Complete Iterativity for Algebras with Effects

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Abstract. Completely iterative algebras (cias) are those algebras in which recursive equations have unique solutions. In this paper we study complete iterativity for algebras with computational effects (described by a monad). First, we prove that for every analytic endofunctor on $\mathbf{Set}$ there exists a canonical distributive law over any commutative monad M, hence a lifting of that endofunctor to the Kleisli category of M. Then, for an arbitrary distributive law λ of an endofunctor H on $\mathbf{Set}$ over a monad M we introduce λ-cias. The cias for the corresponding lifting of H (called Kleisli-cias) form a full subcategory of the category of λ-cias. For various monads of interest we prove that free Kleisli-cias coincide with free λ-cias, and these free algebras are given by free algebras for H. Finally, for three concrete examples of monads we prove that Kleisli-cias and λ-cias coincide and give a characterisation of those algebras.

Key words: iterative algebra, monad, distributive law, initial algebra, terminal coalgebra

1 Introduction

Iterative theories [3] and iterative algebras [14, 15] were introduced to study the semantics of recursive equations at a purely algebraic level without the need to use extra structure like order or metric. Iterative algebras are algebras for a signature Σ in which every guarded system of recursive equations

$$\begin{aligned} x_1 &\approx t_1(x_1, \dots, x_n, a_1 \dots, a_k) \\ &\vdots \\ x_n &\approx t_n(x_1, \dots, x_n, a_1 \dots, a_k) \end{aligned} \tag{1.1}$$

where $X = \{x_1, \dots, x_n\}$ is a set of variables, $a_1, \dots, a_k$ are elements of the algebra (called parameters) and the t_i are Σ-terms that are not single variables $x \in X$, has a unique solution. For example, the algebra $T_\Sigma Y$ of all (not necessarily finite) Σ-trees on the set Y is iterative.

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The notion of iterative algebra was extended and generalised in [1]; there, iterative algebras are studied for finitary endofunctors on $\mathbf{Set}$ (or, more generally, on a locally finitely presentable category). And in [11] completely iterative algebras (cias) for an endofunctor were introduced and studied—the completeness refers to the fact that the set of variables in a system (1.1) may be infinite.

It is the aim of this paper to investigate iterativity of algebras in which the operations have side-effects (e. g. partial operations, non-deterministic ones or composite algebras). Such effects are captured by working in the Kleisli category of a monad, see [13]. More precisely, let M be a monad on $\mathbf{Set}$. The Kleisli category of M has all sets as objects and morphisms from X to Y are maps $f : X \to MY$, which are understood as functions with a side-effect captured by M. To study algebras with effects we will consider set functors H having a lifting to the Kleisli category of the monad M. An algebra with effect is then an algebra for the lifting of H. It is well known that to have such a lifting is the same as to have a distributive law $\lambda : HM \Rightarrow MH$ of the functor H over the monad.

It has been proved by Hasuo, Jacobs and Sokolova [4] that every polynomial endofunctor on $\mathbf{Set}$ yields a canonical distributive law over any commutative monad on $\mathbf{Set}$. Here we extend this result by showing that every analytic functor, see [6, 7], has a canonical distributive law over every commutative monad (Section 2). We then study two different notions of algebras with "effectful" operations for an endofunctor H: Kleisli-cias are just the completely iterative algebras for a lifting $\bar{H}$ of H, and we introduce the notion of completely λ-iterative algebras. Whereas in Kleisli-cias recursive equations may have side-effects, in λ-cias recursion is effect-free, i. e., only parameters of recursive equations are allowed to have side-effects. We show that every Kleisli-cia is a λ-cia, but the converse does not hold in general (Section 3).

Our next result concerns free iterative algebras with effects: it often turns out that a free H-algebra on a set X is at the same time a free $\bar{H}$-algebra, a free λ-cia and a free Kleisli-cia on X. We prove this in the case where the Kleisli category is suitably cpo-enriched, see Assumption 4.1 for details. This is the setting as studied in [4], and our result is a consequence of the fact proved in loc. cit. that the initial H-algebra is a terminal $\bar{H}$-coalgebra (Section 4).

Finally, we prove that for polynomial functors and the concrete monads $(-)+1$ (the maybe monad) and $\mathbb{P}$ (the powerset monad), as well as for every endofunctor and the monad $(-)^E$ (the environment monad) Kleisli-cias and λ-cias coincide. We also give in all three cases a characterisation of λ-cias (or, equivalently, of Kleisli-cias): in the first two cases an $\bar{H}$-algebra is a λ-cia iff its carrier has a well-founded order such that each operation is "strictly increasing" (see Theorems 5.1 and 5.2 for a precise statement) and for the environment monad an $\bar{H}$-algebra is a λ-cia iff each component is a cia for H. This also implies that for the environment monad the free λ-cias (or Kleisli-cias) on the set X are given by the terminal coalgebra TX for $H(-) + X$, if it exists (Section 5).

2 Background—Distributive Laws for Set-Functors

Assumption 2.1. Throughout the paper we let M be a monad on $\mathbf{Set}$, i. e., $M : \mathbf{Set} \to \mathbf{Set}$ is an endofunctor equipped with two natural transformations $\eta : \mathrm{Id} \Rightarrow M$ (the unit) and $\mu : MM \Rightarrow M$ (the multiplication) subject to the axioms $\mu \cdot \eta M = \mu \cdot M\eta = \mathrm{id}$ and $\mu \cdot M\mu = \mu \cdot \mu M$. We shall also consider $\mathbf{Set}$ as a symmetric monoidal category with cartesian product as the monoidal product. We denote by $\mathsf{c}_{X,Y} : X \times Y \to Y \times X$ the symmetry isomorphisms.

Throughout the paper we denote by $\mathsf{inl} : A \to A + B \leftarrow B : \mathsf{inr}$ the injections of any coproduct $A + B$.

The purpose of the monad M is to represent the type of side-effect for recursive equations, for their solutions and for operations in algebras.

Examples 2.2. (1) The *maybe monad* is given by $MX = X + 1$ with the unit $\mathsf{inl} : X \to X + 1$ and the multiplication $\mathrm{id}_X + \nabla_1 : X + 1 + 1 \to X + 1$ (where $\nabla_1 = [\mathrm{id}, \mathrm{id}] : 1 + 1 \to 1$ is the codiagonal).

(2) The *powerset monad* $M = \mathbb{P}$ has as unit the singleton map $\eta_X : X \to \mathbb{P}X$ and as multiplication the union map $\mu_X : \mathbb{P}\mathbb{P}X \to \mathbb{P}X$.

(3) The *subdistribution monad* $M = \mathbb{D}$ assigns to a set X the set $\mathbb{D}X$ of functions $d : X \to [0, 1]$ with finite support and with $\sum_{x \in X} d(x) \leq 1$.

(4) The *environment monad* is given by $MX = X^E$, where E is a fixed set. Its unit $\eta_X : X \to X^E$ assigns to an element $x \in X$ the constant map on x, and the multiplication μ_X assigns to an element of $(X^E)^E$ (i. e., an $|E| \times |E|$ matrix with values in X) its diagonal (considered as a map from E to X).

(5) The *finite-list monad* $MX = X^*$ has the unit given by singleton lists and the multiplication by flattening a list of lists.

(6) The *finite-multiset monad* $M = \mathbb{M}$ assigns to a set X the free commutative monoid on X or, equivalently, the set of all finite multisets on X.

In this paper we are interested in endofunctors on $\mathbf{Set}$ that have a lifting to the Kleisli category of the monad M. In fact, we shall establish that every analytic endofunctor (see [6, 7]) has a canonical lifting to the Kleisli category of every commutative monad (see [8]). This extends a previous result from [4] where this was proved for polynomial endofunctors on $\mathbf{Set}$.

Remark 2.3. We shall not present the formal definition of a commutative monad. It is well known that giving the structure of a commutative monad is equivalent to giving the structure of a symmetric monoidal monad, see [8, 9]. So every commutative monad (M, η, μ) on $\mathbf{Set}$ comes equipped with a *double strength*, i. e., a family of maps $\mathsf{m}_{X,Y} : MX \times MY \to M(X \times Y)$ natural in X and Y such that the following axioms hold:

$$\mathsf{m}_{X,Y} \cdot (\eta_X \times \eta_Y) = \eta_{X \times Y}\ , \quad \mu_{X \times Y} \cdot M\mathsf{m}_{X,Y} \cdot \mathsf{m}_{MX,MY} = \mathsf{m}_{X,Y} \cdot (\mu_X \times \mu_Y)$$

$$\text{and} \quad \mathsf{m}_{X \times Y,Z} \cdot (\mathsf{m}_{X,Y} \times \mathrm{id}_{MZ}) = \mathsf{m}_{X,Y \times Z} \cdot (\mathrm{id}_{MX} \times \mathsf{m}_{Y,Z})\ .$$

Moreover, the monad is symmetric monoidal, i. e., we have $\mathsf{m}_{Y,X} \cdot \mathsf{c}_{MX,MY} = M\mathsf{c}_{X,Y} \cdot \mathsf{m}_{X,Y}$.

Examples 2.4. All monads in Example 2.2, except for the list monad, are commutative monads with a (unique) canonical double strength: see [4] for the first three examples; for the environment monad the double strength is the canonical isomorphism $X^E \times$

$Y^E \cong (X\times Y)^E$; and the finite-multiset monad has $\mathsf{m}_{X,Y} : \mathbb{M}X\times\mathbb{M}Y \to \mathbb{M}(X\times Y)$ given by $\mathsf{m}_{X,Y}(\langle x_1,\dots,x_n\rangle, \langle y_1,\dots,y_m\rangle) = \langle (x_i,y_j) \mid i=1,\dots,n; j=1,\dots,m\rangle$, where the angular brackets denote multisets.

Next we recall the notion of an analytic functor on $\mathbf{Set}$ and a characterisation that we shall subsequently use.

Definition 2.5. (A. Joyal [5, 6]) An endofunctor H on $\mathbf{Set}$ is called *analytic* provided that it is the left Kan extension of a functor from the category $\mathcal{B}$ of natural numbers and bijections to $\mathbf{Set}$ along the inclusion.

Remark 2.6. (1) In fact, Joyal defined analytic functors by explicitly stating what these Kan extensions are. Let $\mathcal{S}_n$ be the symmetric group of all permutations of n. For every subgroup G of $\mathcal{S}_n$ the *symmetrised representable functor* sends each set X to the set X^n/G of orbits under the action of G on X^n by coordinate interchange, i. e., X^n/G is the quotient of X^n modulo the equivalence $\sim_G$ with $(x_1,\dots,x_n) \sim_G (y_1,\dots,y_n)$ iff $(x_{p(1)},\dots,x_{p(n)}) = (y_1,\dots,y_n)$ for some $p\in G$. It is straightforward to work out that an endofunctor on $\mathbf{Set}$ is analytic iff it is a coproduct of symmetrised representables. So every analytic functor H can be written in the form

$$HX = \coprod_{n\in\mathbb{N}, G\leq\mathcal{S}_n} A_{n,G}\times X^n/G\,. \tag{2.1}$$

(2) Notice that by (2.1) an analytic functor is a quotient of the corresponding polynomial functor P with $PX = \coprod_{n,G} A_{n,G}\times X^n$.
(3) Clearly every analytic functor is finitary. Joyal proved in [5, 6] that a finitary endofunctor on $\mathbf{Set}$ is analytic iff it weakly preserves wide pullbacks.

Examples 2.7. (1) Let Σ be a signature of operation symbols with prescribed arity. The associated *signature functor* is the polynomial endofunctor given by $H_\Sigma X = \coprod_{n\in\mathbb{N}} \Sigma_n \times X^n$; the elements of $H_\Sigma X$ are written suggestively as flat terms $\sigma(x_1,\dots,x_n)$. Clearly H_Σ is analytic (take $A_{n,G} = \Sigma_n$ for the trivial subgroup $G=\{\mathrm{id}\}\leq\mathcal{S}_n$ and $A_{n,G}=0$ else).
(2) The functor H assigning to a set X the set of finite multisets over X is analytic, since it arises from putting $HX = \coprod_{n\in\mathbb{N}} X^n/\mathcal{S}_n$.
(3) The functor H assigning to a set X the set of trees (always taken to be rooted and ordered) with nodes labelled in X is analytic. In fact, H is the left Kan extension of $t:\mathcal{B}\to\mathbf{Set}$ assigning to the natural number n the set $t(n)$ of trees with $\{0,\dots,n-1\}$ as the set of nodes.
(4) The finitary powerset functor $\mathbb{P}_{\mathit{fin}}$ is not analytic as it does not preserve weak wide pullbacks.

Recall that liftings of an endofunctor H to the Kleisli category of the monad M are in bijective correspondence with distributive laws of H over M:

Definition 2.8. A *distributive law* of an endofunctor H over a monad M is a natural transformation $\lambda: HM \Rightarrow MH$ such that

$$\lambda\cdot H\eta = \eta H \quad\text{and}\quad \lambda\cdot H\mu = \mu H\cdot M\lambda\cdot\lambda M\,.$$

Theorem 2.9. *Let H be an analytic functor on $\mathbf{Set}$ and let M be a commutative monad. Then there exists a canonical distributive law λ of H over M.*

The proof is given in Appendix A.

