

Guard Your Daggers and Traces: Properties of Guarded (Co-)recursion

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Abstract. Motivated by the recent interest in models of guarded (co-)recursion, we study their equational properties. We formulate axioms for guarded fixpoint operators generalizing the axioms of iteration theories of Bloom and Ésik. Models of these axioms include both standard (e.g., cpo-based) models of iteration theories and models of guarded recursion such as complete metric spaces or the topos of trees studied by Birkedal et al. We show that the standard result on the satisfaction of all Conway axioms by a unique dagger operation generalizes to the guarded setting. We also introduce the notion of guarded trace operator on a category, and we prove that guarded trace and guarded fixpoint operators are in one-to-one correspondence. Our results are intended as first steps leading, hopefully, towards future description of classifying theories for guarded recursion.

1. Introduction

Our ability to describe concisely potentially infinite computations or infinite behaviour of systems relies on recursion, corecursion and iteration. Most programming languages and specification formalisms include a fixpoint operator. In order to give semantics to such operators one usually considers either

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- models based on complete partial orders where fixpoint operators are interpreted by least fixpoints using the Kleene-Knaster-Tarski theorem or
- models based on complete metric spaces and unique fixpoints via Banach’s theorem or
- term models where unique fixpoints arise by unfolding specifications syntactically.

In the last of these cases, one only considers *guarded* (co-)recursive definitions; see e.g. Milner’s solution theorem for CCS [1] or Elgot’s iterative theories [2]. Thus, the fixpoint operator becomes a partial operator defined only on a special class of maps. For a concrete example, consider complete metric spaces which form a category with all non-expansive maps as morphisms, but unique fixpoints are taken only of contractive maps.

Recently, there has been a wave of interest in expressing guardedness by a new type constructor \blacktriangleright , a kind of “later” modality, which allows to make the fixpoint operator total [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. For example, in the case of complete metric spaces, \blacktriangleright can be an endofunctor scaling the metric of any given space by a fixed factor $0 < r < 1$ so that non-expansive maps of type $\blacktriangleright X \rightarrow X$ are precisely r -contractive ones. This allows to define a guarded (parametrized) fixpoint operator on the model that assigns to *every* morphism $e : \blacktriangleright X \times Y \rightarrow X$ a morphism $e^\dagger : Y \rightarrow X$. Languages with a guarded fixpoint operator can be also interpreted in the “topos of trees”, i.e., presheaves on ω^{op} [9] or, more generally, sheaves on complete Heyting algebras with a well-founded basis [13, 9]. Note that by using \blacktriangleright , guarded recursion becomes a generalization of standard recursion (since \blacktriangleright can be the identity functor) rather than a specialization as in previous approaches.

This paper initiates the study of the essential properties of such operators. Iteration theories [14] are known to axiomatize equalities of *unguarded* fixpoint terms in models based on complete partial orders (see also [15]). We make here the first steps towards similar completeness results in the guarded setting.

We begin with formalizing the notion of a guarded fixpoint operator on a cartesian category. We discuss a number of models, including not only all those mentioned above, but also some not mentioned so far in the context of \blacktriangleright -guarded (co-)recursion. In fact, we consider the inclusion of examples such as the lifting functor on CPO and, more broadly, *let-ccc’s with a fixpoint object* [16] (see Examples 2.4.(6)–(7) and Theorem 3.6) or completely iterative monads (Section 2.2) a pleasant by-product of our work and a potentially fruitful connection for future research.

In Section 3, we formulate \blacktriangleright -guarded generalizations of standard axioms of Conway and iteration theories (see, e.g., [14, 15]) and prove their soundness. In particular, models with *unique* guarded fixpoint operators satisfy all our axioms (Theorem 3.4). Without the assumption of uniqueness, some problems appear (notably, Open Problems 3.8, 3.16 and 3.17) and generalizations of several known derivations, like that of the Bekič identity from the Conway axioms (Proposition 3.10) require some ingenuity. We believe these are positive signs: sticking \blacktriangleright in “all the right places” cannot always be done on autopilot and subtle aspects of (co-)recursion invisible to the unguarded eye come to light, even on the purely equational level. For natural examples, however, most properties in question seem to hold even without requiring uniqueness, as witnessed, e.g., by Theorems 3.6 and 3.7.

Hasegawa [17] proved that giving a parametrized fixpoint operator on a cartesian category is equivalent to giving a *traced cartesian structure* [18] on that category.¹ In Section 4, we introduce a natural notion of a *guarded trace operator* on a category, and we prove in Theorem 4.5 that guarded traces and guarded fixpoint operators are in one-to-one correspondence. This extends to an isomorphism between the (2-)categories of guarded traced cartesian categories and guarded Conway categories (Corollary 4.10). Just like in the unguarded case, the notion of trace would make sense in a general monoidal setting (Remarks 4.1 and 4.3). We leave this as an exciting avenue for future research.

Finally, Section 5 concludes and discusses further work.

A few words are due on differences with a previously published extended abstract of this paper [19]. We obviously provide full proof details of all results. We also discuss additional equational properties of guarded fixpoint operators in Sections 3.1 and 3.2. Moreover, Theorem 3.6 concerning let-ccc's is new and so is, e.g., Example 2.8 (the last provided by Aleš Bizjak).

We decided to move some more technical proofs to an appendix in order to make the paper more readable.

1.1. Notational conventions

We assume familiarity with basic notions of category theory. We denote the product of two objects by $A \leftarrow^{\pi_\ell} A \times B \xrightarrow{\pi_r} B$ and $\Delta : A \rightarrow A \times A$ denotes the diagonal. For every functor F we write $\text{can} = \langle F\pi_\ell, F\pi_r \rangle : F(A \times B) \rightarrow FA \times FB$ for the canonical morphism. We denote the terminal object in a cartesian category as 1 and the unique morphism for each X as $! : X \rightarrow 1$. Wherever convenient, we use freely other standard conventions such as identifying X and $1 \times X$ or dropping subscripts of natural transformations if they are clear from the context.

CPO denotes the category of complete partial orders (cpo's), i.e. partially ordered sets (possibly without a least element) having joins of ω -chains. The morphisms of CPO are Scott-continuous maps, i.e. maps preserving joins of ω -chains. CPO_\perp is the full subcategory of CPO given by all cpo's with a least element \perp . We will also consider the category CMS of complete 1-bounded metric spaces and non-expansive maps, i.e. maps $f : X \rightarrow Y$ such that for all $x, y \in X$, $d_Y(fx, fy) \leq d_X(x, y)$; see Krishnaswami and Benton [7, 8] or Birkedal et al. [9, Section 5] and references therein.

Instead of writing “the following square commutes” or “the following diagram commutes”, we write \circlearrowleft in the middle of the diagram in question. We also use objects to denote their identity morphisms. Finally, we sometimes write $X = Y$ to indicate that two objects in a category are isomorphic.

2. Guarded fixpoint operators

In this section we define the notion of a guarded fixpoint operator on a cartesian category and present an extensive list of examples. Some of these examples like the lifting functor $(-)_\perp$ on CPO (see Example 2.4.6) or completely iterative monads (see Section 2.2) do not seem to have been considered as instances of the guarded setting before. We also discuss in detail the connection with *models of guarded fixpoint terms* of Birkedal et al. [9], see Proposition 2.7.

¹Cartesian here refers to the monoidal product being the ordinary categorical product.

2.1. Definition and examples of guarded fixpoint operators

Assumption 2.1. We assume throughout the rest of the paper that $(\mathcal{C}, \blacktriangleright)$ is a pair consisting of a category \mathcal{C} with finite products (also known as a *cartesian category*) and a pointed endofunctor $\blacktriangleright : \mathcal{C} \rightarrow \mathcal{C}$, i.e. we have a natural transformation $p : \text{Id} \rightarrow \blacktriangleright$. The endofunctor \blacktriangleright is called *delay* or the “later” modality.

Remark 2.2. In references like Birkedal et al. [9, 10], much more is assumed about both the underlying category and the delay endofunctor. Modelling simply-typed lambda calculus requires cartesian closure. Dependent types require additional conditions like being a *type-theoretic fibration category* (see, e.g., [10, Definition IV.1]). In such a case, one also wants to impose some limit-preservation or at least finite-limit-preservation condition on the delay endofunctor [9, Definition 6.1]—e.g., to ensure the transfer of the guarded fixpoint operator to slices. We do not impose any of those restrictions because we do not need them in our derivations. For more on the connection with the setting of Birkedal et al. [9], see Proposition 2.7 below.

Definition 2.3. A *guarded fixpoint operator* on $(\mathcal{C}, \blacktriangleright)$ is a family of operations

$$\dagger_{X,Y} : \mathcal{C}(\blacktriangleright X \times Y, X) \rightarrow \mathcal{C}(Y, X)$$

such that for every $f : \blacktriangleright X \times Y \rightarrow X$,

$$\begin{array}{ccc} Y & \xrightarrow{f^\dagger} & X \\ \langle f^\dagger, Y \rangle \downarrow & \circlearrowleft & \uparrow f \\ X \times Y & \xrightarrow{p_{X \times Y}} & \blacktriangleright X \times Y \end{array} \quad (2.1)$$

where (as usual) we drop the subscripts and write $f^\dagger : Y \rightarrow X$ in lieu of $\dagger_{X,Y}(f)$. We call the triple $(\mathcal{C}, \blacktriangleright, \dagger)$ a *guarded fixpoint category*.

Moreover, $(\mathcal{C}, \blacktriangleright, \dagger)$ is called a *unique guarded fixpoint category* if for every $f : \blacktriangleright X \times Y \rightarrow X$, f^\dagger is the unique morphism satisfying (2.1). In this case, we can just write a pair $(\mathcal{C}, \blacktriangleright)$ instead of a triple $(\mathcal{C}, \blacktriangleright, \dagger)$.

