactic concept is given by

Definition B.16. (1) A c -clustered equation over the set X of variables is an expression of the form

$$x_i =_{\varepsilon_i} y_i \ (i \in I) \vdash s =_\varepsilon t \quad (\text{B.7})$$

where (i) I is a set, (ii) $x_i, y_i \in X$ for all i , (iii) s and t are Σ -terms over X , (iv) $\varepsilon_i, \varepsilon \in [0, \infty]$, and (v) X (viewed as a discrete metric space, i.e. with $d(x, x') = \infty$ for $x \neq x'$) is c -clustered so that for each $i \in I$, x_i, y_i lie in the same coproduct component of X . In other words, X can be expressed as a disjoint union $X = \coprod_{j \in J} X_j$ of subsets of size $< c$ such that only relations between elements in the same X_j are mentioned on the left-hand side of (B.7)

(2) A quantitative Σ -algebra A *satisfies* (B.7) if for every map $h: X \rightarrow A$,

$$d_A(h(x_i), h(y_i)) \leq \varepsilon_i \text{ for all } i \in I \quad \text{implies} \quad d_A(h^\sharp(s), h^\sharp(t)) \leq \varepsilon.$$

Here we denote by $h^\sharp: T_\Sigma X \rightarrow A$ the unique Σ -algebra morphism extending h .

Remark B.17. Let us discuss some important special cases:

(1) A 2-clustered equation is called an *unconditional equation* because it contains only trivial conditions of the form $x_i =_{\varepsilon_i} x_i$; thus, it is equivalent to $\emptyset \vdash s =_\varepsilon t$.

(2) Mardare et al. [20] introduced *c -basic conditional equations*, i.e. equations (B.7) with $|I| < c$. This concept is closely related to the one of a c -clustered equation. First, note that every c -basic conditional equation is a c -clustered equation (with a single cluster). Conversely, if κ is an infinite regular cardinal such that every operation symbol in Σ has arity $< \kappa$, and one has $c \geq \kappa$, then every c -clustered equation can be expressed in terms of equivalent c -basic conditional equations. To see this, suppose that a c -clustered equation (B.7) is given. Remove all conditions $x_i =_{\varepsilon_i} y_i$ such that the coproduct component containing x_i, y_i does not contain any variable occurring in s or t . The resulting equation is clearly equivalent to (B.7). Moreover, since s and t contain $< \kappa$ variables, and every cluster of X has size $< c$, it follows that less than $c \cdot \kappa = c$ conditions remain, i.e. we obtain a c -basic conditional equation.

Lemma B.18. *Equations and c -clustered equations are expressively equivalent.*

Proof. (1) Given any equation $e: T_\Sigma X \rightarrow E$, where $X = \coprod_{j \in J} X_j$ with $|X_j| < c$, form the c -clustered equations over X given by

$$x =_{\varepsilon_{x,y}} y \ (j \in J, x, y \in X_j) \vdash s =_{\varepsilon_{s,t}} t \ (s, t \in T_\Sigma X), \quad (\text{B.8})$$

with $\varepsilon_{x,y} = d_X(x, y)$ and $\varepsilon_{s,t} = d_E(e(s), e(t))$. Note that (B.8) is c -clustered because c is regular. Then an algebra $A \in \mathbf{QAlg}(\Sigma)$ satisfies the equation e iff it satisfies all the c -clustered equations (B.8). Indeed, we have

- A satisfies e
- \Leftrightarrow for all $h: X \rightarrow A$ in \mathbf{Met}_∞ , $h^\sharp: T_\Sigma X \rightarrow A$ factorizes through e
 - \Leftrightarrow for all $h: X \rightarrow A$ in \mathbf{Met}_∞ and $s, t \in T_\Sigma X$, one has

$$d_A(h^\sharp(s), h^\sharp(t)) \leq d_E(e(s), e(t))$$
 - \Leftrightarrow for all maps $h: X \rightarrow A$ with $d_A(h(x), h(y)) \leq d_X(x, y)$ for all $x, y \in X$, one has $d_A(h^\sharp(s), h^\sharp(t)) \leq d_E(e(s), e(t))$ for all $s, t \in T_\Sigma X$
 - \Leftrightarrow for all maps $h: X \rightarrow A$ with $d_A(h(x), h(y)) \leq \varepsilon_{x,y}$ for all $j \in J$ and $x, y \in X_j$, one has $d_A(h^\sharp(s), h^\sharp(t)) \leq \varepsilon_{s,t}$ for all $s, t \in T_\Sigma X$
 - $\Leftrightarrow A$ satisfies (B.8).

In the penultimate step, we use that for $x \in X_j$ and $y \in X_k$ with $j \neq k$, the inequality $d_A(h(x), h(y)) \leq d_X(x, y)$ holds trivially because $d_X(x, y) = \infty$.

(2) Conversely, to every c -clustered equation (B.7) over a set X of variables, we associate an equation in two steps:

- Take the pseudometric p on X generated by the relations $x_i =_{\varepsilon_i} y_i$ ($i \in I$), and let $e_p: X \rightarrow X_p$ denote the corresponding quotient.
- Take the congruence q on $T_\Sigma(X_p)$ generated by the single relation $T_\Sigma e_p(s) =_\varepsilon T_\Sigma e_p(t)$, and let $e_q: T_\Sigma(X_p) \rightarrow E_q$ be the corresponding quotient.

We claim that (a) X_p is c -clustered (and thus e_q is an equation), and (b) e_q and (B.7) are equivalent, i.e. satisfied by the same algebras.

For (a), note that since (B.7) is a c -clustered equation, X can be decomposed as a coproduct $X = \coprod X_j$ of subsets of size $< c$ such that for all $i \in I$ one has $x_i, y_i \in X_j$ for some (unique) j . Let p_j be pseudometric on X_j generated by the relations $x_i =_{\varepsilon_i} y_i$ with $i \in I$ and $x_i, y_i \in X_j$. Then we have $X_p = \coprod_j (X_j)_{p_j}$, so X_p is a coproduct of spaces of size $< c$, i.e. a c -clustered space.

In order to prove (b), let r denote the congruence on $T_\Sigma X$ generated by the relations $x_i =_{\varepsilon_i} y_i$ ($i \in I$) and $s =_\varepsilon t$, with corresponding quotient $e_r: T_\Sigma X \rightarrow E_r$. We claim that the quotients $e_q \cdot T_\Sigma e_p$ and e_r are isomorphic. To prove this, we use the homomorphism theorem. We have

$$d_{E_q}(e_q \cdot T_\Sigma e_p(x_i), e_q \cdot T_\Sigma e_p(y_i)) \leq d_{X_p}(T_\Sigma e_p(x_i), T_\Sigma e_p(y_i)) = p(x_i, y_i) \leq \varepsilon_i$$

for each $i \in I$ and, moreover,

$$d_{E_q}(e_q \cdot T_\Sigma e_p(s), e_q \cdot T_\Sigma e_p(t)) = q(T_\Sigma e_p(s), T_\Sigma e_p(t)) \leq \varepsilon.$$

Thus $e_q \cdot T_\Sigma e_p$ factorizes through e_r , i.e. $k \cdot e_r = e_q \cdot T_\Sigma e_p$ for some $k : E_r \rightarrow E_q$.

For the converse, note first that e_r factorizes through $T_\Sigma e_p$ because $r \leq p$. Thus $e_r = f \cdot T_\Sigma e_p$ for some $f : T_\Sigma X_p \rightarrow E_r$. The morphism f factorizes through e_q because

$$d_{E_r}(f \cdot T_\Sigma e_p(s), f \cdot T_\Sigma e_p(t)) = d_{E_r}(e_r(s), e_r(t)) = r(s, t) \leq \varepsilon.$$

Thus $f = l \cdot e_q$ for some $l : E_q \rightarrow E_r$. This yields the commutative diagram below, which proves that k and l are mutually inverse since e_r and $e_q \cdot T_\Sigma e_p$ are epimorphisms:

$$\begin{array}{ccc}
 & T_\Sigma X & \\
 T_\Sigma e_p \swarrow & & \searrow e_r \\
 T_\Sigma X_p & & E_r \\
 e_q \swarrow & f \searrow & \\
 E_q & \xrightarrow{l} & E_r \\
 & \xleftarrow{k} &
 \end{array}$$

Consequently, for every $A \in \mathbf{QAlg}(\Sigma)$,

- A satisfies e_q
- \Leftrightarrow for all $h : X_p \rightarrow A$ in \mathbf{Met}_∞ , $h^\sharp : T_\Sigma X_p \rightarrow A$ factorizes through e_q
- \Leftrightarrow for all $h : X_p \rightarrow A$ in \mathbf{Met}_∞ , $h^\sharp \cdot T_\Sigma e_p$ factorizes through $e_q \cdot T_\Sigma e_p \cong e_r$
- \Leftrightarrow for all $g : X \rightarrow A$, if g^\sharp factorizes through $T_\Sigma e_p$, then g^\sharp factorizes through e_r
- \Leftrightarrow for all $g : X \rightarrow A$ with $d_A(g(x_i), g(y_i)) \leq \varepsilon_i$ ($i \in I$) one has $d_A(g^\sharp(s), g^\sharp(t)) \leq \varepsilon$
- $\Leftrightarrow A$ satisfies (B.7).