Remark 2.10. Let M be a commutative monad and let H be an analytic endofunctor. Write H as a quotient of the polynomial functor P, see Remark 2.6(2), and denote by $\epsilon : P \Rightarrow H$ the natural transformation formed by the canonical surjections. It follows from the proof of Theorem 2.9 that for the canonical distributive laws $\tilde{\lambda} : PM \Rightarrow MP$ and $\lambda : HM \Rightarrow MH$ we have

$$M\epsilon \cdot \tilde{\lambda} = \lambda \cdot \epsilon M : PM \Rightarrow MH\,. \tag{2.2}$$

Examples 2.11. We make the distributive law λ of Theorem 2.9 explicit for some combinations of M and H of interest. In the first three items we consider the signature functor $H = H_\Sigma$.

(1) For the maybe monad $MX = X + 1$, $\lambda_X : H_\Sigma(X + 1) \to H_\Sigma X + 1$ maps $\sigma(x_1, \dots, x_n)$ to itself if all x_i are in X and to the unique element of 1 otherwise.
(2) For $M = \mathbb{P}$, $\lambda_X : H_\Sigma \mathbb{P} X \to \mathbb{P} H_\Sigma X$ acts as follows: $\lambda_X(\sigma(X_1, \dots, X_n)) = \{\sigma(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$ for $\sigma \in \Sigma_n$ and $X_i \subseteq X$, $i = 1, \dots, n$.
(3) For the environment monad $MX = X^E$, the distributive law $\lambda_X : H_\Sigma X^E \to (H_\Sigma X)^E$ acts as follows: $\lambda_X(\sigma(v_1, \dots, v_n)) = i \mapsto \sigma(v_1(i), \dots, v_n(i))$ for $\sigma \in \Sigma_n$ and $v_1, \dots, v_n : E \to X$ and $i \in E$. More generally, there exists a canonical distributive law of every endofunctor H over M as follows: observe that $X^E \cong \prod_{i \in E} X$ with projections $\pi_i^X : X^E \to X$ for each $i \in E$. Define $\lambda_X : H(X^E) \to (HX)^E$ as the unique morphism such that $\pi_i^{HX} \cdot \lambda_X = H\pi_i^X$ for every $i \in E$. It is easy to prove that λ is a distributive law of H over M.
(4) For $H = \mathrm{Id}$ we obtain from Theorem 2.9 the identity transformation $\lambda = \mathrm{id} : M \Rightarrow M$ (which, in fact, is a distributive law for any monad).
(5) Let H be the finite-multiset functor of Example 2.7(2). Its canonical distributive law λ over the powerset monad is given by

$$\lambda_X(\langle X_1, \dots, X_n \rangle) = \{\langle x_1, \dots, x_n \rangle \mid x_i \in X_i, i = 1, \dots, n\}$$

for $X_j \subseteq X$, $j = 1, \dots, n$. In fact, this follows from (2.2) and (2) above, applied to the polynomial functor $PX = \coprod_{n \in \mathbb{N}} X^n$.

3 Iterative Algebras with Effects

Assumption 3.1. In this section we assume that H is an endofunctor on $\mathbf{Set}$ and that λ is a distributive law of H over the monad (M, η, μ).

Notation 3.2. (1) We denote morphisms in the Kleisli category $\mathbf{Set}_M$ by the symbol $\multimap$, i. e., $X \multimap Y$ is a map from X to MY. Moreover, $J : \mathbf{Set} \to \mathbf{Set}_M$ denotes the identity-on-objects functor with $Jf = \eta_Y \cdot f$ for any map $f : X \to Y$. Recall that J has a right adjoint V assigning to every $f : X \multimap Y$ the map $\mu_Y \cdot Mf : MX \to MY$. The counit ε of this adjunction is given by the identity maps on MA considered as morphisms $\varepsilon_A : MA \multimap A$ in $\mathbf{Set}_M$.

(2) The lifting of H introduced by λ is denoted by $\bar{H} : \mathbf{Set}_M \to \mathbf{Set}_M$. It takes a morphism $f : X \to MY$ in $\mathbf{Set}_M$ to $\bar{H}f = \lambda_Y \cdot Hf$.

Remark 3.3. (1) Recall that an algebra for the endofunctor H on $\mathbf{Set}$ is a set A with an algebra structure $\alpha : HA \to A$, and that an H-algebra homomorphism from (A, α) to (B, β) is a map $f : A \to B$ such that $f \cdot \alpha = \beta \cdot Hf$.

(2) Observe that an algebra for the lifting $\bar{H}$ is an algebra whose operations have side-effects described by the monad M. In fact, an $\bar{H}$-algebra is a set A equipped with an algebra structure $\alpha : HA \to MA$. So, for example, for a signature functor H_Σ, an $\bar{H}_\Sigma$-algebra is an algebra for the signature Σ where the operations are partial (in case $MX = X + 1$ is the maybe monad), nondeterministic (in case $M = \mathbb{P}$ is the powerset monad), or are families of operations indexed by elements of E (in case $MX = X^E$ is the environment monad).

Notation 3.4. Let H_Σ be a signature functor, and let $\alpha : H_\Sigma A \to MA$ be an $\bar{H}_\Sigma$-algebra. We denote by $\sigma^A : A^n \to MA$ the component of α corresponding to the n-ary operation symbol σ of Σ.

In the remainder of this paper we shall study algebras with effects in which one can uniquely solve recursive equations. Before we give the formal definition let us discuss a motivating example (without effects first). Let Σ be a signature, and let A be a Σ-algebra. We are interested in unique solutions in A of recursive equation systems (1.1). In fact, it suffices to consider the right-hand side terms of the form $t_i = \sigma(x_{i_1}, \dots, x_{i_k})$, $\sigma \in \Sigma_k$, or $t_i = a \in A$—every system can be turned into one with right-hand sides of this form by introducing new variables. A solution to a system (1.1) consists of elements $s_1, \dots, s_n \in A$ turning the formal equations into identities $s_i = t_i^A(s_1, \dots, s_n, a_1, \dots, a_p)$ in A. For example, let Σ be the signature with one binary operation symbol $*$ and with the constant symbol c. In the Σ-algebra $A = T_\Sigma Y$ of all Σ-trees on Y, i. e., (rooted and ordered) trees so that nodes with $n > 0$ children are labelled in Σ_n and leaves are labelled in $\Sigma_0 + Y$, the following system