If one does not require that f^\dagger is the unique solution of (2.1), one usually assumes that \dagger satisfies further properties. For example, Simpson and Plotkin [15] require that a parametrized fixpoint operator \dagger is natural in Y in the base definition. We will come to the study of properties of \dagger such as naturality in Section 3. Let us begin with a list of examples. Note that in most cases, we do not explicitly mention the action of \blacktriangleright on morphisms whenever it is canonical; for Example (5), it is given in Appendix A.

Examples 2.4. (1) Taking as \blacktriangleright the identity functor on \mathcal{C} and p_X the identity on X we arrive at the special case of categories with an ordinary fixpoint operator $\mathcal{C}(X \times Y, X) \rightarrow \mathcal{C}(Y, X)$ (see e.g. Hasegawa [17, 20] or Simpson and Plotkin [15]). Concrete examples are: the category CPO_\perp with its usual least fixpoint operator or (the dual of) any iteration theory of Bloom and Ésik [14].

- (2) Taking \blacktriangleright to be the constant functor $\blacktriangleright X = 1$ and $p_X = ! : X \rightarrow 1$, a trivial guarded fixpoint operator is given by the family of identity maps on the hom-sets $\mathcal{C}(Y, X)$.
- (3) Take \mathcal{C} to be CMS, $r \in (0, 1)$, $\blacktriangleright_r : \text{CMS} \rightarrow \text{CMS}$ to be the endofunctor keeping the carrier of the space and multiplying all distances by r , and $p_X : X \rightarrow \blacktriangleright_r X$ to be the obvious “contracted identity” mapping. Note that a non-expansive mapping $f : \blacktriangleright_r X \rightarrow X$ is the same as an r -contractive endomap, i.e. an endomap satisfying $d(fx, fy) \leq r \cdot d(x, y)$ for every $x, y \in X$. An application of Banach’s unique fixpoint theorem yields a guarded fixpoint operator: for every $f : \blacktriangleright_r X \times Y \rightarrow X$ we consider the map

$$\Phi_f : \text{CMS}(Y, X) \rightarrow \text{CMS}(Y, X), \quad \Phi_f(m) = f \cdot (p_X \times Y) \cdot \langle m, Y \rangle;$$

notice that $\text{CMS}(Y, X)$ is a complete metric space with the sup-metric

$$d_{Y,X}(m, n) = \sup_{y \in Y} \{d_X(my, ny)\}.$$

Using that non-expansive maps are closed under composition, product and pairing, it is easy to show that Φ_f is an r -contractive map, and so its fixpoint is the unique non-expansive map $f^\dagger : Y \rightarrow X$ satisfying (2.1).

- (4) Let \mathcal{A} be a category with finite products and \mathcal{C} be the presheaf category $\text{presheaf}(\omega, \mathcal{A}) = \mathcal{A}^{\omega^{\text{op}}}$ of ω^{op} -chains in \mathcal{A} . The delay functor is given by $\blacktriangleright X(0) = 1$ and $\blacktriangleright X(n+1) = X(n)$ for $n \geq 0$, whereas p_X is given by $(p_X)_0 = ! : X(0) \rightarrow 1$ and $(p_X)_{n+1} = X(n+1 \geq n) : X(n+1) \rightarrow X(n)$. For every $f : \blacktriangleright X \times Y \rightarrow X$ there is a unique $f^\dagger : Y \rightarrow X$ satisfying (2.1) given by $f_0^\dagger = f_0 : Y(0) \rightarrow X(0)$ and

$$f_{n+1}^\dagger = (Y(n+1) \xrightarrow{\langle f_n^\dagger \cdot Y(n+1 \geq n), Y(n+1) \rangle} X(n) \times Y(n+1) \xrightarrow{f_{n+1}} X(n+1)).$$

Notice that for $\mathcal{A} = \text{Set}$, \mathcal{C} is the “topos of trees” of Birkedal et al. [9].

The next example generalizes this one.

- (5) Assume $\mathfrak{W} = (W, \leq)$ is a well-founded poset, i.e. contains no infinite descending chains. As usual we write $x < y$ whenever $x \leq y$ and $x \neq y$. Furthermore, let \mathcal{D} be a (small) complete category and $\mathcal{C} = \text{presheaf}(\mathfrak{W}, \mathcal{D})$, i.e., $\mathcal{C} = \mathcal{D}^{(W, >)}$. Define $\blacktriangleright X(w)$ to be the limit of the diagram whose nodes are $X(v)$ for $v < w$ and whose arrows are restriction morphisms: $\blacktriangleright X(w) = \lim_{v < w} X(v)$. As restriction mappings from $X(w)$ itself form a cone on that diagram, a natural $p_X : X \rightarrow \blacktriangleright X$ is given by the universal property of the limits. Note that for any minimal element $r \in W$, we have that $(\blacktriangleright X)(r)$ is the terminal object 1 of \mathcal{D} . The \dagger -operation on $f : \blacktriangleright X \times Y \rightarrow X$ is defined by induction on (W, \leq) : assuming that f_v^\dagger is already defined for all $v < w$ let

$$f_w^\dagger = (Y(w) \xrightarrow{\langle k, Y(w) \rangle} \blacktriangleright X(w) \times Y(w) \xrightarrow{f_w} X(w)),$$

where $k : Y(w) \rightarrow \blacktriangleright X(w)$ is the morphism uniquely induced by the cone

$$Y(w) \xrightarrow{Y(w > v)} Y(v) \xrightarrow{f_v^\dagger} X(v) \quad \text{for every } v < w.$$

This includes the case of a minimal element r for which the above definition yields $f_r^\dagger = f_r : Y(r) = 1 \times Y(r) \rightarrow X(r)$.

One can prove that f^\dagger is a unique morphism of presheaves satisfying (2.1); more details can be found in Appendix A. This result generalizes Birkedal et al. [9, Theorem 2.4].

Regarding the examples given by Birkedal et al. [9], see also Proposition 2.7 below, which also establishes that a let-ccc with a fixpoint object with \blacktriangleright and \dagger just defined forms a guarded fixpoint category.

- (6) Let \blacktriangleright be the lifting functor $(-)_\perp$ on CPO, i.e. for any cpo X , X_\perp is the cpo with a newly added least element and the natural transformation $p_X : X \rightarrow X_\perp$ is the embedding of X into X_\perp . Taking least fixpoints yields a guarded fixpoint operator. To see this notice that the hom-sets $\text{CPO}(X, Y)$ are cpos with the pointwise order: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Now any continuous $f : X_\perp \times Y \rightarrow X$ gives rise to a continuous map Φ_f on $\text{CPO}(Y, X_\perp)$:

$$\Phi_f : \text{CPO}(Y, X_\perp) \rightarrow \text{CPO}(Y, X_\perp), \quad \Phi_f(m) = p_X \cdot f \cdot \langle m, Y \rangle.$$

Using the least fixpoint s of Φ_f define $f^\dagger = (Y \xrightarrow{\langle s, Y \rangle} X_\perp \times Y \xrightarrow{f} X)$. As $s = \Phi_f(s)$, one can easily show f^\dagger satisfies (2.1). Just as Example (5) was more general than (4), the present example is also known to be an instance of a more general situation:

- (7) Crole and Pitts [16, 21] define a *let-ccc with a fixpoint object* as a tuple

$$(\mathcal{C}, T, \eta, \mu, s, T\Omega \xrightarrow{\sigma} \Omega, 1 \xrightarrow{\omega} \Omega),$$

where

- \mathcal{C} is a cartesian closed category,
- (T, η, μ, s) is a strong monad on \mathcal{C} , i.e. a monad together with a *strength* viz. a family of morphisms $s_{A,B} : A \times TB \rightarrow T(A \times B)$ natural in A and B and compatible with the monad structure in the obvious way,
- $\sigma : T\Omega \rightarrow \Omega$ is an initial algebra for the functor T and
- ω is the equalizer of $\sigma \cdot \eta_\Omega$ and identity on Ω (i.e., the unique fixpoint of $\sigma \cdot \eta_\Omega$).

Recall from Crole and Pitts [21, 16] (and cf. Manes [22]) that a strong monad can be specified by an object assignment T , a family of morphisms η_A and an operation $(-)^* : \mathcal{C}(A \times B, TC) \rightarrow \mathcal{C}(A \times TB, TC)$ satisfying the following laws:

- (a) for any $f : A \times B \rightarrow TC$, we have $f^* \cdot (A \times \eta_B) = f$;
- (b) $(\eta_B \cdot \pi_r)^* = \pi_r : A \times TB \rightarrow TB$;
- (c) given $f : A \rightarrow A'$ and $g : A' \times B \rightarrow TC$ we have $(g \cdot (f \times B))^* = g^* \cdot (f \times TB)$;
- (d) given $f : A \times B \rightarrow TC$ and $g : A \times C \rightarrow TD$ we have $(g^* \cdot \langle \pi_\ell, f \rangle)^* = g^* \cdot \langle \pi_\ell, f^* \rangle$.