The third step might not be immediately clear, and so we now provide further details. First a general fact about free algebras: let Y be any set, and denote by $\eta_Y : Y \rightarrow T_\Sigma Y$ the universal map. Then we have $(h^\sharp \cdot \eta_Y)^\sharp = h^\sharp$ for every $h : Y \rightarrow A$.

For the “ \Rightarrow ” direction of the third equivalence, suppose that $g^\sharp = k \cdot T_\Sigma e_p$ for some $k : T_\Sigma X_p \rightarrow A$. Let $h = k \cdot \eta_{X_p}$ so that $g^\sharp = h^\sharp \cdot T_\Sigma e_p$, which factorizes through e_r by assumption.

For the converse “ \Leftarrow ”, let $h : X_p \rightarrow A$ be in \mathbf{Met}_∞ . Then $h^\sharp \cdot T_\Sigma e_p = g^\sharp$ where $g = h^\sharp \cdot T_\Sigma e_p \cdot \eta_X : X \rightarrow A$. Then g^\sharp factorizes through $T_\Sigma e_p$ and therefore through e_r , i.e. $h^\sharp \cdot T_\Sigma e_p$ factorizes through e_r as desired. \square

Step 4. From Lemma B.18 and Theorem 3.16, we conclude:

Theorem B.19. *For any regular cardinal $c > 1$, a class of quantitative Σ -algebras is a c -variety (i.e. closed under c -reflexive homomorphic images, subalgebras, and products) if and only if it is axiomatizable by c -clustered equations.*

Remark B.20. The above theorem is closely related to the quantitative HSP theorem in the recent work of Mardare et al. [20]. These authors show that for a signature with finite or countably infinite arities (i.e. $\kappa \in \{\aleph_0, \aleph_1\}$ in the notation of Remark B.17) and for $c \leq \aleph_1$, c -varieties are precisely the classes of quantitative algebras axiomatizable by c -basic conditional equations. By Remark B.17, Theorem B.19 implies this result except for the case $\kappa = \aleph_1$ and $c = \aleph_0$.

Note that our above theorem generalizes the one of Mardare et al. in the sense that we do not impose any restrictions on Σ and c .

Quantitative equational logic. Mardare et al. [19] also proposed a sound and complete deduction system for unconditional equations (i.e. the case $c = 2$, cf. Remark B.17(1)) over a finitary signature Σ . It rests on the following proof rules, where s, t, u, s_i, t_i are Σ -terms over a set X of variables and $\varepsilon, \varepsilon' \in [0, \infty]$.

- (Refl) $\vdash t =_0 t$
- (Sym) $s =_\varepsilon t \vdash t =_\varepsilon s$
- (Triang) $s =_\varepsilon t, t =_{\varepsilon'} u \vdash s =_{\varepsilon + \varepsilon'} u$
- (Max) $s =_\varepsilon t \vdash s =_{\varepsilon'}, t$ for $\varepsilon' > \varepsilon$
- (Arch) $\{s =_{\varepsilon'}, t : \varepsilon' > \varepsilon\} \vdash s =_\varepsilon t$
- (Cong) $s_i =_\varepsilon t_i (i = 1, \dots, n) \vdash \sigma(s_1, \dots, s_n) =_\varepsilon \sigma(t_1, \dots, t_n)$ for all $\sigma \in \Sigma_n$
- (Subst) $s =_\varepsilon t \vdash h(s) =_\varepsilon h(t)$ for all Σ -homomorphisms $h: T_\Sigma X \rightarrow T_\Sigma Y$

Given a set Γ of unconditional equations and an unconditional equation $s =_\varepsilon t$, we write $\Gamma \vdash s =_\varepsilon t$ if $s =_\varepsilon t$ can be proved from the axioms in Γ using the above rules. Note that due to the infinitary rule (Arch), a proof can be transfinite. We write $\Gamma \models s =_\varepsilon t$ if every quantitative Σ -algebra that satisfies all equations in Γ also satisfies $s =_\varepsilon t$. In the following, we demonstrate how to obtain the completeness of this calculus from our general completeness result (Theorem 4.4). As in our treatment of Birkhoff's equational logic in Section B.1, the key lies in the observation that the above rules amount to computing the congruence (or the equational theory, resp.) generated by given a set of equations.

Remark B.21. Since 2-clustered spaces are precisely the discrete spaces (i.e. $d(x, y) = \infty$ for $x \neq y$), the class \mathcal{X} consists of all free algebras $T_\Sigma X$ with $X \in \mathbf{Set}$. Moreover, we have $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$. Thus, by Remark 3.11, in the current setting an equational theory is presented by a family of quotients $(e_X : T_\Sigma X \twoheadrightarrow E_X)_{X \in \mathbf{Set}}$ which is *substitution invariant* in the sense that for every Σ -homomorphism $h : T_\Sigma X \rightarrow T_\Sigma Y$ with $X, Y \in \mathbf{Set}$, the morphism $e_Y \cdot h$ factorizes through e_X .

For any equation $e : T_\Sigma X \twoheadrightarrow E$ we denote by

$$\Gamma_e = \{s =_\varepsilon t : s, t \in T_\Sigma X \text{ and } d_E(e(s), e(t)) \leq \varepsilon\}$$

the set of *unconditional equations associated to e* . More generally, for a family $(e_X : T_\Sigma X \twoheadrightarrow E_X)_{X \in \mathbf{Set}}$ of equations we get an associated family $(\Gamma_{e_X})_{X \in \mathbf{Set}}$ of sets of unconditional equations.

Lemma B.22. (1) *A set of unconditional equations over the set X is associated to some equation iff it is closed under (Refl), (Sym), (Triang), (Max), (Arch), (Cong).*

(2) *A family $(\Gamma_X)_{X \in \mathbf{Set}}$ of sets of unconditional equations is associated to some equational theory iff it is closed under (Refl), (Sym), (Triang), (Max), (Arch), (Cong), (Subst).*

Proof. (1) For the “only if” direction let $e : T_\Sigma X \rightarrow E$ be an equation and let $p(s, t) := d_E(e(s), e(t))$ be the congruence on $T_\Sigma X$ associated to e . That Γ_e is closed under the required rules now follows easily from the congruence properties of p . Indeed, Γ_e is closed under (Refl), (Sym), and (Triang) because p is a pseudometric. For instance, closure under (Triang) is equivalent to the implication

$$p(s, t) \leq \varepsilon \text{ and } p(t, u) \leq \delta \implies p(s, u) \leq \varepsilon + \delta, \quad (\text{B.9})$$

which in turn is equivalent to $p(s, t) + p(t, u) \geq p(s, u)$.

That operations are nonexpansive w.r.t. p is equivalent to the statement that, for all $\sigma \in \Sigma_n$,

$$p(s_i, t_i) \leq \varepsilon \text{ for } i = 1, \dots, n \implies p(\sigma(s_1, \dots, s_n), \sigma(t_1, \dots, t_n)) \leq \varepsilon,$$

which means precisely that Γ_e is closed under (Cong).

Closure under (Max) is clear since $p(s, t) \leq \varepsilon$ implies $p(s, t) \leq \varepsilon'$ for all $\varepsilon' > \varepsilon$, and similarly, to see closure under (Arch), use that if $p(s, t) \leq \varepsilon'$ for all $\varepsilon' > \varepsilon$, then $p(s, t) \leq \varepsilon$.

For the “if” direction, suppose that Γ is a set of unconditional equations that has the required closure properties. Define $p : T_\Sigma X \times T_\Sigma X \rightarrow [0, \infty]$ by

$$p(s, t) = \inf\{\varepsilon \in [0, \infty] : (s =_\varepsilon t) \in \Gamma\}.$$

It is straightforward to verify that p is a congruence on $T_\Sigma X$. To see this note that $T_\Sigma X$ is a discrete space since so is (the set) X . Hence, $p(s, t) \leq d_{T_\Sigma X}(s, t)$ is clear. That p is a pseudometric follow from closure of Γ under (Refl), (Sym), and (Triang). E.g., the triangle inequality is equivalent to the statement that (B.9) holds, and to this end observe that $p(s, t) \leq \varepsilon$ is equivalent to $(s =_{\varepsilon+\varepsilon'} t) \in \Gamma$ for all $\varepsilon' > 0$, and similarly $p(t, u) \leq \delta$ is equivalent to $(t =_{\delta+\delta'} u) \in \Gamma$ for all $\delta' > 0$. Thus,

$$(s =_{\varepsilon+\delta+\varepsilon'+\delta'} u) \in \Gamma \quad \text{for all } \varepsilon', \delta' > 0,$$

and this is equivalent to the right-hand side of the implication in (B.9). That the operations on $T_\Sigma X$ are nonexpansive w.r.t. p follows in a similar way from closure of Γ under (Cong).