$$x_1 \approx x_2 * c \qquad x_2 \approx x_1 * y$$

has as its unique solution the trees

```
        *              *
       / \            / \
      *   c          *   y      .
     / \            / \
    ⋮   y          ⋮   c
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This motivates the following

Definition 3.5. ([11]) Let $\mathcal{A}$ be a category with finite coproducts and let $H : \mathcal{A} \to \mathcal{A}$. A *flat equation morphism* in an object A (of parameters) is a morphism $e : X \to HX + A$. An H-algebra $\alpha : HA \to A$ is called *completely iterative* (or a *cia*, for short) if every flat equation morphism in A has a unique solution, i. e., for every $e : X \to HX + A$ there exists a unique morphism $e^\dagger : X \to A$ such that

$$e^\dagger = (X \xrightarrow{e} HX + A \xrightarrow{He^\dagger + \mathrm{id}_A} HA + A \xrightarrow{[\alpha, \mathrm{id}_A]} A) . \tag{3.1}$$

Examples 3.6. We recall some examples from previous work.

(1) Let TX denote a terminal coalgebra for $H(-)+X$. Its structure map is an isomorphism by Lambek's Lemma [10], and so its inverse yields (by composing with the coproduct injections) an H-algebra $\tau_X : HTX \to TX$ and a morphism $\eta_X : X \to TX$. Then (TX, τ_X) is a free cia on X with the universal arrow η_X, see [11], Theorems 2.8 and 2.10.
(2) Let H_Σ be a signature functor. The terminal coalgebra for $H_\Sigma(-) + X$ is carried by the set $T_\Sigma X$ of all Σ-trees on X. According to the previous item, this is a free cia for H_Σ on X.
(3) The algebra of addition on $\bar{\mathbb{N}} = \{1, 2, 3, \dots\} \cup \{\infty\}$ is a cia for $HX = X \times X$, see [1].
(4) Let $\mathcal{A} = \mathbf{CMS}$ be the category of complete metric spaces and let H be a contracting endofunctor on $\mathbf{CMS}$, see e. g. [2]. Then any non-empty algebra for H is a cia, see [11] for details. For example, let A be the set of non-empty compact subsets of the unit interval $[0, 1]$ equipped with the Hausdorff metric. This complete metric space can be turned into a cia such that the Cantor set arises as the unique solution of a flat equation morphism, see [12], Example 3.3(v).
(5) Unary algebras of $\mathbf{Set}$. Here we take $\mathcal{A} = \mathbf{Set}$ and $H = \mathrm{Id}$. An algebra $\alpha : A \to A$ is a cia iff α has a fixed point a_0 and there is no infinite sequence $a_1, a_2, a_3, \dots$ with $a_i = \alpha(a_{i+1})$, $i = 1, 2, 3, \dots$, except for the one all of whose members are a_0. The second part of this condition can be put more vividly as follows: the graph with node set $A \setminus \{a_0\}$ and with an edge from $\alpha(a) \neq a_0$ to a for all a is well-founded. Since any well-founded graph induces a well-founded (strict) order on its node set, we have yet another formulation: there is a well-founded order on $A \setminus \{a_0\}$ for which α is strictly increasing in the sense that $\alpha(a) \neq a_0$ implies $a < \alpha(a)$ for all $a \in A$.
(6) Classical algebras are seldom cias. For example, a group or a semilattice is a cia (for $HX = X \times X$) iff they contain one element only (consider the unique solution of $x \approx x \cdot 1$ or $x \approx x \vee x$, respectively).

In this paper we consider cias in the two categories $\mathbf{Set}$ and $\mathbf{Set}_M$. To distinguish ordinary cias for an endofunctor H on $\mathbf{Set}$ from those for a lifting $\bar{H}$ we have the following

Definition 3.7. We call a cia for a lifting $\bar{H} : \mathbf{Set}_M \to \mathbf{Set}_M$ a *Kleisli-cia*.

Remark 3.8. If we spell out the definition of a cia in $\mathbf{Set}_M$, we see that a flat equation morphism is a map $e : X \to M(HX + A)$, and a solution of e in the algebra $\alpha : HA \to MA$ is a map $e^\dagger : X \to MA$ such that

$$e^\dagger = (X \xrightarrow{e} M(HX + A) \xrightarrow{M(\lambda \cdot He^\dagger + \eta_A)} M(MHA + MA) \xrightarrow{\mu_{X+A} \cdot M[M\mathsf{inl}, M\mathsf{inr}]} M(HA + A) \xrightarrow{\mu_A \cdot [\alpha, \eta_A]} MA)$$

This means that the algebra (A, α) as well as the recursive equation e and its solution are "effectful". For example, for $M = \mathbb{P}$ the effect is non-determinism. If $H = H_\Sigma$

for a signature Σ with two binary operation symbols $+$ and $*$, the non-deterministic equation

$$x \approx \{x + x, x * x, a\} \quad \text{where } x \in X \text{ and } a \in A \tag{3.2}$$

gives rise to a flat equation morphism. As we shall see in Example 5.4(2), the set $A = \{2, 3, \dots, 100\} \subset \mathbb{N}$ together with the operations $+, * : A \times A \to \mathbb{P}A$ returning the sum and product, respectively, as a singleton set if this is less than or equal to 100 and $\emptyset$ otherwise, is a Kleisli-cia. If we choose $a = 10$ in the above equation, the unique solution of x is $\{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\} \subseteq A$.

Observe that, in general, the notion of a Kleisli-cia automatically connects effectful operations of an algebra with effectful recursive equations. One might want to consider these two separately. To this end we propose λ-cias as a notion of algebras with effectful operations where effects in recursive equations are allowed in the parameters only.

Definition 3.9. For the monad M a (flat) M*-equation morphism* in an object A is a morphism $e : X \to HX + MA$. A solution of e in the $\bar{H}$-algebra $\alpha : HA \multimap A$ is a morphism $e^\dagger : X \multimap A$ such that

$$e^\dagger = (X \xrightarrow{Je} HX + MA \xrightarrow{\bar{H}e^\dagger + \varepsilon_A} HA + A \xrightarrow{[\alpha, \mathrm{id}_A]} A) \tag{3.3}$$

holds in $\mathbf{Set}_M$. The $\bar{H}$-algebra (A, α) is called *completely* λ*-iterative* (or λ*-cia*, for short) if every M-equation morphism in A has a unique solution.

Remark 3.10. Observe that an $\bar{H}$-algebra (A, α) is a λ-cia iff $\mu_A \cdot M\alpha \cdot \lambda_A : HMA \to MA$ is an (ordinary) cia for the endofunctor H on $\mathbf{Set}$. In fact, this is trivial to see by writing Equation (3.3) in $\mathbf{Set}$:

$$e^\dagger = (X \xrightarrow{e} HX + MA \xrightarrow{\lambda_A \cdot He^\dagger + \mathrm{id}_A} MHA + MA \xrightarrow{[\mu_A \cdot M\alpha, \mathrm{id}_A]} MA) .$$

Also notice that in e the monad M is applied only to the second component of the coproduct in the codomain, whereas in a flat equation morphism with respect to a Kleisli-cia M is applied to the whole coproduct, cf. Remark 3.8. Continuing our concrete example for $M = \mathbb{P}$ from Remark 3.8, we see that the formal equation in (3.2) does not give rise to an M-equation morphism in the algebra A, but the system

$$x \approx y * z \qquad y \approx \{2, 3\} \qquad z \approx \{4, 5\}$$

does. Its unique solution assigns to x the set $\{8, 10, 12, 15\}$. In fact, as we shall see in Proposition 3.14, A also is a λ-cia.

Remark 3.11. We analyse the meaning of Equation (3.3) for a signature functor and the distributive laws of Examples 2.11, (1)–(3). Notice that in this case an M-equation morphism e corresponds to a system (1.1) where the right-hand sides are of the form $\sigma(x_{i_1}, \dots, x_{i_k})$, $\sigma \in \Sigma_k$, or are elements of MA. Further notice that we always have $e^\dagger(x) = e(x)$ if $e(x) \in MA$. We describe the meaning of a solution of an equation $x \approx \sigma(x_1, \dots, x_k)$.

(1) For the maybe monad $MX = X + \{\bot\}$, we have $e^\dagger(x) = \bot$ if one of the $e^\dagger(x_i)$ is $\bot$, otherwise $e^\dagger(x) = \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_k)) \in A + \{\bot\}$.

(2) For the powerset monad $M = \mathbb{P}$, we have

$$e^\dagger(x) = \bigcup \{\sigma^A(a_1, \dots, a_k) \mid a_i \in e^\dagger(x_i)\} \, .$$

(3) For the environment monad $MX = X^E$, we have

$$e^\dagger(x)(i) = \sigma^A(e^\dagger(x_1)(i), \dots, e^\dagger(x_k)(i))(i), \quad i \in E \, .$$

Examples 3.12. (1) Consider the maybe monad $MX = X + \{\bot\}$ and the distributive law λ of a signature functor H_Σ over M of Example 2.11(1). Let $F_\Sigma Y$ be the algebra of finite Σ-trees on Y. Consider its structure map as partial, so that it becomes an $\bar{H}$-algebra. As such, it is a λ-cia—in fact, the unique solution of an M-equation morphism $e : X \to H_\Sigma X + F_\Sigma Y + \{\bot\}$ gives its operational semantics, i. e., each variable in X is mapped to the tree unfolding of its recursive definition if this unfolding is finite, and to $\bot$ otherwise.

(2) Analogously, $F_\Sigma Y$ becomes a λ-cia for the distributive law λ of H_Σ over $\mathbb{P}$ of Example 2.11(2). The unique solution of $e : X \to H_\Sigma X + \mathbb{P} F_\Sigma Y$ assigns to a variable x the set of all possible tree unfoldings (taking into account that $e(x') \subseteq F_\Sigma Y$ for some variables x') of the recursive definition of x if all these unfoldings are finite and $\emptyset$ else. For example, for the signature with one binary operation symbol $*$ the system

$$x \approx x_1 * x_2 \qquad x' \approx x' * x_2 \qquad x_1 \approx \{ \begin{array}{c} * \\ / \ \backslash \\ y_1 \quad y_2 \end{array} , y_3 \} \qquad x_2 \approx \{ \begin{array}{c} * \\ / \ \backslash \\ y_3 \quad y_4 \end{array} \}$$

has the unique solution with $e^\dagger(x)$ given by the set of trees with elements

$$\begin{array}{c} * \\ / \quad \backslash \\ \begin{array}{c} * \\ / \, \backslash \\ y_1 \; y_2 \end{array} \quad \begin{array}{c} * \\ / \, \backslash \\ y_3 \; y_4 \end{array} \end{array} \qquad \text{and} \qquad \begin{array}{c} * \\ / \quad \backslash \\ y_3 \quad \begin{array}{c} * \\ / \, \backslash \\ y_3 \; y_4 \end{array} \end{array}$$

and with $e^\dagger(x') = \emptyset$.

(3) Let H be an analytic functor and let λ be the distributive law over one of the monads $MX = X + 1$, $\mathbb{P}X$ or $\mathbb{D}X$ according to Theorem 2.9. Then the initial H-algebra $\phi : H\Phi \to \Phi$ exists (since H is finitary) and $J\phi : H\Phi \to M\Phi$ is a λ-cia. In fact, we prove in Section 4 that Φ is the initial λ-cia.