The initiality of Ω and cartesian closedness of \mathcal{C} yield that for any $f : C \times TA \rightarrow A$ there exists a unique morphism $\text{it}(f) : C \times \Omega \rightarrow A$ such that

$$\begin{array}{ccc}
 C \times \Omega & \xrightarrow{\text{it}(f)} & A \\
 C \times \sigma \uparrow & \circlearrowleft & \uparrow f \\
 C \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta \cdot \text{it}(f))^* \rangle} & C \times TA
 \end{array} \tag{2.2}$$

Now set $\blacktriangleright = T$ and for any $f : Y \times TX \rightarrow X$ define² $f^\dagger = \text{it}(f) \cdot (Y \times \omega)$. To see that this example covers the preceding one, follow [16] and set $T = (-)_\perp$, $\eta_X = p_X$, $(-)^*$ the strict extension (i.e. with $f^*(a, \perp) = \perp$) and $\Omega = \{0 \sqsubset 1 \sqsubset \dots \sqsubset \top\}$. We return to let-ccc's in Theorem 3.6 below.

Remark 2.5. If we replaced initiality of Ω in Example 2.4.(7) by the (parametrized) universal property (2.2) of Ω we could drop cartesian closure of \mathcal{C} and simply assume \mathcal{C} to be a category with products; cf. Crole and Pitts [16, Lemma 2.2] and an explicit claim therein: *if one simply has a category with finite products and a strong monad, the definition of fixpoint object should be strengthened to a parametrised form*. See also Proposition 2.7 below for an analogous discussion.

Birkedal et al. [9] provide a general setting for topos-theoretic examples like 2.4.(4) and 2.4.(5) (the latter restricted to the case of Set-presheaves) by defining a notion of a *model of guarded recursive terms* and showing that *sheaves over complete Heyting algebras with a well-founded basis* proposed by Di Gianantonio and Miculan [13] are instances of this notion. The difference between Definition 6.1 in [9] and our Definition 2.3 is that in the former a) the delay endofunctor \blacktriangleright is also assumed to preserve finite limits, in particular finite products. On other hand b) our equality (2.1) is only postulated in the case where Y is the terminal object, i.e., only a non-parametrized fixpoint identity is assumed but c) the dagger in this less general version of (2.1) is assumed to be unique. Now, one can show that assumptions a) and c) imply our parametrized identity (2.1) *whenever the underlying category is cartesian closed*, in particular whenever \mathcal{C} is a topos. Let us state both the definition and the result formally:

Definition 2.6. A *weak model of guarded fixpoint terms* is a triple $(\mathcal{C}, \blacktriangleright, \ddagger)$, where

- $(\mathcal{C}, \blacktriangleright)$ satisfy our general Assumption 2.1,
- \blacktriangleright preserves finite products with $\text{can}_{X,Y}^{-1} : \blacktriangleright X \times \blacktriangleright Y \rightarrow \blacktriangleright (X \times Y)$ as the witnessing isomorphism,
- \ddagger is a family of operations $\ddagger_X : \mathcal{C}(\blacktriangleright X, X) \rightarrow \mathcal{C}(1, X)$ such that for every $f : \blacktriangleright X \rightarrow X$, f^\ddagger is

²Note that to be consistent with [16] we put $\blacktriangleright X$ in the right-hand product component of the domain of f .

a unique morphism for which

$$\begin{array}{ccc} 1 & \xrightarrow{f^\ddagger} & X \\ f^\ddagger \downarrow & \circlearrowleft & \uparrow f \\ X & \xrightarrow{p_X} & \blacktriangleright X \end{array} \quad (2.3)$$

A model of guarded fixpoint terms [9] is a weak model in which in addition \mathcal{C} has finite *limits* (not just products) and \blacktriangleright preserves them.

Proposition 2.7. If $(\mathcal{C}, \blacktriangleright, \ddagger)$ is a weak model of guarded fixpoint terms and \mathcal{C} is *cartesian closed* with

$$\text{curry}_{Y,Z}^X : \mathcal{C}(X \times Y, Z) \rightarrow \mathcal{C}(X, Z^Y), \quad \text{and} \quad \text{eval}_{Y,Z} : Z^Y \times Y \rightarrow Z,$$

then the operator $\ddagger_{X,Y} : \mathcal{C}(\blacktriangleright X \times Y, X) \rightarrow \mathcal{C}(Y, X)$ defined as

$$f^\ddagger = \text{eval}_{Y,X} \cdot \left(\left[\text{curry}_{Y,X}^{\blacktriangleright(X^Y)} \left(f \cdot \langle \blacktriangleright \text{eval}_{Y,X} \cdot \text{can}_{X^Y,Y}^{-1} \cdot (\blacktriangleright(X^Y) \times p_Y), \pi_r \rangle \right) \right]^\ddagger \times Y \right)$$

is a unique guarded fixpoint operator on $(\mathcal{C}, \blacktriangleright)$.

Proof:

Take $f : \blacktriangleright X \times Y \rightarrow X$. For notational simplicity, let

$$g = f \cdot \langle \blacktriangleright \text{eval}_{Y,X} \cdot \text{can}_{X^Y,Y}^{-1} \cdot (\blacktriangleright(X^Y) \times p_Y), \pi_r \rangle \quad \text{and} \quad \widehat{g} = \text{curry}_{Y,X}^{\blacktriangleright(X^Y)}(g)$$

so that $f^\ddagger = \text{eval}_{Y,X} \cdot (\widehat{g}^\ddagger \times Y)$. We need to show (2.1) for f^\ddagger defined this way, i.e., commutativity of the outside of the diagram below:

$$\begin{array}{ccccc} Y = 1 \times Y & \xrightarrow{\widehat{g}^\ddagger \times Y} & X^Y \times Y & \xrightarrow{\text{eval}} & X \\ \widehat{g}^\ddagger \times Y \downarrow & \circlearrowleft \text{ by (2.3)} & \widehat{g} \times Y \uparrow & \circlearrowleft \text{ by ccc} & \uparrow f \\ X^Y \times Y & \xrightarrow{p \times Y} & \blacktriangleright(X^Y) \times Y & \xrightarrow{\langle \blacktriangleright(X^Y) \times p, \pi_r \rangle} & \blacktriangleright(X^Y) \times \blacktriangleright Y \times Y \quad \circlearrowleft \text{ by def. of } g \\ \langle \text{eval}, \pi_r \rangle \downarrow & & \langle p, \pi_r \rangle \searrow & \downarrow \text{can}^{-1} \times Y & \\ X \times Y & & \blacktriangleright(X^Y \times Y) \times Y & \xrightarrow{\blacktriangleright \text{eval} \times Y} & \blacktriangleright X \times Y \\ & \circlearrowleft \text{ by pointedness} & & & \uparrow p \times Y \end{array}$$

Thus, we need to show the left-hand product component (?) of the inner triangle commutes. We postcompose both sides with the isomorphism $\text{can} : \blacktriangleright(X^Y \times Y) \rightarrow \blacktriangleright(X^Y) \times \blacktriangleright Y$ and obtain

$$\text{can} \cdot p_{X^Y \times Y} = \langle \blacktriangleright \pi_\ell, \blacktriangleright \pi_r \rangle \cdot p_{X^Y \times Y} = \langle p_{X^Y} \cdot \pi_\ell, p_Y \cdot \pi_r \rangle = p_{X^Y} \times p_Y,$$

where the middle equation holds by the naturality of p .

It remains to prove that f^\dagger is the unique solution of (2.1). Suppose that $s : Y \rightarrow X$ satisfies $s = f \cdot (p_X \times Y) \cdot \langle s, Y \rangle$. Let $c = \text{curry}_{Y,X}^1(s) : 1 \rightarrow X^Y$. We show that (2.3) holds with f^\dagger replaced by c , i.e. that $c = \widehat{g} \cdot p_{X^Y} \cdot c$. Equivalently, we show

$$\text{eval}_{Y,X} \cdot (c \times Y) = \text{eval}_{Y,X} \cdot (\widehat{g} \times Y) \cdot (p_{X^Y} \times Y) \cdot (c \times Y).$$

In order to see this first modify the above diagram by replacing \widehat{g}^\dagger in the upper left-hand square by c . Notice that our desired equation corresponds to the (modified) upper left-hand square postcomposed with $\text{eval}_{Y,X}$, i.e., the right-hand morphism of the upper edge. Since $\text{eval}_{Y,X} \cdot (c \times Y) = s$, the outside of the modified diagram commutes by hypothesis. Thus, since all other parts commute as indicated we obtain the desired equation. This implies that $c = \widehat{g}^\dagger$, by the uniqueness of the latter. Hence

$$s = \text{eval}_{Y,X} \cdot (c \times Y) = \text{eval}_{Y,X} \cdot (\widehat{g}^\dagger \times Y) = f^\dagger,$$

which completes the proof. \square

Proposition 2.7 cannot be reversed: Example 2.4.(6) is a guarded fixpoint category, but $(-)_\perp$ clearly fails to preserve even finite products and hence it does not yield a model of guarded recursive terms.

Example 2.8. A counterexample kindly provided by Aleš Bizjak shows that Proposition 2.7 does not hold without the assumption that \mathcal{C} is cartesian closed: there are examples of models of guarded recursive terms which are not guarded fixpoint categories. In fact, the adjective “guarded” can be removed altogether from the previous sentence: \mathcal{C} in the Bizjak counterexample is the category of groups and $\blacktriangleright = \text{Id}_{\mathcal{C}}$. Initial and final objects coincide in \mathcal{C} , which entails two things: 1) cartesian closure fails and 2) there is exactly one canonical choice for \ddagger possible; $f^\ddagger : 1 \rightarrow X$ is the unique morphism from the zero object for every group endomorphism $f : \blacktriangleright X = X \rightarrow X$. However, picking X to be any non-trivial abelian group shows there is no way of defining $\dagger_{X,X}$. It is enough to consider $h : \blacktriangleright X \times X = X \times X \rightarrow X$ as being simply the group operation $+$: Equation (2.1) then yields $h^\dagger(x) + x = h^\dagger(x)$ for every $x \in X$ and using inverses one has $x = 0$ contradicting nontriviality of X .