Furthermore, we have $\Gamma = \Gamma_e$ for the quotient $e : T_\Sigma X \rightarrow E$ corresponding to p . Indeed, we have $d_E(e(s), e(t)) = p(s, t)$ by Lemma B.14. Thus $\Gamma \subseteq \Gamma_e$ is clear. For $\Gamma_e \subseteq \Gamma$ suppose that $(s =_\varepsilon t) \in \Gamma_e$, i.e. $p(s, t) \leq \varepsilon$. By the definition of p we thus have $(s =_{\varepsilon'} t) \in \Gamma$ for all $\varepsilon' > p(s, t)$, whence by the closure of Γ under (Arch), $(s =_{p(s,t)} t) \in \Gamma$. From the closure of Γ under (Max), we conclude that $(s =_\varepsilon t) \in \Gamma$ (if $\varepsilon > p(s, t)$ and for $\varepsilon = p(s, t)$ we were done before).

(2) For the “only if” direction, suppose that $(\Gamma_X)_{X \in \mathbf{Set}}$ is associated to some theory $(e_X: T_\Sigma X \rightarrow E_X)_{X \in \mathbf{Set}}$, so $\Gamma_X = \Gamma_{e_X}$ for all X . By part (1), each Γ_X is closed under (Refl), (Sym), (Triang), (Max), (Arch), (Cong). To show closure under (Subst), let $h: T_\Sigma X \rightarrow T_\Sigma Y$ be a homomorphism. By substitution closure of the theory $(e_X)_X$, the morphism $e_Y \cdot h$ factorizes through e_X , which implies

$$d_{E_Y}(e_Y \cdot h(s), e_Y \cdot h(t)) \leq d_{E_X}(e_X(s), e_X(t)) \quad \text{for all } s, t \in T_\Sigma X. \quad (\text{B.10})$$

by the homomorphism theorem. But this inequality states precisely that for $(s =_\varepsilon t) \in \Gamma_X$ one has $h(s) =_\varepsilon h(t) \in \Gamma_Y$, i.e. closure under (Subst).

For the “if” direction, part (1) implies that each Γ_X is associated to some $e_X: T_\Sigma X \rightarrow E_X$. Moreover, closure under (Subst) states precisely that, for each homomorphism $h: T_\Sigma X \rightarrow T_\Sigma Y$ one has (B.10), which by the homomorphism theorem implies that $e_Y \cdot h$ factorizes through e_X . Thus, $(e_X)_{X \in \mathbf{Set}}$ is a theory.

The completeness proof is now analogous to the proof of Theorem B.3:

Theorem B.23 (Mardare et al. [19]). $\Gamma \models s =_\varepsilon t$ implies $\Gamma \vdash s =_\varepsilon t$.

Proof. We derive this statement from Theorem 4.4. Choose a set X of variables such that all equations in Γ and the equation $s =_\varepsilon t$ are formed over X , and suppose that $\Gamma \models s =_\varepsilon t$. Let $e: T_\Sigma X \rightarrow E_X$ and $e': T_\Sigma X \rightarrow E'_X$ be the quotients corresponding to the congruences generated by the relations in Γ and by $s =_\varepsilon t$, respectively. Then $e \models e'$ by the homomorphism theorem, so by Theorem 4.4 (cf. also Remark B.2), there exists a proof

$$e = e_0 \vdash e_1 \vdash \dots \vdash e_n = e'$$

in our abstract calculus, where $e_i: T_\Sigma X_i \rightarrow E_i$. We show that for every $i = 0, \dots, n$ and $(s' =_{\varepsilon'} t') \in \Gamma_{e_i}$ one has $\Gamma \vdash s' =_{\varepsilon'} t'$; this then implies $\Gamma \vdash s =_\varepsilon t$ by putting $i = n$ and $(s' =_{\varepsilon'} t') = (s =_\varepsilon t)$. The proof is by induction on i . For $i = 0$, we have that the set $\Gamma_{e_0} = \Gamma_e$ corresponds to the congruence generated by Γ , so it is the closure of Γ under the rules (Refl), (Sym), (Triang), (Max), (Arch), (Cong) by Lemma B.22(1). Thus, every equation $s' =_{\varepsilon'} t'$ in Γ_{e_0} can be proved from Γ using these rules. Now suppose that $0 < i < n$. If the step $e_i \vdash e_{i+1}$ is an application of the weakening rule, the statement follows trivially by induction because then $\Gamma_{e_{i+1}} \subseteq \Gamma_{e_i}$. Thus suppose that $e_i \vdash e_{i+1}$ uses the substitution rule. By Lemma B.22(2), the substitution closure of e_i is given by the family of sets of equations $(\overline{\Gamma}_Y)_{Y \in \mathbf{Set}}$ obtained by closing Γ_{e_i} under all the rules (Refl), (Sym), (Triang), (Max), (Arch), (Cong), (Subst). Since $\Gamma_{i+1} = \overline{\Gamma}_{X_{i+1}}$, we have $\Gamma_i \vdash s' =_{\varepsilon'} t'$ for each $(s' =_{\varepsilon'} t') \in \Gamma_{i+1}$. Thus $\Gamma \vdash s' =_{\varepsilon'} t'$ by induction. \square

B.6 Nominal Algebras

In this section, we derive an HSP theorem for algebras in the category of nominal sets. We first recall some terminology; see Pitts [24] for details. Fix a countably infinite set \mathbb{A} of atoms and denote by $\text{Perm}(\mathbb{A})$ the group of all permutations

$\pi: \mathbb{A} \rightarrow \mathbb{A}$ moving only finitely many elements of \mathbb{A} . A *nominal set* is a set X equipped with a group action $\text{Perm}(\mathbb{A}) \times X \rightarrow X$, $(\pi, x) \mapsto \pi \cdot x$, such that every element of X has a finite *support*; that is, for every $x \in X$ there exists a finite set $S \subseteq \mathbb{A}$ such that for every $\pi \in \text{Perm}(\mathbb{A})$ one has

$$[\forall a \in S : \pi(a) = a] \quad \Rightarrow \quad \pi \cdot x = x.$$

This implies that x has a least support $\text{supp}_X(x) \subseteq \mathbb{A}$, viz. the intersection of all supports of x . Every nominal set X can be partitioned into the subsets of the form $\{\pi \cdot x : \pi \in \text{Perm}(\mathbb{A})\}$ ($x \in X$), called the *orbits* of X . An *equivariant map* between nominal sets X and Y is a function $f: X \rightarrow Y$ such that $f(\pi \cdot x) = \pi \cdot f(x)$ for all $x \in X$ and $\pi \in \text{Perm}(\mathbb{A})$. Equivariance implies that $\text{supp}_Y(f(x)) \subseteq \text{supp}_X(x)$ for all $x \in X$. We denote by **Nom** the category of nominal sets and equivariant maps. **Nom** has the factorization system of epimorphisms and monomorphisms (= surjective and injective equivariant maps). The product of a family of nominal sets X_i ($i \in I$) is given by

$$\prod X_i = \{ (x_i)_{i \in I} \in \prod_{i \in I} |X_i| : \bigcup_{i \in I} \text{supp}(x_i) \text{ is finite} \},$$

where $|X_i|$ denotes the underlying set of X_i and the group action is given pointwise. The coproduct $\coprod_{i \in I} X_i$ is formed on the level of underlying sets. A nominal set X is called *strong* if for every element $x \in X$ and $\pi \in \text{Perm}(\mathbb{A})$ one has

$$[\forall a \in \text{supp}_X(x) : \pi(a) = a] \quad \Leftrightarrow \quad \pi \cdot x = x.$$

For any finite set I let $\mathbb{A}^I = \prod_{i \in I} \mathbb{A}$ denote the I -fold power of \mathbb{A} . Then

$$\mathbb{A}^{\#I} = \{ a \in \mathbb{A}^I : a \text{ injective} \},$$

is a strong nominal set with group action $(\pi \cdot a)(i) := \pi(a(i))$ for $\pi \in \text{Perm}(\mathbb{A})$.

Definition B.24. A *supported set* is a set X together with a map $\text{supp}_X: X \rightarrow \mathcal{P}_f(\mathbb{A})$. A *morphism* between supported sets X and Y is a function $f: X \rightarrow Y$ with $\text{supp}_Y(f(x)) \subseteq \text{supp}_X(x)$ for all $x \in X$.

Every nominal set X is a supported set w.r.t. its least-support function supp_X . The following result is a reformulation of [21, Prop. 5.10]:

Lemma B.25. *The forgetful functor from **Nom** to **SuppSet** has a left adjoint.*

Remark B.26. The left adjoint $F: \mathbf{SuppSet} \rightarrow \mathbf{Nom}$ sends a supported set X to the nominal set $FX = \coprod_{x \in X} \mathbb{A}^{\#\text{supp}_X(x)}$, and the universal map $\eta_X: X \rightarrow FX$ maps an element $x \in X$ to the inclusion map $\text{supp}_X(x) \hookrightarrow \mathbb{A}$ in $\mathbb{A}^{\#\text{supp}_X(x)}$.

