

# A New Foundation for Finitary Corecursion and Iterative Algebras

Stefan Milius<sup>a,1</sup>, Dirk Pattinson<sup>b</sup>, Thorsten Wißmann<sup>a,1</sup>

<sup>a</sup>*Friedrich-Alexander-Universität Erlangen-Nürnberg*

<sup>b</sup>*The Australian National University*

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## Abstract

This paper contributes to a theory of the behaviour of “finite-state” systems that is generic in the system type. We propose that such systems are modeled as coalgebras with a finitely generated carrier for an endofunctor on a locally finitely presentable category. Their behaviour gives rise to a new fixpoint of the coalgebraic type functor called *locally finite fixpoint* (LFF). We prove that if the given endofunctor is finitary and preserves monomorphisms then the LFF always exists and is a subcoalgebra of the final coalgebra (unlike the rational fixpoint previously studied by Adámek, Milius, and Velebil). Moreover, we show that the LFF is characterized by two universal properties: (1) as the final locally finitely generated coalgebra, and (2) as the initial fg-iterative algebra. As instances of the LFF we first obtain the known instances of the rational fixpoint, e.g. regular languages, rational streams and formal power-series, regular trees etc. Moreover, we obtain a number of new examples, e.g. (realtime deterministic resp. non-deterministic) context-free languages, constructively  $S$ -algebraic formal power-series (in general, the behaviour of finite coalgebras under the coalgebraic language semantics arising from the generalized powerset construction by Silva, Bonchi, Bonsangue, and Rutten), and the monad of Courcelle’s algebraic trees.

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## 1. Introduction

Coalgebras capture many types of state based system within a uniform and mathematically rich framework [51]. One outstanding feature of the general theory is *final semantics* which gives a fully abstract account of system behaviour, i.e. it provides precisely all the behavioural equivalence classes. For example, the coalgebraic modelling of deterministic automata (without a finiteness restriction on state sets) yields the set of all formal languages as a final model, and restricting to *finite* automata one precisely obtains the regular languages [50]. This correspondence has been generalized to locally finitely presentable categories [12, 26], where *finitely presentable* objects play the role of finite sets, leading to the notion of *rational fixpoint* that provides final semantics to all models with finitely presentable carrier [39]. It is known that the rational fixpoint is fully abstract for these models as long as finitely presentable objects agree with finitely generated objects in the base category [18, Proposition 3.12]. While this is the case in some categories (e.g. sets, posets, graphs, vector spaces, commutative monoids), it is currently unknown in other base categories that are used in the construction of system models, for example in idempotent semirings (used in the treatment of context-free grammars [56]), in algebras for the stack monad (used for modelling configurations of stack machines [30]); or it even fails, for example in the category of finitary monads on sets (used in the categorical study of algebraic trees [9]), or Eilenberg-Moore categories for a monad in general (the target category of generalized determinization [53], in which the above examples live). Coalgebras over a category of Eilenberg-Moore algebras over **Set** in particular provide a paradigmatic setting: automata that describe languages beyond the class of regular languages consist of a finite state set, but their transitions produce side effects such as the manipulation of a stack. These can be described by a monad, so that the (infinite) set of system configurations (machine states plus stack content) is described by a free algebra (for that monad) that is generated by the finite set of machine states. This is formalized by the generalized powerset construction [53] and interacts nicely with the coalgebraic framework we present.

Technically, the shortcoming of the rational fixpoint is due to the fact that finitely presentable objects are not closed under quotients, so that the rational fixpoint itself may fail to be a subcoalgebra of the final coalgebra and so does not identify all behaviourally equivalent states. The main conceptual contribution of this paper is the insight that also in cases where finitely

presentable and finitely generated do not agree, we have a canonical domain for finitely generated behaviour. We introduce the *locally finite fixpoint* which provides a fully abstract model for such behaviour. We support this claim both by general results and concrete examples: we show that under mild assumptions, the locally finite fixpoint always exists, and we give a coalgebraic construction of it (Theorem 3.8); we also prove that it is indeed a subcoalgebra of the final coalgebra (Theorem 3.12). Moreover, we give a characterization of the locally finite fixpoint as the initial fg-iterative algebra (Corollary 4.9). We then instantiate our results to several scenarios studied in the literature.

First, we show that the locally finite fixpoint is universal (and fully abstract) for the class of systems produced by the generalized powerset construction over  $\mathbf{Set}$ : every determinized finite-state system induces a unique homomorphism to the locally finite fixpoint, and the latter contains precisely the finite-state behaviours (Theorem 6.5).

Applied to the coalgebraic treatment of context-free languages, we show that the locally finite fixpoint yields precisely the context-free languages (Theorem 6.8), and real-time deterministic context-free languages (Theorem 6.7), respectively, when their accepting machines are modelled as coalgebras over the category of algebras for the stack monad of [30]. For context-free languages weighted in a semiring  $S$ , or equivalently for constructively  $S$ -algebraic power series [47], the locally finite fixpoint comprises precisely those (Corollary 6.15), by phrasing the results of Winter et al. [57] in terms of the generalized powerset construction.

Our last example shows the applicability of our results to Eilenberg-Moore algebras over categories beyond  $\mathbf{Set}$ , and we characterize the monad of Courcelle’s algebraic trees over a signature [22, 9] as the locally finite fixpoint of an associated functor (on a category of monads) (Corollary 6.25), solving an open problem in [9].

The work extends the conference paper [44]. The present paper is a completely reworked version containing detailed proofs of all our results. In addition, Section 4 on fg-iterative algebras is new.

*Related Work.* The characterization of languages in terms of (co-)algebraic constructions has been carried out for various examples, such as (weighted) context-free languages [58, 30] as well as regular languages [50] where characterization theorems were established on a case-by-case basis. We show that the locally finite fixpoint provides a more general, and conceptual account. We have already mentioned the rational fixpoint [6, 39] that serves a similar

purpose and shares many technical similarities with the locally finite fixpoint, introduced here. Many of the properties of the rational fixpoint in fact hold, *mutatis mutandis*, also for locally finite fixpoint, cf. *op.cit.*

*Outline of the paper.* The rest of this paper is structured as follows. In Section 2 we recall a few basic facts about the central notions of this paper: locally finitely presentable categories, coalgebras, and the rational fixpoint of an endofunctor. Next, in Section 3 we introduce locally finitely generated (lfg) coalgebras, and we prove that a final lfg coalgebra exists, is a fixpoint (called locally finite fixpoint) and a subcoalgebra of the final coalgebra. The new Section 4 provides a characterization of the locally finite fixpoint as an algebra: it is the initial fg-iterative algebra. Then in Section 5 we investigate the relationship of the locally finite fixpoint to the rational fixpoint. Under slightly stronger assumptions than before we prove that the locally finite fixpoint is the image of the rational fixpoint in the final coalgebra. Finally, in Section 6 we consider several examples of the locally finite fixpoint, and Section 7 discusses future work and concludes the paper.

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## 2. Preliminaries and Notation

In this section we briefly recall a number of technical preliminaries needed throughout the paper. We assume that readers are familiar with basic category theory and with algebras and coalgebras for endofunctors.

### 2.1. Eilenberg-Moore-categories

Given a monad  $T: \mathcal{C} \rightarrow \mathcal{C}$ , its Eilenberg-Moore category  $\mathcal{C}^T$  is the category whose objects are the algebras for the monad  $T$ , i.e. pairs  $(A, a)$  where  $A$  is an object of  $\mathcal{C}$  (the *carrier* of the algebra) and  $a: TA \rightarrow A$  a morphism (the *structure* of the algebra) such that  $a \cdot \eta_A = \text{id}_A$  and  $a \cdot Ta = a \cdot \mu_A$ , where  $\eta: \text{Id} \rightarrow T$  and  $\mu: TT \rightarrow T$  are the unit and multiplication of the monad  $T$ . Morphisms of  $T$ -algebras are morphisms of  $\mathcal{C}$  commuting with algebra structures. More precisely, a  $T$ -algebra morphism from  $(A, a)$  to  $(B, b)$  is a morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  such that  $f \cdot a = b \cdot Tf$ . See Awodey [13, Chapter 10] for a more detailed introduction.

*Liftings* are a common way to define endofunctors on  $\mathcal{C}^T$ . Given a functor on the base category  $H: \mathcal{C} \rightarrow \mathcal{C}$ , a lifting of  $H$  is a functor  $H^T: \mathcal{C}^T \rightarrow \mathcal{C}^T$  such that the square below commutes, where  $U: \mathcal{C}^T \rightarrow \mathcal{C}$  denotes the forgetful functor:

$$\begin{array}{ccc} \mathcal{C}^T & \xrightarrow{H^T} & \mathcal{C}^T \\ U \downarrow & & \downarrow U \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C} \end{array}$$

Recall that the forgetful functor  $U$  has a left adjoint given by assigning to an object  $X$  of  $\mathcal{C}$  the free Eilenberg-Moore algebra  $(TX, \mu_X)$ .

The examples of Eilenberg-Moore-categories over **Set** include groups, monoids, (idempotent) semirings, **Set** itself, and moreover any variety of (finitary) algebras, i.e. a class of algebras specified by (finite-arity) operations and equations.

Note that monos in  $\mathbf{Set}^T$  are precisely the injective  $T$ -algebra morphisms. However, epis need not be surjective in  $\mathbf{Set}^T$ ; for example the embedding  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  from the integers to the rationals, each considered as a monoid w.r.t. multiplication, is an epi in the category of monoids. The surjective  $T$ -algebra morphisms are precisely the strong epis.<sup>2</sup>

Recall that, in general, an epi  $e: X \twoheadrightarrow Y$  is called *strong*, if for every mono  $m: A \hookrightarrow B$  and morphisms  $f: X \rightarrow A$ ,  $g: Y \rightarrow B$  with  $g \cdot e = m \cdot f$ , there exists a unique *diagonal fill-in*, i.e. a unique  $d: Y \rightarrow A$  such that:

$$\begin{array}{ccc} X & \xrightarrow{e} \twoheadrightarrow & Y \\ f \downarrow & \exists! d \swarrow \dashrightarrow & \downarrow g \\ A & \xrightarrow{m} \hookrightarrow & B \end{array}$$

Observe that for strong epis we have the same cancellation law as for ordinary epis:

if  $e' \cdot e$  is a strong epi, then  $e'$  is a strong epi.

We will continue to denote monos and strong epis in a category by  $\hookrightarrow$  and  $\twoheadrightarrow$ , respectively.

The coproduct of a family  $(X_i)_{i \in I}$  of objects is denoted by  $(\text{in}_i: X_i \rightarrow \coprod_{i \in I} X_i)_{i \in I}$ , and we call the morphism  $\text{in}_i$  the *coproduct injections*. Further-

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<sup>2</sup>In  $\mathbf{Set}^T$  the classes of strong and regular epimorphisms coincide.

more, for a family of morphisms  $(f_i: X_i \rightarrow Y)_{i \in I}$ , we denote by

$$[f_i]_{i \in I}: \coprod_{i \in I} X_i \rightarrow Y$$

the unique morphism with  $[f_i]_{i \in I} \cdot \text{in}_i = f_i$  for every  $i$ . In the case of binary coproducts we write

$$X \xrightarrow{\text{inl}} X_1 + X_2 \xleftarrow{\text{inr}} X_2 \quad \text{and} \quad X_1 + X_2 \xrightarrow{[f_1, f_2]} Y.$$

Similarly, for every colimit cocone  $(c_i: C_i \rightarrow C)_{i \in I}$  we call the morphisms  $c_i$  the *injections* of the colimit (even though they are not injective maps, in general).

**Example 2.1.** Suppose that  $\mathcal{C}$  is a cocomplete category. For every diagram  $D: \mathcal{D} \rightarrow \mathcal{C}$ , the injections of the colimit cocone  $(d_i: Di \rightarrow \text{colim } D)_{i \in \mathcal{D}}$  yield the strong epi

$$[d_i]_{i \in \mathcal{D}}: \coprod_{i \in \mathcal{D}} Di \rightarrow \text{colim } D.$$

## 2.2. Locally finitely presentable categories

A *filtered colimit* is the colimit of a diagram  $\mathcal{D} \rightarrow \mathcal{C}$  where  $\mathcal{D}$  is a filtered category (i.e. every finite subcategory  $\mathcal{D}_0 \hookrightarrow \mathcal{D}$  has a cocone in  $\mathcal{D}$ ), and a *directed colimit* is a colimit of a diagram having a directed poset as its diagram scheme  $\mathcal{D}$ . *Finitary functors* preserve filtered (equivalently directed) colimits. An object  $C \in \mathcal{C}$  is called *finitely presentable* (fp) if its hom-functor  $\mathcal{C}(C, -)$  is finitary and *finitely generated* (fg) if  $\mathcal{C}(C, -)$  preserves directed colimits of monos (i.e. all connecting morphisms in  $\mathcal{C}$  are monic). Clearly every fp object is fg, but not conversely in general. Moreover, fg objects are closed under strong quotients; here, a strong quotient of an object  $X$  is represented by a strong epi  $X \twoheadrightarrow Y$ , and closure under strong quotients means that  $Y$  is fg whenever  $X$  is. For fp objects this fails in general.

A cocomplete category is called *locally finitely presentable* (lfp) if the full subcategory  $\mathcal{C}_{\text{fp}}$  of finitely presentable objects is essentially small, i.e. is up to isomorphism only a set, and every object  $C \in \mathcal{C}$  is a filtered colimit of a diagram in  $\mathcal{C}_{\text{fp}}$ . We refer to [26, 12] for further details.

It is well known that the categories of sets, posets and graphs are lfp with finitely presentable objects precisely the finite sets, posets, graphs, respectively. The category of vector spaces is lfp with finite-dimensional spaces being the

fp objects. Every finitary variety is lfp. The finitely generated objects are precisely the finitely generated algebras, i.e. those algebras having a finite set of generators, and finitely presentable objects are precisely those algebras specified by finitely many generators and relations. This includes the categories of groups, monoids, (idempotent) semirings, semi-modules, etc. More generally, for every finitary monad  $T$ , i.e. the underlying functor of  $T$  is finitary, on the lfp category  $\mathcal{C}$ , the Eilenberg-Moore category  $\mathcal{C}^T$  is lfp again [12, Remark 2.78].

Every lfp category has (strong epi,mono)-factorizations of morphisms [12, Proposition 1.16], i.e. every morphism  $f: A \rightarrow B$  factorizes as  $f = m \cdot e$  for some mono  $m: \text{Im}(f) \rightarrow B$  and strong epi  $e: A \twoheadrightarrow \text{Im}(f)$ . We call the subobject of  $B$  represented by  $m$  the *image* of  $f$ .

We will subsequently make use of the following technical lemma. Recall that a union of subobjects of some object  $B$  is their join in the poset of all subobjects of  $B$ . In an lfp category, a *directed union* is, equivalently, a directed colimit of monos (see e.g. [5, Lemma 2.3]).

**Lemma 2.2** (Adámek, Milius, Sousa, and Wißmann [5], Lemma 2.11).  
*Images of filtered colimits in the lfp category  $\mathcal{C}$  are directed unions of images.*

More precisely, suppose we have a filtered diagram  $D: \mathcal{D} \rightarrow \mathcal{C}$  with a colimit cocone  $(c_i: Di \rightarrow C)_{i \in \mathcal{D}}$  and a morphism  $f: C \rightarrow B$ . Then the image of  $f$  together with the induced monomorphisms  $d_i$  forms the directed union of the images of the  $f \cdot c_i$ :

$$\begin{array}{ccc}
 Di & \xrightarrow{e_i} & \text{Im}(f \cdot c_i) \\
 c_i \downarrow & & \swarrow d_i \quad \downarrow m_i \\
 C & \xrightarrow{e} & \text{Im}(f) \xrightarrow{m} B \\
 \underbrace{\hspace{10em}}_f & & \uparrow
 \end{array} \tag{1}$$

### 2.3. Coalgebras

Let  $H: \mathcal{C} \rightarrow \mathcal{C}$  be an endofunctor. An  $H$ -*coalgebra* is a pair  $(C, c)$ , where  $C$  is an object of  $\mathcal{C}$  called the *carrier* and  $c: C \rightarrow HC$  is a morphism called the *structure* of the coalgebra. A homomorphism  $f: (C, c) \rightarrow (D, d)$  is a morphism  $f: C \rightarrow D$  of  $\mathcal{C}$  such that the following square commutes:

$$\begin{array}{ccc}
 C & \xrightarrow{c} & HC \\
 f \downarrow & & \downarrow Hf \\
 D & \xrightarrow{d} & HD
 \end{array}$$

Coalgebras and homomorphisms form a category, which we denote by  $\mathbf{Coalg}H$ .

If this category has a final object, then this final  $H$ -coalgebra is denoted by

$$\tau: \nu H \rightarrow H(\nu H).$$

The final coalgebra exists provided  $H$  is a finitary endofunctor on the lfp category  $\mathcal{C}$  (see e.g. [4, Theorem 6.10]).

By the universal property, we have for every coalgebra  $(C, c)$  a unique homomorphism  $c^\dagger: (C, c) \rightarrow (\nu H, \tau)$ . By Lambek's Lemma [36],  $\nu H$  is a fixpoint of  $H$  (i.e.  $\tau$  is an isomorphism). The final coalgebra represents a canonical domain of behaviour of systems of type  $H$ , and the unique homomorphism  $c^\dagger$  provides the semantics for a system  $(C, c)$ . For a concrete category  $\mathcal{C}$ , i.e.  $\mathcal{C}$  is equipped with a faithful functor  $|-|: \mathcal{C} \rightarrow \mathbf{Set}$ , we obtain a notion of semantic equivalence called *behavioural equivalence*: given two coalgebras  $(C, c)$  and  $(D, d)$ , two states  $x \in |C|$  and  $y \in |D|$  are called behavioural equivalent (notation:  $x \sim y$ ) whenever  $|c^\dagger(x)| = |d^\dagger(y)|$ .