However, to apply Proposition 2.7, it is enough that $(\mathcal{C}, \blacktriangleright, \ddagger)$ is a *full subcategory* of a cartesian closed model of guarded recursive terms such that, moreover, the inclusion functor preserves products and \blacktriangleright .

Example 2.9. Monads provide perhaps the most natural and well-known examples of pointed endofunctors. Among delay endofunctors in Example 2.4, (1), (2), (6) and (7) happen to be monads. In (3), i.e. the CMS example, the type $\blacktriangleright \blacktriangleright A \rightarrow \blacktriangleright A$ is still inhabited (by any constant mapping), but one can easily show that monad laws cannot hold whatever candidate for monad multiplication is postulated. In the remaining (i.e., topos-theoretic) examples, monad laws fail more dramatically: $\blacktriangleright \blacktriangleright A \rightarrow \blacktriangleright A$ is not even always inhabited.

Section 2.2 below discusses a class of examples of guarded fixpoint categories with unique dagger, where the delay endofunctor arises from (a module for) a monad. Later on, in Example 2.15 and Theorem 3.6, we will return to Examples 2.4.(6)–(7): we will see that while they do not possess uniqueness, they do enjoy other properties introduced in Section 3 below.

2.2. Completely iterative theories

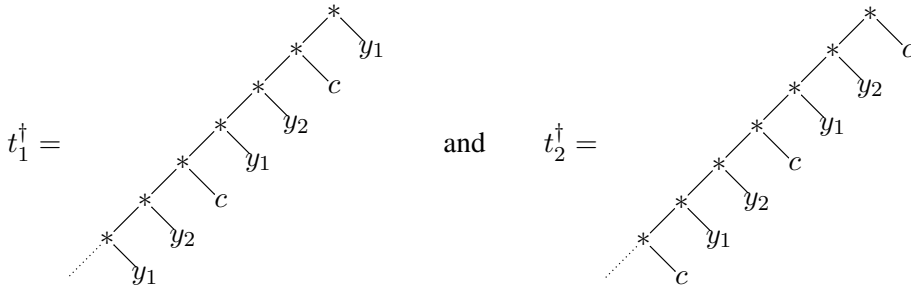
In this subsection we will explain how categories with guarded fixpoint operator capture a classical setting in which guarded recursive definitions are studied—Elgot’s (completely) iterative theories [2, 23]. The connection to guarded fixpoint operators is most easily seen if we consider monads in lieu of Lawvere theories, and so we follow the presentation of *completely iterative monads* by Milius [24]. First we recall details of their motivating example: infinite trees on a signature Σ , i.e. a sequence $(\Sigma_n)_{n < \omega}$ of sets of operation symbols with prescribed arity n . A Σ -tree t on a set X of generators is a rooted and ordered (finite or infinite) tree whose nodes with $n > 0$ children are labelled by n -ary operation symbols from Σ and a leaf is labelled by a constant symbol from Σ_0 or by a generator from X . One considers systems of mutually recursive equations of the form

$$x_i \approx t_i(\vec{x}, \vec{y}) \quad i \in I,$$

where $X = \{x_i \mid i \in I\}$ is a set of recursion variables and each t_i is a Σ -tree on $X + Y$ with Y a set of parameters (i.e. generators that do not occur on the left-hand side of a recursive equation). A system of recursive equations is *guarded* if none of the trees t_i is only a recursion variable $x \in X$. Every guarded system has a unique *solution*, which assigns to every recursion variable $x_i \in X$ a Σ -tree $t_i^\dagger(\vec{y})$ on Y such that $t_i^\dagger(\vec{y}) = t_i[\sigma]$, where σ is the substitution replacing each x_j by $t_j^\dagger(\vec{y})$. For a concrete example, let Σ consist of a binary operation symbol $*$ and a constant symbol c , i.e. $\Sigma_0 = \{c\}$, $\Sigma_2 = \{*\}$ and $\Sigma_n = \emptyset$ else. Then the following system

$$x_1 \approx x_2 * y_1 \quad x_2 \approx (x_1 * y_2) * c,$$

where y_1 and y_2 are parameters, has the following unique solution:



For any set X , let $T_\Sigma(X)$ be the set of Σ -trees on X . It has been realized by Badouel [25] that T_Σ is the object part of a monad. A system of equations is then nothing but a map

$$f : X \rightarrow T_\Sigma(X + Y),$$

and a solution is a map $f^\dagger : X \rightarrow T_\Sigma Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{f^\dagger} & T_\Sigma Y \\ f \downarrow & \circlearrowleft & \uparrow \mu_Y \\ T_\Sigma(X + Y) & \xrightarrow{T_\Sigma[f^\dagger, \eta_Y]} & T_\Sigma T_\Sigma Y \end{array}$$

where η and μ are the unit and multiplication of the monad T_Σ , respectively.

It is clear that the notion of equation and solution can be formulated for every monad S . However, the notion of guardedness requires one to speak about *non-variables* in S . This is enabled by Elgot’s notion of *ideal theory* [2], which for a finitary monad on Set is equivalent to the notion recalled in the following definition. We assume for the rest of this subsection that \mathcal{A} is a category with finite coproducts such that coproduct injections are monomorphic.

Definition 2.10. ([26])

An *ideal monad* on \mathcal{A} is a six-tuple $(S, \eta, \mu, S', \sigma, \mu')$ consisting of a monad (S, η, μ) on \mathcal{A} , a subfunctor $\sigma : S' \hookrightarrow S$ and a natural transformation $\mu' : S'S \rightarrow S'$ such that

- (1) $S = S' + \text{Id}$ with coproduct injections σ and η , and
- (2) μ restricts to μ' along σ , i.e.,

$$\begin{array}{ccc}
 S'S & \xrightarrow{\mu'} & S' \\
 \sigma S \downarrow & \circlearrowleft & \downarrow \sigma \\
 SS & \xrightarrow{\mu} & S
 \end{array} \tag{2.4}$$

The subfunctor S' of an ideal monad S allows us to formulate the notion of a guarded equation system abstractly; this leads to the notion of completely iterative theory of Elgot et al. [23] for which we here present the formulation with monads from Milius [24]:

Definition 2.11. Let $(S, \eta, \mu, S', \sigma, \mu')$ be an ideal monad on \mathcal{A} .

1. An *equation morphism* is a morphism $f : X \rightarrow S(X + Y)$ in \mathcal{A} , where X is an object (“of variables”) and Y is an object (“of parameters”).
2. A *solution* of f is a morphism $f^\dagger : X \rightarrow SY$ such that

$$\begin{array}{ccc}
 X & \xrightarrow{f^\dagger} & SY \\
 f \downarrow & \circlearrowleft & \uparrow \mu_Y \\
 S(X + Y) & \xrightarrow{S[f^\dagger, \eta_Y]} & SSY
 \end{array} \tag{2.5}$$

3. The equation morphism f is called *guarded* if it factors through the summand $S'(X + Y) + Y$ of $S(X + Y) = S'(X + Y) + X + Y$:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & S(X + Y) \\
 \circlearrowleft & & \uparrow [\sigma_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \\
 & \searrow & S'(X + Y) + Y
 \end{array} \tag{2.6}$$

4. The given ideal monad is called *completely iterative* if every guarded equation morphism has a unique solution.

Examples 2.12. We only briefly mention two examples of completely iterative monads. More can be found in literature [26, 24, 27].

- (1) The monad T_Σ of Σ -trees is a completely iterative monad.
- (2) A more general example is given by parametrized final coalgebras. Let $H : \mathcal{A} \rightarrow \mathcal{A}$ be an endofunctor such that for every object X of \mathcal{A} a final coalgebra TX for $H(-) + X$ exists. Then T is the object assignment of a completely iterative monad; in fact, T is the free completely iterative monad on H (see [24]).

We will now explain how a completely iterative monad S yields a guarded fixpoint category. Namely, let us show that the dual of its Kleisli category $\mathcal{C} = (\mathcal{A}_S)^{op}$ is equipped with a guarded fixpoint operator. First, since \mathcal{A}_S has coproducts given by the coproducts in \mathcal{A} , we see that \mathcal{C} has products. Next we need to obtain the endofunctor \blacktriangleright on \mathcal{C} . This will be given as the dual of an extension of the subfunctor $S' : \mathcal{A} \rightarrow \mathcal{A}$ of S to the Kleisli category \mathcal{A}_S . Indeed, it is well-known that to have an extension of S' to \mathcal{A}_S is equivalent to having a distributive law of the functor S' over the monad S (see Mulry [28]).

One easily verifies that the natural transformation $S'S \xrightarrow{\mu'} S' \xrightarrow{\eta^{S'}} SS'$ satisfies the two required laws and thus yields a distributive law. The corresponding extension of S' maps X to $S'X$ and a morphism $X \rightarrow SY$ of \mathcal{A}_S to

$$S'X \xrightarrow{S'f} S'SY \xrightarrow{\mu'_Y} S'Y \xrightarrow{\eta_{S'Y}} SS'Y.$$

Moreover, the endofunctor $\blacktriangleright^{op} = S'$ on \mathcal{A}_S is copointed, i.e. we have a natural transformation p from S' to $\text{Id} : \mathcal{A}_S \rightarrow \mathcal{A}_S$; indeed, its components at X are given by the coproduct injections $\sigma_X : S'X \rightarrow SX$, and it is not difficult to verify that this is a natural transformation; thus, \blacktriangleright is a pointed endofunctor on \mathcal{C} .