Proof. Let X be a supported set and let Y be a nominal set. We need to show that every morphism $h: X \rightarrow Y$ in **SuppSet** uniquely extends to an equivariant

map $\bar{h}: FX \rightarrow Y$ with $\bar{h} \cdot \eta_X = h$. Note that every element of FX is of the form $\pi \cdot \eta_X(x)$ for a (unique) $x \in X$ and some $\pi \in \text{Perm}(\mathbb{A})$. Thus the formula

$$\bar{h}(\pi \cdot \eta_X(x)) := \pi \cdot h(x) \quad (\pi \in \text{Perm}(\mathbb{A}))$$

gives a total function $\bar{h}: FX \rightarrow Y$, provided that we can prove it to be well-defined. To this end, suppose that $\pi \cdot \eta_X(x) = \sigma \cdot \eta_X(x)$ for $x \in X$ and $\pi, \sigma \in \text{Perm}(\mathbb{A})$. Since FX is strong, π and σ agree on $\text{supp}_{FX}(\eta_X(x)) = \text{supp}_X(x)$. In particular, they agree on $\text{supp}_Y(h(x)) \subseteq \text{supp}_X(x)$, which implies $\pi \cdot h(x) = \sigma \cdot h(x)$. Thus \bar{h} is a well-defined map.

From its definition it is immediately clear that \bar{h} is equivariant and satisfies $\bar{h} \cdot \eta_X(x) = h(x)$ for all $x \in X$. Moreover, since the elements $\eta_X(x)$ ($x \in X$) meet every orbit of FX , the map \bar{h} is unique with this property. \square

Corollary B.27. (1) *For each nominal set Z , there exists a strong nominal set X and a surjective equivariant map $e: X \rightarrow Z$ preserving least supports, i.e. with $\text{supp}_Z(e(x)) = \text{supp}_X(x)$ for all $x \in X$.*

(2) *Every strong nominal set is isomorphic to FY for some $Y \in \mathbf{SuppSet}$.*

Proof. (1) Choose a subset $Y \subseteq Z$ containing exactly one element of every orbit of Z . Then Y is a supported set, with supp_Y being the restriction of supp_Z . By Lemma B.25, the inclusion map $Y \hookrightarrow Z$ uniquely extends to an equivariant map $e: FY \rightarrow Z$. The map e is surjective because its image meets every orbit of Z . Moreover, it preserves least supports: for all $y \in Y$ and $\pi \in \text{Perm}(\mathbb{A})$, one has

$$\begin{aligned} \text{supp}_Z(e(\pi \cdot \eta_Y(y))) &= \pi \cdot \text{supp}_Z(e(\eta_Y(y))) = \pi \cdot \text{supp}_Y(y) = \pi \cdot \text{supp}_{FY}(\eta_Y(y)) \\ &= \text{supp}_{FY}(\pi \cdot \eta_Y(y)), \end{aligned}$$

where the middle equation in the first line follows since $e \cdot \eta_Y$ is the inclusion map $Y \hookrightarrow Z$.

(2) Suppose that Z is a strong nominal set. We show that the map $e: FY \rightarrow Z$ constructed in part (1) of the proof is injective, and thus an isomorphism. By the choice of $Y \subseteq Z$, the map e sends elements of distinct orbits of FY to distinct orbits of Z . It therefore suffices to verify that e does not merge any two elements of FY that belong to the same orbit. Thus let $y \in Y$ and $\pi, \sigma \in \text{Perm}(\mathbb{A})$ with $e(\pi \cdot \eta_Y(y)) = e(\sigma \cdot \eta_Y(y))$, i.e. $\pi \cdot y = \sigma \cdot y$. Since Z is strong, π and σ agree on $\text{supp}_Z(y) = \text{supp}_{FY}(\eta_Y(y))$. Thus $\pi \cdot \eta_Y(y) = \sigma \cdot \eta_Y(y)$, which proves that e is injective. \square

Fix a finitary signature Σ . A *nominal Σ -algebra* is a nominal set A with a Σ -algebra structure such that all operations $\sigma: A^n \rightarrow A$ ($\sigma \in \Sigma$) are equivariant. Morphisms of nominal Σ -algebras are equivariant Σ -homomorphisms. The forgetful functor from the category $\mathbf{NomAlg}(\Sigma)$ of nominal Σ -algebras to \mathbf{Nom} has a left adjoint associating to each $X \in \mathbf{Nom}$ the term algebra $T_\Sigma X$, with group action inherited from the one of X . To get an HSP theorem for nominal Σ -algebras, we follow the four steps indicated at the beginning of Section 5.

Step 1. We choose the parameters of our setting as follows:

- $\mathcal{A} = \mathcal{A}_0 = \mathbf{NomAlg}(\Sigma)$;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective morphisms, injective morphisms})$;
- $\Lambda = \text{all cardinal numbers}$;
- $\mathcal{X} = \{T_\Sigma X : X \text{ is a strong nominal set}\}$.

The quotients in $\mathcal{E}_{\mathcal{X}}$ are characterized as follows:

Lemma B.28. *A quotient $e: A \twoheadrightarrow B$ belongs to $\mathcal{E}_{\mathcal{X}}$ if and only if for every $b \in B$ there exists $a \in A$ with $e(a) = b$ and $\text{supp}_A(a) = \text{supp}_B(b)$.*

In the following, a quotient with this property is called *support-reflecting*.

Proof. By Remark B.1 applied to the adjunction $\mathbf{NomAlg}(\Sigma) \xrightarrow{\overline{\quad} \top} \mathbf{Nom}$ with $\mathcal{X}' = \text{strong nominal sets}$ and $\mathcal{E}' = \text{surjective equivariant maps}$, it suffices to consider the case where the signature Σ is empty, i.e. $\mathcal{A} = \mathbf{Nom}$ and $\mathcal{X} = \text{strong nominal sets}$.

(\Rightarrow) Suppose that $e: A \twoheadrightarrow B$ lies in $\mathcal{E}_{\mathcal{X}}$. Choose a strong nominal set X and a quotient $h: X \twoheadrightarrow B$ preserving least supports, see Corollary B.27. Since X is projective w.r.t. e , there exists an equivariant map $g: X \rightarrow A$ with $e \cdot g = h$. To prove that e is support-reflecting, let $b \in B$. Choose $x \in X$ with $h(x) = b$, and put $a := g(x)$. Then $\text{supp}_A(a) \subseteq \text{supp}_X(x) = \text{supp}_B(h(x)) = \text{supp}_B(b)$. Moreover, $\text{supp}_B(b) = \text{supp}_B(e(a)) \subseteq \text{supp}_A(a)$ because e is equivariant. Thus $\text{supp}_A(a) = \text{supp}_B(b)$ and $e(a) = b$, which shows that e is support-reflecting.

(\Leftarrow) Suppose that $e: A \twoheadrightarrow B$ is support-reflecting, and let $h: X \twoheadrightarrow B$ be an equivariant map whose domain X is a strong nominal set. By Corollary B.27, we may assume that $X = FY$ for some $Y \in \mathbf{SuppSet}$. For each $y \in Y$, choose an element $g(y) \in A$ with $e(g(y)) = h(\eta_Y(y))$ and $\text{supp}_A(g(y)) = \text{supp}_B(h(\eta_Y(y)))$, using that e is support-reflecting. This defines a map $g: Y \rightarrow A$ with $e \cdot g = h \cdot \eta_Y$. Moreover, g is a morphism in $\mathbf{SuppSet}$ because

$$\text{supp}_A(g(y)) = \text{supp}_B(e(g(y))) = \text{supp}_B(h(\eta_Y(y))) \subseteq \text{supp}_X(\eta_Y(y)) = \text{supp}_Y(y).$$

By Lemma B.25, g extends uniquely to an equivariant map $\bar{g}: X \rightarrow A$ with $\bar{g} \cdot \eta_Y = g$. Then also $e \cdot \bar{g} = h$, since this holds when precomposed with the universal map η_Y ; see the diagram below.

$$\begin{array}{ccc} Y & \xrightarrow{g} & A \\ \eta_Y \downarrow & \nearrow \bar{g} & \downarrow e \\ X & \xrightarrow{h} & B \end{array}$$

This proves that each $X \in \mathcal{X}$ is projective w.r.t. e , that is, $e \in \mathcal{E}_{\mathcal{X}}$. \square

It follows that our data satisfies the Assumptions 3.1. For (1) use that products in $\mathbf{NomAlg}(\Sigma)$ are formed in \mathbf{Nom} . (2) holds trivially. For (3), let A be a

nominal Σ -algebra, and express A as a quotient $e: X \rightarrow A$ in **Nom** preserving least supports, with X a strong nominal set; see Corollary B.27. Then the unique extension $e^\#: T_\Sigma X \rightarrow A$ to a morphism in **NomAlg**(Σ) is support-reflecting. Indeed, given $a \in A$, choose $x \in X$ with $e(x) = a$. Then $e^\#(x) = a$ and $\text{supp}_{T_\Sigma X}(x) = \text{supp}_X(x) = \text{supp}_A(e(x)) = \text{supp}_A(a)$.

Step 2. The exactness property of **NomAlg**(Σ) is a straightforward generalization of the one of **Alg**(Σ), see (3.1). An *equivariant congruence relation* on a nominal Σ -algebra A is a congruence relation $\equiv \subseteq A \times A$ that forms an equivariant subset of $A \times A$, i.e., $a \equiv a'$ implies $\pi \cdot a \equiv \pi \cdot a'$ for all $\pi \in \text{Perm}(\mathbb{A})$.