Next, we recall a few categorical properties of  $\mathbf{Coalg}H$ . The forgetful functor  $\mathbf{Coalg}H \rightarrow \mathcal{C}$  creates colimits and reflects monos and epis. A morphism  $f$  in  $\mathbf{Coalg}H$  is *mono-carried* (resp. *strong epi-carried*) if the underlying morphism in  $\mathcal{C}$  is monic (resp. a strong epi). A *directed union of coalgebras* is a directed colimit of a diagram in  $\mathbf{Coalg}H$  whose connecting morphisms are mono-carried. Furthermore, by a *subcoalgebra* we mean a subobject in  $\mathbf{Coalg}H$  represented by a mono-carried homomorphism, and a *quotient coalgebra* is represented by a strong epi-carried homomorphism  $(C, c) \twoheadrightarrow (D, d)$ , and we say that  $(D, d)$  is a quotient of  $(C, c)$ .

#### 2.4. Non-empty Monos

Recall that endofunctors on  $\mathbf{Set}$  preserve all non-empty monomorphisms (because they are split monos in  $\mathbf{Set}$ ). We will assume a similar property for functors on general lfp categories. Recall that an initial object  $0$  is called *strict*, if every morphism  $I \rightarrow 0$  is an isomorphism.

**Definition 2.3.** A monomorphism  $m: X \rightarrow Y$  is called *empty* if its domain  $X$  is a strict initial object.

That means if  $\mathcal{C}$  has no initial object or a non-strict one, all monos are non-empty. Among the categories that have a strict initial object are: sets, posets, graphs, topological spaces, nominal sets and all Grothendieck toposes.



In fact, every extensive category in the sense of Carboni, Lack and Walters [20] has a strict initial object. Categories of algebras (over  $\mathbf{Set}$ ) have a strict initial object if the empty set carries an algebra.

**Lemma 2.4.** *Let  $T: \mathcal{C} \rightarrow \mathcal{C}$  be a monad and let  $H: \mathcal{C} \rightarrow \mathcal{C}$  be an endofunctor. Then if  $H$  preserves non-empty monos so does every lifting  $H^T: \mathcal{C}^T \rightarrow \mathcal{C}^T$ .*

*Proof.* The right-adjoint  $U: \mathcal{C}^T \rightarrow \mathcal{C}$  preserves and reflects monos. Given a non-empty monomorphism  $m: (A, a) \rightarrow (B, b)$  in  $\mathcal{C}^T$ ,  $Um: A \rightarrow B$  is monic as well.

Furthermore,  $Um: A \rightarrow B$  is non-empty; for if it were not, i.e. if  $A$  were a strict initial object in  $\mathcal{C}$ , then by  $a: TA \rightarrow A$ ,  $TA \cong A$  is initial in  $\mathcal{C}$ , and it follows that  $(A, a)$  is a strict initial object in  $\mathcal{C}^T$ , a contradiction.

Thus,  $UH^Tm = HUm$  is monic, and therefore so is  $H^Tm$ .  $\square$

**Lemma 2.5.** *If  $H: \mathcal{C} \rightarrow \mathcal{C}$  preserves non-empty monos, then the (strong epi,mono)-factorizations lift from  $\mathcal{C}$  to (strong epi-carried,mono-carried)-factorizations in  $\mathbf{Coalg}H$ .*

*Proof.* Given a coalgebra homomorphism  $h: (C, c) \rightarrow (D, d)$  and its factorization in  $\mathcal{C}$ :

$$\begin{array}{ccc} & \xrightarrow{h} & \\ C & \xrightarrow{e} \twoheadrightarrow I \xrightarrow{m} \rightarrow & D \end{array}$$

If  $I$  is a strict initial object, then  $e$  is an isomorphism,  $h$  is monic and so  $h = h \cdot \text{id}_C$  is the factorization in  $\mathbf{Coalg}H$ . Otherwise, the mono  $m$  is non-empty and thus preserved by  $H$ . By the diagonal fill-in property for the strong epi  $e$  and the mono  $Hm$ , we obtain a unique coalgebra structure on  $I$  making  $e$  and  $m$  into homomorphisms:

$$\begin{array}{ccccc} & & \xrightarrow{h} & & \\ & \xrightarrow{e} \twoheadrightarrow & I & \xrightarrow{m} \rightarrow & D \\ c \downarrow & & \exists! \downarrow & & \downarrow d \\ HC & \xrightarrow{He} \twoheadrightarrow & HI & \xrightarrow{Hm} \rightarrow & HD \\ & & \xrightarrow{Hh} & & \end{array} \quad \square$$

## 2.5. The Rational Fixpoint

Let  $H: \mathcal{C} \rightarrow \mathcal{C}$  finitary on the lfp category  $\mathcal{C}$ . We denote by  $\mathbf{Coalg}_{\text{fp}}H$  the full subcategory of  $\mathbf{Coalg}H$  of coalgebras with fp carrier, and by  $\mathbf{Coalg}_{\text{lfp}}H$

the full subcategory of  $\mathbf{Coalg}H$  of coalgebras that arise as filtered colimits of coalgebras with fp carrier [39, Corollary III.13]. The coalgebras in  $\mathbf{Coalg}_{\text{fp}}H$  are called *lfp coalgebras*, and for  $\mathcal{C} = \mathbf{Set}$  those are precisely the locally finite coalgebras (i.e. those coalgebras where every element is contained in a finite subcoalgebra).

The final lfp coalgebra  $(\rho H, r)$  exists and is the colimit of the inclusion  $\mathbf{Coalg}_{\text{fp}}H \hookrightarrow \mathbf{Coalg}H$ . Moreover, it is a fixpoint of  $H$  (see [6, Lemma 3.4]) called the *rational fixpoint* of  $H$ . Here are some examples: for the functor  $2 \times (-)^\Sigma$ , where  $\Sigma$  is some input alphabet, the rational fixpoint is the set of regular languages over  $\Sigma$ ; the rational fixpoint of a polynomial set functor associated to a finitary signature  $\Sigma$  is the set of rational  $\Sigma$ -trees [6], i.e. finite and infinite  $\Sigma$ -trees having, up to isomorphism, finitely many subtrees only [28]; one obtains rational weighted languages for Noetherian semirings  $S$  for a functor on the category of  $S$ -semimodules [18], and rational  $\lambda$ -trees for a functor on the category of presheaves on finite sets [10] or for a related functor on nominal sets [45].

If the classes of fp and fg objects coincide in  $\mathcal{C}$ , then the rational fixpoint is a subcoalgebra of the final coalgebra [18, Theorem 3.12]. This is the case in the above examples, but not in general, see [18, Example 3.15] for a concrete example where the rational fixpoint does not identify behaviourally equivalent states. However, even if the classes of fp and fg objects differ, the rational fixpoint can be a subcoalgebra, e.g. for every constant endofunctor.

## 2.6. Iterative Algebras

One important property of the rational fixpoint  $\rho H$  is that, besides being the final lfp coalgebra, it is also characterized by a universal property as an algebra for  $H$ .

Let  $H: \mathcal{C} \rightarrow \mathcal{C}$  be finitary on the lfp category  $\mathcal{C}$  once again. An  $H$ -algebra  $(A, a: HA \rightarrow A)$  is called *iterative* if every *flat equation morphism*  $e: X \rightarrow HX + A$  where  $X$  is an fp object has a unique *solution*, i.e. there exists a unique morphism  $e^\dagger: X \rightarrow A$  such that

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ e \downarrow & & \uparrow [a, \text{id}_A] \\ HX + A & \xrightarrow{He^\dagger + A} & HA + A \end{array}$$

Morphisms of iterative algebras are the usual  $H$ -algebra homomorphisms.

The rational fixpoint  $\varrho H$  is characterized as the initial iterative algebra for  $H$  [6, Theorem 3.3].

Note that this result is the starting point of the coalgebraic approach to Elgot's iterative theories [24] and to the iteration theories of Bloom and Ésik [16, 6, 7, 8]. For a well-motivated and much more detailed introduction to iterative algebras as well as examples see [6].

### 3. The Locally Finite Fixpoint

The locally finite fixpoint can be characterized similarly to the rational fixpoint, but with respect to coalgebras with finitely generated (not finitely presentable) carrier. We show that the locally finite fixpoint always exists, and is a subcoalgebra of the final coalgebra, i.e. identifies all behaviourally equivalent states. As a consequence, the locally finite fixpoint provides a fully abstract domain of finitely generated behaviour.

**Assumption 3.1.** Throughout the rest of the paper we assume that  $\mathcal{C}$  is an lfp category and that  $H: \mathcal{C} \rightarrow \mathcal{C}$  is finitary and preserves non-empty monos.

Recall that the last assumption is met by every lifted functor  $H^T: \mathbf{Set}^T \rightarrow \mathbf{Set}^T$  on a finitary variety  $\mathbf{Set}^T$ .

As for the rational fixpoint, we denote the full subcategory of  $\mathbf{Coalg}H$  comprising all coalgebras with finitely generated carrier by  $\mathbf{Coalg}_{\text{fg}}H$  and have the following notion of locally finitely generated coalgebra.

**Definition 3.2.** A coalgebra  $X \xrightarrow{x} HX$  is called *locally finitely generated (lfg)* if for all  $f: S \rightarrow X$  with  $S$  finitely generated, there exist a coalgebra  $p: P \rightarrow HP$  in  $\mathbf{Coalg}_{\text{fg}}H$ , a homomorphism  $h: (P, p) \rightarrow (X, x)$  and some  $f': S \rightarrow P$  such that  $h \cdot f' = f$ :

$$\begin{array}{ccccc}
 S & \xrightarrow{f} & X & \xrightarrow{x} & HX \\
 & \searrow f' & \uparrow h & & \uparrow Hp \\
 & & P & \xrightarrow{p} & HP
 \end{array}$$

$\mathbf{Coalg}_{\text{lfg}}H \subseteq \mathbf{Coalg}H$  denotes the full subcategory of lfg coalgebras.

Equivalently, one can characterize lfg coalgebras in terms of subobjects and subcoalgebras, making it a generalization of *local finiteness* in  $\mathbf{Set}$ , i.e. the property of a coalgebra that every element is contained in a finite subcoalgebra.

**Lemma 3.3.**  $X \xrightarrow{x} HX$  is an lfg coalgebra iff for all fg subobjects  $S \xrightarrow{f} X$ , there exist a subcoalgebra  $h: (P, p) \rightrightarrows (X, x)$  and a mono  $f': S \rightrightarrows P$  with  $h \cdot f' = f$ , i.e.  $S$  is a subobject of  $P$ .

*Proof.* ( $\Rightarrow$ ) Given a mono  $f: S \rightrightarrows X$ , consider the induced factor  $f': S \rightarrow P$  and factorize the induced homomorphism  $h: (P, p) \rightarrow (X, x)$  into a strong epi-carried homomorphism  $e$  followed by a mono-carried one  $m$ . Then  $\text{Im}(h)$  is fg since fg objects are closed under strong quotients, and  $e \cdot f': S \rightarrow \text{Im}(h)$  is the desired factor, which is monic since  $f = h \cdot f' = m \cdot (e \cdot f')$  is so.

( $\Leftarrow$ ) Factor  $f: S \rightarrow X$  into a strong epi  $e: S \rightarrow \text{Im}(f)$  and a mono  $g: \text{Im}(f) \rightrightarrows X$ . By hypothesis we obtain a subcoalgebra  $h: (P, p) \rightrightarrows (X, x)$  and  $g': \text{Im}(f) \rightrightarrows P$  with  $h \cdot g' = g$ . Then  $f' = e \cdot g'$  is the desired factor of  $f$ .  $\square$

Evidently, every coalgebra with a finitely generated carrier is lfg. Moreover, we will prove that the lfg coalgebras are precisely the filtered colimits of coalgebras from  $\text{Coalg}_{\text{fg}}H$ .

**Proposition 3.4.** *Every filtered colimit of coalgebras from  $\text{Coalg}_{\text{fg}}H$  is lfg.*

*Proof.* (1) Observe first that the statement of Lemma 2.2 holds for  $\text{Coalg}H$  in lieu of  $\mathcal{C}$  (despite the fact that  $\text{Coalg}H$  is not lfp in general). Indeed, this follows since in  $\text{Coalg}H$  one works with the lifted (strong-epi carried, mono-carried) factorizations and using that the forgetful functor  $\text{Coalg}H \rightarrow \mathcal{C}$  creates colimits.

(2) We prove that every directed union of coalgebras from  $\text{Coalg}_{\text{fg}}H$  is an lfg coalgebra.

Let  $D: (I, \leq) \rightarrow \text{Coalg}H$  be a diagram of coalgebras from  $\text{Coalg}_{\text{fg}}H$  and of mono-carried coalgebra homomorphisms, where  $(I, \leq)$  is a directed poset. For each  $i \in I$  denote  $D_i = (D_i, d_i)$  and let  $c_i: (D_i, d_i) \rightarrow (A, a)$  denote the colimit cocone in  $\text{Coalg}H$ . In order to verify the condition in Definition 3.2, let  $f: S \rightarrow A$  be a morphism in  $\mathcal{C}$  where  $S$  is fg. Recall that colimits in  $\text{Coalg}H$  are created by the forgetful functor to  $\mathcal{C}$ . Hence, the object  $A$  is a directed colimit of the objects  $D_i$  in  $\mathcal{C}$ , and since  $S$  is an fg object in  $\mathcal{C}$  we obtain the desired factorization:

$$\begin{array}{ccc}
 S & \xrightarrow{f} & A \\
 \searrow f' & & \nearrow c_i \\
 & D_i &
 \end{array}$$

(3) Now let  $c_i: (X_i, x_i) \rightarrow (X, x)$  be a colimit cocone of a filtered diagram with  $(X_i, x_i)$  in  $\mathbf{Coalg}_{\text{fg}}H$ . Take the (strong epi,mono)-factorizations

$$c_i = (X_i \xrightarrow{e_i} T_i \xrightarrow{m_i} X)$$

to obtain the subcoalgebras  $(T_i, t_i)$  of  $(X, x)$ . By Lemma 2.2,  $f = \text{id}_X: X \rightarrow X$  is the directed union of the  $m_i$ , and therefore  $\text{Im}(f) = X$  is the directed colimit of the diagram formed by the  $T_i$ , both in  $\mathcal{C}$  and in  $\mathbf{Coalg}H$ . The coalgebras  $(T_i, t_i)$  are in  $\mathbf{Coalg}_{\text{fg}}H$  since strong quotients of finitely generated objects are finitely generated. Hence, according to (2),  $(X, x)$  is lfg.  $\square$

**Proposition 3.5.** *Every lfg coalgebra  $(X, x)$  is a directed colimit of its subcoalgebras from  $\mathbf{Coalg}_{\text{fg}}H$ .*

*Proof.* Recall [12, Proof I of Theorem 1.70] that  $X$  is the directed colimit of the diagram of all its finitely generated subobjects. Now the subdiagram given by all subcoalgebras of  $X$  is cofinal. Indeed, this follows directly from the fact that  $(X, x)$  is an lfg coalgebra: for every subobject  $S \rightarrow X$ ,  $S$  fg, we have a subcoalgebra of  $(X, x)$  in  $\mathbf{Coalg}_{\text{fg}}H$  containing  $S$ .  $\square$

**Corollary 3.6.** *The lfg coalgebras are precisely the filtered colimits, or equivalently directed unions, of coalgebras with fg carrier.*

As a consequence, a coalgebra is final in  $\mathbf{Coalg}_{\text{lfg}}H$  if there is a unique morphism from every coalgebra with finitely generated carrier:

**Proposition 3.7.** *An lfg coalgebra  $(L, \ell)$  is final in  $\mathbf{Coalg}_{\text{lfg}}H$  if and only if for every coalgebra  $(X, x)$  in  $\mathbf{Coalg}_{\text{fg}}H$  there exists a unique coalgebra morphism from  $(X, x)$  to  $(L, \ell)$ .*

The proof is analogous to Milius' proof [39, Theorem 3.14]:

*Proof.* The direction from left to right is clear, as  $\mathbf{Coalg}_{\text{fg}}H$  is a full subcategory of  $\mathbf{Coalg}_{\text{lfg}}H$ . For the converse, let  $(S, s)$  be some lfg coalgebra. By Proposition 3.5, it is the directed union of all its subcoalgebras with finitely generated carrier. For every subcoalgebra  $\text{in}_p: (P, p) \rightarrow (S, s)$ , there exists a unique coalgebra homomorphism  $p^\dagger: (P, p) \rightarrow (L, \ell)$ . By the uniqueness of  $p^\dagger$  it follows that  $L$  together with the  $p^\dagger$  form a cocone. Hence there exists a unique coalgebra homomorphism  $u: (S, s) \rightarrow (L, \ell)$  such that  $u \cdot \text{in}_p = p^\dagger$  for every subcoalgebra  $(P, p)$  of  $(S, s)$ . Moreover, for every coalgebra homomorphism  $\bar{u}: (S, s) \rightarrow (L, \ell)$  the equation  $\bar{u} \cdot \text{in}_p = p^\dagger$  must hold as well, due to the uniqueness of  $p^\dagger$ . Since the colimit injections  $\text{in}_p$  are jointly epic, one obtains  $\bar{u} = u$  so that  $u$  is the unique homomorphism from  $(S, s)$  to  $(L, \ell)$ .  $\square$

Cocompleteness of  $\mathcal{C}$  ensures that the final lfg coalgebra always exists:

**Theorem 3.8.** *The category  $\text{Coalg}_{\text{lfg}}H$  has a final object, and the final lfg coalgebra is the colimit of the inclusion  $\text{Coalg}_{\text{fg}}H \hookrightarrow \text{Coalg}_{\text{lfg}}H$ .*

*Proof.* By Corollary 3.6, the colimit of the inclusion  $\text{Coalg}_{\text{fg}}H \hookrightarrow \text{Coalg}_{\text{lfg}}H$  is the same as the (large) colimit of the entire category  $\text{Coalg}_{\text{lfg}}H$ , and the latter is clearly the final object of  $\text{Coalg}_{\text{lfg}}H$ .  $\square$

**Notation 3.9.** We denote the final lfg coalgebra by  $\vartheta H \xrightarrow{\ell} H(\vartheta H)$ , and for every lfg coalgebra  $(X, x)$  we write

$$x^\dagger: (X, x) \rightarrow (\vartheta H, \ell)$$

for the unique coalgebra homomorphism.