Now observe that $\mathcal{C}(\blacktriangleright X \times Y, X)$ is just $\mathcal{A}(X, S(S'X + Y))$. We are ready to describe the guarded fixpoint operator on \mathcal{C} .

Construction 2.13. For any morphism $f : X \rightarrow S(S'X + Y)$ form the following morphism

$$\bar{f} = (X \xrightarrow{f} S(S'X + Y) \xrightarrow{S(\sigma_X + \eta_Y)} S(SX + SY) \xrightarrow{S\text{can}} SS(X + Y) \xrightarrow{\mu_{X+Y}} S(X + Y)),$$

where $\text{can} = [S\text{inl}, S\text{inr}] : SX + SY \rightarrow S(X + Y)$. We shall verify that \bar{f} is a guarded equation morphism for S which allows defining $f^\dagger : X \rightarrow SY$ as the unique solution of \bar{f} .

Proposition 2.14. For every f , f^\dagger from Construction 2.13 is the unique morphism satisfying (2.1).

Proof:

We first verify that f^\dagger is well-defined, i.e. \bar{f} is guarded with the following factor \bar{f}_0 :

$$\begin{array}{ccccc} X & \xrightarrow{f} & S(S'X + Y) & \xrightarrow{[\sigma, \eta]^{-1}} & S'(S'X + Y) + S'X + Y \\ \bar{f}_0 \downarrow & & \swarrow [\mu', S'\text{inl}] + Y & & \downarrow S'(\sigma + \eta) + S'X + Y \\ S'(X + Y) + Y & \xleftarrow{\quad} & S'S(X + Y) + S'X + Y & \xleftarrow{S'\text{can} + S'X + Y} & S'(SX + SY) + S'X + Y \end{array}$$

Notice that both \bar{f} and \bar{f}_0 start with f . Thus, in order to prove $\bar{f} = [\sigma_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \cdot \bar{f}_0$ (see (2.6)) we can remove f . Then it suffices to prove that the two remaining morphisms are equal when precomposed with the isomorphism $[\sigma, \eta] : S'(S'X + Y) + S'X + Y \rightarrow S(S'X + Y)$, i.e. we verify

$$\begin{aligned} & [\sigma_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \cdot ([\mu'_{X+Y}, S'\text{inl}] + Y) \cdot (S'\text{can} + S'X + Y) \cdot (S'(\sigma_X + \eta_Y) + S'X + Y) \\ &= \mu_{X+Y} \cdot \text{Scan} \cdot S(\sigma_X + \eta_Y) \cdot [\sigma_{S'X+Y}, \eta_{S'X+Y}]. \end{aligned}$$

We consider the two coproduct components separately and compute for the left-hand component

$$\begin{aligned} \sigma_{X+Y} \cdot \mu'_{X+Y} \cdot S'\text{can} \cdot S'(\sigma_X + \eta_Y) &\stackrel{(2.4)}{=} \mu_{X+Y} \cdot \sigma_{S(X+Y)} \cdot S'\text{can} \cdot S'(\sigma_X + \eta_Y) \\ &= \mu_{X+Y} \cdot \text{Scan} \cdot S(\sigma_X + \eta_Y) \cdot \sigma_{S'X+Y}, \end{aligned}$$

where the second step uses naturality of σ twice; for the right-hand component we obtain

$$\begin{aligned} \mu_{X+Y} \cdot \text{Scan} \cdot S(\sigma_X + \eta_Y) \cdot \eta_{S'X+Y} &= \mu_{X+Y} \cdot \eta_{S(X+Y)} \cdot \text{can} \cdot (\sigma_X + \eta_Y) \quad (\text{by nat. of } \eta) \\ &= \underbrace{[S'\text{inl}, S'\text{inr}]}_{=\text{can}} \cdot (\sigma_X + \eta_Y) \quad (\text{since } \mu \cdot \eta S = \text{id}) \\ &= [S'\text{inl} \cdot \sigma_X, S'\text{inr} \cdot \eta_Y] \\ &= [\sigma_{X+Y} \cdot S'\text{inl}, \eta_{X+Y} \cdot \text{inr}] \quad (\text{by nat. of } \sigma \text{ and } \eta) \\ &= [\sigma_{X+Y}, \eta_{X+Y} \cdot \text{inr}] \cdot (S'\text{inl} + Y). \end{aligned}$$

This finishes the proof of guardedness of \bar{f} .

Now observe that (2.1) can equivalently be expressed in \mathcal{A} (using $\mathcal{C} = (\mathcal{A}_S)^{op}$) as the outside square of the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{f^\dagger} & SY & & \\ & \searrow \bar{f} & \uparrow \mu & & \\ & & SSY & & \\ & & \uparrow S[f^\dagger, \eta_Y] & & \\ S(S'X + Y) & \xrightarrow{S(\sigma+\eta)} & S(SX + SY) & \xrightarrow{S\text{can}} & SS(X + Y) & \xrightarrow{\mu_{X+Y}} & S(X + Y) \end{array}$$

cf. (2.5)

This outside commutes iff the upper right-hand triangle commutes iff f^\dagger is a solution of \bar{f} . Since the latter exists uniquely we see that f^\dagger is the desired unique morphism satisfying (2.1). \square

Examples 2.15. Several items in Examples 2.4 are unique guarded fixpoint categories; this holds for Examples 2.4.(2)–(5), and also for the example of completely iterative monads in Section 2.2. However, Example 2.4.(6) is not a unique guarded fixpoint category: for let $X = \{0, 1\}$ be the two-chain, $Y = 1$ the one element cpo and $f : X_\perp = X_\perp \times Y \rightarrow X$ be the map with $f(0) = f(\perp) = 0$ and $f(1) = 1$. Then both $0 : 1 \rightarrow X$ and $1 : 1 \rightarrow X$ make (2.1) commutative.

3. Properties of guarded fixpoint operators

In this section, we study properties of guarded fixpoint operators. Except for uniformity, these properties are purely equational. They are generalizing analogous properties of Bloom and Ésik's iteration theories [14]; more precisely, they would collapse to the original, unguarded counterparts when \blacktriangleright is instantiated to the identity endofunctor as in Example 2.4.(1). Just like these original counterparts, they are all satisfied whenever the operator assigns a unique fixpoint (Theorem 3.4), but this is not a necessary condition for them to hold, as witnessed by Theorems 3.6 and 3.7 concerning Examples 2.4.(6)–(7). However, the standard notion of dinaturality turns out to behave surprisingly enough in the guarded setting to merit a separate Subsection 3.2. As a prerequisite, we also discuss the so-called Bekič identity in Section 3.1.

Definition 3.1. We define the following possible properties of a guarded fixpoint category $(\mathcal{C}, \blacktriangleright, \dagger)$:

(1) **Fixpoint Identity** (\dagger). For every $f : \blacktriangleright X \times Y \rightarrow X$, (2.1) holds. This is built into the definition of guarded fixpoint categories and only mentioned here again for the sake of completeness.

(2) **Parameter Identity** (P). For every $f : \blacktriangleright X \times Y \rightarrow X$ and every $h : Z \rightarrow Y$,

$$Z \xrightarrow{h} Y \xrightarrow{f^\dagger} X = (\blacktriangleright X \times Z \xrightarrow{\blacktriangleright X \times h} \blacktriangleright X \times Y \xrightarrow{f} X)^\dagger.$$

(3) **(Simplified) Composition Identity** (C). Given $f : \blacktriangleright X \times Y \rightarrow Z$ and $g : Z \rightarrow X$,

$$(\blacktriangleright X \times Y \xrightarrow{f} Z \xrightarrow{g} X)^\dagger = (Y \xrightarrow{(f \cdot (\blacktriangleright g \times Y))^\dagger} Z \xrightarrow{g} X).$$

(4) **Double Dagger Identity** ($\dagger\dagger$). For every $f : \blacktriangleright X \times \blacktriangleright X \times Y \rightarrow X$,

$$(Y \xrightarrow{f^\dagger\dagger} X) = (\blacktriangleright X \times Y \xrightarrow{\Delta \times Y} \blacktriangleright X \times \blacktriangleright X \times Y \xrightarrow{f} X)^\dagger.$$

(5) **Uniformity** (U). Given $f : \blacktriangleright X \times Y \rightarrow X$, $g : \blacktriangleright X' \times Y \rightarrow X'$ and $h : X \rightarrow X'$,

$$\begin{array}{ccc} \blacktriangleright X \times Y & \xrightarrow{f} & X \\ \blacktriangleright h \times Y \downarrow & \circlearrowleft & \downarrow h \\ \blacktriangleright X' \times Y & \xrightarrow{g} & X' \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} & & X \\ Y & \xrightarrow{f^\dagger} & \downarrow h \\ & \circlearrowleft & \\ Y & \xrightarrow{g^\dagger} & X' \end{array}$$

We call the first four properties (1)–(4) the *Conway axioms*.

Notice that the Conway axioms are equational properties while (5) is quasiequational, i.e., an implication between equations.

Next we shall show that in the presence of certain of the above properties the natural transformation $p : \text{Id} \rightarrow \blacktriangleright$ is a derived structure. Let $(\mathcal{C}, \blacktriangleright)$ be equipped with an operator \dagger *not necessarily satisfying (2.1)*. For every object X of \mathcal{C} define $q_X : X \rightarrow \blacktriangleright X$ as follows: consider

$$f_X = (\blacktriangleright(\blacktriangleright X \times X) \times X \xrightarrow{\blacktriangleright \pi_r \times X} \blacktriangleright X \times X)$$

and form

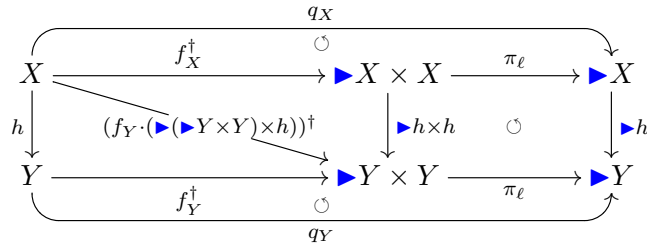
$$q_X = (X \xrightarrow{f_X^\dagger} \blacktriangleright X \times X \xrightarrow{\pi_\ell} \blacktriangleright X).$$

Lemma 3.2.