Examples 3.13. Unary λ-cias. Here we consider $H = \mathrm{Id}$ and $\lambda = \mathrm{id} : M \Rightarrow M$. From Remark 3.10 we see that $\alpha : A \to MA$ is a λ-cia iff $\mu_A \cdot M\alpha$ is a cia for H, i. e., iff $\mu_A \cdot M\alpha$ has a unique fixed point a_0 and $MA \setminus \{a_0\}$ has a well-founded order for which $\mu_A \cdot M\alpha$ is strictly increasing (see Example 3.6(5)).

(1) $\eta_A : A \to MA$ is a λ-cia iff MA is a singleton set since $\mu_A \cdot M\eta_A = \mathrm{id}_A$.

(2) For the maybe monad $MX = X + \{\bot\}$, the fixed point of MA must be $\bot$. Thus, $\alpha : A \to A + \{\bot\}$ is a λ-cia iff A has a well-founded order for which α is strictly increasing, i. e., $\alpha(a) \neq \bot$ implies $a < \alpha(a)$.

(3) For $M = \mathbb{P}$, an $\bar{H}$-algebra $\alpha : A \to \mathbb{P}A$ can be considered as a directed graph with node set A (i. e., a binary relation on A) where there is an edge from v to w iff $v \in \alpha(w)$, and vice versa. Then (A, α) is a λ-cia iff this graph is well-founded, see Corollary 5.3. We could add two equivalent formulations; cf. Example 3.6(5).

(4) For the environment monad $MX = X^E$, $\alpha : A \to A^E$ is a λ-cia iff for each $i \in E$ the map $\pi_i \cdot \alpha : A \to A$ is a unary cia. In fact, this is easy to see by extending (the appropriately simplified version of) Equation (3.3) by each π_i, see also Theorem 5.5.

(5) Let $M = F_\Sigma$ be the monad assigning to a set X the set of all finite Σ-trees on X. If Σ consists of one constant symbol, then M is the maybe monad. In all other cases there are no unary λ-cias. In order to see this, assume that $\alpha : A \to MA$ is a λ-cia. Then $\mu_A \cdot M\alpha$ has a unique fixed point t. Observe that the action of $\mu_A \cdot M\alpha$ is that of replacing in all trees of MA each leaf labelled by $a \in A$ with $\alpha(a)$. This implies that for every leaf of the tree t labelled by an element $a \in A$ we have that $\alpha(a)$ is the single-node tree labelled by a. But then every Σ-tree whose leaves have labels that also appear as leaf labels of t is a fixed point of $\mu_A \cdot M\alpha$. Hence, t does not contain a leaf labelled in A. On the other hand, each tree with no such leaves is a fixed point of $\mu_A \cdot M\alpha$. Thus there must be a unique constant in Σ and no other operation symbols, and so M is the maybe monad.

Proposition 3.14. *Every Kleisli-cia is a λ-cia.*

Proof. Let $\alpha : HA \to MA$ be a Kleisli-cia and let $e : X \to HX + MA$ be an M-equation morphism. We form a flat equation morphism $\bar{e} : X \multimap \bar{H}X + A$ as follows:

$$\bar{e} = (X \xrightarrow{Je}\!\!\!\circ\; \bar{H}X + MA \xrightarrow{\mathrm{id}_{\bar{H}X} + \varepsilon_A}\!\!\!\circ\; \bar{H}X + A)\,.$$

It is clear from the definitions of solutions (see Definitions 3.5 and 3.9) that a morphism $s : X \multimap A$ is a solution of $\bar{e}$ in the Kleisli-cia A iff it is a solution of e. Since the former exists uniquely, so does the latter; thus (A, α) is a λ-cia. □

Example 3.15. This example demonstrates that the converse of Proposition 3.14 does not hold in general. Let $H = \mathrm{Id}$, let $MX = X^*$ be the monad of finite lists on X, and let $\lambda = \mathrm{id} : M \Rightarrow M$. Consider the algebra $A = \{0, 1\}$ with the structure $\alpha : A \to A^*$ given by $\alpha(0) = [1]$ and $\alpha(1) = [1, 1]$, where the square brackets denote lists. Then (A, α) is a λ-cia; in fact, $\mu_A \cdot M\alpha$ has as its unique fixed point the empty list, every list starting with 0 is mapped to a list starting with 1, and every list starting with 1 is mapped to a longer list. So an appropriate well-founded order on A^* such that $\mu_A \cdot M\alpha$ is strictly increasing on non-empty lists is given by putting $v < w$ if either v is shorter than w or the lengths agree and v goes before w lexicographically (this is even a well-order on A^*).

To see that (A, α) is not a Kleisli-cia, consider the formal equation $x \approx [x, 1]$ as a flat equation morphism $e : X \to (X + A)^*$. It is not difficult to check that for a solution $e^\dagger(x) = [a_1, \ldots, a_n]$ we have $[a_1, \ldots, a_n] = \alpha(a_1) \cdot \cdots \cdot \alpha(a_n) \cdot [1]$ where $\cdot$ denotes concatenation of lists. Thus $a_n = 1$, and therefore, since $\alpha(a_n) = [1, 1]$ we have $a_{n-1} = 1$ etc., so that we have $a_i = 1$ for all $i = 1, \ldots, n$. But then the two sides of the above equation are lists of different length, a contradiction. So there is no solution of e in A, and A is no Kleisli-cia.

The λ-cias for the endofunctor H together with the usual $\bar{H}$-algebra homomorphisms form a category, which we denote by λ-$\mathbf{Cia}_H$; and Kleisli-cias and $\bar{H}$-algebra homomorphisms form the category $\mathbf{Cia}_{\bar{H}}$. From Proposition 3.14 we see that $\mathbf{Cia}_{\bar{H}}$ is a full subcategory of λ-$\mathbf{Cia}_H$ which in turn is a full subcategory of the category $\mathbf{Alg}_{\bar{H}}$ of all algebras for $\bar{H}$:

$$\mathbf{Cia}_{\bar{H}} \hookrightarrow \lambda\text{-}\mathbf{Cia}_H \hookrightarrow \mathbf{Alg}_{\bar{H}} .$$

Our next result is that our choice of morphisms for λ-cias is appropriate. A similar result holds for cias, see [11], Proposition 2.3.

Notation 3.16. For any M-equation morphism $e : X \to HX + MA$ and any morphism $f : A \to MB$ we denote by $f \bullet e$ the M-equation morphism $(\mathrm{id}_{HX} + Vf) \cdot e : X \to HX + MB$ (see Notation 3.2(1)), where the parameters in e are "renamed by f".

Proposition 3.17. *Let $f : A \multimap B$ be a morphism, and let $\alpha : \bar{H}A \multimap A$ and $\beta : \bar{H}B \multimap B$ be λ-cias. Then the following statements are equivalent:*

(1) f is an $\bar{H}$-algebra homomorphism from (A, α) to (B, β).
(2) f preserves solutions, i. e., for all M-equation morphisms $e : X \to HX + MA$ we have $(f \bullet e)^\dagger = f \cdot e^\dagger$ (here $\cdot$ is composition in $\mathbf{Set}_M$).

The proof is given in Appendix B.

4 Free λ-cias

We have already stated in Example 3.12(3) that in a number of concrete cases an initial algebra for the functor H is an initial λ-cia. In this section we shall establish that free H-algebras yield free λ-cias whenever $\mathbf{Set}_M$ is suitably cpo-enriched and the lifting $\bar{H}$ is locally monotone. Our results here are an application of the work of Hasuo, Jacobs and Sokolova [4] and of the work in [11].

Recall that a cpo is a partially ordered set with a least element and with joins of ω-chains. A category $\mathcal{A}$ is *cpo-enriched* if each hom-set carries the structure of a cpo such that composition is continuous (i. e., preserves joins of ω-chains). Furthermore, recall that an endofunctor H on the cpo-enriched category $\mathcal{A}$ is *locally monotone* (*locally continuous*) if each derived function $\mathcal{A}(X, Y) \to \mathcal{A}(HX, HY)$ is monotone (continuous).

Assumption 4.1. Throughout this section we assume that

(1) H is a finitary endofunctor on $\mathbf{Set}$,
(2) λ is a distributive law of H over the monad M (equivalently, $\bar{H}$ is a lifting of H to $\mathbf{Set}_M$),
(3) $\mathbf{Set}_M$ is cpo-enriched and the composition in $\mathbf{Set}_M$ is left-strict, i. e., for each $f : X \multimap Y$ the maps $- \cdot f$ preserve the least element,
(4) $\bar{H}$ is locally monotone.

Remark 4.2. (1) Notice that the coproducts of $\mathbf{Set}_M$ are cpo-enriched. In fact, it is easy to see that the copairing map

$$[-, -] : \mathbf{Set}_M(X, Z) \times \mathbf{Set}_M(Y, Z) \to \mathbf{Set}_M(X + Y, Z)$$

is continuous (and hence monotone). Clearly, for every object X we have the distributive law $[M\mathsf{inl}, M\mathsf{inr}] \cdot (\lambda + \eta) : H_X M \Rightarrow M H_X$, where $H_X = H(-) + X$. It follows that $\overline{H_X} = \bar{H}_X = \bar{H}(-) + X$ is locally monotone (locally continuous) whenever $\bar{H}$ itself is.

(2) Since the functor H is finitary, it has free algebras. We denote by $\phi_X : HFX \to FX$ the structure of a free H-algebra on X and by $u_X : X \to FX$ the universal map. Notice that a free H-algebra on X is the same as an initial algebra for the functor H_X. So by Lambek's Lemma [10], $[\phi_X, u_X]$ is an isomorphism. Finally, for the initial H-algebra we use the notation $\phi : H\Phi \to \Phi$.

Examples 4.3. (1) In this example, we let M be one of the monads $(-) + 1$, $\mathbb{P}$ or $\mathbb{D}$. Assume that P is a polynomial functor; then, as shown in [4], Assumption 4.1 is satisfied. In fact, $\mathbf{Set}_M$ is cpo-enriched with a left-strict composition since in all three cases we have a cpo structure $\sqsubseteq$ on each MY: for $(-) + 1$ take the flat cpo structure (i. e., $x \sqsubseteq y$ iff $x = \bot$ or $x = y$), for $\mathbb{P}$ take inclusion, and for $\mathbb{D}$ take the pointwise order ($d \sqsubseteq d'$ iff $d(x) \leq d'(x)$ for all $x \in Y$). This yields a cpo structure $\sqsubseteq$ on $\mathbf{Set}_M(X, Y)$ in a pointwise fashion: $f \sqsubseteq g$ iff for all $x \in X$ we have $f(x) \sqsubseteq g(x)$. It is proved in [4] that there is a canonical distributive law $\tilde{\lambda} : PM \Rightarrow MP$ and that the lifting $\bar{P}$ is locally continuous.

(2) The lifting of every analytic functor H is locally continuous. We know from Theorem 2.9 that there is a canonical distributive law $\lambda : HM \Rightarrow MH$ since the three monads of interest are commutative. Furthermore, recall that H is the quotient of a polynomial functor P via some $\epsilon : P \Rightarrow H$ such that Equation (2.2) holds. Then it is easy to see that $\bar{H}$ is locally continuous: in fact, due to the naturality of ϵ and Equation (2.2) we have for every $f : X \multimap Y$ in $\mathbf{Set}_M$ a commutative square (written in $\mathbf{Set}$)

$$
\begin{array}{ccccc}
 & & \xrightarrow{\quad \bar{P}f \quad} & & \\
PX & \xrightarrow{Pf} & PMY & \xrightarrow{\tilde{\lambda}_Y} & MPY \\
\downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \epsilon_{MY}} & & \downarrow{\scriptstyle M\epsilon_Y} \\
HX & \xrightarrow{Hf} & HMY & \xrightarrow{\lambda_Y} & MHY \\
 & & \xrightarrow{\quad \bar{H}f \quad} & &
\end{array}
\tag{4.1}
$$

showing that ϵ amounts to a natural transformation $\bar{\epsilon} : \bar{P} \Rightarrow \bar{H}$ with the components $\bar{\epsilon}_X = J\epsilon_X : \bar{P}X \to \bar{H}X$. Suppose that $(f_n)_{n\in\mathbb{N}}$ is an ω-chain in $\mathbf{Set}_M(X, Y)$. Then we have

$$
\begin{aligned}
\bar{H}(\bigsqcup f_n) \cdot \bar{\epsilon}_X &= \bar{\epsilon}_Y \cdot \bar{P}(\bigsqcup f_n) && \text{(naturality of } \bar{\epsilon}) \\
&= \bar{\epsilon}_Y \cdot \bigsqcup \bar{P} f_n && \text{(local continuity of } \bar{P}) \\
&= \bigsqcup (\bar{\epsilon}_Y \cdot \bar{P} f_n) && \text{(continuity of } \cdot \,) \\
&= \bigsqcup (\bar{H} f_n \cdot \bar{\epsilon}_X) && \text{(naturality of } \bar{\epsilon}) \\
&= \bigsqcup \bar{H} f_n \cdot \bar{\epsilon}_X && \text{(continuity of } \cdot \,)\,.
\end{aligned}
$$

Since ϵ_X is a surjective map and the left adjoint J preserves epimorphisms, we can cancel $\bar{\epsilon}_X$ to obtain $\bar{H}(\bigsqcup f_n) = \bigsqcup \bar{H} f_n$, as desired.

Theorem 4.4. *([4]) Under Assumption 4.1, the initial H-algebra $\phi : H\Phi \to \Phi$ yields a terminal $\bar{H}$-coalgebra $J(\phi^{-1}) : \Phi \multimap \bar{H}\Phi$ and an initial $\bar{H}$-algebra $J\phi : \bar{H}\Phi \multimap \Phi$.*

Theorem 4.5. *Under Assumption 4.1, the free H-algebra $\phi_X : HFX \to FX$ on X with universal map $u_X : X \to FX$ yields a free $\bar{H}$-algebra $J\phi_X : \bar{H}FX \multimap FX$ on X with universal map $Ju_X : X \multimap FX$, and this is a Kleisli-cia.*

Proof. By Remark 4.2(2), $[\phi_X, u_X] : HFX + X \to FX$ is an initial algebra for $H_X = H(-) + X$. Since $\bar{H}_X = \bar{H}(-) + X$ is locally continuous (see Remark 4.2(1)), hence locally monotone, we can apply Theorem 4.4 to see that $J[\phi_X, u_X] : \bar{H}FX + X \multimap FX$ is an initial algebra and $J([\phi_X, u_X]^{-1})$ a terminal coalgebra for $\bar{H}_X$ on $\mathbf{Set}_M$. Then from Example 3.6(1) we see that $(FX, J\phi_X)$ is a (free) Kleisli-cia on X with universal map Ju_X. □

Corollary 4.6. *Under Assumption 4.1, a free H-algebra yields a free Kleisli-cia and a free λ-cia.*

Example 4.7. For a signature functor H_Σ and $M \in \{(-) + 1, \mathbb{P}, \mathbb{D}\}$, the algebra $F_\Sigma X$ of all finite Σ-trees on X yields a free $\bar{H}_\Sigma$-algebra on X and also a free Kleisli- and a free λ-cia on X.

5 Characterisation of λ-cias

In this section we consider three concrete examples of monads: the maybe monad $MX = X + 1$, the powerset monad $M = \mathbb{P}$ and the environment monad $MX = X^E$. We show that for these monads the Kleisli-cias and the λ-cias for a signature functor H_Σ coincide. In fact, in the case of the environment monad we allow arbitrary endofunctors H on $\mathbf{Set}$. In each case we give a characterisation of the λ- or Kleisli-cias.

Theorem 5.1. *Let $MX = X + 1$ be the maybe monad, let H_Σ be a signature functor, and let $\lambda : H_\Sigma M \Rightarrow MH_\Sigma$ be the distributive law of Example 2.11*(1). *Then the following three conditions are equivalent:*

(1) (A, α) is λ-cia,
(2) (A, α) is Kleisli-cia,
(3) (A, α) is an $\bar{H}_\Sigma$-algebra, and there exists a well-founded order $>$ on A such that every n-ary algebra operation σ^A is strictly increasing in the sense that for all $a_1, \dots, a_n \in A$ with $\sigma^A(a_1, \dots, a_n) \neq \bot$ we have $\sigma^A(a_1, \dots, a_n) > a_i$ for $i = 1, \dots, n$.

The proof is given in Appendix C.

Theorem 5.2. *Let $M = \mathbb{P}$ be the powerset monad, let H_Σ be a signature functor, and let $\lambda : H_\Sigma M \Rightarrow MH_\Sigma$ be the distributive law of Example 2.11*(2). *Then the following three conditions are equivalent:*

(1) (A, α) *is* λ*-cia,*
(2) (A, α) *is Kleisli-cia,*
(3) (A, α) *is an* $\bar{H}_\Sigma$*-algebra, and there exists a well-founded order* $>$ *on* A *such that every* n*-ary algebra operation* σ^A *is strictly increasing in the sense that for all* $a_1, \dots, a_n \in A$ *with* $\sigma^A(a_1, \dots, a_n) = B$ *we have* $b > a_i$ *for all* $b \in B$ *and* $i = 1, \dots, n$*.*

The proof is given in Appendix C.

Corollary 5.3. *For* $M = \mathbb{P}$*, an* $\overline{\mathrm{Id}}$*-algebra is a* λ*-cia iff the corresponding graph (cf. Example 3.13*(3)*) is well-founded.*

In fact, if a graph is well-founded, then the induced (strict) order is, and conversely, a subgraph of a well-founded graph is well-founded.

Notice that this result also shows that even though Kleisli- and λ-cias coincide, and free λ-cias (or Kleisli-cias) coincide with free $\bar{H}$-algebras, not every $\bar{H}$-algebra needs to be a λ-cia; in fact, every directed graph yields an $\overline{\mathrm{Id}}$-algebra.

Examples 5.4. (1) Let $A = \{a_0, a_1, a_2\}$, and define a binary operation $*$ by

$$\begin{array}{lll} a_0 * a_0 = \{a_1, a_2\} & a_1 * a_0 = \emptyset & a_2 * a_0 = \emptyset \\ a_0 * a_1 = \{a_2\} & a_1 * a_1 = \{a_2\} & a_2 * a_1 = \emptyset \\ a_0 * a_2 = \emptyset & a_1 * a_2 = \emptyset & a_2 * a_2 = \emptyset \end{array}$$

Here we have the well-founded order $a_2 > a_1 > a_0$ for which the operation $*$ is strictly increasing.

(2) Let $A = \{2, 3, \dots, 100\} \subset \mathbb{N}$ be the $\bar{H}$-algebra of Remark 3.8. For the usual order on A the operations are clearly strictly increasing, and well-foundedness follows from finiteness of A, i. e., A satisfies condition (3) in Theorem 5.2. Observe that for $\mathbb{P}$-equation morphisms $e : X \to HX + \mathbb{P}A$ an *infinitely unfolding variable* x, i. e., one for which there exists an infinite sequence $x_0, x_1, x_2, \dots$ with $x_0 = x$ and each x_{i+1} appearing in a flat term $e(x_i) \in HX$, is always solved to $\emptyset$. However, for flat equation morphisms $e' : X \multimap HX + A$ infinitely unfolding variables may be solved to non-empty sets, see Remark 3.8.

Theorem 5.5. *Let* $MX = X^E$ *be the environment monad, let* H *be an endofunctor on* $\mathbf{Set}$*, and let* $\lambda : HM \Rightarrow MH$ *be the distributive law of Example 2.11*(3)*. Then the following three conditions are equivalent:*

(1) (A, α) *is a* λ*-cia,*
(2) (A, α) *is a Kleisli-cia,*
(3) (A, α) *is an* $\bar{H}$*-algebra such that for every* $i \in E$ *the* H*-algebra* $(A, \pi_i \cdot \alpha)$ *is a cia in* $\mathbf{Set}$*.*

The proof is given in Appendix D.

Corollary 5.6. *Let* M*,* H *and* λ *be as in Theorem 5.5, and let* TX *be a terminal coalgebra for* $H(-) + X$*. Then* TX *yields a free* λ*-cia (and Kleisli-cia) on* X*. More precisely, the inverse of the coalgebra structure yields an* H*-algebra structure* $\tau_X : HTX \to TX$ *and a map* $u_X : X \to TX$*, and* $(TX, J\tau_X)$ *is a free* λ*-cia with universal arrow* Ju_X*.*

This result follows from the facts that, firstly, TX is a free cia on X for H (see [11]), and that, secondly, $\bar{H}$-algebras are families $(\alpha_i : HA \to A)_{i\in E}$ of H-algebras with the same carrier A, and similarly, $\bar{H}$-algebra homomorphisms are E-indexed families of H-algebra homomorphisms (see Lemma D.1 in the appendix).

6 Conclusions

We have proved that every analytic endofunctor on $\mathbf{Set}$ admits a canonical distributive law over every commutative monad M. This extends previous work by Hasuo, Jacobs and Sokolova [4]. We then applied this result in our study of algebras whose operations have computational effects (described by the monad M) and that admit unique solutions of recursive equations: we introduced λ-cias and proved that every Kleisli-cia is a λ-cia. We also proved that for the maybe and powerset monads and the respective canonical distributive laws of a signature functor over these monads Kleisli- and λ-cias coincide, and that for the environment monad this is true even for an arbitrary endofunctor on $\mathbf{Set}$. It should be interesting to see whether this coincidence can be extended to non-polynomial functors for the maybe and powerset monads, and it would be desirable to have a uniform proof of Theorems 5.1 and 5.2. We leave this for future work.

Concerning free λ-cias we have the following result: for a monad M such that $\mathbf{Set}_M$ is suitably cpo-enriched and for a finitary endofunctor H with a distributive law $\lambda : HM \Rightarrow MH$ such that the associated lifting $\bar{H}$ is locally monotone we proved that free H-algebras (which always exist for a finitary functor) yield free $\bar{H}$-algebras, which at the same time are free λ-cias and free Kleisli-cias.

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A Proof of Theorem 2.9

Throughout this section we sometimes omit the subscripts indicating components of natural transformations for better readability.

Remark A.1. From [4] we know several distributive laws of $\mathbf{Set}$-endofunctors H over commutative monads M:

(1) Let $H = C_A$ be the constant functor with value A. A distributive law $\lambda : C_A M \Rightarrow MC_A$ can be given by $\lambda_X = \eta_A$.
(2) Let $H = H_1 \times H_2$ where H_1 and H_2 are functors with distributive laws $\lambda_1 : H_1 M \Rightarrow MH_1$ and $\lambda_2 : H_2 M \Rightarrow MH_2$. Then we have a distributive law λ of $H_1 \times H_2$ over M given by

$$(H_1 \times H_2)M = H_1 M \times H_2 M \xRightarrow{\lambda_1 \times \lambda_2} MH_1 \times MH_2 \xRightarrow{\mathsf{m}} M(H_1 \times H_2).$$

Here we abuse notation and write m for the natural transformation with components $\mathsf{m}_{H_1 X, H_2 X} : MH_1 X \times MH_2 X \to M(H_1 X \times H_2 X)$.
(3) Let $H = \coprod_{i \in I} H_i$ where each endofunctor H_i has a distributive law $\lambda_i : H_i M \Rightarrow MH_i$ over the monad M. Then we have a distributive law of H over M as follows:

$$(\coprod_{i \in I} H_i)M = \coprod_{i \in I} H_i M \xRightarrow{\coprod_{i \in I} \lambda_i} \coprod_{i \in I} MH_i \xRightarrow{[M\mathsf{in}_i]} M(\coprod_{i \in I} H_i)$$

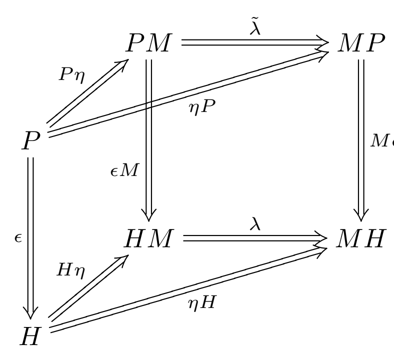
where $\mathsf{in}_i : H_i \Rightarrow \coprod_{i \in I} H_i$ denote the coproduct injections and $[M\mathsf{in}_i]$ is the unique natural transformation with $[M\mathsf{in}_i] \cdot \mathsf{in}_j = M\mathsf{in}_j$ for all $j \in I$.
(4) We have distributive laws λ_n of the functors $H = Q_n$ over M where $Q_n X = X^n$ defined by induction on n as follows: $\lambda = \mathrm{id} : \mathrm{Id}M \Rightarrow M\mathrm{Id}$ is the trivial distributive law and

$$\lambda_{n+1} = (Q_{n+1} M = Q_n M \times M \xRightarrow{\lambda_n \times \mathrm{id}} MQ_n \times M \xRightarrow{\mathsf{m}} M(Q_n \times \mathrm{Id}) = MQ_{n+1})$$

is a distributive law by (2) above.

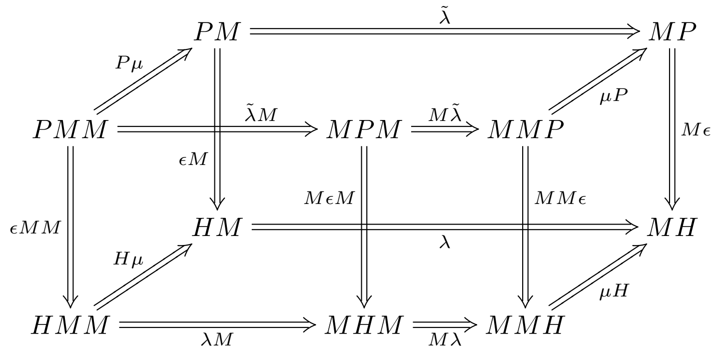
Lemma A.2. *Let H be a quotient of the polynomial endofunctor P via $\epsilon : P \Rightarrow H$. Let $\tilde{\lambda} : PM \Rightarrow MP$ be a distributive law and let $\lambda : HM \Rightarrow MH$ be a natural transformation such that $M\epsilon \cdot \tilde{\lambda} = \lambda \cdot \epsilon M$. Then λ is a distributive law.*

Proof. We verify that λ is a distributive law. For the unit axiom consider the diagram below:



The upper triangle commutes by the unit axiom for $\tilde{\lambda}$. The squares commute by naturality of η, by naturality of ϵ and by hypothesis. Hence, the desired lower triangle commutes when precomposed with ϵ. Thus, this triangle commutes since ϵ is epimorphic.

For the multiplication axiom consider the following diagram



Here the upper square commutes by the multiplication axiom for $\tilde{\lambda}$. The squares forming the cuboid sides commute due to naturality of μ (right-hand side), to naturality of ϵ (left-hand side) and due to the hypothesis (the other three sides). This proves that the desired lower square commutes when precomposed with the epimorphism ϵMM, whence this square commutes. □

In order to prove Theorem 2.9, we shall construct a canonical distributive law of any symmetrised representable endofunctor over M. Then by using Remark A.1, (1)–(3) we obtain a distributive law for every endofunctor of the form (2.1) over M, which is the desired result.

Let H be the symmetrised representable functor $HX = X^n/G$ for some natural number n and some group $G \leq \mathcal{S}_n$. Recall that each permutation $p \in G$ is a composite of transpositions $(i\,i{+}1)$ of neighbouring elements. Let γ_{p_X} denote the bijection $X^n \to X^n$ corresponding to p. For $p = (i\,i{+}1)$ this is the map swapping the i-th and $(i+1)$-st components of the product, and we simply write γ_{i_X}. For the endofunctor $Q_n X = X^n$ on $\mathbf{Set}$ we see that $\gamma_p, \gamma_i : Q_n \Rightarrow Q_n$ are natural transformations. Moreover, H is a quotient of Q_n via $\epsilon : Q_n \Rightarrow H$, where each component ϵ_X is the coequaliser of all the automorphisms γ_{p_X}, $p \in G$, on X^n.

Recall the canonical distributive laws $\lambda_n : Q_n M \Rightarrow MQ_n$ from Remark A.1(4). We prove that for any n and any γ_p the following square commutes:

$$\begin{array}{ccc} Q_n M & \stackrel{\lambda_n}{\Longrightarrow} & MQ_n \\ {\scriptstyle \gamma_p M}\Big\Downarrow & & \Big\Downarrow{\scriptstyle M\gamma_p} \\ Q_n M & \stackrel{\lambda_n}{\Longrightarrow} & MQ_n \end{array} \tag{A.1}$$

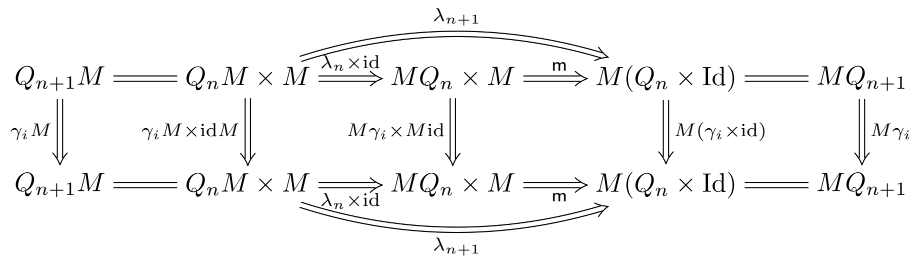
In fact, it is sufficient to prove this for each γ_i, $1 \leq i < n$, because every γ_p is a composite of maps of the form γ_i. Notice that each component of γ_i is of the form

$\mathrm{id}^{i-1} \times \mathsf{c} \times \mathrm{id}^{n-i-1}$. We prove that Diagram (A.1) commutes for each γ_i by induction on n.

Base cases: for $n = 1$ there is nothing to prove; for $n = 2$ there are two cases: the case $\gamma_i = \mathrm{id}$ is obvious, and for $\gamma_{iX} = \mathsf{c}_{X,X}$ on $X \times X$ we simply use that M is symmetric monoidal (cf. Remark 2.3): we have $\lambda_2 = \mathsf{m} : M(-)\times M(-) \Rightarrow M(-\times-)$ and so the desired square in (A.1) simply is the commutative diagram

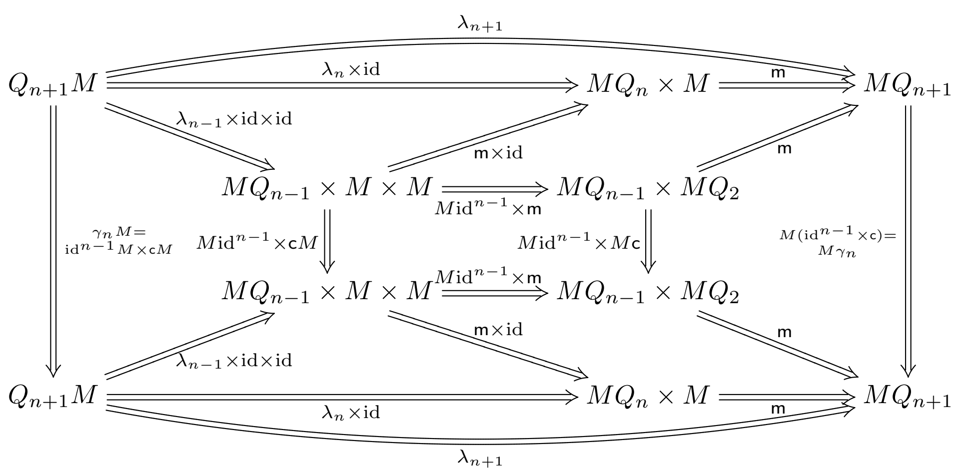
$$\begin{array}{ccc} Q_2 M & \overset{\mathsf{m}}{\Longrightarrow} & MQ_2 \\ {\scriptstyle \mathsf{c}M}\Downarrow & & \Downarrow{\scriptstyle M\mathsf{c}} \\ Q_2 M & \overset{\mathsf{m}}{\Longrightarrow} & MQ_2 \end{array}$$

Induction step (from n to $n+1$, $n \geq 2$): here we distinguish two cases. For the case $i < n$ consider the diagram



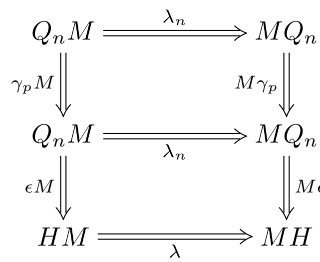
The left-hand square commutes by the induction hypothesis, the right-hand one by naturality of m and the upper and lower triangles by definition of λ_{n+1}.

For the case $i = n$ consider the diagram



The four triangles commute by definition of λ_n. The left-hand square commutes due to naturality of λ_{n-1}, the middle one due to symmetric monoidality of M and the right-hand one due to naturality of m. The remaining two squares commute since m is associative, see Remark 2.3. This completes the verification of (A.1).

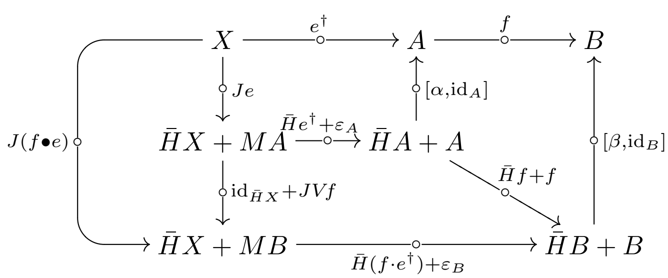
To complete the proof as well consider the diagram



The upper squares are instances of the commutative squares in (A.1) for each permutation $p \in G$. Thus from the universality of the coequaliser ϵ_{MX} we obtain the unique map λ_X such that the lower square commutes. It is not difficult to see that λ is a natural transformation, whence it is a distributive law by Lemma A.2.

B Proof of Proposition 3.17

(1) ⇒ (2): Consider an $\bar{H}$-algebra homomorphism $f : A \multimap B$ from $\alpha : \bar{H}A \multimap A$ to $\beta : \bar{H}B \multimap B$. To prove that f preserves solutions we must show that the outside of the diagram

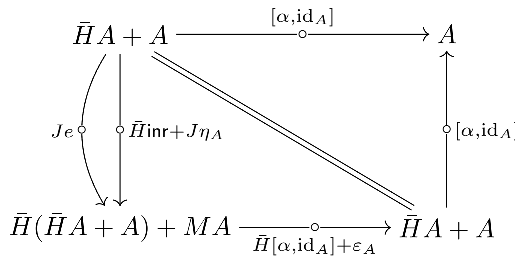


commutes. The right-hand part commutes since f is an $\bar{H}$-algebra homomorphism, and the upper square commutes since $e^\dagger$ is a solution of the M-equation morphism e. The left-hand part commutes since J preserves composition and coproducts. Finally, consider the lower part componentwise: the left-hand component clearly commutes and the right-hand one commutes due to the naturality of the counit ε (cf. Notation 3.2(1)).

(2) ⇒ (1): Let f be solution preserving and recall that for the lifting $\bar{H}$ of H we have $\bar{H}J = JH$. Consider the M-equation morphism

$$e = (HA + A \xrightarrow{H\mathsf{inr} + \eta_A} H(HA + A) + MA) .$$

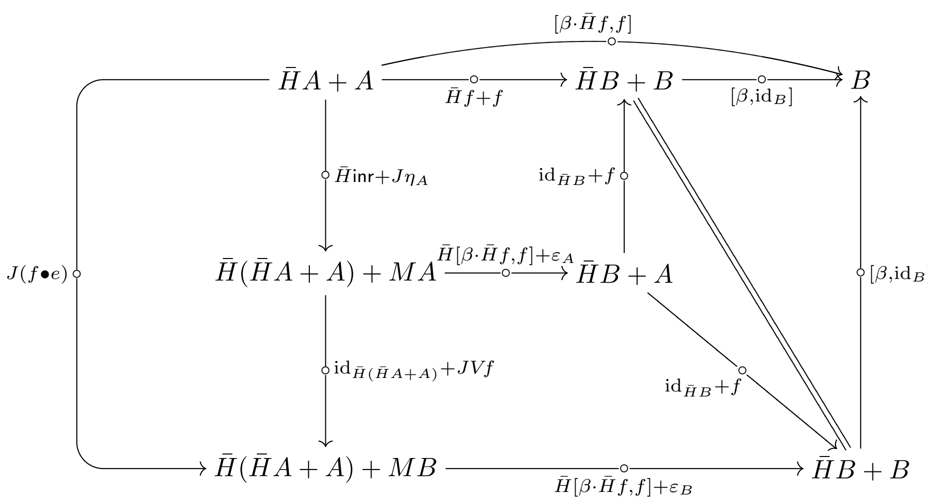
Its unique solution is displayed in the diagram



This does indeed commute: for the equality of the two left-hand arrows use that J preserves coproducts and that $\bar{H}J = JH$, for the non-trivial component of the lower left-hand triangle use that J is identity-on-objects and the adjunction equation $\varepsilon_{JA} \cdot J\eta_A = \mathrm{id}_{JA}$, and the remaining triangle is trivial. This proves $e^\dagger = [\alpha, \mathrm{id}_A]$, and since f is solution preserving we get

$$(f \bullet e)^\dagger = f \cdot e^\dagger = [f \cdot \alpha, f]\,. \tag{B.1}$$

Next we show that $[\beta \cdot \bar{H}f, f] : HA + A \to B$ is a solution of $f \bullet e$. To this end we establish that the diagram



commutes. For the commutativity of the lower part use the naturality of the counit ε for the right-hand component, the left-hand one is trivial; the upper left-hand square commutes by the adjunction equation $\varepsilon J \cdot J\eta = \mathrm{id}_J$. The left-hand part is again due to J preserving coproducts, and the remaining triangles commute obviously. Thus

$$(f \bullet e)^\dagger = [\beta \cdot \bar{H}f, f]\,. \tag{B.2}$$

We complete the proof by observing that that the left-hand components of (B.1) and (B.2) give $f \cdot \alpha = \beta \cdot \bar{H}f$ as desired.

C Proofs of Theorems 5.1 and 5.2

We first state some results in a more general setting which is common for the maybe and powerset monads. Using these results, we then prove Theorem 5.1 (characterisation of λ-cias where H_Σ is a signature functor and $M = (-) + 1$ is the maybe monad) and Theorem 5.2 (characterisation of λ-cias where H_Σ is a signature functor and $M = \mathbb{P}$ is the powerset monad) separately.

C.1 Common Results for the Maybe and Powerset Monads

Notation C.1. For any set A we denote by $!_A : A \to 1$ the unique map from A to the singleton set and by $¡_A : \emptyset \to A$ the unique map from the empty set to A.

Assumption C.2. Throughout this subsection let

(1) H_Σ be a signature functor on $\mathbf{Set}$,
(2) (M, η, μ) be a nontrivial monad (i. e., not the monad $(C_1, !, \mathrm{id}_1)$),
(3) MA carry a cpo-structure $(\sqsubseteq_A, \bot_A)$, for each set A, such that for the least elements we have $\bot_A = M¡_A(\bot_\emptyset)$,
(4) $(\sqsubseteq_{X,A}, \bot_{X,A})$ be the cpo-structure on $\mathbf{Set}(X, MA)$ that arises from the cpo-structure $(\sqsubseteq_A, \bot_A)$ on MA by pointwise definition, for each pair of sets X, A,
(5) λ be a distributive law of H_Σ over the monad M which is strict w. r. t. the cpos $(\sqsubseteq_A, \bot_A)$, i. e., for all A and terms $\sigma(\dots, \bot_A, \dots) \in H_\Sigma MA$ we have

$$\lambda_A(\sigma(\dots, \bot_A, \dots)) = \bot_{H_\Sigma A}\ , \tag{C.1}$$

(6) the map (cf. Equation (3.3))

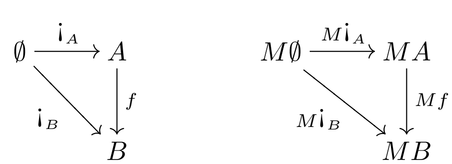
$$F : s \mapsto [\alpha, \mathrm{id}_A] \cdot (\bar{H}_\Sigma s + \varepsilon_A) \cdot Je \tag{C.2}$$

on $\mathbf{Set}(X, MA)$ be continuous w. r. t. the cpo $(\sqsubseteq_{X,A}, \bot_{X,A})$, for each $\bar{H}_\Sigma$-algebra $\alpha : \bar{H}_\Sigma A \multimap A$ and each M-equation morphisms $e : X \to H_\Sigma X + MA$.

Notation C.3. From now on we shall drop the subscripts of orders and least elements and simply write $\sqsubseteq$ and $\bot$.

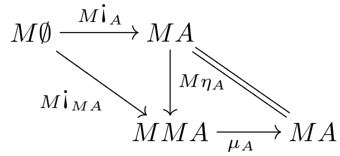
Lemma C.4. *For any map $f : A \to B$ the map Mf preserves the least element. Furthermore, for all sets A the map μ_A preserves the least element and $\bot \notin \eta_A[A]$.*