**Corollary 3.10.** *If in  $\mathcal{C}$  the classes of fp- and fg-objects coincide, then the final lfg coalgebra coincides with the rational fixpoint, i.e. we have  $\vartheta H \cong \rho H$ .*

Indeed, the colimit constructions of both coalgebras are the same (cf. Section 2.5).

Theorem 3.8 provides a construction of the final lfg coalgebra collecting precisely the behaviours of the coalgebras with fg carriers. In the following we shall show that this construction does indeed identify precisely behaviourally equivalent states. In other words, the final lfg coalgebra is always a subcoalgebra of the final coalgebra. First we show that since fg objects are closed under strong quotients – in contrast to fp objects – we have a similar property of lfg coalgebras:

**Lemma 3.11.** *Lfg coalgebras are closed under quotients, i.e. for every strong epi carried coalgebra homomorphisms  $X \twoheadrightarrow Y$ , if  $X$  is lfg then so is  $Y$ .*

*Proof.* Consider some quotient  $q: (X, x) \twoheadrightarrow (Y, y)$  where  $(X, x)$  is lfg. As  $(X, x)$  is the directed colimit of its subcoalgebras with fg carrier, we have that  $(Y, y)$  – the codomain of the strong epi-carried  $q$  – is the directed union of the images of these subcoalgebras by Lemma 2.2 applied to  $f = q$ . These images are coalgebras with a finitely generated carrier since fg object are closed under strong quotients, whence  $(Y, y)$  is lfg as desired.  $\square$

The failure of the corresponding closure property for lfp coalgebras is the reason why the rational fixpoint is not necessarily a subcoalgebra of the final coalgebra. In particular the rational fixpoint given in [18, Example 3.15] is an lfp coalgebra having a quotient which is not lfp. However, for the final lfg coalgebra we have the following result.

**Theorem 3.12.** *The final lfg  $H$ -coalgebra is a subcoalgebra of the final  $H$ -coalgebra.*

*Proof.* Let  $(L, \ell)$  be the final lfg coalgebra. Consider the unique coalgebra morphism  $L \rightarrow \nu H$  and take its (strong epi, mono) factorization:

$$\ell^\dagger = \text{id} \circ \begin{array}{c} \dashrightarrow \\ \dashleftarrow \\ \dashrightarrow \end{array} (L, \ell) \xrightarrow[\text{\scriptsize } i^\dagger]{\text{\scriptsize } e} (I, i) \xrightarrow{\text{\scriptsize } m} (\nu H, \tau) , \quad \text{with } e \text{ strong epi in } \mathcal{C}.$$

By Lemma 3.11,  $I$  is an lfg coalgebra and so by finality of  $L$  we have the coalgebra morphism  $i^\dagger$  such that  $\text{id}_L = i^\dagger \cdot e$ . It follows that  $e$  is monic and therefore an iso.  $\square$

In other words, the final lfg  $H$ -coalgebra collects precisely the finitely generated behaviours from the final  $H$ -coalgebra. We now show that the final lfg coalgebra is a fixpoint of  $H$  which hinges on the following:

**Lemma 3.13.** *For every lfg coalgebra  $C \xrightarrow{c} HC$ , the coalgebra  $HC \xrightarrow{Hc} HHC$  is lfg.*

*Proof.* Let  $(C, c)$  be an lfg coalgebra and consider any morphism  $f: S \rightarrow HC$  with  $S$  finitely generated. By case distinction, one can show that  $(C, c)$  is a directed union of subcoalgebras  $\text{in}_p: (P, p) \rightarrow (C, c)$  with  $(P, p)$  in  $\text{Coalg}_{\text{fg}} H$  and such that this colimit is preserved by  $H$ :

- If  $C$  is a strict initial object, then it is an fg object, and  $(C, c)$  is the directed colimit of the diagram consisting of itself only. Hence, the directed colimit is preserved by  $H$ .
- If  $C$  is not a strict initial object, then by Proposition 3.5  $(C, c)$  is the directed union of all its subcoalgebras from  $\text{Coalg}_{\text{fg}} H$ . Since  $C$  is not a strict initial object, it is also the directed union of all its non-empty subcoalgebras (i.e. the inclusions and connecting morphisms are carried by non-empty monos). The latter directed union is preserved by  $H$ .

Hence,  $HC \xrightarrow{Hc} HHC$  is a directed colimit with colimit injections

$$Hin_p: (HP, Hp) \rightarrow (HC, Hc).$$

Now, since  $S$  is an fg object, the morphism  $f: S \rightarrow HC$  factorizes through one of these colimit injections, i.e. we have  $HP \xrightarrow{Hp} HHC$  with  $(P, p) \in \text{Coalg}_{\text{fg}} H$  and  $f': S \rightarrow HP$  with  $Hin_p \cdot f' = f$ . This allows us to construct a coalgebra with fg carrier:

$$S + P \xrightarrow{[f', p]} HP \xrightarrow{Hin_r} H(S + P)$$

and a coalgebra homomorphism  $Hin_p \cdot [f', p]: S + P \rightarrow HC$ ; in fact, in the following diagram every part trivially commutes:

$$\begin{array}{ccccc}
S & \xrightarrow{f} & HC & \xrightarrow{Hc} & HHC \\
& \searrow^{f'} & \uparrow^{Hin_p} & & \uparrow^{HHin_p} \\
& & HP & \xrightarrow{Hp} & HHP \\
& \searrow^{inl} & \uparrow^{[f', p]} & \parallel & \uparrow^{H[f', p]} \\
& & S + P & \xrightarrow{[f', p]} & HP \xrightarrow{Hin_r} H(S + P)
\end{array}
\quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} H(Hin_p \cdot [f', p])$$

This provides the desired factorization of  $f$ .  $\square$

Hence with a proof in virtue to Lambek's Lemma [36, Lemma 2.2], we obtain the desired fixpoint:

**Theorem 3.14.** *The final lfg  $H$ -coalgebra is a fixpoint of  $H$ .*

*Proof.* Let  $(C, c)$  be a final lfg  $H$ -coalgebra. Then  $(HC, Hc)$  is an lfg coalgebra by Lemma 3.13. Denote by  $d: (HC, Hc) \rightarrow (C, c)$  the unique coalgebra homomorphism. Then  $d \cdot c$  is a coalgebra homomorphism:

$$\begin{array}{ccc}
C & \xrightarrow{c} & HC \\
c \downarrow & & \downarrow Hc \\
HC & \xrightarrow{Hc} & HHC \\
d \downarrow & & \downarrow Hd \\
C & \xrightarrow{c} & HC
\end{array}$$

Thus,  $d \cdot c = \text{id}$  by the finality of  $(C, c)$ . Therefore  $c \cdot d = Hd \cdot Hc = H(d \cdot c) = H\text{id} = \text{id}$  using the commutativity of the upper square.  $\square$



In the light of Theorem 3.14 we will call the final lfg coalgebra the *locally finite fixpoint* (LFF) of  $H$ . In particular, the LFF always exists under Assumption 3.1, and its finality provides a finitary corecursion/coinduction principle.

#### 4. Iterative Algebras

Recall from [6, 39] that the rational fixpoint of a functor  $H$  has a universal property both as a coalgebra and as an algebra for  $H$ . This situation is completely analogous for the LFF. We already established its universal property as a coalgebra in Theorem 3.8. Now we turn to study the LFF as an algebra for  $H$ .

**Definition 4.1.** A (flat fg-) *equation morphism*  $e$  in an object  $A$  is a morphism  $X \rightarrow HX + A$ , where  $X$  is a finitely generated object. If  $A$  is the carrier of an algebra  $\alpha: HA \rightarrow A$ , we call the morphism  $e^\dagger: X \rightarrow A$  a *solution* of  $e$  if the diagram below commutes:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ e \downarrow & & \uparrow [\alpha, A] \\ HX + A & \xrightarrow{He^\dagger + A} & HA + A \end{array} \quad (2)$$

An  $H$ -algebra  $A$  is called *fg-iterative* if every equation morphism in  $A$  has a unique solution.

Note that we are overloading the  $\dagger$ -notation from Notation 3.9. This is justified by the fact, established in Proposition 4.5 below, that  $\vartheta H$  is an fg-iterative algebra whose operation of taking a unique solution of a flat equation morphism extends the final semantics operation  $\dagger$  from Notation 3.9, as explained in Remark 4.6.

**Example 4.2** (Milius [38, Example 2.5 (iii)]). The final  $H$ -coalgebra (considered as an algebra for  $H$ ) is fg-iterative. In fact, in this algebra even morphisms  $X \rightarrow HX + \nu H$  where  $X$  is not necessarily an fg object have a unique solution.

**Definition 4.3.** For fg-iterative algebras  $A$  and  $B$ , an equation morphism  $e: X \rightarrow HX + A$  and a morphism  $h: A \rightarrow B$  of  $\mathcal{C}$  define an equation morphism  $h \bullet e$  in  $B$  as

$$X \xrightarrow{e} HX + A \xrightarrow{HX+h} HX + B.$$

We say that  $h$  preserves the solution  $e^\dagger$  of  $e$  if

$$\begin{array}{ccc} & X & \\ e^\dagger \swarrow & & \searrow (h \bullet e)^\dagger \\ A & \xrightarrow{h} & B \end{array}$$

The morphism  $h$  is called *solution preserving* if it preserves the solution of every equation morphism  $e$ .

**Proposition 4.4.** *Let  $(A, \alpha)$  and  $(B, \beta)$  be fg-iterative algebras. Then a morphism  $h: A \rightarrow B$  is solution preserving iff it is an algebra homomorphism.*

The proof is identical to the one for ordinary iterative algebras [6, Proposition 2.18]; we leave it as an easy exercise for the reader. It follows that fg-iterative algebras form a full subcategory of the category of all  $H$ -algebras.

**Proposition 4.5.** *The locally finite fixpoint is fg-iterative.*

*Proof.* Let  $e: X \rightarrow HX + \vartheta H$  be an equation morphism. If  $X$  is an initial object, we are done because the unique morphism  $X \rightarrow \vartheta H$  is the desired unique solution of  $e$ .

So suppose that  $X$  is non-initial. In the following, we first define an lfg coalgebra structure  $\bar{e}$  on  $HX + \vartheta H$ , then take the unique coalgebra homomorphism  $\bar{e}^\dagger: HX + \vartheta H \rightarrow \vartheta H$  into the final lfg coalgebra, and obtain the unique solution of  $e$  as  $\bar{e}^\dagger \cdot e$ .

(1) We show that the following coalgebra is lfg:

$$\bar{e} = (HX + \vartheta H \xrightarrow{[He, Hinr \cdot \ell]} H(HX + \vartheta H)).$$

Consider an fg object  $S$  and a monomorphism  $f: S \rightarrow HX + \vartheta H$ . The carrier  $HX + \vartheta H$  is the directed colimit of the following diagram of monos:

- The diagram scheme  $\mathcal{D}$  is the product category containing pairs  $(T \xrightarrow{t} HX, v^\dagger: (V, v) \rightarrow (\vartheta H, \ell))$  consisting of an fg subobject of  $HX$  and a subcoalgebra of  $(\vartheta H, \ell)$  where  $V$  is fg.  $\mathcal{D}$  is (essentially) a directed poset, because both of its product components are essentially small and directed posets.
- The diagram  $D: \mathcal{D} \rightarrow \mathcal{C}$  is defined by

$$D(t, v) = \text{lm}(t + v^\dagger: T + V \rightarrow HX + \vartheta H)$$

on objects and by diagonal fill-in on morphisms. This implies that  $D(t, v)$  is  $\text{fg}$ , since  $\text{fg}$  objects are closed under coproducts and strong quotients, and that all connecting morphisms are monic.

That  $HX + \vartheta H$  is indeed the colimit of  $D$  is seen as follows. The object  $HX$  is the directed colimit of all its  $\text{fg}$  subobjects  $t: T \twoheadrightarrow HX$  (see [12, Proof I of Theorem 1.70]), and  $(\vartheta H, \ell)$  is the directed colimit of its subcoalgebras in  $\text{Coalg}_{\text{fg}} H$  by Proposition 3.5. Since colimits commute with coproducts,  $HX + \vartheta H$  is thus a directed colimit with the injection  $t + v^\dagger: T + V \rightarrow HX + \vartheta H$  in  $\mathcal{C}$ . By Lemma 2.2 applied to  $f$  being the identity morphism on  $HX + \vartheta H$ , we see that this object is the directed colimit of the diagram  $D$  of monos.

Because  $X + S$  is  $\text{fg}$ , the morphism  $[e, f]: X + S \rightarrow HX + \vartheta H$  factors through one of the colimit injections, i.e. we obtain a mono  $m: W \twoheadrightarrow HX + \vartheta H$ ,  $W$   $\text{fg}$ , and a morphism  $[e', f']: X + S \rightarrow W$  such that  $m \cdot [e', f'] = [e, f]$ . We know that  $W$  is not a strict initial object; for otherwise  $e': X \rightarrow W$  would imply that  $X$  is a strict initial object. Furthermore, choose some  $t: T \twoheadrightarrow HX$  and  $v: V \rightarrow HV$  from  $\mathcal{D}$  such that  $W = D(t, v)$  as shown in the diagram below:

$$\begin{array}{ccc}
 & T + V & \\
 & \downarrow [e_T, e_V] & \\
 & W & \\
 [e', f'] \nearrow & \downarrow m & \\
 X + S \xrightarrow{[e, f]} & HX + \vartheta H & \leftarrow
 \end{array}
 \quad \left. \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} t + v^\dagger
 \tag{3}$$

Since  $T + V$  is  $\text{fg}$ , so is its strong quotient  $W$ . The intermediate object  $W$  carries a coalgebra structure by the diagonal fill-in property (using that the mono  $m$  is non-empty and therefore  $Hm$  is monic):

$$\begin{array}{ccccc}
 & T + V & \xrightarrow{t+v} & HX + HV & \\
 [e_T, e_V] \swarrow & \downarrow & & \downarrow & [He', He_V] \searrow \\
 W & & & & HW \\
 m \swarrow & \downarrow t+v^\dagger & & \downarrow & \swarrow Hm \\
 HX + \vartheta H & \xrightarrow{[He, Hinr \cdot \ell]} & H(HX + \vartheta H) & & 
 \end{array}$$

Indeed, the left-hand component of the inner square above commutes trivially, and its right-hand component commutes because  $v^\dagger$  is a  $H$ -coalgebra homomorphism. The two triangles commute by the previous diagram (3).

Therefore we obtain a morphism  $w: W \rightarrow HW$  making  $m$  a coalgebra homomorphism from  $(W, w)$  to  $(HX + \vartheta H, \bar{e})$ . Thus  $m$  is the desired subcoalgebra through which  $f$  factorizes (see (3)).

(2) We take the unique coalgebra homomorphism

$$\bar{e}^\dagger: (HX + \vartheta H, \bar{e}) \longrightarrow (\vartheta H, \ell)$$

and put

$$s = (X \xrightarrow{e} HX + \vartheta H \xrightarrow{\bar{e}^\dagger} \vartheta H).$$

Clearly,  $\text{inr}: \vartheta H \rightarrow HX + \vartheta H$  is a coalgebra homomorphism from  $(\vartheta H, \ell)$  to  $(HX + \vartheta H, \bar{e})$ . Therefore, we see that  $\bar{e}^\dagger \cdot \text{inr} = \text{id}_{\vartheta H}$ .

We proceed to prove that  $s$  is a solution of the equation morphism  $e$ , i.e. diagram (2) commutes:

$$\begin{array}{ccccc}
 & & & & s \\
 & & & & \downarrow \\
 X & \xrightarrow{e} & HX + \vartheta H & \xrightarrow{\bar{e}^\dagger} & \vartheta H \\
 \downarrow e & & \downarrow He + \vartheta H & \searrow \bar{e} & \uparrow \ell^{-1} \\
 & & & H(HX + \vartheta H) & \xrightarrow{H\bar{e}^\dagger} & H(\vartheta H) & \left[ \ell^{-1}, \vartheta H \right] \\
 & & & \uparrow [\text{id}, H\text{inr} \cdot \ell] & (*) & \uparrow [\text{id}, \ell] \\
 HX + \vartheta H & \xrightarrow{He + \vartheta H} & H(HX + \vartheta H) + \vartheta H & \xrightarrow{H\bar{e}^\dagger + \vartheta H} & H(\vartheta H) + \vartheta H & \left[ \ell^{-1}, \vartheta H \right] \\
 & & \downarrow Hs + \vartheta H & & \uparrow & \\
 & & & & & 
 \end{array}$$

For the commutativity of the part  $(*)$ , we consider the coproduct components separately. The left-hand component trivially commutes, and for the right-hand one use  $\bar{e}^\dagger \cdot \text{inr} = \text{id}_{\vartheta H}$ . Since all other parts clearly commute, so does the desired outside of the diagram.