Lemma B.29. *For each nominal Σ -algebra A , there is an isomorphism of complete lattices*

$$\text{quotients of } A \cong \text{equivariant congruences on } A$$

mapping $e: A \rightarrow B$ to its kernel $\equiv_e \subseteq A \times A$, given by $a \equiv_e a'$ iff $e(a) = e(a')$.

Proof. This follows immediately from the corresponding statement for ordinary Σ -algebras, together with the observation that an equivalence relation $\equiv \subseteq A \times A$ on a nominal set A is equivariant iff the corresponding surjection $e: A \rightarrow A/\equiv$ is equivariant. \square

Step 3. By Remark 3.4, in our current setting an equation can be presented as a single quotient $e: T_\Sigma X \rightarrow E_X$ in **NomAlg**(Σ). The corresponding syntactic concept is the following:

Definition B.30. Let Y be a set of variables.

(1) A *nominal Σ -term* over Y is an element of $T_\Sigma(\text{Perm}(\mathbb{A}) \times Y)$. Every map $h: Y \rightarrow A$ into a nominal Σ -algebra A extends to a Σ -algebra homomorphism

$$\hat{h} = (T_\Sigma(\text{Perm}(\mathbb{A}) \times Y) \xrightarrow{T_\Sigma(\text{Perm}(\mathbb{A}) \times h)} T_\Sigma(\text{Perm}(\mathbb{A}) \times A) \xrightarrow{T_\Sigma(- \cdot -)} T_\Sigma A \xrightarrow{id^\#} A)$$

where $id^\#$ the unique extension of the identity map $id: A \rightarrow A$.

(2) A *nominal equation* over Y is an expression of the form

$$\text{supp}_Y \vdash s = t$$

where $\text{supp}_Y: Y \rightarrow \mathcal{P}_f(\mathbb{A})$ is a function and s and t are nominal Σ -terms over Y . A nominal Σ -algebra A *satisfies* the equation $\text{supp}_Y \vdash s = t$ if for every map $h: (Y, \text{supp}_Y) \rightarrow (A, \text{supp}_A)$ of supported sets one has $\hat{h}(s) = \hat{h}(t)$.

Lemma B.31. *Equations and nominal equations are expressively equivalent.*

Proof. (1) To every equation $e: T_\Sigma X \rightarrow E$, with X a strong nominal set, we associate a set of nominal equations as follows. By Corollary B.27, we may assume that $X = FY$ for some supported set Y . For notational simplicity, we

identify Y with a subset of FY and the universal map $\eta_Y: Y \rightarrow FY$ with the inclusion. Form the nominal equations over Y given by

$$\text{supp}_Y \vdash s = t \quad (s, t \in T_\Sigma(\text{Perm}(\mathbb{A}) \times Y) \text{ and } e \cdot T_\Sigma m(s) = e \cdot T_\Sigma m(t)), \quad (\text{B.11})$$

where the map $m: \text{Perm}(\mathbb{A}) \times Y \rightarrow X$ is given by $(\pi, y) \mapsto \pi \cdot y$. It follows from the definition of F in Remark B.26 that the map m is surjective, thus so is $e \cdot T_\Sigma m$. We claim that, for every nominal Σ -algebra A ,

$$A \text{ satisfies } e \iff A \text{ satisfies (B.11)}.$$

To prove (\Leftarrow) , suppose that A satisfies the nominal equations (B.11), and let $h: X \rightarrow A$ be an equivariant map. Then the restriction $g: Y \rightarrow A$ of h satisfies $\text{supp}_A(g(y)) \subseteq \text{supp}_Y(y)$ for all $y \in Y$, that is, it is a map of supported sets. Thus, since A satisfies (B.11), the kernel of $e \cdot T_\Sigma m$ is contained in the kernel of \hat{g} . It follows that there exists $k: E \rightarrow A$ with $k \cdot e \cdot T_\Sigma m = \hat{g}$, i.e., the outside of the diagram below commutes:

$$\begin{array}{ccc} T_\Sigma(\text{Perm}(\mathbb{A}) \times Y) & & \\ \downarrow T_\Sigma m & \searrow \hat{g} & \\ T_\Sigma X & \xrightarrow{h^\#} & A \\ \downarrow e & \nearrow k & \\ E & & \end{array} \quad (\text{B.12})$$

The upper triangle also commutes because, for all $(\pi, y) \in \text{Perm}(\mathbb{A}) \times Y$,

$$h^\# \cdot T_\Sigma m(\pi, y) = h^\#(\pi \cdot y) = \pi \cdot h^\#(y) = \pi \cdot h(y) = \pi \cdot g(y) = \hat{g}(\pi, y) \quad (\text{B.13})$$

and both $h^\# \cdot T_\Sigma m$ and \hat{g} are Σ -algebra homomorphisms. Since $T_\Sigma m$ is an epimorphism, it follows that the lower triangle commutes, i.e., $h^\#$ factors through e . Thus A satisfies e .

For the proof of (\Rightarrow) , suppose that A satisfies e , and let $g: Y \rightarrow A$ be a map of supported sets. By Lemma B.25, g extends uniquely to an equivariant map $h: X \rightarrow A$. Since A satisfies e , we have $h^\# = k \cdot e$ for some $k: E \rightarrow A$. Then the diagram (B.12) commutes: the lower triangle commutes by definition, and the upper one by (B.13). Therefore, for all $s, t \in T_\Sigma(\text{Perm}(\mathbb{A}) \times Y)$ with $e \cdot T_\Sigma m(s) = e \cdot T_\Sigma m(t)$ one has

$$\hat{g}(s) = k \cdot e \cdot T_\Sigma m(s) = k \cdot e \cdot T_\Sigma m(t) = \hat{g}(t),$$

i.e. A satisfies (B.11).

(2) To every nominal equation $\text{supp}_Y \vdash s = t$ over the set Y we associate an equation as follows. Put $X = FY$; as before, we view Y as a subset of X . Form the nominal congruence generated by the pair $(T_\Sigma m(s), T_\Sigma m(t))$ (viz. the

intersection of all nominal congruences containing this pair), and let $e: T_\Sigma X \rightarrow E$ be the corresponding quotient. Then for every nominal Σ -algebra A one has

$$A \text{ satisfies } e \iff A \text{ satisfies } \text{supp}_Y \vdash s = t.$$

To prove (\Rightarrow) , note that $\text{supp}_Y \vdash s = t$ is one of the nominal equations (B.11) associated to e , and we have already shown in part (1) that every algebra that satisfies e also satisfies its associated nominal equations.

For (\Leftarrow) , suppose that A satisfies $\text{supp}_Y \vdash s = t$, and let $h: X \rightarrow A$ be an equivariant map. Then its restriction $g: Y \rightarrow A$ is a map of supported sets, and $h^\# \cdot T_\Sigma m = \hat{g}$ by (B.13). Then

$$h^\#(T_\Sigma m(s)) = \hat{g}(s) = \hat{g}(t) = h^\#(T_\Sigma m(t)),$$

which implies that the kernel of e (being generated by $(T_\Sigma m(s), T_\Sigma m(t))$) is contained in the kernel of $h^\#$. It follows that $h^\#$ factorizes through e . Thus A satisfies e . \square

Step 4. From the previous lemma and Theorem 3.16, we deduce:

Theorem B.32 (Nominal HSP Theorem). *A class of nominal Σ -algebras is a variety (i.e. closed under support-reflecting quotients, subalgebras and products) iff it is axiomatizable by nominal equations.*

The above theorem is a special case of a result of Kurz and Petrişan [17], who in lieu of Σ -algebras considered algebras for an endofunctor on **Nom** with a suitable finitary presentation.

B.7 Continuous Σ -algebras

In this section, we derive the HSP theorem for continuous Σ -algebras proved by Adámek, Nelson, and Reiterman [4]. Let us first recall some terminology. An ω -cpo is a poset with a least element \perp and suprema of ω -chains. A monotone map $h: A \rightarrow B$ between ω -cpo's is *continuous* if it preserves all suprema of ω -chains, and *strict continuous* if it additionally preserves \perp . We denote by $\omega\mathbf{CPO}$ the category of ω -cpos and strict continuous maps. Given an ω -cpo B , a subset $A \subseteq B$ is called *closed* if it is closed under ω -suprema, that is, for every ω -chain $a_0 \leq a_1 \leq a_2 \leq \dots$ in A one has $\bigvee_{n < \omega} a_n \in A$. The *closure* of a subset $A \subseteq B$ is the least closed subset containing A , i.e. $\bar{A} = \bigcap \{A' \subseteq B : A \subseteq A' \text{ closed}\}$. The closure can be computed by transfinitely closing A under ω -suprema. More precisely, one has $\bar{A} = \bigcup_i A_i$, where i ranges over all ordinal numbers and the sets $A_i \subseteq B$ are inductively defined by

- $A_0 = A$;
- $A_i = \{ \bigvee_{n < \omega} a_n : (a_n)_{n < \omega} \text{ } \omega\text{-chain in } A_{i-1} \}$ if i is a successor ordinal;
- $A_i = \bigcup_{j < i} A_j$ if i is a limit ordinal.