1. If \dagger satisfies the parameter identity and uniformity, then $q : \text{Id} \rightarrow \blacktriangleright$ is a natural transformation.
2. If \dagger satisfies the fixpoint identity, then $q_X = p_X$ for all X .

Proof:

1. For every morphism $h : X \rightarrow Y$ we have the following diagram:



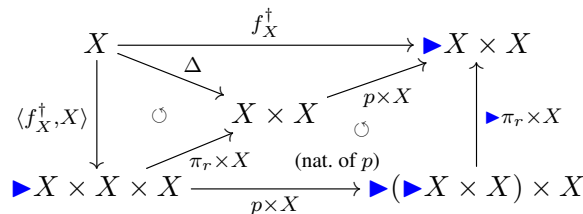
Of the left-hand square, the lower left-hand triangle commutes by the parameter identity and the upper right-hand triangle by uniformity since we have

$$\begin{array}{ccc} \blacktriangleright(\blacktriangleright X \times X) \times X & \xrightarrow{f_X =} & \blacktriangleright X \times X \\ \downarrow \blacktriangleright(h \times h) \times X & \searrow \blacktriangleright(h \times h) \times h & \downarrow h \times h \\ \blacktriangleright(\blacktriangleright Y \times Y) \times X & \xrightarrow{\blacktriangleright(\blacktriangleright Y \times Y) \times h} & \blacktriangleright(\blacktriangleright Y \times Y) \times Y \xrightarrow{f_Y =} \blacktriangleright Y \times Y \end{array}$$

2. Notice first that from the fixpoint identity for f_X^\dagger we have:

$$\begin{aligned} \pi_r \cdot f_X^\dagger &\stackrel{(\dagger)}{=} \pi_r \cdot f_X \cdot (p_{\blacktriangleright X \times X} \times X) \cdot \langle f_X^\dagger, X \rangle \\ &= \pi_r \cdot (\blacktriangleright \pi_r \times X) \cdot (p_{\blacktriangleright X \times X} \times X) \cdot \langle f_X^\dagger, X \rangle \\ &= \text{id}_X \end{aligned}$$

Then we consider the following diagram:



Since its outside commutes by the fixpoint identity so does the upper inner triangle. It follows that we have $q_X = \pi_\ell \cdot f_X^\dagger = \pi_\ell \cdot (p_X \times X) \cdot \Delta = p_X$. \square

Definition 3.3. A guarded fixpoint category $(\mathcal{C}, \blacktriangleright, \dagger)$ satisfying the Conway axioms (i.e. fixpoint, parameter, composition and double dagger identities) is called a *guarded Conway category*.

If in addition uniformity is satisfied, we call $(\mathcal{C}, \blacktriangleright, \dagger)$ a *uniform guarded Conway category*.

Note that an example of a guarded fixpoint category that is not a guarded Conway category and an example of a guarded Conway category that is not a uniform guarded Conway category exist already in the realm of iteration theories of Bloom and Ésik (cf. Examples 2.4.1). See Ésik [29], Model 2 and Section 3, respectively.

Theorem 3.4. A unique guarded fixpoint category $(\mathcal{C}, \blacktriangleright)$ is a uniform guarded Conway category.

Proof:

We shall prove that \dagger satisfies the Conway axioms and uniformity.

- (1) The fixpoint identity for \dagger is satisfied by definition of a unique guarded fixpoint category.
- (2) Parameter identity. Given $f : \blacktriangleright X \times Y \rightarrow X$ and $h : Z \rightarrow Y$ we have

$$\begin{array}{ccccc}
 Z & \xrightarrow{h} & Y & \xrightarrow{f^\dagger} & X \\
 \downarrow \langle f^\dagger \cdot h, Z \rangle & & \downarrow \langle f^\dagger, Y \rangle & \circlearrowleft \text{ by } (\dagger) & \uparrow f \\
 & & X \times Y & \xrightarrow{p_X \times Y} & \blacktriangleright X \times Y \\
 & \nearrow X \times h & \circlearrowleft & & \uparrow \blacktriangleright X \times h \\
 X \times Z & \xrightarrow{p_X \times Z} & \blacktriangleright X \times Z & &
 \end{array}$$

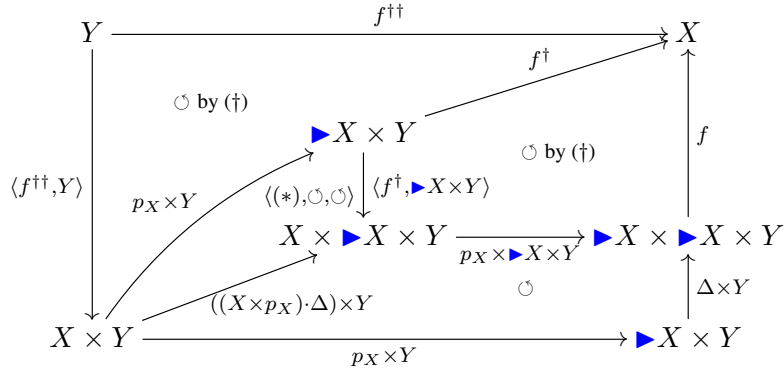
Thus, $f^\dagger \cdot h$ fits square (2.1) for $f \cdot (\blacktriangleright X \times h)$, and thus we have the desired equation by the uniqueness of $(f \cdot (\blacktriangleright X \times h))^\dagger$.

- (3) Composition Identity. Let f and g be as in the definition of the identity. Then we have

$$\begin{array}{ccccc}
 Y & \xrightarrow{(f \cdot (\blacktriangleright g \times Y))^\dagger} & Z & \xrightarrow{g} & X \\
 \downarrow \langle (f \cdot (\blacktriangleright g \times Y))^\dagger, Y \rangle & & \uparrow f & \circlearrowleft & \uparrow g \\
 & \circlearrowleft \text{ by } (\dagger) & \blacktriangleright X \times Y & \xrightarrow{f} & Z \\
 & & \uparrow \blacktriangleright g \times Y & & \uparrow f \\
 Z \times Y & \xrightarrow{p_Z \times Y} & \blacktriangleright Z \times Y & \circlearrowleft & \\
 \downarrow g \times Y & & \downarrow \blacktriangleright g \times Y & \circlearrowleft \text{ by nat. of } p & \\
 X \times Y & \xrightarrow{p_X \times Y} & \blacktriangleright X \times Y & &
 \end{array}$$

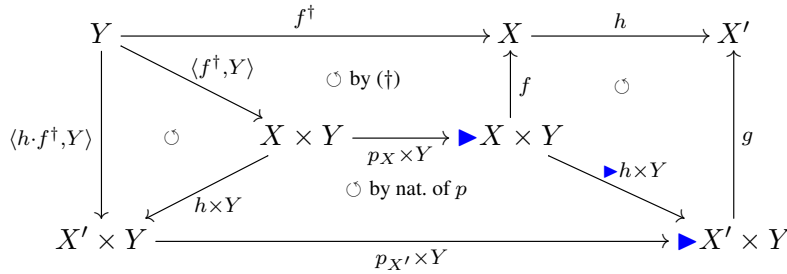
Since the outside commutes we obtain the desired equation by the unicity of $(g \cdot f)^\dagger$.

(4) Double Dagger Identity. Let $f : \blacktriangleright X \times \blacktriangleright X \times Y \rightarrow X$. Then we have



We do not claim that part $(*)$ commutes, but it commutes when precomposed with $\langle f^{\dagger\dagger}, Y \rangle$. This is because the lower passage yields simply $f^{\dagger\dagger}$ and the upper passage yields $f^{\dagger} \cdot (p_X \times Y) \cdot \langle f^{\dagger\dagger}, Y \rangle$, which is equal to $f^{\dagger\dagger}$ by the fixpoint identity. We conclude that the outside of the diagram commutes and so we obtain the desired equality by the unicity of $(f \cdot (\Delta \times Y))^{\dagger}$.

(5) Uniformity. Let f, g and h be as in the definition of uniformity. Then we have



Since the outside commutes we obtain $h \cdot f^{\dagger} = g^{\dagger}$ by unicity of g^{\dagger} . □

Example 3.5. Coming back to Example 2.15 we see that the unique \dagger in Examples 2.4.(2)–(5) and the one for the example of completely iterative monads in Section 2.2 satisfy all the properties in Definition 3.1. But Example 2.15 also entails that for let-ccc’s with fixpoint objects of Example 2.4.(7) one cannot hope for uniqueness. So how about all the properties of Definition 3.1?

Theorem 3.6. Let $(\mathcal{C}, T, \eta, \mu, s, T\Omega \xrightarrow{\sigma} \Omega, 1 \xrightarrow{\omega} \Omega)$ be a let-ccc with a fixpoint object, let $\blacktriangleright = T$ and for any $f : Y \times TX \rightarrow X$ define $f^{\dagger} = \text{it}(f) \cdot (Y \times \omega)$, as in Example 2.4.(7). Then \blacktriangleright satisfies all properties introduced in Definition 3.1, except, possibly, the double dagger identity $(\dagger\dagger)$.