To verify uniqueness of solutions, suppose that  $s': X \rightarrow \vartheta H$  is any solution of  $e$ , i.e. we have

$$s' = [\ell^{-1} \cdot Hs', \text{id}_{\vartheta H}] \cdot e. \quad (4)$$

Then  $[\ell^{-1} \cdot Hs', \text{id}_{\vartheta H}]$  is a coalgebra homomorphism from  $(HX + \vartheta H, \bar{e})$  to

$(\vartheta H, \ell)$ :

$$\begin{array}{ccc}
 HX + \vartheta H & \xrightarrow{[\ell^{-1} \cdot Hs', \vartheta H]} & \vartheta H \\
 \downarrow [He, Hinr \cdot \ell] & \searrow [Hs', \ell] & \downarrow \ell \\
 H(HX + \vartheta H) & \xrightarrow{H[\ell^{-1} \cdot Hs', \vartheta H]} & H\vartheta H
 \end{array}
 \quad (4)$$

Hence  $[\ell^{-1} \cdot Hs', \text{id}_{\vartheta H}] = \bar{e}^\dagger$  so that we obtain

$$s' = [\ell^{-1} \cdot Hs', \text{id}_{\vartheta H}] \cdot e = \bar{e}^\dagger \cdot e = s. \quad \square$$

**Remark 4.6.** Every coalgebra  $e: X \rightarrow HX$  in  $\text{Coalg}_{\text{fg}} H$  canonically defines an equation morphism  $\text{inl} \cdot e: X \rightarrow HX + \vartheta H$ , and its solution in  $\vartheta H$  is just the unique coalgebra homomorphism from  $(X, e)$  to  $(\vartheta H, \ell)$ . To see this consider the diagram below:

$$\begin{array}{ccc}
 X & \xrightarrow{s} & \vartheta H \leftarrow \\
 e \downarrow & & \uparrow \ell \\
 HX & \xrightarrow{Hs} & H(\vartheta H) \\
 \text{inl} \downarrow & & \downarrow \text{inl} \\
 HX + \vartheta H & \xrightarrow{Hs + \vartheta H} & H(\vartheta H) + \vartheta H
 \end{array}
 \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} [\ell, \text{id}]$$

Since the lower square and the right-hand part trivially commute, we see that the upper square commutes iff so does the outside of the diagram. This shows that the operation  $\dagger$  of the fg-iterative algebra  $\vartheta H$  extends the final semantics operation of Notation 3.9 and so justifies our overloading of this notation.

**Lemma 4.7.** *Let  $\alpha: HA \rightarrow A$  be an fg-iterative algebra and  $e: X \rightarrow HX$  a coalgebra from  $\text{Coalg}_{\text{fg}} H$ . Then there exists a unique coalgebra-to-algebra morphism from  $X$  to  $A$ , i.e. a unique morphism  $u_e: X \rightarrow A$  such that  $u_e = \alpha \cdot Hu_e \cdot e$ .*

$$\begin{array}{ccc}
 X & \overset{\exists! u_e}{\dashrightarrow} & A \\
 e \downarrow & \circlearrowleft & \alpha \uparrow \\
 HX & \xrightarrow{Hu_e} & HA
 \end{array}$$

*Proof.* Consider the equation morphism  $\text{inl} \cdot e: X \rightarrow HX + A$ . Let  $s: X \rightarrow A$  be any morphism and consider the diagram below:

$$\begin{array}{ccccc}
 X & \xrightarrow{\quad s \quad} & A & \longleftarrow & \\
 \downarrow e & & \uparrow [\alpha, A] & & \\
 HX & \xrightarrow{\text{inl}} & HX + A & \xrightarrow{Hs+A} & HA + A & \circlearrowleft \alpha \\
 & & \circlearrowleft & & \uparrow \text{inl} \\
 & & & & HA & \longleftarrow \\
 & \xrightarrow{Hs} & & & & 
 \end{array}$$

Its lower and right-hand parts always commute. The upper square expresses that  $s$  is a solution of  $\text{inl} \cdot e$ , and we see that this square commutes if and only if the outside of the diagram commutes. Thus, solutions of  $\text{inl} \cdot e$  are equivalently, coalgebra-to-algebra morphisms from  $X$  to  $A$ . Hence, since the former exists uniquely so does the latter.  $\square$

**Theorem 4.8.** *Let  $\alpha: HA \rightarrow A$  be an fg-iterative algebra and  $e: X \rightarrow HX$  an lfg coalgebra. Then there exists a unique coalgebra-to-algebra morphism from  $X$  to  $A$ .*

*Proof.* By Proposition 3.5,  $e: X \rightarrow HX$  is the union of the diagram  $D$  of its subcoalgebras  $s: S \rightarrow HS$  with  $S$  finitely generated. Denote the corresponding colimit injections by  $\text{in}_s: (S, s) \rightarrow (X, e)$ . By Lemma 4.7, each such  $s$  induces a unique morphism  $u_s: S \rightarrow A$  with

$$u_s = \alpha \cdot Hu_s \cdot s. \quad (5)$$

For every coalgebra homomorphism  $h: (R, r) \rightarrow (S, s)$  in  $\text{Coalg}_{\text{fg}}H$  the diagram below commutes:

$$\begin{array}{ccccc}
 R & \xrightarrow{h} & S & \xrightarrow{u_s} & A \\
 r \downarrow & & s \downarrow & & \uparrow \alpha \\
 HR & \xrightarrow{Hh} & HS & \xrightarrow{Hu_s} & HA
 \end{array}$$

Hence  $u_r = u_s \cdot h$ . In other words,  $A$  together with the morphisms  $u_s: S \rightarrow A$  form a cocone on  $D$ . Thus, we obtain a unique morphism  $u_e: X \rightarrow A$  such that  $u_e \cdot \text{in}_s = u_s$  holds for every  $(S, s)$  in  $\text{Coalg}_{\text{fg}}H$ .

We now prove that  $u_e$  is an coalgebra-to-algebra morphism. For this we consider the following diagram:

$$\begin{array}{ccccc}
 & & \overset{u_s}{\circlearrowleft} & & \\
 & & \downarrow & & \\
 S & \xrightarrow{\text{in}_s} & X & \xrightarrow{u_e} & A \\
 \downarrow s & & \downarrow e & & \uparrow \alpha \\
 (i) & & (ii) & & \\
 HS & \xrightarrow{H\text{in}_s} & HX & \xrightarrow{Hu_e} & HA \\
 & & \circlearrowleft & & \\
 & & \underset{Hu_s}{\uparrow} & & 
 \end{array}$$

Indeed, the outside and all inner parts except, perhaps, part (ii) commute. This shows that part (ii) commutes when precomposed by every colimit injection  $\text{in}_s$ . Since these colimit injections are jointly epic, we have that (ii) commutes as desired.

To see the desired uniqueness assume that  $u_e$  is any coalgebra-to-algebra morphism, i.e. part (ii) of the above diagram commutes. Since part (i) also commutes, we see that  $u_e \cdot \text{in}_s$  is a coalgebra-to-algebra morphism from  $(S, s)$  to  $(A, \alpha)$ . Thus  $u_e \cdot \text{in}_s = u_s$  by the uniqueness of the latter (see Lemma 4.7).  $\square$

**Corollary 4.9.** *The locally finite fixpoint is the initial fg-iterative algebra.*

## 5. Relation to the Rational Fixpoint

There are examples, where the rational fixpoint is not a subcoalgebra of the final coalgebra (e.g. [18, Example 3.15] and [41, Example 2.18]). In categories, where the classes of fp and fg objects coincide, the rational fixpoint and the LFF are isomorphic (see Corollary 3.10). In this section we will see, under slightly stronger assumptions, that fg-carried coalgebras are quotients of fp-carried coalgebras, and in particular the locally finite fixpoint is the image of the rational fixpoint in the final coalgebra, i.e. we have the following picture:

$$\varrho F \twoheadrightarrow \vartheta F \twoheadrightarrow \nu F.$$

Recall that an object  $X$  of  $\mathcal{C}$  is called *projective* if for every strong epi  $e: A \twoheadrightarrow B$  and every morphism  $f: X \rightarrow A$  there exists a morphism  $f': X \rightarrow A$  such that  $e \cdot f' = f$ :

$$\begin{array}{ccc}
 X & \overset{\exists f'}{\dashrightarrow} & A \\
 \searrow f & & \downarrow e \\
 & & B
 \end{array}$$

**Assumption 5.1.** In addition to our standing Assumption 3.1, we assume that in the base category  $\mathcal{C}$ , every finitely presentable object is a strong quotient of a finitely presentable projective object and that the endofunctor  $H$  preserves strong epis.

Note that the related condition for arbitrary objects, i.e. that every object is the strong quotient of a projective object is phrased as *having enough projectives* [19, Definition 4.6.5]. Assumption 5.1 is relatively strong but still is met in many situations:

**Example 5.2.** (1) In categories in which all (strong) epis are split, every object is projective and every endofunctor preserves epis, e.g. in  $\mathbf{Set}$  or the category of vector spaces over a fixed field. In such categories  $\mathbf{fp}$  and  $\mathbf{fg}$  objects coincide.

(2) In the category  $\mathbf{Fun}_f(\mathbf{Set})$  of finitary endofunctors on sets, every polynomial functor is projective. This is easy to see for the polynomial functor  $PX = X^n$  associated to the signature  $\Sigma$  with a single  $n$ -ary operation symbol. Indeed, this follows from the Yoneda Lemma, since  $P \cong \mathbf{Set}(n, -)$ : given a natural transformation  $q: K \rightarrow L$  with surjective components, a natural transformation  $f: P \rightarrow L$  corresponds to an element of  $Ln$ , and we find its inverse image (under  $q_n$ ) in  $Kn$ . This gives us  $f': P \rightarrow K$  such that  $q \cdot f' = f$ . If  $\Sigma$  has more symbols, apply Yoneda Lemma to each of them separately and use that  $P$  is the coproduct of the corresponding hom-functors.

Furthermore, note that the finitely presentable functors are precisely the quotients of polynomial functors  $H_\Sigma$ , where  $\Sigma$  is a finite signature [5].

(3) In the Eilenberg-Moore category  $\mathbf{Set}^T$  for a finitary monad  $T$ , strong epis are surjective  $T$ -algebra homomorphisms, and thus preserved by every endofunctor  $H^T$  lifting the endofunctor  $H$  on  $\mathbf{Set}$ . Moreover, in  $\mathbf{Set}^T$ , every free algebra  $TX$  is projective; this is easy to see using the projectivity of  $X$  in  $\mathbf{Set}$ . Every finitely generated object of  $\mathbf{Set}^T$  is a strong quotient of some free algebra  $TX$  with  $X$  finite. Eilenberg-Moore algebras for set monads are the setting of the generalized powerset construction (see Section 6.1).

**Proposition 5.3.** *Every coalgebra in  $\mathbf{Coalg}_{\mathbf{fg}}H$  is a strong quotient of a coalgebra with finitely presentable carrier.*

*Proof.* Take a coalgebra  $(X, x)$  with finitely generated carrier. Recall that in an  $\mathbf{lfp}$  category an object is  $\mathbf{fg}$  if and only if it is a strong quotient of some  $\mathbf{fp}$  object [12, Proposition 1.69]. Hence  $X$  is the strong quotient of some



fp object  $X'$  via  $q: X' \twoheadrightarrow X$ . By assumption,  $X'$  is the strong quotient of a projective fp object  $X''$  via  $q': X'' \twoheadrightarrow X'$ . Since  $H$  preserves strong epis, the projectivity of  $X''$  induces a coalgebra structure  $x''$  such that  $q \cdot q'$  is a homomorphism:

$$\begin{array}{ccc}
X'' & \overset{x''}{\dashrightarrow} & HX'' \\
q' \downarrow & & \downarrow Hq' \\
X' & & HX' \\
q \downarrow & & \downarrow Hq \\
X & \xrightarrow{x} & HX
\end{array}
\quad \square$$

**Theorem 5.4.** *The locally finite fixpoint  $\vartheta H$  is the image of the rational fixpoint  $\varrho H$  in the final coalgebra.*

*Proof.* Consider the factorization  $(\varrho H, r) \xrightarrow{e} (B, b) \xrightarrow{m} (\nu H, \tau)$ . Since  $\varrho H$  is the colimit of all fp carried  $H$ -coalgebras it is an lfg coalgebra by Proposition 3.4 using that fp objects are also fg. Hence, by Lemma 3.11 the coalgebra  $B$  is lfg, too. By Proposition 3.7 it now suffices to show that from every  $(X, x) \in \mathbf{Coalg}_{\text{fg}} H$  there exists a unique coalgebra morphism into  $(B, b)$ . Given  $(X, x)$  in  $\mathbf{Coalg}_{\text{fg}} H$ , it is the quotient  $q: (P, p) \twoheadrightarrow (X, x)$  of an fp-carried coalgebra by Proposition 5.3. Hence, we obtain a unique coalgebra morphism  $p^\dagger: (P, p) \rightarrow (\varrho H, r)$ . By finality of  $\nu H$ , we have  $m \cdot e \cdot p^\dagger = x^\ddagger \cdot q$ , with  $x^\ddagger: (X, x) \rightarrow (\nu H, \tau)$  the unique homomorphism. So the diagonal fill-in property induces a homomorphism  $(X, x) \rightarrow (B, b)$ . By the finality of  $\nu H$  and because  $m$  is monic, this is the unique homomorphism  $(X, x) \rightarrow (B, b)$ .  $\square$

## 6. Instances of the Locally Finite Fixpoint

We will now present a number of instances of the LFF. First note, that all the instances of the rational fixpoint mentioned in previous work (see e.g. [6, 18, 39]) are also instances of the locally finite fixpoint, because in all those cases the classes of fp and fg objects coincide. For example, the class of regular languages is the rational fixpoint of  $2 \times (-)^\Sigma$  on  $\mathbf{Set}$ . In this section, we will study further instances of the LFF that are not known to be instances of the rational fixpoint and which – to the best of our knowledge – have not been characterized by a universal property yet:

(1) Behaviours of finite-state machines with side-effects as considered by the generalized powerset construction (cf. Section 6.1), in particular the following:

- (a) Deterministic and ordinary context-free languages obtained as the behaviours of deterministic and non-deterministic stack-machines, respectively.
  - (b) Constructively  $S$ -algebraic formal power series, i.e. the “context-free” subclass of weighted languages with weights from a semiring  $S$ , obtained from weighted context-free grammars.
- (2) The monad of Courcelle’s algebraic trees [22].

### 6.1. Generalized Powerset Construction

The determinization of a non-deterministic automaton using the powerset construction is an instance of a more general construction, described by Silva, Bonchi, Bonsangue, and Rutten [53] based on an observation by Bartels [15] (see also Jacobs [33]). In that *generalized powerset construction*, an automaton with side-effects is turned into an ordinary automaton by internalizing the side-effects in the states. The LFF interacts well with this construction, because it precisely captures the behaviours of finite-state automata with side effects. The notion of side-effect is formalized by a monad, which induces the category, in which the LFF is considered.

**Notation 6.1.** Given a monad  $(T, \eta^T, \mu^T)$  on  $\mathcal{C}$  and an Eilenberg-Moore algebra  $a: TA \rightarrow A$  we denote for any morphism  $f: X \rightarrow A$  by  $f^\sharp: TX \rightarrow A$  the unique  $T$ -algebra morphism from the free Eilenberg-Moore algebra  $(TX, \mu_X)$  to  $(A, a)$  extending  $f$ , i.e. such that  $f^\sharp \cdot \eta_X = f$ .

**Example 6.2.** In Sections 6.4 and 6.6 we are going to make use of Moggi’s exception monad transformer (see e.g. [21]). Let us recall that for a fixed object  $E$ , the finitary functor  $(-)+E$  together with the unit  $\eta_X = \text{inl}: X \rightarrow X+E$  and multiplication  $\mu_X = \text{id}_X + [\text{id}_E, \text{id}_E]: X+E+E \rightarrow X+E$  forms a finitary monad, the *exception monad*. Its algebras are  $E$ -pointed objects, i.e. objects  $X$ , together with a morphism  $E \rightarrow X$ , and homomorphisms are morphisms preserving the pointing. So the induced Eilenberg-Moore category is just the slice category  $E/\mathcal{C} \cong \mathcal{C}^{(-)+E}$ .

Now, given any monad  $T$  we obtain a new monad  $T(-+E)$  with obvious unit and multiplication. An Eilenberg-Moore algebra for  $T(-+E)$  consists of an Eilenberg-Moore algebra for  $T$  and an  $E$ -pointing, and homomorphisms are  $T$ -algebra homomorphisms preserving the pointing [32].

An automaton with side-effects is modelled as an  $HT$ -coalgebra, where  $T$  is a finitary monad on  $\mathcal{C}$  providing the type of side-effect. For example,

for  $HX = 2 \times X^\Sigma$ , where  $\Sigma$  is an input alphabet,  $2 = \{0, 1\}$  and  $T$  the finite powerset monad on  $\mathbf{Set}$ ,  $HT$ -coalgebras are non-deterministic automata. However, the coalgebraic semantics using the final  $HT$ -coalgebra does not yield the usual language semantics of non-deterministic automata. This is obtained by turning the  $HT$ -coalgebra into a coalgebra for a lifting of  $H$  on  $\mathcal{C}^T$  via the generalized powerset construction that we now recall. We work under the following

**Assumption 6.3.** We assume that  $\mathcal{C}$  is an lfp category,  $T$  a finitary monad on  $\mathcal{C}$  and  $H$  a finitary endofunctor on  $\mathcal{C}$  preserving non-empty monos and  $H^T : \mathcal{C}^T \rightarrow \mathcal{C}^T$  is a lifting of  $H$ , i.e.  $H \cdot U = U \cdot H^T$ , where  $U : \mathcal{C}^T \rightarrow \mathcal{C}$  is the canonical forgetful functor.