Proof. For every $f : A \to B$ initiality of $\emptyset$ makes the left-hand one of the two triangles



commute. Applying M yields the commutativity of the right-hand triangle, and this yields $Mf(\bot) = \bot$, as desired.

Next, for all A the left-hand triangle in the diagram



commutes since it is the special case of the previous one with $f = \eta_A$. The right-hand triangle exhibits the unit law of the monad M. Thus the outside of the diagram commutes. This implies $\mu_A(\bot) = \bot$.

Assume $\bot \in \eta_A[A]$ for some set A, i. e., there exists $a \in A$ with $\eta_A(a) = \bot$. Then $\eta_B(b) = \bot$ for all sets B and all $b \in B$: in fact, let $c_b : A \to B$ be the constant function with value b; by naturality of η and the fact that Mc_b preserves the least element we have

$$\bot = Mc_b(\bot) = (Mc_b \cdot \eta_A)(a) = (\eta_B \cdot c_b)(a) = \eta_B(b).$$

From the unit law $\mu_B \cdot \eta_{MB} = \mathrm{id}_{MB}$ it follows $MB \cong 1$ and $\mu_B \cong \mathrm{id}_1$ for all non-empty sets B. For $B = M\emptyset$ (which is non-empty since $\bot \in M\emptyset$) this yields $MM\emptyset \cong 1$, and using the unit law again we conclude $M\emptyset \cong 1$ and $\mu_\emptyset \cong \mathrm{id}_1$. Altogether this means that (M, η, μ) is the trivial monad $(C_1, !, \mathrm{id}_1)$, a contradiction to our assumptions. Thus, finally, $\bot \notin \eta_A[A]$ for all sets A. □

Definition C.5. Let $e : X \to H_\Sigma X + MA$ be an M-equation morphism. A variable $x \in X$ is called *infinitely unfolding* if there exists an infinite list $x_0, x_1, \dots$ of variables $x_i \in X$ with $x_0 = x$ and with x_{j+1} appearing in the term $e(x_j) \in H_\Sigma X$ for all $j \in \mathbb{N}$.

Lemma C.6. *In every $\bar{H}_\Sigma$-algebra every M-equation morphism e has the least solution s, and this satisfies $s(x) = \bot$ for every infinitely unfolding variable x of e.*

Proof. By continuity of F (cf. Assumption C.2) it follows from the Kleene Fixpoint Theorem that F has a least fixed point s, i. e., a solution of e in $\alpha : H_\Sigma A \to MA$, and we have $s = \bigsqcup_{i<\omega} s_i$, where $s_0 = \bot$ and $s_{i+1} = Fs_i$. Since the cpo-structure on $\mathbf{Set}(X, MA)$ is the pointwise one, we have $s(x) = \bigsqcup_{i<\omega} s_i(x)$ for all $x \in X$. We now prove that $s_i(x) = \bot$ for every infinitely unfolding variable x and every $i < \omega$; it then follows that $s(x) = \bot$ as desired.

We use induction on i. The base case is obvious. For the induction step we assume that $s_i(y) = \bot$ for all infinitely unfolding variables y. Now let x be one infinitely unfolding variable. Then $e(x) = \sigma(\dots, y, \dots) \in H_\Sigma X$ for another one, y. We obtain

$$\begin{aligned}
s_{i+1}(x) &= (Fs_i)(x) \\
&= ([\mu_A, \mathrm{id}_{MA}] \cdot (M\alpha + \mathrm{id}_{MA}) \cdot (\lambda_A + \mathrm{id}_{MA}) \cdot (H_\Sigma s_i + \mathrm{id}_{MA}) \cdot e)(x) \\
&= ([\mu_A \cdot M\alpha \cdot \lambda_A, \mathrm{id}_{MA}] \cdot (H_\Sigma s_i + \mathrm{id}_{MA}))(\sigma(\dots, y, \dots)) \\
&= [\mu_A \cdot M\alpha \cdot \lambda_A, \mathrm{id}_{MA}](\sigma(\dots, s_i(y), \dots)) \\
&= [\mu_A \cdot M\alpha \cdot \lambda_A, \mathrm{id}_{MA}](\sigma(\dots, \bot, \dots)) \\
&= (\mu_A \cdot M\alpha)(\bot) \\
&= \bot
\end{aligned}$$

The second equation holds by the definition of F written in $\mathbf{Set}$, the last but one equation holds by (C.1), and the last equation holds since $M\alpha$ and μ_A (by Lemma C.4) preserve the least element. $\square$

Theorem C.7. *Under Assumption C.2 there exists for every λ-cia $\alpha : H_\Sigma A \to MA$ a well-founded order $>$ on A such that all operations of α are strictly increasing in the sense that $a_0 > a_1$ whenever $\eta_A(a_0) \sqsubseteq \sigma^A(\dots, a_1, \dots)$ for some algebra operation σ^A.*

Proof. Let $\alpha : H_\Sigma A \to MA$ be a λ-cia. From the cpo-structure on MA we construct a well-founded order $>$ on A in two steps:

(1) Whenever $\eta_A(a_0) \sqsubseteq \sigma^A(b_1, \dots, b_n)$ for some algebra operation σ^A and some values $a_0, b_1, \dots, b_n \in A$ we let $a_0 > b_i$ for all $1 \le i \le n$.
(2) We take the reflexive transitive closure.

We only need to verify that the relation defined in (1) is well-founded, i. e., there exists no infinitely descending chain $a_0 > a_1 > a_2 > \cdots$. Then it follows that $>$ is a well-founded order.

Suppose that we have an infinitely descending chain $a_0 > a_1 > a_2 > \cdots$. This means that we have an infinite list

$$\begin{aligned} \eta_A(a_0) &\sqsubseteq \sigma_0^A(b_{0,1}, \dots, a_1, \dots, b_{0,n_0}) \\ \eta_A(a_1) &\sqsubseteq \sigma_1^A(b_{1,1}, \dots, a_2, \dots, b_{1,n_1}) \\ \eta_A(a_2) &\sqsubseteq \sigma_2^A(b_{2,1}, \dots, a_3, \dots, b_{2,n_2}) \\ &\vdots \end{aligned}$$

From this list we construct an M-equation morphism $e : X \to H_\Sigma X + MA$ (written as a system of equations) as follows:

$$\begin{array}{llll} x_0 \approx \sigma_0(x_{b_{0,1}}, \dots, x_1, \dots, x_{b_{0,n_0}}) & x_{b_{0,0}} \approx \eta_A(b_{0,1}) & x_{b_{1,0}} \approx \eta_A(b_{1,1}) & \\ x_1 \approx \sigma_1(x_{b_{1,1}}, \dots, x_2, \dots, x_{b_{1,n_1}}) & \qquad\vdots & \qquad\vdots & \cdots \\ x_2 \approx \sigma_2(x_{b_{2,1}}, \dots, x_3, \dots, x_{b_{2,n_2}}) & x_{b_{0,n_0-1}} \approx \eta_A(b_{0,n_0}) & x_{b_{1,n_1-1}} \approx \eta_A(b_{1,n_1}) & \\ \quad\vdots & & & \end{array}$$

Observe that this system has the infinitely unfolding variable x_0 (note that one may have $b_{i,j} = b_{i',j'}$ where $i \ne i'$ or $j \ne j'$; in this case we have $x_{b_{i,j}} = x_{b_{i',j'}}$). We obtain a solution s' of e in $\alpha : H_\Sigma A \to MA$ as follows: we define a function $s'_0 : X \to MA$ by $s'_0(x_i) = \eta_A(a_i)$ for every $i \in \mathbb{N}$ and by $s'_0(x_{b_{i,j}}) = \eta_A(b_{i,j})$ for all other variables.

We have $s_0' \sqsubseteq Fs_0'$ since for every $i \in \mathbb{N}$ we have

$$\begin{aligned}
(Fs_0')(x_i) &= (\mu_A \cdot M\alpha \cdot \lambda_A)(\sigma_i(s_0'(x_{b_{i,1}}), \ldots, s_0'(x_{i+1}), \ldots, s_0'(x_{b_{i,n_i}}))) \\
&= (\mu_A \cdot M\alpha \cdot \lambda_A)(\sigma_i(\eta_A(b_{i,1}), \ldots, \eta_A(a_{i+1}), \ldots, \eta_A(b_{i,n_i}))) \\
&= (\mu_A \cdot M\alpha \cdot \lambda_A \cdot H_\Sigma \eta_A)(\sigma_i(b_{i,1}, \ldots, a_{i+1}, \ldots, b_{i,n_i})) \\
&= (\mu_A \cdot M\alpha \cdot \eta_{H_\Sigma A})(\sigma_i(b_{i,1}, \ldots, a_{i+1}, \ldots, b_{i,n_i})) \\
&= \sigma_i^A(b_{i,1}, \ldots, a_{i+1}, \ldots, b_{i,n_i}) \\
&\sqsupseteq \eta_A(a_i) \\
&= s_0'(x_i)
\end{aligned}$$

and for all other variables we have $Fs_0'(x_{b_{i,j}}) = s_0'(x_{b_{i,j}}) = \eta_A(b_{i,j}) \in MA$. By assumption, F is continuous, thus we have the chain $s_0' \sqsubseteq Fs_0' \sqsubseteq F^2 s_0' \sqsubseteq \cdots$ and can define s' to be its join. Then s' is a fixed point for F, and consequently s' is a solution of e in $\alpha : H_\Sigma A \to MA$. Notice that $\eta_A(a_0) \sqsubseteq s'(x_0)$, and since we have $\eta_A(a_0) \neq \bot$ by Lemma C.4, we see that $s'(x_0) \neq \bot$. By Lemma C.6 the least solution s of e in $\alpha : H_\Sigma A \to MA$ has $s(x) = \bot$ for every infinitely unfolding variable. So we have $s(x_0) = \bot$. Thus there are two different solutions s and s' of e in $\alpha : H_\Sigma A \to MA$, which is a contradiction to (A, α) being a λ-cia. So the assumption that there exists an infinitely descending chain is false, and we have proved well-foundedness of the order $>$. □

C.2 Proof of Theorem 5.1

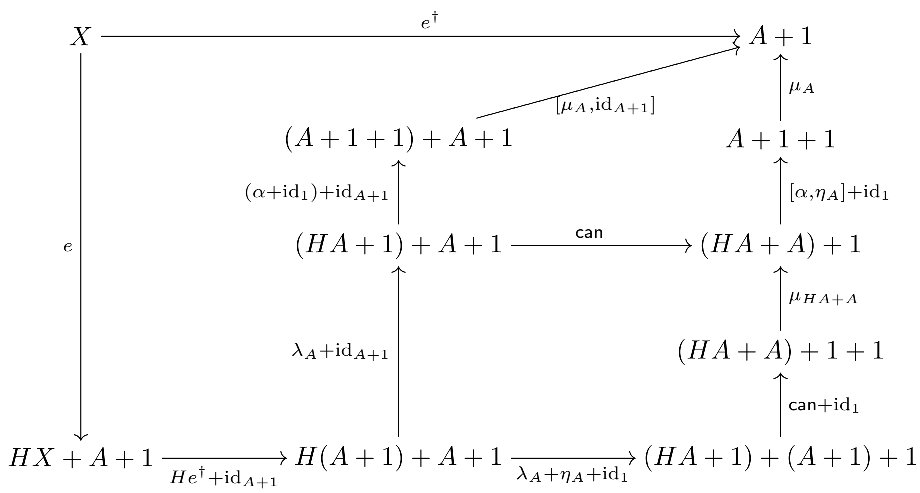
Lemma C.8. *Let H be an endofunctor with a distributive law λ over the maybe monad $M = (-) + 1$. Then an $\bar{H}$-algebra is a λ-cia iff it is a Kleisli-cia.*

Proof. Let (A, α) be an $\bar{H}$-algebra. Observe that for the maybe monad M we have

$$M(HX + A) = HX + A + 1 = HX + MA\,,$$

and so M-equation morphisms (cf. Definition 3.9) and flat equation morphisms in $\mathbf{Set}_M$ (cf. Definition 3.5) coincide. Now it is straightforward to verify that Equation

(3.3) holds for a map $e^\dagger : X \multimap A$ iff Equation (3.1) holds for $e^\dagger$: consider the diagram



Here $\mathsf{can} = [M\mathsf{inl}, M\mathsf{inr}] : MY + MZ \to M(Y + Z)$ is the canonical map. The outside square is Equation (3.1), and the left-hand part is Equation (3.3) (both written in $\mathbf{Set}$). The two right-hand parts are easily seen to commute using $\mu_Y = \mathrm{id}_Y + \nabla_1$ and $\eta_Y = \mathsf{inl} : Y \to Y + 1$, see Example 2.2(1). Thus, $e^\dagger$ is a solution of the M-equation morphism $e : X \to HX + MA$ iff $e^\dagger$ is a solution of (the same) flat equation morphism $e : X \to M(HX + A)$. $\square$

Notation C.9. Let $\alpha : H_\Sigma A \to A+1$ be an $\bar{H}_\Sigma$-algebra. We extend $\sigma^A : A^n \to A+1$ from Notation 3.4 to $A + 1$ strictly, that means that $\sigma^A(s_1, \ldots, s_n) = \bot$ if $s_i = \bot$ for some $1 \leq i \leq n$ (this is the component of $\mu_A \cdot M\alpha \cdot \lambda_A : H_\Sigma MA \to MA$ corresponding to the n-ary operation symbol σ of Σ).

We are now ready to prove Theorem 5.1. From Lemma C.8 we obtain (1) $\Leftrightarrow$ (2). Furthermore, in the current setting Assumption C.2 is fulfilled with $\sqsubseteq_A$ taken to be the flat cpo-structure: the map F of (C.2) is continuous since $\bar{H}_\Sigma$ is locally continuous (cf. Assumption 4.1 and Example 4.3(1)); all other assumptions are clearly fulfilled. Clearly, the conditions on the orders of Theorems 5.1(3) and C.7 are the same: in fact, $\eta_A(a_0) \sqsubseteq \sigma^A(\ldots, a_1, \ldots)$ means $a_0 = \sigma^A(\ldots, a_1, \ldots)$ here. Thus, we have (1) $\Rightarrow$ (3) by Theorem C.7.