We will see in Theorem 3.7 below that for the special case where $\mathcal{C} = \text{CPO}$ and $T = (-)_{\perp}$ (see Example 2.4.(6)) more can be shown.

Proof:

Recall first the 4 axioms (a)–(d) of the operation $(-)^*$ from Example 2.4.(7). Further observe that the action of T on a morphism $g : Z = 1 \times Z \rightarrow X$ is defined by

$$Tg = (\eta_X \cdot g)^* : TZ = 1 \times TZ \rightarrow TX. \tag{3.1}$$

Notice that axioms (a) and (d) imply for $f : B \rightarrow TC$ and $g : C \rightarrow TD$ (i.e. the special case for $A = 1$) the usual extension laws of Manes [22, Definition 2.13]:

$$f^* \cdot \eta_B = f \quad \text{and} \quad (g^* \cdot f)^* = g^* \cdot f^*. \quad (3.2)$$

Finally, recall that for let-ccc's we write $T = \blacktriangleright$ on the right-hand side of product components. The fixpoint identity (\dagger). Take $f : Y \times TX \rightarrow X$. Then:

$$\begin{array}{ccccc}
 & & & & f^\dagger \\
 & & & & \circlearrowleft \\
 Y = Y \times 1 & \xrightarrow{Y \times \omega} & Y \times \Omega & \xrightarrow{\text{it}(f)} & X \\
 \langle \pi_\ell, f^\dagger \rangle \downarrow & \searrow^{Y \times \omega} & \circlearrowleft \text{ by def. of } \omega & \nearrow^{Y \times \sigma} & \uparrow f \\
 & & Y \times \Omega & \xrightarrow{Y \times \eta} & Y \times T\Omega & \circlearrowleft \text{ by (2.2)} \\
 & & \circlearrowleft & \circlearrowleft \text{ by axiom (a)} & \langle \pi_\ell, (\eta \cdot \text{it}(f))^* \rangle \\
 & & \langle \pi_\ell, \text{it}(f) \rangle & & \\
 Y \times X & \xrightarrow{Y \times \eta} & Y \times TX & &
 \end{array}$$

The parameter identity (P). Take $f : Y \times TX \rightarrow X$ and $h : Z \rightarrow Y$, and define $g = f \cdot (h \times TX)$. Then

$$\begin{array}{ccccccc}
 & & & & & & (f \cdot (h \times TX))^\dagger \\
 & & & & & & \circlearrowleft \\
 Z = Z \times 1 & \xrightarrow{Z \times \omega} & Z \times \Omega & \xrightarrow{\text{it}(g)} & X \\
 \downarrow h = h \times 1 & \searrow^{Z \times \omega} & \circlearrowleft \text{ by def. of } \omega & \nearrow^{Z \times \sigma} & \circlearrowleft \text{ by (2.2)} & \searrow^{g} & \\
 & & Z \times \Omega & \xrightarrow{Z \times \eta} & Z \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta_X \cdot \text{it}(g))^* \rangle} & Z \times TX \\
 & & \circlearrowleft & \circlearrowleft & \langle \circlearrowleft, (*) \rangle & \langle \circlearrowleft, (**) \rangle & \downarrow h \times TX \\
 & & & & Y \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta_X \cdot \text{it}(f))^* \rangle} & Y \times TX \\
 & & & & \downarrow Y \times \sigma & \circlearrowleft \text{ by (2.2)} & \downarrow f \\
 Y = Y \times 1 & \xrightarrow{Y \times \omega} & Y \times \Omega & \xrightarrow{\text{it}(f)} & X \\
 & & \circlearrowleft & \circlearrowleft & \circlearrowleft \\
 & & & & f^\dagger
 \end{array}$$

The part (*) by itself does not need to commute, but it does when precomposed with ω . The task reduces then to showing (**), viz. the equation

$$(\eta_X \cdot \text{it}(f))^* \cdot (h \times T\Omega) = (\eta_X \cdot \text{it}(f \cdot (h \times TX)))^*.$$

By axiom (c), this reduces to showing

$$(\eta_X \cdot \text{it}(f) \cdot (h \times \Omega))^* = (\eta_X \cdot \text{it}(f \cdot (h \times TX)))^*.$$

To show this it is sufficient to prove

$$\text{it}(f) \cdot (h \times \Omega) = \text{it}(f \cdot (h \times TX)).$$

A proof of this relies on $\text{it}(-)$ being the unique morphism satisfying a suitable instance of (2.2):

$$\begin{array}{ccccc}
 Z \times \Omega & \xrightarrow{h \times \Omega} & Y \times \Omega & \xrightarrow{\text{it}(f)} & X \\
 \uparrow Z \times \sigma & \circlearrowleft & \uparrow Y \times \sigma & \circlearrowleft \text{ by (2.2)} & \uparrow f \\
 & & Y \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta \cdot \text{it}(f))^* \rangle} & Y \times TX \\
 & \nearrow h \times T\Omega & & \circlearrowleft & \uparrow h \times TX \\
 Z \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta \cdot \text{it}(f)) \cdot (h \times \Omega)^* \rangle} & & & Z \times TX \\
 & & & \circlearrowleft (\circlearrowleft, (\star)) &
 \end{array}$$

and we get (\star) by axiom (c) again.

Composition Identity (C). Assume $f : Y \times TX \rightarrow Z$ and $g : Z \rightarrow X$. We want to show:

$$\text{it}(g \cdot f) \cdot (Y \times \omega) = g \cdot \text{it}(f \cdot (Y \times Tg)) \cdot (Y \times \omega).$$

For this, it is enough to show

$$\text{it}(g \cdot f) = g \cdot \text{it}(f \cdot (Y \times Tg)).$$

We again use the fact that $\text{it}(-)$ is the unique morphism satisfying a suitable instance of (2.2), which in this case is:

$$\begin{array}{ccccc}
 Y \times \Omega & \xrightarrow{\text{it}(f \cdot (Y \times Tg))} & Z & \xrightarrow{g} & X \\
 \uparrow Y \times \sigma & \circlearrowleft \text{ by (2.2)} & \uparrow f \cdot (Y \times Tg) & \circlearrowleft & \uparrow g \\
 & & Y \times TZ & \xrightarrow{Y \times Tg} & Z \\
 & \nearrow \langle \pi_\ell, (\eta \cdot \text{it}(f \cdot (Y \times Tg)))^* \rangle & & \circlearrowleft & \uparrow f \\
 Y \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta \cdot g \cdot \text{it}(f \cdot (Y \times Tg)))^* \rangle} & & & Y \times TX \\
 & & & \circlearrowleft (\circlearrowleft, (\star)) &
 \end{array}$$

For part (\star) we compute

$$\begin{aligned}
 Tg \cdot (\eta_Z \cdot \text{it}(f \cdot (Y \times Tg)))^* &= (\eta_X \cdot g)^* \cdot (\eta_Z \cdot \text{it}(f \cdot (Y \times Tg)))^* && \text{by (3.1)} \\
 &= ((\eta_X \cdot g)^* \cdot \eta_Z \cdot \text{it}(f \cdot (Y \times Tg)))^* && \text{by (3.2)} \\
 &= (\eta_X \cdot g \cdot \text{it}(f \cdot (Y \times Tg)))^* && \text{by (3.2)}.
 \end{aligned}$$

Uniformity (U). Assume $f : Y \times TX \rightarrow X$, $g : Y \times TX' \rightarrow X'$ and $h : X \rightarrow X'$ are such that $h \cdot f = g \cdot (Y \times Th)$ holds. Our goal is to show $h \cdot f^\dagger = g^\dagger$, for which it is sufficient to show $\text{it}(g) = h \cdot \text{it}(f)$. Once again, we rely on initiality property (2.2), i.e., we need to show:

$$\begin{array}{ccccc}
 Y \times \Omega & \xrightarrow{\text{it}(f)} & X & \xrightarrow{h} & X' \\
 \uparrow Y \times \sigma & \circlearrowleft \text{ by (2.2)} & \uparrow f & \circlearrowleft \text{ by assumption} & \uparrow g \\
 & & Y \times TX & \xrightarrow{Y \times Th} & Y \times TX' \\
 & \nearrow \langle \pi_\ell, (\eta \cdot \text{it}(f))^* \rangle & & \circlearrowleft (\circlearrowleft, (\star)) & \\
 Y \times T\Omega & \xrightarrow{\langle \pi_\ell, (\eta \cdot h \cdot \text{it}(f))^* \rangle} & & & Y \times TX'
 \end{array}$$

For (*) we reason as follows:

$$\begin{aligned}
Th \cdot (\eta_X \cdot \text{it}(f))^* &= (\eta_{X'} \cdot h)^* \cdot (\eta_X \cdot \text{it}(f))^* && \text{by (3.1)} \\
&= ((\eta_{X'} \cdot h)^* \cdot \eta_X \cdot \text{it}(f))^* && \text{by (3.2)} \\
&= (\eta_{X'} \cdot h \cdot \text{it}(f))^* && \text{by (3.2).} \quad \square
\end{aligned}$$

Theorem 3.7. The category CPO with $\blacktriangleright = (-)_\perp$ and the dagger given by least fixpoint as in Example 2.4.(6) satisfies all the properties of Definition 3.1.

Proof:

In the light of Theorem 3.6, we only need to show $(\dagger\dagger)$. We use the notation of Example 2.4.(6). In addition, for any $f : X_\perp \times Y \rightarrow X$, we define continuous functions $s_n : Y \rightarrow X_\perp$, $n \in \mathbb{N}$ as

$$s_0 = \lambda y. \perp, \quad s_{n+1} = \Phi_f(s_n)$$

so that the least fixpoint of Φ_f is $s = \bigsqcup_{n \in \mathbb{N}} s_n$.