The generalized powerset construction transforms an  $HT$ -coalgebra into an  $H^T$ -coalgebra on  $\mathcal{C}^T$ : For a coalgebra  $x : X \rightarrow HTX$ ,  $HTX$  carries an Eilenberg-Moore algebra, and one uses freeness of the Eilenberg-Moore algebra  $TX$  to obtain a canonical  $T$ -algebra homomorphism  $x^\sharp : (TX, \mu_X^T) \rightarrow H^T(TX, \mu_X^T)$ . The *coalgebraic language semantics* of  $(X, x)$  is then given by composing the unique coalgebra morphism induced by  $x^\sharp$  with  $\eta_X$ :

$$\begin{array}{ccccc}
 X & \xrightarrow{\eta_X} & TX & \xrightarrow{x^\sharp} & \nu H \\
 x \downarrow & \swarrow x^\sharp & & & \downarrow \tau \\
 HTX & \xrightarrow{Hx^\sharp} & & & H\nu H
 \end{array}$$

This construction yields a functor

$$T' : \mathbf{Coalg}(HT) \rightarrow \mathbf{Coalg}H^T$$

mapping coalgebras  $X \xrightarrow{x} HTX$  to  $x^\sharp$  and homomorphisms  $f$  to  $Tf$  (see e.g. [18, Proof of Lemma 3.27] for a proof).

Note that since the right adjoint  $U$  preserves monos and is faithful, we know that  $H^T$  preserves monos, and since  $T$  is finitary, filtered colimits in  $\mathcal{C}^T$  are created by the forgetful functor to  $\mathcal{C}$ , and we therefore see that  $H^T$  is finitary. Thus, by Theorem 3.8,  $\vartheta H^T$  exists and is a subcoalgebra of  $\nu H^T$ . Furthermore, recall from [48] and [15, Corollary 3.4.19] that  $\nu H^T$  is carried by  $\nu H$  equipped with a canonical  $T$ -algebra structure, see e.g. [18, Notation 3.22].

In the remainder of this section we will assume that  $\mathcal{C} = \mathbf{Set}$ . It is our aim to show that the LFF of  $H^T$  characterizes precisely the coalgebraic language

semantics of all finite  $HT$ -coalgebras. Formally, the coalgebraic language semantics of all finite  $HT$ -coalgebras is collected by forming the colimit

$$(K, k) = \operatorname{colim} \left( \operatorname{Coalg}_{\text{fg}} HT \xrightarrow{T'} \operatorname{Coalg} H^T \xrightarrow{U} \operatorname{Coalg} H \right).$$

Note that this is a filtered colimit because the category  $\operatorname{Coalg}_{\text{fg}} H$  is closed under finite colimits and therefore filtered.

The coalgebra  $K$  is not yet a subcoalgebra of  $\nu H$  (that means, not all behaviourally equivalent states are identified in  $K$ ), but taking its image in  $\nu H$  we obtain the LFF:

**Proposition 6.4.** *The image of the unique coalgebra morphism  $k^\ddagger: K \rightarrow \nu H^T$  is precisely the locally finite fixpoint of the lifting  $H^T$ .*

*Proof.* Let us denote by  $\operatorname{in}_x: (TX, x^\sharp) \rightarrow (K, k)$  the colimit injection of the above colimit. For every finite  $X$ ,  $(TX, \mu_X)$  is finitely generated in  $\mathbf{Set}^T$ , and hence  $(TX, x^\sharp)$  is in  $\operatorname{Coalg}_{\text{fg}} H^T$ . Therefore we have the unique coalgebra homomorphism  $x^{\sharp\dagger}: (TX, x^\sharp) \rightarrow (\vartheta H^T, \ell)$ . By finality of  $(\nu H, \tau)$ , we see that the outside of the square below commutes:

$$\begin{array}{ccc} \coprod_{(X,x) \in \operatorname{Coalg}_{\text{fg}}(HT)} (TX, x^\sharp) & \xrightarrow{[\operatorname{in}_x]} & (K, k) \\ \downarrow [x^{\sharp\dagger}] & \swarrow w & \downarrow k^\ddagger \\ (\vartheta H^T, \ell) & \xrightarrow{n} & (\nu H, \tau) \end{array} \quad \text{in } \operatorname{Coalg} H$$

Recall from Example 2.1 that  $[\operatorname{in}_x]_{(X,x)}$  is a strong epi in  $\operatorname{Coalg} H$ . Since  $n$  is a mono in  $\operatorname{Coalg} H$ , we obtain a diagonal  $w: (K, k) \rightarrow (\vartheta H^T, \ell)$ . To prove that  $(\vartheta H^T, \ell)$  is indeed the image of  $k^\ddagger$ , it remains to show that  $w$  is a strong epi in  $\mathbf{Set}$  (cf. Lemma 2.5), i.e. a surjective map.

To see that  $w$  is surjective we first establish that every coalgebra  $(Y, f)$  in  $\operatorname{Coalg}_{\text{fg}} H^T$  is the quotient of some  $(TX, x^\sharp)$  with  $X$  finite. Indeed, given  $f: Y \rightarrow HY$  where  $Y$  is a finitely generated  $T$ -algebra, we know that it is the quotient of some free  $T$ -algebra  $TX$ ,  $X$  finite, via  $q: TX \twoheadrightarrow Y$ , say. We know that  $H^T$  preserves surjective  $T$ -algebra morphism since it is a lifting and every set functor  $H$  preserves surjections. Thus, we can use projectivity of the free algebra  $TX$  to obtain a coalgebra structure  $e: TX \rightarrow HTX$  such

that  $q$  is a coalgebra homomorphism:

$$\begin{array}{ccc} TX & \xrightarrow{e} & HTX \\ \downarrow q & & \downarrow Hq \\ Y & \xrightarrow{f} & HY \end{array}$$

Note that  $e$  is of the desired form  $x^\sharp$  for  $x = e \cdot \eta_X$ . Now since the  $f^\dagger: Y \rightarrow \vartheta H$ ,  $(Y, f)$  in  $\mathbf{Coalg}_{\text{fg}} H^T$  are jointly surjective, it follows that so are the  $x^\sharp$ , whence  $x^\sharp$  is a jointly surjective family. Thus,  $w$  is surjective as desired.  $\square$

One can also directly take the union of all desired behaviours:

**Theorem 6.5.** *The locally finite fixpoint of the lifting  $H^T$  comprises precisely the images of determinized HT-coalgebras:*

$$\vartheta H^T = \bigcup_{\substack{x: X \rightarrow HTX \\ X \text{ finite}}} x^\sharp[TX] = \bigcup_{\substack{x: X \rightarrow HTX \\ X \text{ finite}}} x^\sharp \cdot \eta_X^T[X] \subseteq \nu H^T. \quad (6)$$

*Proof.* Combining the previous Proposition 6.4 together with Lemma 2.2 proves the first equality. For the second equality, consider any element  $t \in TX$  and define a new coalgebra on  $X + 1$  by

$$(Y, y) = (X + 1 \xrightarrow{[x, x^\sharp(t)]} HTX \xrightarrow{HT\text{inl}} HT(X + 1)).$$

It is not difficult to see that  $[\eta_X, t]^\sharp: TY \rightarrow TX$  is a  $H^T$ -coalgebra homomorphism; indeed, to see that the following square of  $T$ -algebra morphisms commutes

$$\begin{array}{ccc} T(X + 1) & \xrightarrow{y^\sharp} & HT(X + 1) \\ [\eta_X, t]^\sharp \downarrow & & \downarrow H[\eta_X, t]^\sharp \\ TX & \xrightarrow{x^\sharp} & HTX \end{array}$$

one uses the universal property of the free  $T$ -algebra  $(TY, \mu_Y)$ , i.e. it suffices to see that the square commutes when precomposed with  $\eta_Y: Y \rightarrow TY$ . This is easily done by considering the coproduct components of  $Y = X + 1$  separately.

Furthermore, we clearly have  $t \in y^\sharp \cdot \eta_Y^T[Y]$ , and we are done.  $\square$

This result shows that the locally finite fixpoint  $\vartheta H^T$  captures precisely the behaviour of finite  $HT$ -coalgebras, i.e. it is a fully abstract domain for finite state behaviour w.r.t. the coalgebraic language semantics.

In the following subsections, we instantiate the general theory with examples from the literature to characterize several well-known notions as LFF.

## 6.2. The Languages of Non-deterministic Automata

Let us start with a simple standard example. We already mentioned that non-deterministic automata are coalgebras for the functor  $X \mapsto 2 \times \mathcal{P}_f(X)^\Sigma$ . Hence they are  $HT$ -coalgebras for  $H = 2 \times (-)^\Sigma$  and  $T = \mathcal{P}_f$  the finite powerset monad on **Set**. The above generalized powerset construction then instantiates as the usual powerset construction that assigns to a given non-deterministic automaton its determinization.

Now note that the final coalgebra for  $H$  is carried by the set  $\mathcal{L} = \mathcal{P}(\Sigma^*)$  of all formal languages over  $\Sigma$  with the coalgebra structure given by  $o: \mathcal{L} \rightarrow 2$  with  $o(L) = 1$  iff  $L$  contains the empty word and  $t: \mathcal{L} \rightarrow \mathcal{L}^\Sigma$  with  $t(L)(s) = \{w \mid sw \in L\}$  the left language derivative. The functor  $H$  has a canonical lifting  $H^T$  on the Eilenberg-Moore category of  $\mathcal{P}_f$ , viz. the category of join semi-lattices. The final coalgebra  $\nu H^T$  is carried by all formal languages with the join semi-lattice structure given by union and  $\emptyset$  and with the above coalgebra structure. Furthermore, the coalgebraic language semantics of  $x: X \rightarrow HTX$  assigns to every state of the non-deterministic automaton  $X$  the language it accepts. Observe that join semi-lattices form a so-called *locally finite variety*, i.e. the finitely presentable algebras are precisely the finite ones. Hence, Theorem 6.5 states that the LFF  $\vartheta H^T$  is precisely the subcoalgebra of  $\nu H^T$  formed by all languages accepted by finite non-deterministic automata, i.e. regular languages.

Note that in this example the LFF and the rational fixpoint coincide since both fp and fg join semi-lattices are simply the finite ones. Similar characterizations of the coalgebraic language semantics of finite coalgebras follow from Theorem 6.5 in other instances of the generalized powerset construction (cf. e.g. the treatment of the behaviour of finite weighted automata in [18]).

We now turn to examples that, to the best of our knowledge, cannot be treated using the rational fixpoint.

### 6.3. The Behaviour of Stack Machines

Push-down automata are finite state machines with infinitely many configurations. It is well-known that deterministic and non-deterministic pushdown automata recognize different classes of context-free languages. We will characterize both as instances of the locally finite fixpoint, using results on stack machines [30]; they are finite state machines which can push or read *multiple* elements to or from their stack in a single transition, respectively.

That is, a transition of a stack machine in a certain state consists of reading an input character, going to a successor state based on the stack's topmost elements and of modifying the topmost elements of the stack. These stack operations are captured by the stack monad.

**Definition 6.6** (Stack monad, [29, Proposition 5]). For a finite set of stack symbols  $\Gamma$ , the *stack monad* is the submonad  $T$  of the store monad  $(- \times \Gamma^*)^{\Gamma^*}$  for which the elements  $\langle r, t \rangle$  of  $TX \subseteq (X \times \Gamma^*)^{\Gamma^*} \cong X^{\Gamma^*} \times (\Gamma^*)^{\Gamma^*}$  satisfy the following restriction: there exists  $k$  depending on  $r, t$  such that for every  $w \in \Gamma^k$  and  $u \in \Gamma^*$ ,  $r(wu) = r(w)$  and  $t(wu) = t(w)u$ .

Note that the parameter  $k$  gives a bound on how many of the topmost stack cells the machine can access in one step.

Using the stack monad, stack machines are  $HT$ -coalgebras, where  $H = B \times (-)^{\Sigma}$  is the Moore automaton functor for the finite input alphabet  $\Sigma$  and the set  $B$  of all predicates on (initial) stack configurations which depend only on the topmost  $k$  elements on the stack:

$$B = \{p \in 2^{\Gamma^*} \mid \exists k \in \mathbb{N}_0 : \forall w, u \in \Gamma^*, |w| \geq k : p(wu) = p(w)\} \subseteq 2^{\Gamma^*}.$$

The final coalgebra  $\nu H$  is carried by  $B^{\Sigma^*}$  which is (isomorphic to) a set of functions  $\Gamma^* \rightarrow 2^{\Sigma^*}$ , mapping stack configurations to formal languages. Goncharov et al. [30] show that  $H$  lifts to  $\mathbf{Set}^T$  and that finite-state  $HT$ -coalgebras can be understood as a coalgebraic version of *deterministic* pushdown automata without spontaneous transitions. The languages accepted by those automata are precisely the *real-time deterministic context-free languages*; this notion goes back to Harrison and Havel [31]. We obtain the following, with  $\gamma_0$  playing the role of an initial symbol on the stack:

**Theorem 6.7.** *The locally finite fixpoint  $\vartheta H^T$  is carried by the set of all maps  $f \in B^{\Sigma^*}$  such that for every fixed  $\gamma_0 \in \Gamma$ ,  $\{w \in \Sigma^* \mid f(w)(\gamma_0) = 1\}$  is a real-time deterministic context-free language.*

*Proof.* By [30, Theorem 5.5], a language  $L$  is a real-time deterministic context-free language iff there exists some  $x: X \rightarrow HTX$ ,  $X$  finite, with its determinization  $x^\sharp: TX \rightarrow HTX$  and there exist  $s \in X$  and  $\gamma_0 \in \Gamma$  such that  $f = x^\sharp \cdot \eta_X^T(s) \in B^{\Sigma^*}$  and  $f(w)(\gamma_0) = 1$  for all  $w \in \Sigma^*$ . The rest follows by (6).  $\square$

Just as for pushdown automata, the expressiveness of stack machines increases when equipping them with non-determinism. Technically, this is done by considering the *non-deterministic stack monad*  $T'$ , i.e.  $T'$  denotes a submonad of the non-deterministic store monad  $\mathcal{P}_\dagger(- \times \Gamma^*)^{\Gamma^*}$  [30, Section 6]. In the non-deterministic setting, a similar property holds, namely that the determinized  $HT'$ -coalgebras with finite carrier describe precisely the context-free languages [30, Theorem 6.5]. Combining this with (6) we obtain:

**Theorem 6.8.** *The locally finite fixpoint  $\vartheta H^{T'}$  is carried by the set of all maps  $f \in B^{\Sigma^*}$  such that for every fixed  $\gamma_0 \in \Gamma$ ,  $\{w \in \Sigma^* \mid f(w)(\gamma_0) = 1\}$  is a context-free language.*

#### 6.4. Context-Free Languages and Constructively $S$ -Algebraic Power Series

One generalizes from formal (resp. context-free) languages to weighted formal (resp. context-free) languages by assigning to each word a weight from a fixed semiring. More formally, a weighted language – a.k.a. *formal power series* – over an input alphabet  $X$  is defined as a map  $X^* \rightarrow S$ , where  $S$  is a semiring. The set of all formal power series is denoted by  $S\langle\langle X \rangle\rangle$ . Ordinary formal languages are formal power series over the boolean semiring  $\mathbb{B} = \{0, 1\}$ , i.e. maps  $X^* \rightarrow \{0, 1\}$ .

An important class of formal power series is that of *constructively  $S$ -algebraic* formal power series. We show that this class arises precisely as the LFF of the standard functor  $H = S \times (-)^\Sigma$  for deterministic Moore automata on a finitary variety, i.e. an Eilenberg-Moore category of a finitary set monad. As a special case, constructively  $\mathbb{B}$ -algebraic formal power series are precisely the context-free languages and they form the LFF of the functor  $\mathbb{B} \times (-)^\Sigma$  on the category of idempotent semirings.

The original definition of constructively  $S$ -algebraic formal power series goes back to Fliess [25], see also [23]. Here, we use the equivalent coalgebraic characterization by Winter et al. [57].

Let  $S\langle X \rangle \subseteq S\langle\langle X \rangle\rangle$  be the subset of those maps  $X^* \rightarrow S$  having finite support, i.e. which map all but finitely many  $w \in X^*$  to 0. If  $S$  is commutative,



then  $S\langle - \rangle$  yields a finitary monad and therefore we also have the monad  $T = S\langle - + \Sigma \rangle$  by Example 6.2. Note that the monad  $S\langle - \rangle$  is a composition of two monads: we have  $S\langle X \rangle = S_\omega^{(X^*)}$  where  $X \mapsto X^*$  is the free monoid monad and  $X \mapsto S_\omega^{(X)}$  maps a set  $X$  to the free  $S$ -semimodule on  $X$ , which is carried by the set of finite support functions  $X \rightarrow S$ .

Recall that the algebras for the monad  $S\langle - \rangle$  are the associative  $S$ -algebras (over the commutative semiring  $S$ ), i.e. (left)  $S$ -modules  $A$  together with a monoid structure  $(A, *, 1_A)$  that is *bilinear*, i.e. an  $S$ -module morphism in both of its arguments separately. We write  $(A, +, 0_A)$  for the commutative monoid and  $(s, x) \mapsto s.x$  for every  $s \in S$  and  $x \in A$  for the action of the semiring  $S$  on  $A$  which together form the module structure on  $A$ . Note that  $S$  itself is an  $S$ -algebra where the scalar and monoid multiplication are just the semiring multiplication of  $S$ . Moreover, for every  $S$ -algebra  $A$  there is the  $S$ -algebra morphism

$$i: S \rightarrow A \quad \text{with} \quad i(s) = s.1_A. \quad (7)$$

The algebras for  $T$  are  $\Sigma$ -pointed  $S$ -algebras. The following notions are special instances of  $S$ -algebras:

**Example 6.9.** (1) Idempotent semirings for  $S = \mathbb{B} = \{0, 1\}$  the Boolean semiring.

(2) Semirings for  $S = \mathbb{N}$  the semiring of natural numbers (with the usual addition and multiplication).

(3) Rings for  $S = \mathbb{Z}$  the semiring of integers (again with the usual addition and multiplication).