Note that $A_i = A_{\omega_1}$ for all $i \geq \omega_1$, so the closure process terminates after ω_1 steps. We say that $A \subseteq B$ is a *dense* subset if $\overline{A} = B$. By extension, a continuous map $h: A \rightarrow B$ is called *closed/dense* if its image $h[A] \subseteq B$ is a closed/dense subset of B . The category $\omega\mathbf{CPO}$ has a factorization system given by dense continuous maps and closed continuous order-embeddings. The factorization of $h: A \rightarrow B$ is given by $h = (A \xrightarrow{e} \overline{h[A]} \xrightarrow{m} B)$, where e is the codomain restriction of h to the closure of its image $h[A] \subseteq B$, and m is the embedding of the subspace $\overline{h[A]}$ into B .

A *continuous Σ -algebra* is a Σ -algebra with an ω -cpo structure on its underlying set and continuous operations. Note that the operations are not required to be strict. We denote by $\omega\mathbf{Alg}(\Sigma)$ the category of continuous Σ -algebras and strict continuous Σ -homomorphisms.

Lemma B.33. *The factorization system of $\omega\mathbf{CPO}$ lifts to $\omega\mathbf{Alg}(\Sigma)$.*

Proof. (1) For each cpo B and each Σ -subalgebra $A \subseteq B$, the closure $\overline{A} \subseteq B$ forms a Σ -subalgebra. To see this, it suffices to show that each of the sets A_i defined above is a subalgebra. For $i = 0$, this holds by assumption since $A_0 = A$. If i is a limit ordinal, the claim is clear by induction because directed unions of subalgebras are subalgebras. Thus suppose that i is a successor ordinal, let $\sigma \in \Sigma$ be an n -ary operation symbol and $a_1, \dots, a_n \in A_i$. Thus, for each $j = 1, \dots, n$ one has $a_j = \bigvee_{k < \omega} a_k^j$ for some chain $(a_k^j)_{k < \omega}$ in A_{i-1} . Since $\sigma: B^n \rightarrow B$ is continuous, we have that

$$\sigma(a_1, \dots, a_n) = \bigvee_{k < \omega} \sigma(a_k^1, \dots, a_k^n)$$

is an element on A_i , using that $\sigma(a_k^1, \dots, a_k^n) \in A_{i-1}$ for all $k < \omega$ by induction.

(2) Now let $h: A \rightarrow B$ be a morphism of continuous Σ -algebras. Its canonical factorization $A \rightarrow \overline{h[A]} \rightarrow B$ in $\omega\mathbf{CPO}$ is also one in $\omega\mathbf{Alg}(\Sigma)$ because $\overline{h[A]}$ is a Σ -subalgebra of B by part (1). Moreover, given a commutative square $h \cdot e = m \cdot g$ in $\omega\mathbf{Alg}(\Sigma)$ with e dense and m a closed embedding, the unique diagonal fill-in d in $\omega\mathbf{CPO}$ with $d \cdot e = g$ and $m \cdot d = h$ is also a Σ -homomorphism because m and h are Σ -homomorphisms and m is injective. \square

The forgetful functor from the category $\omega\mathbf{Alg}(\Sigma)$ to \mathbf{Set} has a left adjoint mapping to each set X the *free continuous Σ -algebra* $T_\Sigma(X_\perp)$. The latter is carried by the set of all finite or infinite Σ -trees with leaves labelled in $X \cup \{\perp\}$ [15]. To establish the continuous HSP theorem, we follow our four-step procedure:

Step 1. Choose the following parameters:

- $\mathcal{A} = \mathcal{A}_0 = \omega\mathbf{Alg}(\Sigma)$;
- $\Lambda =$ all cardinal numbers;
- $(\mathcal{E}, \mathcal{M}) =$ (dense morphisms, closed order-embeddings);
- $\mathcal{X} =$ all free algebras $T_\Sigma(X_\perp)$ with $X \in \mathbf{Set}$;

Note that, in contrast to all applications discussed in the previous sections, the morphisms in \mathcal{E} are not necessarily surjective. However, we have

Lemma B.34. *$\mathcal{E}_{\mathcal{X}}$ consists precisely of the surjective morphisms.*

Proof. As usual (cf. the proofs of Lemma B.12 and Lemma B.28), it suffices to consider the case of an empty signature, i.e. where $\mathcal{A} = \omega\mathbf{CPO}$. To this end, just observe that for every set X and every ω -cpo A there is a bijective correspondence between maps $X \rightarrow A$ and strict continuous maps $X_{\perp} \rightarrow A$. Thus, the statement of the lemma follows from the fact that in **Set**, a map e is surjective iff every set X is projective w.r.t. e . \square

We conclude that our Assumptions 3.1 are satisfied by our data. For (1), use that products $\omega\mathbf{Alg}(\Sigma)$ are formed on the level of underlying sets, with Σ -algebra structure and partial order computed pointwise. Condition (2) is trivial. For (3), let $A \in \omega\mathbf{Alg}(\Sigma)$, and choose a surjective map $e: X \twoheadrightarrow A$ for some set X . Then the unique extension $e^{\sharp}: T_{\Sigma}(X_{\perp}) \twoheadrightarrow A$ to a nonexpansive map is also surjective. Moreover, $e^{\sharp} \in \mathcal{E}_{\mathcal{X}}$ by the above lemma and $T_{\Sigma}(X_{\perp}) \in \mathcal{X}$ by definition of \mathcal{X} .

Step 2/3. By Remark 3.4, in our current setting an equation can be presented as a single quotient $e: T_{\Sigma}X_{\perp} \rightarrow E$ in $\omega\mathbf{Alg}(\Sigma)$. The corresponding syntactic concept involves terms endowed with formal join operations. Given a set X of variables, put $S_{\Sigma}(X) = \bigcup_i S_{\Sigma,i}(X)$ where i ranges over all ordinal numbers and $S_{\Sigma,i}(X)$ is defined by transfinite induction as follows:

- $S_{\Sigma,0}(X)$ = set of all Σ -terms in the variables $X_{\perp} = X \cup \{\perp\}$;
- $S_{\Sigma,i}(X) = \bigcup_{j < i} S_{\Sigma,j}(X)$ for i a limit ordinal.
- $S_{\Sigma,i}(X) = \{ \bigvee_{k < \omega} t_k \ : \ t_k \in S_{\Sigma,i-1}(X) \text{ for all } k < \omega \}$ for i a successor ordinal.

Note that $S_{\Sigma,i}(X) = S_{\Sigma,\omega_1}(X)$ for all $i \geq \omega_1$, so $S_{\Sigma}(X)$ is a set. Every map $h: X \rightarrow A$ into a continuous Σ -algebra A extends to a *partial* map $\widehat{h}: S_{\Sigma}(X) \rightarrow A$, defined by structural induction as follows:

- For $t \in S_{\Sigma,0}(X)$, let $\widehat{h}(t)$ be the evaluation of the term t in A ;
- If $t = \bigvee_{k < \omega} t_k$, all the values $\widehat{h}(t_k)$ are defined, and $(\widehat{h}(t_k))_{k < \omega}$ forms a ω -chain in A , put

$$\widehat{h}(t) = \bigvee_{k < \omega} \widehat{h}(t_k), \quad \text{otherwise } \widehat{h}(t) \text{ is undefined.}$$

The above definition of $S_{\Sigma}(X)$ and \widehat{h} is due to Adámek et al. [4].

Lemma B.35. *Let $e: A \rightarrow B$ be a morphism of continuous Σ -algebras.*

- (1) *For every map $h: X \rightarrow A$ one has $e \cdot \widehat{h} = \widehat{e \cdot h}$. More precisely, for all $t \in S_{\Sigma}(X)$ such that $\widehat{h}(t)$ is defined, the value $\widehat{e \cdot h}(t)$ is defined and $e \cdot \widehat{h}(t) = \widehat{e \cdot h}(t)$. If moreover e is an order-embedding, then $\widehat{h}(t)$ is defined iff $\widehat{e \cdot h}(t)$ is defined.*
- (2) *The image of $\widehat{e}: S_{\Sigma}(A) \rightarrow B$ is equal to the closure $\overline{e[B]} \subseteq B$.*

Proof. Obvious by structural induction. \square

Lemma B.36 (Homomorphism Theorem). *Let $e: A \rightarrow B$ and $h: A \rightarrow C$ be morphisms in $\omega\mathbf{Alg}(\Sigma)$ with e dense. Then the following are equivalent:*

- (1) *There exists a morphism $g: B \rightarrow C$ with $g \cdot e = h$.*
- (2) *For every pair of terms $t, t' \in S_\Sigma(A)$, if both $\widehat{e}(t)$ and $\widehat{e}(t')$ are defined and $\widehat{e}(t) \leq_A \widehat{e}(t')$, then also $\widehat{h}(t)$ and $\widehat{h}(t')$ are defined and $\widehat{h}(t) \leq_B \widehat{h}(t')$.*