It remains to prove (3) $\Rightarrow$ (1). So let (A, α) be an $\bar{H}_\Sigma$-algebra satisfying (3), and let $e : X \to H_\Sigma X + MA$ be an M-equation morphism. By Lemma C.6 the least solution s of e in (A, α) has $s(x) = \bot$ for every infinitely unfolding variable $x \in X$. Next we prove that s is the only solution of e in (A, α). So suppose we have another solution $s' \neq s$. Since $s \sqsubseteq s'$, there exists a variable x with $s(x) = \bot$ and $s'(x) = a_0$ for some $a_0 \in A$. We must have $e(x) \in H_\Sigma X$, since otherwise Equation (3.3), applied to the solutions s and s', would yield $s(x) = e(x) = s'(x)$. Thus $e(x) = \sigma(x_1, \ldots, x_n)$ for some n-ary operation symbol σ of Σ, and hence we obtain from Equation (3.3)

$$\begin{aligned} s(x) &= (\mu_A \cdot (\alpha + \mathrm{id}_1) \cdot \lambda_A \cdot H_\Sigma s)(\sigma(x_1, \dots, x_n)) \\ &= \sigma^A(s(x_1), \dots, s(x_n)) \,; \end{aligned}$$

and analogously for s'. Thus we have

$$a_0 = \sigma^A(s'(x_1), \dots, s'(x_n)) \quad \text{and} \quad \bot = \sigma^A(s(x_1), \dots, s(x_n)) \,.$$

This implies that there exists a variable x_i, $1 \leq i \leq n$, with $s(x_i) \neq s'(x_i)$. Moreover, $s'(x_i) \neq \bot$ because otherwise we would have

$$a_0 = \sigma^A(s'(x_1), \dots, s'(x_n)) = \sigma^A(\dots, \bot, \dots) = \bot \,.$$

Thus $s'(x_i) = a_1$ for some $a_1 \in A$, and $a_0 = \sigma^A(\dots, a_1, \dots)$ yields $a_0 > a_1$, because the algebra operations are all strictly increasing.

Having $s(x_i) \neq s'(x_i)$, we can repeat the whole argument with x_i in lieu of x; continuing in this way we obtain an infinite sequence $a_0 > a_1 > a_2 > \cdots$, a contradiction to our assumption of well-foundedness of the order $>$. We conclude that s is indeed the only solution of e in (A, α). Thus (A, α) is a λ-cia.

C.3 Proof of Theorem 5.2

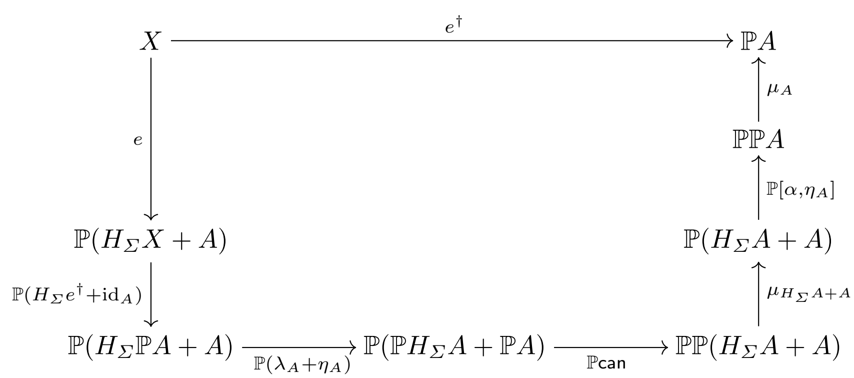
In this subsection we consider the powerset monad $M = \mathbb{P}$, a signature functor H_Σ and the corresponding canonical distributive law λ.

Notation C.10. Let $\alpha : H_\Sigma A \to \mathbb{P}A$ be an $\bar{H}_\Sigma$-algebra. We extend $\sigma^A : A^n \to \mathbb{P}A$ from Notation 3.4 to subsets of A as usual: for subsets $A_1, \dots, A_n \subseteq A$ we write

$$\sigma^A(A_1, \dots, A_n) = \bigcup \{\sigma^A(a_1, \dots, a_n) \mid a_i \in A_i \text{ for all } i = 1, \dots, n\} \,.$$

Notice that for each n-ary operation symbol σ from Σ the extended σ^A is the component of $\mu_A \cdot M\alpha \cdot \lambda_A : H_\Sigma MA \to MA$ corresponding to σ.

Remark C.11. Observe that for a flat equation morphism $e : X \multimap \bar{H}_\Sigma X + A$, Equation (3.1) yields the following commutative diagram for a solution $e^\dagger : X \multimap A$ in the $\bar{H}_\Sigma$-algebra (A, α):



The commutativity of this diagram is equivalent to the following equation for every variable $x \in X$:

$$e^\dagger(x) = (e(x) \cap A) \cup \bigcup_{\substack{\sigma(x_1,\dots,x_n) \\ \in (e(x) \cap H_\Sigma X)}} \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_n)) \tag{C.3}$$

We are now ready to prove Theorem 5.2. We have (2) ⇒ (1) by Proposition 3.14. Furthermore, in the current setting Assumption C.2 is fulfilled with $\sqsubseteq_A$ taken to be set inclusion: the map F of (C.2) is continuous since $\bar{H}_\Sigma$ is locally continuous (cf. Assumption 4.1 and Example 4.3(1)); all other assumptions are clearly fulfilled. Clearly, the conditions on the orders of Theorems 5.2(3) and C.7 are the same: in fact, $\eta_A(a_0) \sqsubseteq \sigma^A(\dots, a_1, \dots)$ means $a_0 \in \sigma^A(\dots, a_1, \dots)$ here. Thus, we have (1) ⇒ (3) by Theorem C.7.

It remains to prove (3) ⇒ (2). So let $\alpha : \bar{H}_\Sigma A \multimap A$ be an $\bar{H}_\Sigma$-algebra satisfying (3), and let $e : X \multimap \bar{H}_\Sigma X + A$ be a flat equation morphism (in $\mathbf{Set}_\mathbb{P}$). Analogously to the map F of (C.2) we obtain from Equation (3.1) a map

$$G : f \mapsto [\alpha, \mathrm{id}_A] \cdot (\bar{H}_\Sigma f + \mathrm{id}_A) \cdot e$$

on $\mathbf{Set}(X, \mathbb{P}A)$, which is clearly continuous. Let $s : X \to \mathbb{P}A$ be the least fixed point of G, i. e., s is the least solution of e in A. We shall prove that s is the only solution of e in A. So suppose we have another solution with $s' \neq s$. Since $s \sqsubseteq s'$, there exists a variable x and an element $a_0 \in A$ such that $a_0 \notin s(x)$ but $a_0 \in s'(x)$. From Equation (C.3) we obtain

$$s(x) = (e(x) \cap A) \cup \bigcup_{\substack{\sigma(x_1,\dots,x_n) \\ \in (e(x) \cap H_\Sigma X)}} \sigma^A(s(x_1), \dots, s(x_n))$$

for s, and analogously for s'. Thus,

$$a_0 \in \bigcup_{\substack{\sigma(x_1,\dots,x_n) \\ \in (e(x) \cap H_\Sigma X)}} \sigma^A(s'(x_1), \dots, s'(x_n))$$

and

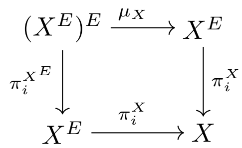
$$a_0 \notin \bigcup_{\substack{\sigma(x_1,\dots,x_n) \\ \in (e(x) \cap H_\Sigma X)}} \sigma^A(s(x_1), \dots, s(x_n)).$$

From these equations we know that there must be some $\sigma(x_1, \dots, x_n) \in e(x) \cap H_\Sigma X$ with $a_0 \in \sigma^A(s'(x_1), \dots, s'(x_n))$ but $a_0 \notin \sigma^A(s(x_1), \dots, s(x_n))$. Consequently, we have some variable x_i, $1 \leq i \leq n$, such that there exists an element $a_1 \in A$ with $a_1 \in s'(x_i)$ and $a_1 \notin s(x_i)$. From $a_0 \in \sigma^A(\dots, a_1, \dots)$ we conclude $a_0 > a_1$, because the algebra operations are all strictly increasing.

We can repeat the argument for a_1 and x_i in lieu of a_0 and x and obtain an infinite sequence $a_0 > a_1 > a_2 > \cdots$. This is a contradiction to our assumption of well-foundedness of the order $>$. Thus A is a Kleisli-cia, which completes the proof.

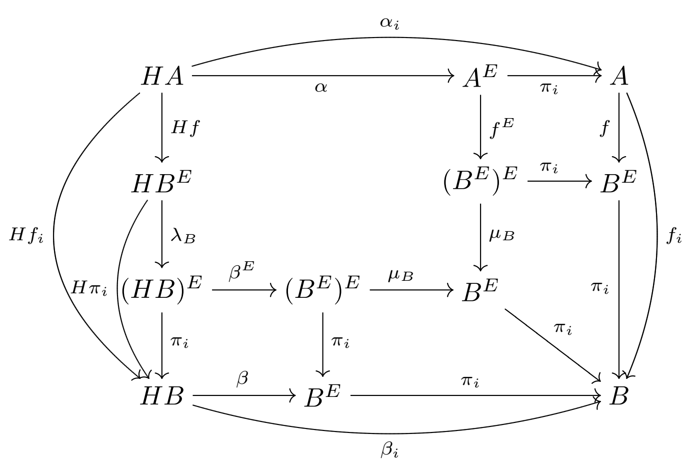
D Proof of Theorem 5.5

In this section, $MX = X^E$ is the environment monad, H denotes an arbitrary endofunctor on $\mathbf{Set}$, and $\lambda : HM \Rightarrow MH$ is the distributive law of Example 2.11(3). Recall that the unit of M is $\eta_X = \Delta_X : X \to X^E$ and that the multiplication μ_X makes the following squares commute for every $i \in E$:



Lemma D.1. *An $\bar{H}$-algebra is, equivalently, a pair $(A, (\alpha_i)_{i\in E})$ where each α_i : $HA \to A$ is an H-algebra. Similarly, an $\bar{H}$-algebra homomorphism from $(A, (\alpha_i)_{i\in E})$ to $(B, (\beta_i)_{i\in E})$ is, equivalently, a family $(f_i)_{i\in E}$ of H-algebra homomorphisms from (A, α_i) to (B, β_i), $i \in E$.*

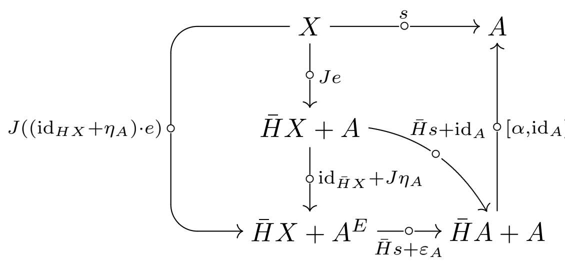
Proof. Given an $\bar{H}$-algebra $\alpha : HA \to A^E$, we put $\alpha_i = \pi_i \cdot \alpha : HA \to A$, and, conversely, every family $(\alpha_i)_{i\in E}$ induces a unique α. Similarly, we shall now prove that every $\bar{H}$-algebra homomorphism f from (A, α) to (B, β) induces a family $(f_i)_{i\in E}$ of homomorphisms, and conversely. Indeed, given the homomorphism f, in the following diagram



the big inner square commutes. Then the outside commutes for every $i \in E$, and so $(\pi_i \cdot f)_{i\in E}$ is the desired family. Conversely, the family $(f_i)_{i\in E}$ such that the outside commutes for each $i \in E$ induces the unique morphism f into the product B^E, and then the inner square commutes. $\square$

We proceed with the proof of Theorem 5.5. The implication (2) $\Rightarrow$ (1) is settled by Proposition 3.14.

We prove (1) $\Rightarrow$ (3): let $\alpha : HA \to A^E$ be a λ-cia. We are going to show that each $\alpha_i = \pi_i \cdot \alpha : HA \to A$ is a cia for H. Let $e : X \to HX + A$ be a flat equation morphism. Form the M-equation morphism $(\mathrm{id}_{HX}+\eta_A)\cdot e : X \to HX + A^E$ and take its unique solution $s : X \to A^E$. Then the outside of the diagram below commutes:

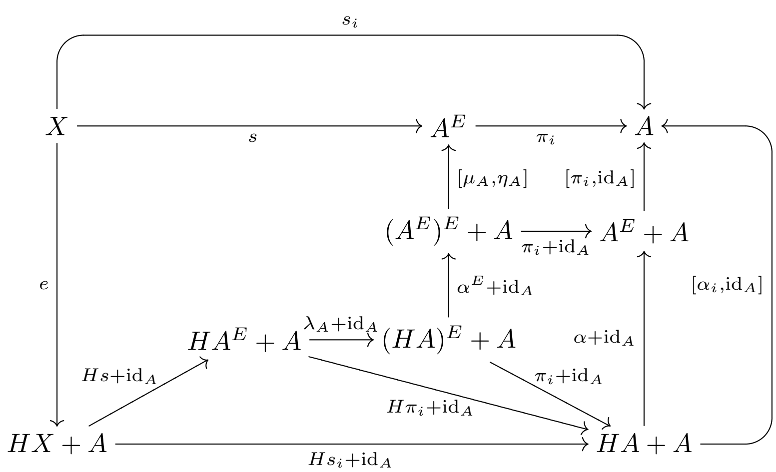


(D.1)

Observe that the left-hand part commutes by the fact that J preserves coproducts and composition, and the lower triangle commutes by the adjunction equation $\varepsilon_{JA} \cdot J\eta_A = \mathrm{id}$. We define for each $i \in E$

$$s_i = (X \xrightarrow{s} A^E \xrightarrow{\pi_i} A)$$

and prove that s_i is a solution of e in (A, α_i) for all $i \in E$. To this end consider the following diagram:



(D.2)

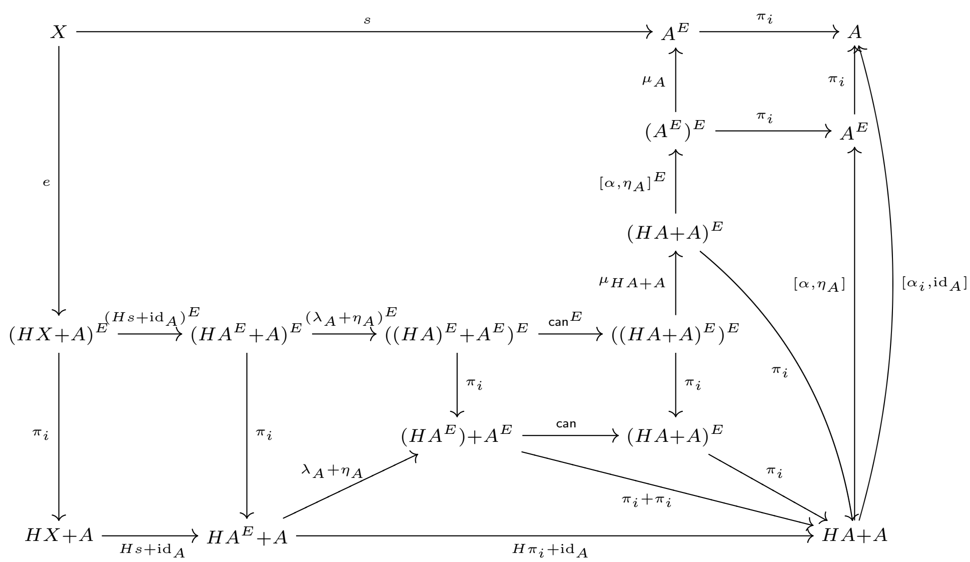
The big inner part is the upper right-hand part of (D.1) written in $\mathbf{Set}$, so it commutes. All other inner parts are also easily seen to commute; use the definitions of μ, η, λ, α_i and s_i. Thus the outside of (D.2) commutes, showing s_i to be a solution of e in (A, α_i).

To prove the uniqueness, suppose that s'_j is one particular solution of e in (A, α_j) for some $j \in E$. Define a morphism $s : X \to A^E$ by

$$\pi_j \cdot s = s'_j \quad \text{and} \quad \pi_i \cdot s = s_i \quad \text{for all } i \neq j\ .$$

Then for each $i \in E$ Diagram (D.2) commutes. Hence, Diagram (D.1) commutes, too, and so s is the unique solution of $(\mathrm{id}_{HX} + \eta_A) \cdot e$ in the λ-cia (A, α). Thus, we have $s'_j = \pi_j \cdot s = s_j$, as desired.

Finally, we prove (3) ⇒ (2): let each $\alpha_i = \pi_i \cdot \alpha : HA \to A$ be a cia. We are going to show that $\alpha : HA \to A^E$ is a Kleisli-cia. Let $e : X \multimap HX + A$ be a flat equation morphism (in $\mathbf{Set}_M$), i. e., $e : X \to (HX{+}A)^E$. Then for every $i \in E$ we have the flat equation morphism $\pi_i \cdot e : X \to HX + A$ which has the unique solution $s_i : X \to A$ in the cia (A, α_i). We define s by $\pi_i \cdot s = s_i$ for all $i \in E$ and prove that s is a unique solution of e in (A, α). We first show that $[\alpha, \mathrm{id}_A] \cdot (\bar{H}s + \mathrm{id}_A) \cdot e = s$ in $\mathbf{Set}_M$. This means, we show that the big upper left-hand square in the diagram below commutes:



The outside commutes for each $i \in E$ since $s_i = \pi_i \cdot s$ solves $\pi_i \cdot e$, and all the other inner parts also clearly commute.

If s is any solution of e in (A, α), then the upper left-hand square commutes, whence the outside commutes for each $i \in E$. This means that $\pi_i \cdot s$ is a solution of $\pi_i \cdot e$, thus we have $s_i = \pi_i \cdot s$ by the uniqueness of solutions in (A, α_i). This shows the uniqueness of s and thus completes the proof.