Winter et al. [57, Proposition 4] show that the final  $H$ -coalgebra is carried by  $S\langle\langle \Sigma \rangle\rangle$  and that constructively  $S$ -algebraic series are precisely those elements of  $S\langle\langle \Sigma \rangle\rangle$  that arise as the behaviours of finite coalgebras  $c: X \rightarrow HS\langle X \rangle$ , after determinizing them to some  $\hat{c}: S\langle X \rangle \rightarrow HS\langle X \rangle$  (see [57, Theorem 23]).

However, this determinization is not directly an instance of the generalized powerset construction. We shall show that the same behaviours can be obtained by using the standard generalized powerset construction with an appropriate lifting of  $H$  to the category of  $T$ -algebras.

### 6.5. A Lifting of $S \times (-)^\Sigma$ to $S$ -algebras

Let  $\Sigma$  be a fixed input alphabet. Given an  $S$ -algebra structure on  $A$  and a  $\Sigma$ -pointing  $j: \Sigma \rightarrow A$ , we will define an  $S$ -algebra structure and  $\Sigma$ -pointing

Structure	Operation	in $S$	in $A^\Sigma$
$S$ -Module	0	$0_S$	$\sigma \mapsto 0_A$
	$(o_1, \delta_1) + (o_2, \delta_2)$	$o_1 + o_2$	$\sigma \mapsto \delta_1(\sigma) + \delta_2(\sigma)$
	$s.(o_1, \delta_1)$	$s \cdot o_1$	$\sigma \mapsto s.\delta_1(\sigma)$
Monoid	1	$1_S$	$\sigma \mapsto 0_A$
	$(o_1, \delta_1) * (o_2, \delta_2)$	$o_1 \cdot o_2$	$\sigma \mapsto \delta_1(\sigma) * [o_2, \delta_2] + i(o_1) * \delta_2(\sigma)$
$\Sigma$ -pointing	$\sigma \in \Sigma$	$0_S$	$\chi_\sigma : \sigma \mapsto 1_A, \quad \tau \mapsto 0_A, \tau \neq \sigma$

Table 1:  $S\langle - + \Sigma \rangle$ -algebra structure on  $S \times A^\Sigma$

on  $HA = S \times A^\Sigma$ . While the  $S$ -module structure is given by the usual componentwise operations on the product, a bit of care is needed for the monoid multiplication on  $HA$ . To this end we first define the operation  $[-]: S \times A^\Sigma \rightarrow A$  by

$$[o, \delta] := i(o) + \sum_{\tau \in \Sigma} (j(\tau) * \delta(\tau)),$$

where  $i: S \rightarrow A$  is the morphism from (7). The idea is that  $[o, \delta]$  acts like a state with output  $o$  and ‘next states’  $\delta$ .

Table 1 shows the definition of the  $S$ -algebra operations and  $\Sigma$ -pointing on  $HA$  (given separately on the product components  $S$  and  $A^\Sigma$ ). Since these operations only make use of the operations from  $S$  (seen as an  $S$ -algebra), the  $S$ -algebra  $A$  and its  $\Sigma$ -pointing  $j$ , we immediately see that for every  $S\langle - + \Sigma \rangle$ -algebra morphism  $h: A \rightarrow B$ , the morphism  $Hh = \text{id}_S \times h^\Sigma : S \times A^\Sigma \rightarrow S \times B^\Sigma$  is an  $S\langle - + \Sigma \rangle$ -algebra morphism again.

Thus, in order to see that we have defined a lifting  $H^T$  of  $H$  it suffices to prove that  $HA$  with the operations defined in Table 1 is an  $S\langle - + \Sigma \rangle$ -algebra. To this end we first prove first that  $[-]: S \times A^\Sigma \rightarrow A$  is an  $S\langle - + \Sigma \rangle$ -algebra morphism.

**Lemma 6.10.** *The map  $[-]: S \times A^\Sigma \rightarrow A$  preserves the operations defined in Table 1.*

*Proof.* First, we show that  $[-]$  preserves the  $S$ -module operations:

$$(1) \text{ Zero: } [0_S, \sigma \mapsto 0_A] = i(0_S) + \sum_{\tau \in \Sigma} j(\tau) \cdot 0_A = 0_A + \sum_{\tau \in \Sigma} 0_A = 0_A.$$

$$(2) \text{ Addition: } \begin{aligned} [o_1 + o_2, \sigma \mapsto \delta_1(\sigma) + \delta_2(\sigma)] \\ &= i(o_1 + o_2) + \sum_{\tau \in \Sigma} (j(\tau) * (\delta_1(\sigma) + \delta_2(\sigma))) \\ &= i(o_1) + i(o_2) + \sum_{\tau \in \Sigma} (j(\tau) * \delta_1(\sigma)) + \sum_{\tau \in \Sigma} (j(\tau) * \delta_2(\sigma)) \\ &= [o_1, \delta_1] + [o_2, \delta_2]. \end{aligned}$$

(3) Scalar multiplication:

$$\begin{aligned} [s \cdot o, \sigma \mapsto s \cdot \delta(\sigma)] &= i(s \cdot o) + \sum_{\tau \in \Sigma} (j(\tau) * (s \cdot \delta(\tau))) \\ &= s \cdot i(o) + \sum_{\tau \in \Sigma} s \cdot (j(\tau) * \delta(\tau)) = s \cdot \left( i(o) + \sum_{\tau \in \Sigma} (j(\tau) * \delta(\tau)) \right) = s \cdot [o, \delta]. \end{aligned}$$

Now note that for every  $s \in S$  and  $x \in A$  we have

$$i(s) * x = x * i(s), \quad (8)$$

because we can compute as follows:

$$i(s) * x = (s \cdot 1_A) * x = s \cdot (1_A * x) = s \cdot x = s \cdot (x * 1_A) = x * (s \cdot 1_A) = x * i(s).$$

Using (8), we see that the monoid operations are preserved by  $[-]$ :

$$(1) \text{ One: } [1_S, \sigma \mapsto 0_A] = i(1_S) + \sum_{\tau \in \Sigma} j(\tau) * 0_A = i(1_S) = 1_A.$$

(2) Multiplication:

$$\begin{aligned} &[o_1, \delta_1] * [o_2, \delta_2] \\ &= \left( i(o_1) + \sum_{\tau \in \Sigma} j(\tau) * \delta_1(\tau) \right) * [o_2, \delta_2] \\ &= i(o_1) * [o_2, \delta_2] + \sum_{\tau \in \Sigma} j(\tau) * \delta_1(\tau) * [o_2, \delta_2] \\ &= i(o_1) * \left( i(o_2) + \sum_{\tau \in \Sigma} j(\tau) * \delta_2(\tau) \right) + \sum_{\tau \in \Sigma} j(\tau) * \delta_1(\tau) * [o_2, \delta_2] \\ &\stackrel{(8)}{=} i(o_1 \cdot o_2) + \sum_{\tau \in \Sigma} j(\tau) * i(o_1) * \delta_2(\tau) + \sum_{\tau \in \Sigma} j(\tau) * \delta_1(\tau) * [o_2, \delta_2] \end{aligned}$$

$$\begin{aligned}
&= i(o_1 \cdot o_2) + \sum_{\tau \in \Sigma} j(\tau) * (i(o_1) * \delta_2(\tau) + \delta_1(\tau) * [o_2, \delta_2]) \\
&= [o_1 \cdot o_2, \sigma \mapsto i(o_1) * \delta_2(\sigma) + \delta_1(\sigma) * [o_2, \delta_2]] = [(o_1, \delta_1) * (o_2, \delta_2)].
\end{aligned}$$

Finally, the  $\Sigma$ -pointing is also preserved:

$$[0_S, \chi_\sigma] = i(0_S) + \sum_{\tau \in \Sigma} (j(\tau) * \chi_\sigma(\tau)) = j(\sigma) * 1_A = j(\sigma). \quad \square$$

**Lemma 6.11.** *For every  $S\langle - + \Sigma \rangle$ -algebra  $A$ ,  $HA = S \times A^\Sigma$  equipped with the operations in Table 1 is an  $S\langle - + \Sigma \rangle$ -algebra.*

*Proof.* Recall from Example 6.2 that an  $S\langle - + \Sigma \rangle$ -algebra is an  $S\langle - \rangle$ -algebra together with a  $\Sigma$ -pointing. Thus, it suffices to prove that  $HA$  is an associative  $S$ -algebra. It is clear that  $HA$  satisfies the axioms of an  $S$ -module because the  $S$ -module operations in Table 1 are just the usual coordinatewise operations on the product.

Next we prove that  $(HA, 1, *)$  is a monoid.

Unit axioms:

$$\begin{aligned}
(1_S, \sigma \mapsto 0_A) * (o, \delta) &= (1_S \cdot o, \sigma \mapsto 0_A * [o, \delta] + i(1_S) * \delta(\sigma)) = (o, \sigma \mapsto \delta(\sigma)), \\
(o, \delta) * (1_S, \sigma \mapsto 0_A) &= (o \cdot 1_S, \sigma \mapsto \delta(\sigma) * [1_S, \sigma \mapsto 0_A] + i(o) * 0_A) \\
&= (o, \sigma \mapsto \delta(\sigma) * 1_A + 0_A) = (o, \delta)
\end{aligned}$$

Associativity:

$$\begin{aligned}
&((o_1, \delta_1) * (o_2, \delta_2)) * (o_3, \delta_3) \\
&= (o_1 \cdot o_2, \sigma \mapsto \delta_1(\sigma) * [o_2, \delta_2] + i(o_1) * \delta_2(\sigma)) * (o_3, \delta_3) \\
&= (o_1 \cdot o_2 \cdot o_3, \sigma \mapsto \\
&\quad (\delta_1(\sigma) * [o_2, \delta_2] + i(o_1) * \delta_2(\sigma)) * [o_3, \delta_3] + i(o_1 \cdot o_2) * \delta_3(\sigma)) \\
&= (o_1 \cdot o_2 \cdot o_3, \sigma \mapsto \\
&\quad \delta_1(\sigma) * [o_2, \delta_2] * [o_3, \delta_3] + i(o_1) * \delta_2(\sigma) * [o_3, \delta_3] + i(o_1 \cdot o_2) * \delta_3(\sigma)) \\
&= (o_1 \cdot o_2 \cdot o_3, \sigma \mapsto \\
&\quad \delta_1(\sigma) * [(o_2, \delta_2) * (o_3, \delta_3)] + i(o_1) * (\delta_2(\sigma) * [o_3, \delta_3] + i(o_2) * \delta_3(\sigma))) \\
&= (o_1, \delta_1) * ((o_2, \delta_2) * (o_3, \delta_3)).
\end{aligned}$$

So  $S \times A^\Sigma$  is both a monoid and a  $S$ -module. We still need to establish the bilinearity of  $*$  with respect to the  $S$ -Module structure. For linearity of  $*$  in the first argument, we use the same property in  $A$ :

$$\begin{aligned}
& ((o_1, \delta_1) + (o_2, \delta_2)) * (o_3, \delta_3) \\
&= (o_1 + o_2, \sigma \mapsto \delta_1(\sigma) + \delta_2(\sigma)) * (o_3, \delta_3) \\
&= ((o_1 + o_2) \cdot o_3, \sigma \mapsto (\delta_1(\sigma) + \delta_2(\sigma)) * [o_3, \delta_3] + i(o_1 + o_2) * \delta_3(\sigma)) \\
&= (o_1 \cdot o_3 + o_2 \cdot o_3, \sigma \mapsto \\
&\quad \delta_1(\sigma) * [o_3, \delta_3] + \delta_2(\sigma) * [o_3, \delta_3] + (i(o_1) + i(o_2)) * \delta_3(\sigma)) \\
&= (o_1, \delta_1) * (o_3, \delta_3) + (o_2, \delta_2) * (o_3, \delta_3),
\end{aligned}$$

$$\begin{aligned}
(0_S, \sigma \mapsto 0_A) * (o, \delta) &= (0_S \cdot o, \sigma \mapsto 0_A * [o, \delta] + i(0_S) * \delta(\sigma)) \\
&= (0_S \cdot o, \sigma \mapsto 0_A * [o, \delta] + 0_A * \delta(\sigma)) = (0_S, \sigma \mapsto 0_A),
\end{aligned}$$

$$\begin{aligned}
(s \cdot (o_1, \delta_1)) * (o_2, \delta_2) &= (s \cdot o_1 \cdot o_2, \sigma \mapsto s \cdot \delta_1(\sigma) * [o_2, \delta_2] + i(s \cdot o_1) * \delta_2(\sigma)) \\
&= (s \cdot o_1 \cdot o_2, \sigma \mapsto s \cdot (\delta_1(\sigma) * [o_2, \delta_2] + i(o_1) * \delta_2(\sigma))) \\
&= s \cdot ((o_1, \delta_1) * (o_2, \delta_2)).
\end{aligned}$$

Finally, linearity in the second argument of  $*$  is proved using the identities for  $[-]$ :

$$\begin{aligned}
& (o_1, \delta_1) * ((o_2, \delta_2) + (o_3, \delta_3)) \\
&= (o_1 \cdot (o_2 + o_3), \sigma \mapsto \delta_1(\sigma) * [o_2 + o_3, \sigma \mapsto \delta_2(\sigma) + \delta_3(\sigma)] \\
&\quad + i(o_1) * (\delta_2(\sigma) + \delta_3(\sigma))) \\
&= (o_1 \cdot o_2 + o_1 \cdot o_3, \sigma \mapsto \delta_1(\sigma) * [o_2, \delta_2] + \delta_1(\sigma) * [o_3, \delta_3] \\
&\quad + i(o_1) * \delta_2(\sigma) + i(o_1) * \delta_3(\sigma)) \\
&= (o_1, \delta_1) * (o_2, \delta_2) + (o_1, \delta_1) * (o_3, \delta_3),
\end{aligned}$$

$$\begin{aligned}
(o, \delta) * (0_S, \sigma \mapsto 0_A) &= (o \cdot 0_S, \sigma \mapsto \delta(\sigma) * [0_S, \sigma \mapsto 0_A] + i(o) * 0_A) \\
&= (o \cdot 0_S, \sigma \mapsto \delta(\sigma) * 0_A + 0_A) = (0_S, \sigma \mapsto 0_A),
\end{aligned}$$

$$\begin{aligned}
& (o_1, \delta_1) * (s \cdot (o_2, \delta_2)) = (o_1, \delta_1) * (s \cdot o_2, \sigma \mapsto s \cdot \delta_2(\sigma)) \\
&= (o_1 \cdot (s \cdot o_2), \delta_1(\sigma) * [s \cdot o_2, \sigma \mapsto s \cdot \delta_2(\sigma)] + i(o_1) * (s \cdot \delta_2(\sigma))) \\
&= (o_1 \cdot (s \cdot o_2), \delta_1(\sigma) * (s \cdot [o_2, \delta_2]) + i(o_1) * (s \cdot \delta_2(\sigma))) \\
&= (s \cdot (o_1 \cdot o_2), s \cdot (\delta_1(\sigma) * [o_2, \delta_2]) + s \cdot (i(o_1) * \delta_2(\sigma))) = s \cdot ((o_1, \delta_1) * (o_2, \delta_2)).
\end{aligned}$$

This completes the proof.  $\square$

We now prove that applying  $[-]$  does not change the behaviour of states after unfolding them using the coalgebra structure.

**Lemma 6.12.** *Let  $c: A \rightarrow H^T A$  be a coalgebra in  $\mathbf{Set}^T$ , and let  $w \in A$ . Then  $w$  and  $[c(w)]$  are behaviourally equivalent w.r.t.  $H$  on  $\mathbf{Set}$ .*

*Proof.* We show that  $h = [c(-)]: A \rightarrow A$  is an  $H$ -coalgebra homomorphism. This implies that  $c^\ddagger = c^\ddagger \cdot h$ , for which we obtain the desired result:  $c^\ddagger(w) = c^\ddagger \cdot h(w) = c^\ddagger([c(w)])$ .