Proof. (1) \Rightarrow (2) follows immediately from the first part of the previous lemma. For the converse, assume that (2) holds and let $b \in B$. Since e is dense, we have $\overline{e[A]} = B$, so by the previous lemma, there exists $t \in S_\Sigma(A)$ with $\widehat{e}(t) = b$. Put $g(b) := \widehat{h}(t)$. By (2), this gives a well-defined monotone map $g: B \rightarrow C$ with $g \cdot e = h$. To see that g preserves ω -suprema, let $(b_k)_{k < \omega}$ be an ω -chain in B , and choose $t_k \in S_\Sigma(A)$ with $\widehat{e}(t_k) = b_k$ for all $k < \omega$. Then the value $\widehat{e}(\bigvee_{k < \omega} t_k)$ is defined, so by (2) the value $\widehat{h}(\bigvee_{k < \omega} t_k)$ is also defined. This implies

$$g\left(\bigvee_k b_k\right) = g\left(\widehat{e}\left(\bigvee_k t_k\right)\right) = \widehat{h}\left(\bigvee_k t_k\right) = \bigvee_k \widehat{h}(t_k) = \bigvee_k g \cdot \widehat{e}(t_k) = \bigvee_k g(b_k).$$

To see that g is a Σ -homomorphism, let $\sigma \in \Sigma$ be n -ary and $b_1, \dots, b_n \in B$. Choose $t_i \in S_\Sigma(A)$ with $\widehat{e}(t_i) = b_i$. Let j be the least ordinal number such that $t_i \in S_{\Sigma, j}(A)$ for all $i = 1, \dots, n$. If $j = 0$, then all t_i lie in $S_{\Sigma, 0}(A) \subseteq T_\Sigma(A_\perp)$. Let $q: T_\Sigma(A_\perp) \rightarrow A$ be the extension to a continuous Σ -homomorphism of the identity map on A . Let $a_i = q(t_i)$. Then we clearly have

$$b_i = \widehat{e}(t_i) = e \cdot q(t_i) = e(a_i).$$

Moreover we obtain

$$\begin{aligned} g \cdot \sigma^B(b_1, \dots, b_n) &= g \cdot \sigma^B(e(a_1), \dots, e(a_n)) \\ &= g \cdot e(\sigma^A(a_1, \dots, a_n)) \\ &= h(\sigma^A(a_1, \dots, a_n)) \\ &= \sigma^C(h(a_1), \dots, h(a_n)) \\ &= \sigma^C(g \cdot e(a_1), \dots, g \cdot e(a_n)) \\ &= \sigma^C(g(b_1), \dots, g(b_n)). \end{aligned}$$

For the induction step assume that $t_i = \bigvee_k s_k$ for some i . Wlog, we assume that $i = 1$. Then we compute

$$\begin{aligned} g(\sigma^B(b_1, \dots, b_n)) &= g(\sigma^B(\widehat{e}(t_1), \dots, \widehat{e}(t_n))) \\ &= g(\sigma^B(\bigvee_k \widehat{e}(s_k), \widehat{e}(t_2), \dots, \widehat{e}(t_n))) \\ &= \bigvee_k g(\sigma^B(\widehat{e}(s_k), \widehat{e}(t_2), \dots, \widehat{e}(t_n))) \end{aligned}$$

$$\begin{aligned}
 &= \bigvee_k \sigma^C(g \cdot \widehat{e}(s_k), g \cdot \widehat{e}(t_2), \dots, g \cdot \widehat{e}(t_n)) \\
 &= \bigvee_k \sigma^C(\widehat{h}(s_k), \widehat{h}(t_2), \dots, \widehat{h}(t_n)) \\
 &= \sigma^C(\bigvee_k \widehat{h}(s_k), \widehat{h}(t_2), \dots, \widehat{h}(t_n)) \\
 &= \sigma^C(\widehat{h}(t_1), \widehat{h}(t_2), \dots, \widehat{h}(t_n)) \\
 &= \sigma^C(g \cdot \widehat{e}(t_1), \dots, g \cdot \widehat{e}(t_n)) \\
 &= \sigma^C(g(b_1), \dots, g(b_n)). \quad \square
 \end{aligned}$$

Remark B.37. If $A_0 \subseteq A$ is a set of generators of the continuous Σ -algebra A , i.e. A is the closure of A_0 under Σ -operations and ω -suprema, then it suffices to check condition (2) for terms $t, t' \in S_\Sigma(A_0)$. More precisely, let $e_0 : A_0 \rightarrow B$ and $h_0 : A_0 \rightarrow C$ be the restrictions of e and h . Then the condition (2) holds for e and h if it holds for e_0 and h_0 .

Definition B.38. A *continuous inequality* over a set X of variables is a pair of terms s, t in $S_\Sigma(X)$, denoted as $s \leq t$. A continuous Σ -algebra A *satisfies* the inequality $s \leq t$ if for every map $h : X \rightarrow A$, both $\widehat{h}(s)$ and $\widehat{h}(t)$ are defined and one has $\widehat{h}(s) \leq \widehat{h}(t)$.

Lemma B.39. *Equations and continuous inequalities are expressively equivalent.*

Proof. In the following, for any equation $e : T_\Sigma(X_\perp) \rightarrow E$ we denote by $e_0 : X \rightarrow E$ its restriction to the generators.

(1) Given an equation $e : T_\Sigma(X_\perp) \rightarrow E$, define Γ_e to be the set of continuous inequalities $s \leq t$ over X such that both $\widehat{e}_0(s)$ and $\widehat{e}_0(t)$ are defined and $\widehat{e}_0(s) \leq \widehat{e}_0(t)$. Then a continuous Σ -algebra A satisfies the equation e iff it satisfies all the continuous inequalities in Γ_e :

(\Rightarrow) Suppose that A satisfies e , let $s \leq t$ be a continuous inequality in Γ_e , and let $h : X \rightarrow A$. By the universal property of $T_\Sigma(X_\perp)$, the map h extends uniquely to a continuous Σ -homomorphism $\bar{h} : T_\Sigma(X_\perp) \rightarrow A$. Since A satisfies e , there exists a continuous Σ -homomorphism $k : E \rightarrow A$ with $k \cdot e = \bar{h}$, which implies that $h = k \cdot e_0$. Now suppose that $\widehat{e}_0(s)$ and $\widehat{e}_0(t)$ are defined and $\widehat{e}_0(s) \leq \widehat{e}_0(t)$. By Lemma B.35, it follows that $\widehat{h}(s)$ and $\widehat{h}(t)$ are defined and

$$\widehat{h}(s) = k \cdot \widehat{e}_0(s) \leq k \cdot \widehat{e}_0(t) = \widehat{h}(t).$$

Thus, A satisfies $s \leq t$.

(\Leftarrow) Suppose that A satisfies every inequality in Γ_e , and let $h : T_\Sigma(X_\perp) \rightarrow A$ be a continuous Σ -homomorphism and $h_0 : X \rightarrow A$ its restriction to X . To show that h factorizes through e , we apply the homomorphism theorem (Lemma B.36). Since the continuous Σ -algebra $T_\Sigma(X_\perp)$ is generated by the subset X , it suffices to verify condition (2) of the theorem for all terms $t, t' \in S_\Sigma(X)$ (see

Remark B.37). Thus suppose that $\widehat{e}(t)$ and $\widehat{e}(t')$ are defined and $\widehat{e}(t) \leq \widehat{e}(t')$. This means that $t \leq t'$ lies in Γ_e . Since A satisfies all inequalities in Γ_e , it follows that $\widehat{h}_0(t)$ and $\widehat{h}_0(t')$ are defined and $\widehat{h}_0(t) \leq \widehat{h}_0(t')$. The homomorphism theorem yields the desired factorization of h through e . Thus A satisfies e .

(2) Given a continuous inequality $s \leq t$ over the set X , let $e_i: T_\Sigma(X_\perp) \rightarrow E_i$ ($i \in I$) be the family of all quotients of $T_\Sigma(X_\perp)$ such that $\widehat{e}_{i,0}(s)$ and $\widehat{e}_{i,0}(t)$ are defined and $\widehat{e}_{i,0}(s) \leq \widehat{e}_{i,0}(t)$, where $e_{i,0}: X \rightarrow E_i$ denotes the restriction of e_i to X . Form the subdirect product $e: T_\Sigma(X_\perp) \twoheadrightarrow E$ of the e_i 's, obtained by factorizing the continuous Σ -homomorphism $\langle e_i \rangle: T_\Sigma(X_\perp) \rightarrow \prod_i E_i$ into a dense morphism $e: T_\Sigma(X_\perp) \twoheadrightarrow E$ followed by an order-embedding $m: E \rightarrow \prod_i E_i$. By Lemma B.35(1) and since m is an order-embedding, it follows that $\widehat{e}_0(s)$ and $\widehat{e}_0(t)$ are defined and $\widehat{e}_0(s) \leq \widehat{e}_0(t)$, where $e_0: X \rightarrow E$ is the restriction of e to X . In other words, e is the least quotient among the e_i 's. We claim that a continuous Σ -algebra A satisfies $s \leq t$ iff it satisfies e .

(\Rightarrow) Suppose that A satisfies $s \leq t$ and let $h: T_\Sigma(X_\perp) \rightarrow A$. To show that h factorizes through e , we may assume wlog. that h is dense, i.e. a quotient. By assumption, we have that $\widehat{h}(s)$ and $\widehat{h}(t)$ are defined and $\widehat{h}(s) \leq \widehat{h}(t)$. Thus $h = e_i$ for some $i \in I$, and since e is the subdirect product of all e_i 's, we have that e_i factorizes through e . This shows that A satisfies e .