First we use that  $c = \langle o, \delta \rangle: A \rightarrow S \times A^\Sigma$  is an  $S\langle - + \Sigma \rangle$ -algebra morphism to see that for every  $s \in S$  we have that

$$c(i(s)) = c(s.1_A) = s.(c(1_A)) = s.(1_S, \sigma \mapsto 0_A) = (s, \sigma \mapsto 0_A). \quad (9)$$

Furthermore, for every  $\tau \in \Sigma$  and  $v \in A$  we prove that

$$c(j(\tau)) * c(v) = (0_S, \sigma \mapsto \chi_\tau(\sigma) * [c(v)]), \quad (10)$$

where recall that  $\chi_\tau: \Sigma \rightarrow A$  with  $\chi_\tau(\tau) = 1$  and  $\chi_\tau(\sigma) = 0$  for  $\sigma \neq \tau$ . Indeed, we compute

$$\begin{aligned} c(j(\tau)) * c(v) &= (0_S, \chi_\tau) * c(v) = (0_S, \chi_\tau) * (o(v), \delta(v)) \\ &= (0_S \cdot o(v), \sigma \mapsto \chi_\tau(\sigma) * [o(v), \delta(v)] + i(0_S) * \delta(v)(\sigma)) \\ &= (0_S, \sigma \mapsto \chi_\tau(\sigma) * [o(v), \delta(v)]) \\ &= (0_S, \sigma \mapsto \chi_\tau(\sigma) * [c(v)]). \end{aligned}$$

We now prove that  $h$  is a coalgebra homomorphism. Let  $w \in A$  and compute:

$$\begin{aligned}
c \cdot h(w) &= c([c(w)]) = c([o(w), \delta(w)]) \\
&= c \left( i(o(w)) + \sum_{\tau \in \Sigma} (j(\tau) * \delta(w)(\tau)) \right) \\
&= c(i(o(w))) + \sum_{\tau \in \Sigma} c(j(\tau)) * c(\delta(w)(\tau)) \\
&\stackrel{(9),(10)}{=} (o(w), \sigma \mapsto 0_A) + \sum_{\tau \in \Sigma} (0_S, \sigma \mapsto \chi_\tau(\sigma) * [c(\delta(w)(\tau))]) \\
&= (o(w), \sigma \mapsto 0_A) + \left( 0_S, \sigma \mapsto \sum_{\tau \in \Sigma} \chi_\tau(\sigma) * [c(\delta(w)(\tau))] \right) \\
&= (o(w), \sigma \mapsto 0_A) + (0_S, \sigma \mapsto [c(\delta(w)(\sigma))]) \\
&= (o(w), \sigma \mapsto [c(\delta(w)(\sigma))]) \\
&= (\text{id}_S \times h^\Sigma)(o(w), \delta(w)) \\
&= Hh \cdot c(w).
\end{aligned}$$

This completes the proof.  $\square$

Given a coalgebra  $c: X \rightarrow HS\langle X \rangle$ , Winter et al. [57, Section 4] determine  $c$  to the coalgebra  $\hat{c} = \langle \hat{o}, \hat{\delta} \rangle: S\langle X \rangle \rightarrow HS\langle X \rangle$  defined as follows: first, one extends  $\langle o, \delta \rangle$  to  $\langle \bar{o}, \bar{\delta} \rangle: X^* \rightarrow HS\langle X \rangle$  by the following inductive definition (in the following we will often write  $\delta$  and its relatives in uncurried form):

$$\begin{aligned}
\bar{o}(\epsilon) &= 1 & \bar{\delta}(\epsilon, \sigma) &= 0 \\
\bar{o}(xu) &= o(x) \cdot \bar{o}(u) & \bar{\delta}(xu, \sigma) &= \delta(x, \sigma) * u + i(o(x)) * \bar{\delta}(u, \sigma),
\end{aligned}$$

where  $x \in X$ ,  $u \in X^*$ ,  $\sigma \in \Sigma$ , and  $i: S \rightarrow S\langle X \rangle$ . Second, one uses that  $HS\langle X \rangle = S \times S\langle X \rangle^A$  is an  $S$ -module with the usual coordinatewise structure on the product and freely extends  $\langle \bar{o}, \bar{\delta} \rangle$  to the free  $S$ -module  $S_\omega^{(X^*)} = S\langle X \rangle$  on  $X^*$  to obtain  $\hat{c} = \langle \hat{o}, \hat{\delta} \rangle$ . It follows that  $\hat{c}$  is an  $S$ -module homomorphism, and moreover it is shown in *loc. cit.* that for every  $v, w \in S\langle X \rangle$ :

$$\hat{o}(v * w) = \hat{o}(v) \cdot \hat{o}(w) \text{ and } \hat{\delta}(v * w, \sigma) = \hat{\delta}(v, \sigma) * w + i(\hat{o}(v)) * \hat{\delta}(w, \sigma). \quad (11)$$

However, for the given coalgebra  $(X, c)$  we may also form the coalgebra

$$X \xrightarrow{c} HS\langle X \rangle \xrightarrow{S\langle \text{inl} \rangle} HS\langle X + \Sigma \rangle$$

and obtain (abusing notation slightly) a coalgebra  $c^\sharp: S\langle X + \Sigma \rangle \rightarrow HS\langle X + \Sigma \rangle$  by performing the generalized powerset construction w.r.t.  $T$ .

In Lemma 6.14, we show that the property (11) together with Lemma 6.12 and the definition of  $*$  imply that  $\hat{c}$  and  $c^\sharp$  are essentially the same coalgebra structures.

**Remark 6.13.** Recall from Section 2.3 the notion of behavioural equivalence. One way to establish behavioural equivalence of two states is via a bisimulation. For a set functor  $H$ , a bisimulation between two  $H$ -coalgebras  $(C, c)$  and  $(D, d)$  is a relation  $R \subseteq C \times D$  such that  $R$  carries a coalgebra structure  $r: R \rightarrow HR$  such that the two projections maps  $\pi_0: R \rightarrow C$  and  $\pi_1: R \rightarrow D$  are coalgebra homomorphisms. The greatest bisimulation on a given coalgebra is called *bisimilarity*.

Whenever two states  $x \in C$  and  $y \in D$  are *bisimilar*, i.e. contained in any bisimulation  $R \subseteq C \times D$ , then they are behaviourally equivalent. The converse holds for every functor  $H$  preserving weak pullbacks.

In Lemma 6.14, we will actually use a more refined bisimulation proof method, namely, bisimulation up to behavioural equivalence. Up-to-techniques such as this one have been studied by Rot et al. [49]. Here one considers a function  $f: \mathcal{P}(C \times D) \rightarrow \mathcal{P}(C \times D)$ , and a *bisimulation up to  $f$*  is a relation  $R \subseteq C \times D$  such that there is a map  $r: R \rightarrow H(f(R))$  making the following diagram commutative:

$$\begin{array}{ccccc}
 C & \xleftarrow{\pi_0} & R & \xrightarrow{\pi_1} & D \\
 \downarrow c & & \downarrow r & & \downarrow d \\
 HC & \xleftarrow{H\pi_0} & H(f(R)) & \xrightarrow{H\pi_1} & HD
 \end{array}$$

We are interested in the function  $f$  defined by  $f(R) = \sim R \sim$ , where  $\sim$  denotes the behavioural equivalences on  $C$  and  $D$ , respectively. Let us spell out the meaning of the above diagram for the case of  $HX = S \times X^\Sigma$  on  $\mathbf{Set}$ . Given two coalgebras  $\langle o, \delta \rangle: X \rightarrow S \times X^\Sigma$  and  $\langle o', \delta' \rangle: X' \rightarrow S \times (X')^\Sigma$  a bisimulation up to behavioural equivalence is a relation  $R \subseteq X \times X'$  such that for all  $x R x'$  we have

$$o(x) = o'(x') \quad \text{and} \quad \delta(x, \sigma) \sim R \sim \delta'(x', \sigma) \text{ for every } \sigma \in \Sigma, \quad (12)$$

with  $\delta$  and  $\delta'$  written in their uncurried form.

It follows from the results of Rot et al. [49] that whenever two states  $x \in C$  and  $y \in D$  are contained in a bisimulation up to behavioural equivalence then they are behaviourally equivalent.



**Lemma 6.14.** *For every coalgebra  $c: X \rightarrow HS\langle X \rangle$  in **Set**,  $u$  in  $(S\langle X \rangle, \hat{c})$  and  $S\langle \text{inl} \rangle(u)$  in  $(S\langle X + \Sigma \rangle, c^\sharp)$  are behaviourally equivalent.*

*Proof.* We use that the free  $S$ -algebra  $S\langle X \rangle$  is a quotient of the algebra  $MX$  of terms for the signature of  $S$ -algebras via the surjective map  $q_X: MX \rightarrow S\langle X \rangle$ . Similarly, we have  $q_{X+\Sigma}: M(X + \Sigma) \rightarrow S\langle X + \Sigma \rangle$ . In fact,  $q: M \rightarrow S\langle - \rangle$  is a natural transformation (even a monad morphism), and therefore we have the following commutative square:

$$\begin{array}{ccc} MX & \xrightarrow{q_X} & S\langle X \rangle \\ \downarrow M\text{inl} & & \downarrow S\langle \text{inl} \rangle \\ M(X + \Sigma) & \xrightarrow{q_{X+\Sigma}} & S\langle X + \Sigma \rangle \end{array}$$

Note that  $M\text{inl}$  is the embedding of terms over  $X$  into the terms over  $X + \Sigma$ . We now prove that for every term  $t \in MX$ , its equivalence classes in  $S\langle X \rangle$  and  $S\langle X + \Sigma \rangle$  are behavioural equivalent. This is done by showing that the relation

$$R = \{(q_X(t), S\langle \text{inl} \rangle \cdot q_{X+\Sigma}(t)) \mid t \in MX\} \subseteq S\langle X \rangle \times S\langle X + \Sigma \rangle$$

is a bisimulation up to behavioural equivalence (see Remark 6.13). We will abuse notation and often denote the equivalence class of a term  $t \in MX$  in  $S\langle X \rangle$  or  $S\langle X + \Sigma \rangle$  by  $t$  again. Put  $c = \langle o, \delta \rangle$  and  $c^\sharp = \langle o^\sharp, \delta^\sharp \rangle$ .

Note first that  $R$  is nothing but the map  $S\langle \text{inl} \rangle: S\langle X \rangle \rightarrow S\langle X + \Sigma \rangle$  considered as a relation. Hence, since this map is a morphism of  $S$ -algebras we have that  $R$  is a congruence (w.r.t.  $S$ -algebra operations). Now we proceed by induction over the terms  $t \in MX$ :

(1) *Base Case:* For every  $x \in X$ , we have that  $\hat{o}(x) = o(x) = o^\sharp(x)$  and  $\hat{\delta}(x, \sigma) = \delta(x, \sigma)$  whereas  $\delta^\sharp(x, \sigma) = S\langle \text{inl} \rangle(\delta(x, \sigma))$ . Thus, (12) holds for  $t = x$ .

(2) *Induction step for the  $S$ -module structure:* The definition of  $\hat{c} = \langle \hat{o}, \hat{\delta} \rangle$  on  $S$ -Module operations is coordinatewise [57, Sect. 3+4] and thus identical to the definition of  $c^\sharp = \langle o^\sharp, \delta^\sharp \rangle$ . Hence, (12) holds for  $t = t_1 + t_2$ ,  $t = 0$ , and  $t = s.t'$  for every  $s \in S$ .

(3) *Induction step for the monoid structure:* The neutral element is mapped by  $\hat{c}$  to  $(1, \sigma \mapsto 0)$  [57, Sect. 4], and this is identical to the definition  $c^\sharp$ . Thus, (12) holds for  $t = 1$ .

Now suppose that  $v, w \in S\langle X \rangle$  and  $v', w' \in S\langle X + \Sigma \rangle$ , and assume that  $v R v', w R w'$ . By induction hypothesis, we have for every  $\sigma \in \Sigma$ ,

$$\begin{aligned}\hat{o}(v) &= o^\sharp(v'), & \hat{\delta}(v, \sigma) &\sim R \sim \delta^\sharp(v', \sigma), \\ \hat{o}(w) &= o^\sharp(w'), & \hat{\delta}(w, \sigma) &\sim R \sim \delta^\sharp(w', \sigma).\end{aligned}$$

Then we clearly have, using the induction hypothesis in the second step and the definition of  $*$  in the last one, that

$$\hat{o}(v * w) \stackrel{(11)}{=} \hat{o}(v) \cdot \hat{o}(w) \stackrel{\text{IH}}{=} o^\sharp(v') \cdot o^\sharp(w') = o^\sharp(v' * w').$$

Moreover, for every  $\sigma \in \Sigma$  we have

$$\begin{aligned}\hat{\delta}(v * w, \sigma) &\stackrel{(11)}{=} \hat{\delta}(v, \sigma) * w + i(\hat{o}(v)) * \hat{\delta}(w, \sigma) \\ &\sim R \sim \delta^\sharp(v', \sigma) * w' + i(o^\sharp(v')) * \delta^\sharp(w', \sigma) \\ &\stackrel{\text{Lemma 6.12}}{\sim} \delta^\sharp(v', \sigma) * [o^\sharp(w'), \delta^\sharp(w')] + i(o^\sharp(v')) * \delta^\sharp(w', \sigma) \\ &= \delta^\sharp(v' * w', \sigma),\end{aligned}$$

where the second step uses the induction hypothesis as well as the fact that  $R$  and  $\sim$  are congruences of  $S$ -algebras, and the last equation uses the definition of  $*$  again. Thus, (12) holds for  $t = t_1 * t_2$ , which completes the proof.  $\square$

**Corollary 6.15.** *The locally finite fixpoint  $\vartheta H^T$  is carried by the set of all constructively  $S$ -algebraic power-series.*

*Proof.* From Lemma 6.14 we conclude that  $\hat{c}^\ddagger = c^{\sharp\sharp} \cdot S\langle \text{inl} \rangle$  and thus their images in  $\nu H$  are identical.

By [57, Theorem 23], a formal power series is constructively  $S$ -algebraic if and only if it is in the image of some

$$\hat{c}^\ddagger \cdot \eta_X = c^{\sharp\sharp} \cdot S\langle \text{inl} \rangle \cdot \eta_X = c^{\sharp\sharp} \cdot \eta_X^T,$$

where  $X$  is finite and  $\eta_X: X \rightarrow S\langle X \rangle$  is the unit of the monad  $S\langle - \rangle$ .

The desired result now follows by (6).  $\square$

## 6.6. Courcelle's Algebraic Trees

For a fixed signature  $\Sigma$  of so called *givens*, a *recursive program scheme* (or *rps*, for short) contains mutually recursive definitions of new operations  $\varphi_1, \dots, \varphi_k$  (with respective arities  $n_1, \dots, n_k$ ). The recursive definition of  $\varphi_i$  may involve symbols from  $\Sigma$ , operations  $\varphi_1, \dots, \varphi_k$  and  $n_i$  variables  $x_1, \dots, x_{n_i}$ .

The (uninterpreted) solution of an rps is obtained by unfolding these recursive definitions, producing a possibly infinite  $\Sigma$ -tree over  $x_1, \dots, x_{n_i}$  for each operation  $\varphi_i$ . The following example shows an rps over the signature  $\Sigma = \{\star/0, \times/2, +/2\}$  and its solution:

$$\varphi(z) = z + \varphi(\star \times z)$$

In general, an *algebraic  $\Sigma$ -tree* is a  $\Sigma$ -tree which is definable by an rps over  $\Sigma$  (see Courcelle [22]). Generalizing from a signature to a finitary endofunctor  $H: \mathcal{C} \rightarrow \mathcal{C}$  on an lfp category, Adámek et al. [9] describe an rps as a coalgebra for a functor  $\mathcal{H}_f$  on the category  $H/\mathbf{Mnd}_f(\mathcal{C})$  whose objects are finitary  $H$ -pointed monads on  $\mathcal{C}$ , i.e. finitary monads  $M$  together with a natural transformation  $H \rightarrow M$ . They introduce the *context-free monad*  $C^H$  of  $H$ , which is an  $H$ -pointed monad that is a subcoalgebra of the final coalgebra for  $\mathcal{H}_f$  and which is the monad of Courcelle’s algebraic  $\Sigma$ -trees in the special case where  $\mathcal{C} = \mathbf{Set}$  and  $H$  is the polynomial functor associated to the signature  $\Sigma$ . We will now prove that this monad is the LFF of  $\mathcal{H}_f$ , and thereby we characterize it by a universal property; this solves an open problem in [9].

The setting is again an instance of the generalized powerset construction, but this time with the category of finitary endofunctors on  $\mathcal{C}$  as the base category in lieu of  $\mathbf{Set}$ .

**Assumption 6.16.** We assume that  $\mathcal{C}$  is an lfp category in which the co-product injections are monic and a coproduct of two monos is also monic. Moreover,  $H: \mathcal{C} \rightarrow \mathcal{C}$  is a finitary mono-preserving endofunctor.

Denote by  $\mathbf{Fun}_f(\mathcal{C})$  the category of finitary endofunctors on  $\mathcal{C}$ , which is an lfp category (see [12]).

Then  $H$  induces an endofunctor  $H \cdot (-) + \text{Id}$  on  $\mathbf{Fun}_f(\mathcal{C})$ , denoted  $\dot{H}$  and mapping an endofunctor  $V$  to the functor  $X \mapsto HVX + X$ . This functor  $\dot{H}$  gets precomposed with a monad on  $\mathbf{Fun}_f(\mathcal{C})$  as we now explain.

**Proposition 6.17** (Free monad, [2, 14]). *For every object  $X$  of  $\mathcal{C}$  there exists a free  $H$ -algebra  $F^H X$  on  $X$ . Moreover, the object assignment  $X \mapsto F^H X$  gives rise to a finitary monad on  $\mathcal{C}$ , and this monad is the free monad on  $H$ .*

For example, if  $H$  is the polynomial functor associated to a signature  $\Sigma$ , then  $F^H X$  is the usual term algebra that contains all finite  $\Sigma$ -trees over the set of generators  $X$ . Proposition 6.17 implies that  $H \mapsto F^H$  is the object assignment of a monad on  $\mathbf{Fun}_f(\mathcal{C})$ . Moreover, it is not difficult to show, using Beck's theorem (see e.g. [37]), that the Eilenberg-Moore category of this monad is  $\mathbf{Mnd}_f(\mathcal{C})$ , the category of finitary monads on  $\mathcal{C}$ . In addition, we have the following

**Lemma 6.18.** *The monad  $H \mapsto F^H$  is finitary.*

*Proof.* Note that for every finitary functor  $H: \mathcal{C} \rightarrow \mathcal{C}$ ,  $- \cdot H$  on  $\mathbf{Fun}_f(\mathcal{C})$  preserves all colimits, and  $H \cdot -$  preserves filtered colimits. It follows from Kelly's result [34, Theorem 23.3] that the free monad  $F^H$  on the finitary functor  $H$  is the initial algebra for  $\dot{H} = H \cdot (-) + \text{Id}$  on  $\mathbf{Fun}_f(\mathcal{C})$ . This initial algebra can be constructed as the colimit of the chain of the functors  $H_i$ ,  $i < \omega$ , where  $H_0 = \text{Id}$  and  $H^{i+1} = H \cdot H^i + \text{Id}$ . It follows that the monad  $H \mapsto F^H$  is finitary.  $\square$

As a consequence, we see that  $\mathbf{Mnd}_f(\mathcal{C})$  is an lfp category (see [12, Remark 2.78]).

**Remark 6.19.** As shown by Adámek et al. [11, Theorem 2.16] (see also [5, Corollary 3.27]) that in  $\mathbf{Fun}(\mathbf{Set})$  fp and fg objects coincide. Moreover, every fp endofunctor on  $\mathbf{Set}$  is the quotient of the polynomial endofunctor associated to a finite signature [5, Lemma 3.23].

However, in  $\mathbf{Mnd}_f(\mathbf{Set})$  the classes of fp and fg objects differ [5, Corollary 4.12]. This means that the rational fixpoint of a finitary functor on  $\mathbf{Mnd}_f(\mathbf{Set})$  may not be fully abstract, and therefore its LFF is needed.

Let us now proceed to presenting the category and endofunctor whose LFF will turn out to be the monad of Courcelle's algebraic trees. Similarly as in the case of context-free languages, we will work with the monad  $E^{(-)} = F^{H+(-)}$  (cf. Example 6.2). Its category of Eilenberg-Moore algebras is isomorphic to the category  $H/\mathbf{Mnd}_f(\mathcal{C})$  of  $H$ -pointed finitary monads on  $\mathcal{C}$ .

Notice that this category is equivalent to a slice category: the universal property of the monad  $F^H$  states, that for every finitary monad  $B$  the natural transformations  $H \rightarrow B$  are in one-to-one correspondence with monad morphisms  $F^H \rightarrow B$ . Hence, the category  $H/\mathbf{Mnd}_f(\mathcal{C})$  of finitary  $H$ -pointed monads on  $\mathcal{C}$  is isomorphic to the slice category  $F^H/\mathbf{Mnd}_f(\mathcal{C})$ . This finishes

the description of the base category and we now lift the functor  $\dot{H}$  to this category.

Consider an  $H$ -pointed monad  $(B, \beta: H \rightarrow B) \in H/\mathbf{Mnd}_f(\mathcal{C})$ . As shown by Ghani et al. [27], the endofunctor  $H \cdot B + \text{Id}$  carries a canonical monad structure with the unit  $\text{inr}: \text{Id} \rightarrow H \cdot B + \text{Id}$  and the multiplication

$$\begin{array}{c}
(HB + \text{Id})(HB + \text{Id}) \\
\parallel \\
HB(HB + \text{Id}) + HB + \text{Id} \\
\downarrow_{HB[\mu \cdot \beta B, \eta] + HB + \text{Id}} \\
HBB + HB + \text{Id} \\
\downarrow_{[H\mu, HB] + \text{Id}} \\
HB + \text{Id},
\end{array}$$

where  $\eta: \text{Id} \rightarrow B$  and  $\mu: B \cdot B \rightarrow B$  are the unit and multiplication of the monad  $B$ . Furthermore, we have an obvious  $H$ -pointing

$$H \xrightarrow{\text{inl} \cdot H\eta} H \cdot B + \text{Id}.$$

Milius and Moss [43] proved that this defines an endofunctor on the category of  $H$ -pointed monads,  $\mathcal{H}_f: H/\mathbf{Mnd}_f(\mathcal{C}) \rightarrow H/\mathbf{Mnd}_f(\mathcal{C})$ , which is a lifting of  $\dot{H}$ . In order to verify that  $\mathcal{H}_f$  is finitary, we first need to know how filtered colimits are formed in  $H/\mathbf{Mnd}_f(\mathcal{C})$ .

It is a straightforward exercise to prove that the forgetful functor  $U: \mathbf{Mnd}_f(\mathcal{C}) \rightarrow \mathbf{Fun}_f(\mathcal{C})$  is finitary. Since  $U$  is also monadic, i.e.  $\mathbf{Mnd}_f(\mathcal{C})$  is isomorphic to the Eilenberg-Moore category for the monad  $H \mapsto F^H$  on  $\mathbf{Fun}_f(\mathcal{C})$ , we see that  $U$  creates filtered colimits.

Clearly, the canonical projection functor  $H/\mathbf{Mnd}_f(\mathcal{C}) \rightarrow \mathbf{Mnd}_f(\mathcal{C})$  creates filtered colimits, too. Therefore, filtered colimits in the slice category  $H/\mathbf{Mnd}_f(\mathcal{C})$  are formed on the level of  $\mathbf{Fun}_f(\mathcal{C})$ , i.e. objectwise. The functor  $\dot{H}$  is finitary on  $\mathbf{Fun}_f(\mathcal{C})$  and thus also its lifting  $\mathcal{H}_f$  is finitary (see Section 6.1). Hence, we see that all requirements from Assumption 3.1 are met: we have a finitary endofunctor  $\mathcal{H}_f$  on the lfp category  $H/\mathbf{Mnd}_f(\mathcal{C})$ , and by [9, Proposition 2.23] the monos in  $H/\mathbf{Mnd}_f(\mathcal{C})$  are those monad morphisms in that category whose components are monic in  $\mathcal{C}$ . Hence  $\mathcal{H}_f$  preserves monos: given any monomorphism  $m: B \rightarrow B'$  in  $H/\mathbf{Mnd}_f(\mathcal{C})$  we know that  $Hm_X$  is monic since  $H$

preserves monos and then  $Hm_X + \text{id}_X$  is monic since monos are assumed to be closed under coproduct in  $\mathcal{C}$ . Thus, by Theorem 3.8, we obtain

**Corollary 6.20.** *The functor  $\mathcal{H}_f: H/\text{Mnd}_f(\mathcal{C}) \rightarrow H/\text{Mnd}_f(\mathcal{C})$  has a locally finite fixpoint.*

**Remark 6.21.** The final  $\mathcal{H}_f$ -coalgebra is not of interest to us, but that of a related functor is.  $\mathcal{H}_f$  generalizes to a functor  $\mathcal{H}: H/\text{Mnd}_c(\mathcal{C}) \rightarrow H/\text{Mnd}_c(\mathcal{C})$  on  $H$ -pointed countably accessible<sup>3</sup> monads. For every object  $X \in \mathcal{C}$ , the finitary endofunctor given by  $X \mapsto HX + X$  has a final coalgebra  $TX$ . Then  $X \mapsto TX$  is the object assignment of a monad [1], the monad  $T$  is countably accessible [9], and it carries the final  $\mathcal{H}$ -coalgebra [43].

Adámek et al. [9] characterize a (guarded) recursive program scheme as a natural transformation

$$V \rightarrow H \cdot F^{H+V} + \text{Id} \quad \text{with } V \text{ fp (in } \text{Fun}_f(\mathcal{C})),$$

or equivalently, via the generalized powerset construction w.r.t. the monad  $E^{(-)} = F^{H+(-)}$  as an  $\mathcal{H}_f$ -coalgebra

$$E^V \rightarrow \mathcal{H}_f(E^V) \quad (\text{in } H/\text{Mnd}_f(\mathcal{C})).$$

These  $\mathcal{H}_f$ -coalgebras on carriers  $E^V$  where  $V \in \text{Fun}_f(\mathcal{C})$  is fp form the full subcategory  $\text{EQ} \subseteq \text{Coalg}\mathcal{H}_f$ . *Op. cit.* provides two equivalent ways of constructing the monad of Courcelle’s algebraic trees: one works with  $\mathcal{H}_f$  for a polynomial endofunctor  $H_\Sigma$  on  $\mathcal{C} = \text{Set}$  and obtains the monad of algebraic  $\Sigma$ -trees

- (1) as the image of  $\text{colim EQ}$  in the final coalgebra  $T$  of Remark 6.21, and
- (2) as the colimit of  $\text{EQ}_2$ , where  $\text{EQ}_2$  is the closure of  $\text{EQ}$  under strong quotients.

We now provide a third characterization, and show that the monad of Courcelle’s algebraic trees is the locally finite fixpoint of  $\mathcal{H}_f$ .

To this end it suffices to show that  $\text{EQ}_2$  is precisely the diagram of  $\mathcal{H}_f$ -coalgebras with an fg carrier. This is established with the help of the following technical lemmas. We now assume that  $\mathcal{C} = \text{Set}$ .

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<sup>3</sup>A colimit is *countably filtered* if its diagram has for every countable subcategory a cocone. A functor is *countably accessible* if it preserves countably filtered colimits.

**Lemma 6.22.**  $\mathcal{H}_f$  maps strong epis to morphisms carried by strong epi natural transformations.

*Proof.* Strong epis in slice categories are carried by strong epis, so consider a strong epi  $q: M \twoheadrightarrow N$  in  $\mathbf{Mnd}_f(\mathbf{Set})$ . Consider the (strong epi,mono)-factorizations of the components  $q_X: MX \rightarrow NX$  in  $\mathbf{Set}$ . This yields a (strong epi, mono)-factorization of  $q$  in  $\mathbf{Fun}_f$ :

$$\begin{array}{c} \xrightarrow{\quad q \quad} \\ M \xrightarrow{e} \twoheadrightarrow I \xrightarrow{m} \rightarrow N \end{array}$$

The factorization lifts further to  $\mathbf{Mnd}_f(\mathbf{Set})$ , i.e. we have factorized the monad morphism  $q$  into an epi  $e$  and a mono  $m$  in  $\mathbf{Mnd}_f(\mathbf{Set})$ . Since every strong epi is an extremal epi (see e.g. [3]), we get that  $m$  is an isomorphism. Hence  $q$  has epic components. Every endofunctor on  $\mathbf{Set}$  preserves (strong) epis, so  $Hq_X + \text{Id}$  is epic for every set  $X$ . Therefore, so is the natural transformation  $Hq + \text{Id}$ .  $\square$

**Remark 6.23.** We recall a few properties of finitary monads and endofunctors on sets that we shall use in the proof of the next lemma.

(1) Every fg object  $B$  in  $\mathbf{Mnd}_f(\mathbf{Set})$  is the strong quotient of a free monad  $F^P$  where  $P$  is the polynomial functor associated to a finite signature. To see this, recall that the category  $\mathbf{Mnd}_f(\mathbf{Set})$  is finitary monadic over  $\mathbf{Fun}_f(\mathbf{Set})$ , i.e.  $\mathbf{Mnd}_f(\mathbf{Set})$  is (isomorphic to) the category of Eilenberg-Moore algebras for the finitary monad  $H \mapsto F^H$  on  $\mathbf{Fun}_f(\mathbf{Set})$  (cf. Lemma 6.18). By [5, Theorem 3.5], we thus have that the fg object  $B$  is a strong quotient of  $F^V$ , where  $V$  is an fp object in  $\mathbf{Fun}_f(\mathbf{Set})$ . In the latter category, the fp objects are precisely the quotients of the polynomial functors on a finite signature (see Remark 6.19). Hence, we have some polynomial functor  $P$  and strong quotient  $P \twoheadrightarrow V$  in  $\mathbf{Fun}_f(\mathbf{Set})$ . Since the left-adjoint  $F^{(-)}$  preserves strong epis we obtain the desired strong quotient in  $\mathbf{Mnd}_f(\mathbf{Set})$ :

$$F^P \twoheadrightarrow F^V \twoheadrightarrow B.$$

(2) We conclude that every  $H$ -pointed monad  $(B, \beta)$  is the strong quotient in  $H/\mathbf{Mnd}_f(\mathbf{Set})$  of  $E^P = (F^{H+P}, \kappa \cdot \text{inl})$  where  $\kappa: H + P \rightarrow F^{H+P}$  denotes the universal natural transformation of the free monad and  $P$  is a polynomial functor on a finite signature. Indeed, given  $\beta: H \rightarrow B$  take a strong quotient  $q: F^P \twoheadrightarrow B$  in  $\mathbf{Mnd}_f(\mathbf{Set})$  and the monad morphism  $m: F^H \rightarrow B$  induced by

$\beta$ . Observing that  $F^{H+P}$  is the coproduct of  $F^H$  and  $F^P$  in  $\mathbf{Mnd}_f(\mathbf{Set})$  and that copairing the strong epi  $q$  with  $m$  yields a strong epi again, we obtain the desired strong quotient  $[m, q]: (F^{H+P}, \kappa \cdot \text{inl}) \rightarrow (B, \beta)$ .

(3) Recall from Example 5.2(2) that the polynomial endofunctors on  $\mathbf{Set}$  are projective.

We obtain the following variation of Proposition 5.3:

**Lemma 6.24.** *Every  $\mathcal{H}_f$ -coalgebra  $b: (B, \beta) \rightarrow \mathcal{H}_f(B, \beta)$ , with  $B$  fg, is the strong quotient of a coalgebra from EQ.*

*Proof.* By Remark 6.23,  $(B, \beta)$  is the strong quotient of  $(F^{H+P}, \kappa \cdot \text{inl})$ , where  $P$  a polynomial functor associated to a finite signature and therefore a projective object in  $\mathbf{Fun}_f(\mathcal{C})$ . The following morphism in  $H/\mathbf{Mnd}_f(\mathbf{Set})$

$$E^P = (F^{H+P}, \kappa \cdot \text{inl}) \xrightarrow{q} (B, \beta) \xrightarrow{b} \mathcal{H}(B, \beta)$$

corresponds to a natural transformation  $\overline{b \cdot q}: P \rightarrow HB + \text{Id}$  (using that  $E^P$  is the free Eilenberg-Moore algebra on  $P$ ). Since  $P$  is projective and, by Lemma 6.22,  $\mathcal{H}_f q$  is epic as a natural transformation, we obtain a natural transformation  $p: P \rightarrow HF^{H+P} + \text{Id}$  such that the triangle on the left below commutes; equivalently, the square on the right below commutes using again the universal property of  $E^P$  as a free Eilenberg-Moore algebra:

$$\begin{array}{ccc} P \xrightarrow{p} HF^{H+P} + \text{Id} & & (F^{H+P}, \kappa \cdot \text{inl}) \xrightarrow{p^\sharp} \mathcal{H}_f(F^{H+P}, \hat{\kappa} \cdot \text{inl}) \\ \searrow_{\overline{b \cdot q}} \quad \downarrow_{Hq + \text{Id}} & \iff & \downarrow_q \quad \downarrow_{Hq + \text{Id}} \\ HB + \text{Id} & & (B, \beta) \xrightarrow{b} \mathcal{H}_f(B, \beta) \end{array}$$

Thus we see that the coalgebra  $b$  is the strong quotient of the coalgebra  $p^\sharp$ , which is a coalgebra in EQ.  $\square$

It follows from Lemma 6.24 that  $\mathbf{Coalg}_{\text{fg}} \mathcal{H}_f$  is the same category as EQ<sub>2</sub>; thus their colimits in  $\mathbf{Coalg} \mathcal{H}_f$  are isomorphic and we conclude:

**Corollary 6.25.** *Let  $H_\Sigma: \mathbf{Set} \rightarrow \mathbf{Set}$  be polynomial endofunctor on  $\mathbf{Set}$ . Then the locally finite fixpoint of  $\mathcal{H}_f: H_\Sigma/\mathbf{Mnd}_f(\mathbf{Set}) \rightarrow H_\Sigma/\mathbf{Mnd}_f(\mathbf{Set})$  is the monad of Courcelle's algebraic trees, mapping a set to the algebraic  $\Sigma$ -trees over it.*



## 7. Conclusions and Future Work

We have introduced the locally finite fixpoint of a finitary mono-preserving endofunctor on an lfp category. We proved that this fixpoint is characterized by two universal properties: it is the final lfg coalgebra and the initial fg-iterative algebra for the given endofunctor. Moreover, we have seen many instances where the LFF is the domain of behaviour of finite-state and finite-equation systems. In particular, all previously known instances of the rational fixpoint are also instances of the LFF, and we have obtained a number of interesting further instances not captured by the rational fixpoint.

On a more technical level, the LFF solves a problem that sometimes makes the rational fixpoint hard to apply. The latter identifies behaviourally equivalent states (i.e. is a subcoalgebra of the final coalgebra) if the classes of fp and fg objects coincide. This condition, however, may be false or unknown (and sometimes non-trivial to establish) in a given lfp category. But the LFF always identifies behaviourally equivalent states.

There are a number of interesting topics for further work concerning the LFF. First, it should be interesting to obtain further instances of the LFF, e.g. analyzing the behaviour of tape machines [30] might lead to a description of the recursively enumerable languages by the LFF. Second, syntactic descriptions of the LFF are of interest. In works such as [54, 52, 18, 46] Kleene type theorems and axiomatizations of the behaviour of finite systems are studied. Completeness of an axiomatization is then established by proving that expressions modulo axioms form the rational fixpoint. It is an interesting question whether the theory of the LFF we presented here may be of help as a tool for syntactic descriptions and axiomatizations of further system types.

As we have mentioned already the rational fixpoint is the starting point for the coalgebraic study of Elgot's iterative [24] and Bloom and Ésik's iteration theories [16]. A similar path could be followed based on the LFF and this should lead to new coalgebraic iteration/recursion principles, in particular in instances such as context-free languages or constructively  $S$ -algebraic formal power series.

Another approach to more powerful recursive definition principles are abstract operational rules (see [35] for an overview). It has been shown that certain rule formats define operations on the rational fixpoint [17, 42], and it should be investigated whether a similar theory can be developed based on the LFF.

Furthermore, in the special setting of Eilenberg-Moore categories, one can base the study of finite systems on *free* finitely generated algebras (rather than all fp or all fg algebras). Urbat [55] recently proved that this yields a third fixpoint  $\varphi H^T$  besides the rational fixpoint and the LFF, and Milius [40] investigated the relationship of the three fixpoints obtaining the following picture for a lifting  $H^T$  on an Eilenberg-Moore category  $\mathbf{Set}^T$ :

$$\varphi H^T \rightarrow \varrho H^T \rightarrow \vartheta H^T \rightarrow \nu H^T.$$

In addition, *op. cit.* establishes sufficient conditions when the three fixpoints on the left are isomorphic, i.e.  $\varphi H^T \cong \varrho H^T \cong \vartheta H^T$ . This is the most desired situation where the rational fixpoint is fully abstract and determined by the  $H^T$ -coalgebras  $TX$ , where  $X$  is a finite set, i.e. precisely the targets of the generalized powerset construction.

Finally, the parallelism in the technical development between rational fixpoint and LFF indicates that there should be a general theory that is parametric in a class of “finite objects” and that allows to obtain results about the rational fixpoint, the LFF and other possible “finite behaviour domains” as instances. This has also been studied by Urbat [55], and he obtains a uniform theory which yields some results about the four fixpoints above as special instances.

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