(\Leftarrow) Suppose that A satisfies e , and let $h_0: X \rightarrow A$. Extend h_0 to a continuous Σ -homomorphism $h: T_\Sigma(X_\perp) \rightarrow A$. By assumption, there exists $g: E \rightarrow A$ with $h = g \cdot e$. This implies $h_0 = g \cdot e_0$. Since $\widehat{e}_0(s)$ and $\widehat{e}_0(t)$ are defined and $\widehat{e}_0(s) \leq \widehat{e}_0(t)$, Lemma B.35(1) shows that $\widehat{h}_0(s) = g \cdot \widehat{e}_0(s) \leq g \cdot \widehat{e}_0(t) = \widehat{h}_0(t)$. Thus, A satisfies $s \leq t$. \square

Step 4. From the above lemma and Theorem 3.16, we obtain the following result of Adámek, Nelson, and Reiterman:

Theorem B.40 (Continuous HSP Theorem [4]). *A class of continuous Σ -algebras is a variety (i.e. closed under homomorphic images with respect to surjective maps, subalgebras, and products) iff it is axiomatizable by continuous inequalities.*

B.8 Algebras for a Monad

In this section, we show how to recover Manes's HSP theorem [18] for algebras for an arbitrary monad $\mathbb{T} = (T, \mu, \eta)$ on **Set**. Choose the parameters

- $\mathcal{A} = \mathcal{A}_0 = \mathbf{Set}^{\mathbb{T}}$, the category of \mathbb{T} -algebras and \mathbb{T} -homomorphisms;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjective } \mathbb{T}\text{-homomorphisms, injective } \mathbb{T}\text{-homomorphisms})$;
- $\Lambda =$ all cardinal numbers;
- $\mathcal{X} =$ all free \mathbb{T} -algebras $TX = (TX, \mu_X)$ with $X \in \mathbf{Set}$.

Since all sets are projective, we get $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$ (again by Remark B.1). Thus our Assumptions 3.1 are satisfied: for (1), use that products of \mathbb{T} -algebras are formed

on the level of sets. (2) is trivially satisfied, and (3) is obvious. Instantiating Definition 3.6, a *variety of \mathbb{T} -algebras* is a class of \mathbb{T} -algebras closed under quotient algebras, subalgebras, and products. *Quotient monads* of \mathbb{T} are represented by monad morphisms $q: \mathbb{T} \rightarrow \mathbb{T}'$ with surjective components. The following result is an easy consequence of our general correspondence between varieties and equational theories (see Theorem 3.15):

Theorem B.41 (Manes). *Varieties of \mathbb{T} -algebras correspond bijectively to quotient monads of \mathbb{T} .*

Remark B.42. Recall from Remark 3.11 that in the current setting an equational theory is given by a family of single quotients $(e_X : TX \rightarrow E_X)_{X \in \mathbf{Set}}$ which is *substitution invariant* in the sense that for every \mathbb{T} -homomorphism $h : TX \rightarrow TY$ there exists a \mathbb{T} -homomorphism $\bar{h} : E_X \rightarrow E_Y$ with $\bar{h} \cdot e_X = e_Y \cdot h$.

Proof. In view of Theorem 3.15, we only need to verify that equational theories correspond to quotient monads of \mathbb{T} .

(1) Every quotient monad $q: \mathbb{T} \rightarrow \mathbb{T}'$ induces an equational theory $(q_X : TX \rightarrow T'X)_{X \in \mathbf{Set}}$, where the \mathbb{T} -algebra structure on $T'X$ is given by

$$TT'X \xrightarrow{q_{T'X}} T'T'X \xrightarrow{\mu'_X} T'X.$$

Indeed, let $h : TX \rightarrow TY$ be a \mathbb{T} -homomorphism. Then the map $q_Y \cdot h \cdot \eta_X : X \rightarrow T'Y$ uniquely extends to a \mathbb{T}' -homomorphism $\bar{h} : T'X \rightarrow T'Y$ with $\bar{h} \cdot \eta'_X = q_Y \cdot h \cdot \eta_X : X \rightarrow T'Y$. By the naturality of q , \bar{h} is then also a \mathbb{T} -homomorphism. It follows that the square of \mathbb{T} -homomorphisms below commutes, as it commutes when precomposed with the universal map $\eta_X : X \rightarrow TX$:

$$\begin{array}{ccc} TX & \xrightarrow{h} & TY \\ q_X \downarrow & & \downarrow q_Y \\ T'X & \xrightarrow{\bar{h}} & T'Y \end{array}$$

Thus $(q_X)_{X \in \mathbf{Set}}$ is an equational theory.

(2) Conversely, suppose that $(q_X : TX \rightarrow T'X)_{X \in \mathbf{Set}}$ is an equational theory. Let us denote by $\alpha'_X : TT'X \rightarrow T'X$ the \mathbb{T} -algebra structure on $T'X$. We show that the object map $X \mapsto T'X$ can be extended to a monad $\mathbb{T}' = (T', \mu', \eta')$ on \mathcal{A} such that $q: \mathbb{T} \rightarrow \mathbb{T}'$ is a monad morphism. The action of T' on morphisms, the unit and the multiplication of \mathbb{T}' are uniquely determined by the commutative diagrams below:

$$\begin{array}{ccc} TX \xrightarrow{Th} TY & X \xrightarrow{\eta_X} TX & TT'X \xrightarrow{\alpha_X} TX \\ q_X \downarrow & \eta'_X \searrow & q_{T'X} \downarrow & \alpha'_X \searrow & q_X \downarrow \\ T'X \xrightarrow{T'h} T'Y & T'X & T'T'X \xrightarrow{\mu'_X} T'X \end{array} \tag{B.14}$$

In more detail:

- (a) For each map $h: X \rightarrow Y$, by substitution invariance, there exists a (necessarily unique) \mathbb{T} -homomorphism $T'h: T'X \rightarrow T'Y$ making the left-hand square commute. This makes $T': \mathbf{Set} \rightarrow \mathbf{Set}$ a functor and $q: T \rightarrow T'$ a natural transformation.
- (b) The unit η' is defined by $\eta' := q \cdot \eta$.
- (c) To define the multiplication $\mu': T'T' \rightarrow T'$, note that for every set X , the map α'_X is a \mathbb{T} -homomorphism $\alpha'_X: TT'X \rightarrow T'X$ by the associative law of the \mathbb{T} -algebra $(T'X, \alpha'_X)$. By projectivity of $TT'X$ there exists some \mathbb{T} -homomorphism $\alpha_X: TT'X \rightarrow TX$ with $q_X \cdot \alpha_X = \alpha'_X$, and thus substitution invariance gives a (necessarily unique) μ'_X making the outside of the right-hand diagram commute. Note that μ'_X is independent of the choice of α'_X because $\mu'_X \cdot q_{T'X} = \alpha'_X$ and $q_{T'X}$ is epimorphic.

Using that $q_X: TX \rightarrow T'X$ is a \mathbb{T} -homomorphism we furthermore obtain the following commutative diagram:

$$\begin{array}{ccc}
 TT'X & \xrightarrow{\mu_X} & TX \\
 \downarrow Tq_X & & \downarrow q_X \\
 TT'X & \searrow \alpha'_X & T'X \\
 \downarrow q_{T'X} & & \downarrow \\
 T'T'X & \xrightarrow{\mu'_X} & T'X
 \end{array}$$

From the commutativity of this diagram, the left-hand and middle diagram in (B.14), and using that q_X , Tq_X and $q_{T'X}$ are epimorphic, it is now a straightforward calculation to prove that η and μ are natural transformations, that they satisfy the monad laws, and that q is a monad morphism. We leave this easy task to the reader.

- (3) Finally, the two constructions described in (1) and (2) are easily seen to be mutually inverse (using again that q_X is epimorphic to see that one gets back to μ' when going from (1) to (2) and then back). □

B.9 Banaschewski-Herrlich Theorem

Let \mathcal{A} be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$, and suppose that \mathcal{A} (1) has products, (2) is \mathcal{E} -co-wellpowered, and (3) has enough \mathcal{E} -projectives, i.e. every object is a quotient of some \mathcal{E} -projective object. Choose the parameters of our framework as follows:

- $\mathcal{A}_0 = \mathcal{A}$;
- $\Lambda =$ all cardinal numbers;
- $\mathcal{X} =$ all \mathcal{E} -projectives.

By definition we $\mathcal{E}_{\mathcal{X}} = \mathcal{E}$, and our Assumptions 3.1 are clearly satisfied. Recall from Remark 3.4 that in this case an equation is given by a single quotient $e: X \rightarrow E$ with $X \in \mathcal{X}$. Theorem 3.16 then gives the following classical result:

Theorem B.43 (Banaschewski and Herrlich [7]). *Let \mathcal{A} be a category with a proper factorization system satisfying (1), (2), (3). Then a subclass $\mathcal{V} \subseteq \mathcal{A}$ is equationally presentable iff it is closed under quotients, subalgebras and products.*