Now suppose we are given $f : X_\perp \times X_\perp \times Y \rightarrow X$. To prove $(\dagger\dagger)$ we will first show that the least fixpoints s of $\Phi_{f\dagger}$ and s' of $\Phi_{f \cdot (\Delta \times Y)}$ coincide, i.e. we prove (a) $s \sqsubseteq s'$ and (b) $s' \sqsubseteq s$.

For (a), it suffices to show that s' is a prefixpoint of $\Phi_{f\dagger}$, i.e.

$$p_X \cdot f^\dagger \cdot \langle s', Y \rangle \sqsubseteq s'. \quad (3.3)$$

To see this let s'' be the least fixpoint of Φ_f . We will prove that

$$s'' \cdot \langle s', Y \rangle = s'. \quad (3.4)$$

by showing the two inequalities below by induction on n :

$$s''_n \cdot \langle s', Y \rangle \sqsubseteq s' \quad \text{and} \quad s'_n \sqsubseteq s'' \cdot \langle s', Y \rangle. \quad (3.5)$$

Note that the left-hand inequalities above already imply (3.3) using that

$$p_X \cdot f^\dagger = p_X \cdot f \cdot \langle s'', Y \rangle = \Phi_f(s'') = s''.$$

The right-hand inequalities in (3.5) will be used at the end of our proof.

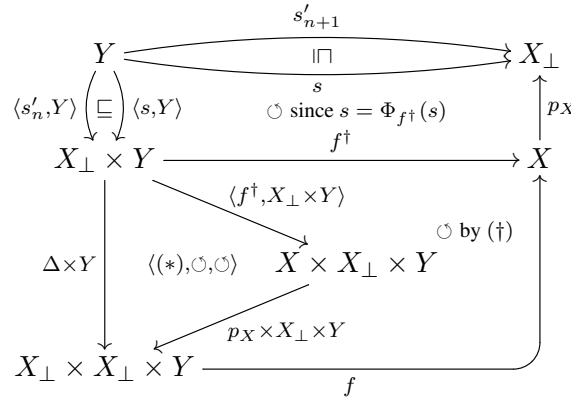
For the induction proofs the base cases are obvious: $\perp \cdot \langle s', Y \rangle = \perp \sqsubseteq s'$ and $\perp \sqsubseteq s'' \cdot \langle s', Y \rangle$. For the induction steps we obtain

$$\begin{aligned}
s''_{n+1} \cdot \langle s', Y \rangle &= p_X \cdot f \cdot \langle s''_n, X_\perp \times Y \rangle \cdot \langle s', Y \rangle && \text{since } s''_{n+1} = \Phi_f(s''_n) \\
&\sqsubseteq p_X \cdot f \cdot \langle s', X_\perp \times Y \rangle \cdot \langle s', Y \rangle && \text{by induction hypothesis} \\
&= p_X \cdot f \cdot \langle s', s', Y \rangle \\
&= p_X \cdot f \cdot (\Delta \times Y) \cdot \langle s', Y \rangle \\
&= s' && \text{since } s' = \Phi_{f \cdot (\Delta \times Y)}(s')
\end{aligned}$$

and

$$\begin{aligned}
 s'_{n+1} &= p_X \cdot f \cdot (\Delta \times Y) \cdot \langle s'_n, Y \rangle && \text{since } s'_{n+1} = \Phi_{f \cdot (\Delta \times Y)}(s'_n) \\
 &= p_X \cdot f \cdot \langle s'_n, s'_n, Y \rangle \\
 &\sqsubseteq p_X \cdot f \cdot \langle s'' \cdot \langle s', Y \rangle, s', Y \rangle && \text{by induction hypothesis and } s'_n \sqsubseteq s' \\
 &= p_X \cdot f \cdot \langle s'', X_\perp \times Y \rangle \cdot \langle s', Y \rangle \\
 &= s'' \cdot \langle s', Y \rangle && \text{since } s'' = \Phi_f(s'').
 \end{aligned}$$

For inequality (b) we prove by induction on n that $s_n \sqsubseteq s'$ holds for all n . The base case is again trivial: $\perp \sqsubseteq s'$. For the induction step suppose that $s'_n \sqsubseteq s$. Then we consider the following diagram



Its outside commutes since $s'_{n+1} = \Phi_{f \cdot (\Delta \times Y)}(s'_n)$ and $(*)$ commutes when extended by $\langle s, Y \rangle$ since s is a fixpoint of Φ_{f^\dagger} . The equalities in the diagram together with the inequality obtained from the induction hypothesis in the upper left-hand corner yield the desired inequality in the top row.

We are now ready to prove the desired equality $f^{\dagger\dagger} = (f \cdot (\Delta \times Y))^{\dagger\dagger}$:

$$\begin{aligned}
 f^{\dagger\dagger} &= f^\dagger \cdot \langle s, Y \rangle && \text{by def. of } f^{\dagger\dagger} \\
 &= f \cdot \langle s'', Y \rangle \cdot \langle s, Y \rangle && \text{by def. of } f^\dagger \\
 &= f \cdot \langle s'', Y \rangle \cdot \langle s', Y \rangle && \text{since } s = s' \\
 &= f \cdot \langle s', s', Y \rangle && \text{by (3.4)} \\
 &= f \cdot (\Delta \times Y) \cdot \langle s', Y \rangle \\
 &= (f \cdot (\Delta \times Y))^{\dagger\dagger} && \text{by def. of } (-)^{\dagger\dagger}
 \end{aligned}$$

This completes the proof. □

Open Problem 3.8. Do let-ccc's with fixpoint objects with the dagger defined as in Example 2.4.(7) satisfy the double dagger property $(\dagger\dagger)$?

We do not see how an argument using two inequalities as in (a) and (b) as well as in (3.5) generalizes to let-ccc's. However, as the flagship Example 2.4.(6) satisfies $(\dagger\dagger)$, we believe that a counterexample might be intricate.

3.1. The Bekič identity

We generalize here the known fact that the double dagger identity can be replaced by the *Bekič identity* (also known as the *pairing identity*) among axioms of unguarded Conway theories (see, e.g., [30], [14, Ch. 6.2, 6.8–6.9], [31], [20, Ch. 7.1] and references therein). We will make use of this in our discussion of another property, dinaturality, in Section 3.2 and also in the discussion of trace operators in Section 4.

Definition 3.9. We introduce the following possible property of a guarded fixpoint category $(\mathcal{C}, \blacktriangleright, \dagger)$:

Bekič identity (Bč). For any $f : \blacktriangleright X \times \blacktriangleright Y \times A \rightarrow X$ and $g : \blacktriangleright X \times \blacktriangleright Y \times A \rightarrow Y$,

$$(\blacktriangleright(X \times Y) \times A \xrightarrow{\text{can} \times A} \blacktriangleright X \times \blacktriangleright Y \times A \xrightarrow{\langle f, g \rangle} X \times Y)^\dagger = \langle e_L^\dagger, e_R^\dagger \rangle,$$

where

$$e_R = (\blacktriangleright Y \times A \xrightarrow{\langle p_X \cdot f^\dagger, \blacktriangleright Y \times A \rangle} \blacktriangleright X \times \blacktriangleright Y \times A \xrightarrow{g} Y),$$

$$e_L = (\blacktriangleright X \times A \xrightarrow{\blacktriangleright X \times \langle p_Y \cdot e_R^\dagger, A \rangle} \blacktriangleright X \times \blacktriangleright Y \times A \xrightarrow{f} X).$$

Proposition 3.10. The Bekič identity holds in each guarded Conway category $(\mathcal{C}, \blacktriangleright, \dagger)$.

Proof:

First observe that $\text{can} = (\blacktriangleright \pi_\ell \times \blacktriangleright \pi_r) \cdot \Delta$:

$$\begin{array}{ccc} & \blacktriangleright(X \times Y) & \\ & \xrightarrow{\quad \quad \quad} & \\ \blacktriangleright(X \times Y) & \xrightarrow{\blacktriangleright(\Delta)} \blacktriangleright((X \times Y) \times (X \times Y)) & \xrightarrow{\blacktriangleright(\pi_\ell \times \pi_r)} \blacktriangleright(X \times Y) \\ & \searrow \Delta & \downarrow \text{can} \\ & \blacktriangleright(X \times Y) \times \blacktriangleright(X \times Y) & \xrightarrow{\blacktriangleright \pi_\ell \times \blacktriangleright \pi_r} \blacktriangleright X \times \blacktriangleright Y \\ & \downarrow \text{can} & \\ & \blacktriangleright X \times \blacktriangleright Y & \end{array}$$

Next we compute:

$$\begin{aligned} (\langle f, g \rangle \cdot (\text{can} \times A))^\dagger &= (\langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright \pi_r \times A) \cdot (\Delta \times A))^\dagger \\ &\stackrel{(\dagger\dagger)}{=} (\langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright \pi_r \times A))^\dagger \\ &= (\langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright Y \times A) \cdot (\blacktriangleright(X \times Y) \times \blacktriangleright \pi_r \times A))^\dagger \\ &\stackrel{(P)}{=} ((\langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright Y \times A))^\dagger \cdot (\blacktriangleright \pi_r \times A))^\dagger \end{aligned}$$

Now let $h = \langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright Y \times A) : \blacktriangleright(X \times Y) \times \blacktriangleright Y \times A \rightarrow X \times Y$. Then we have

$$\begin{aligned} \pi_\ell \cdot h^\dagger &= \pi_\ell \cdot (\langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright Y \times A))^\dagger \\ &\stackrel{(C)}{=} (\pi_\ell \cdot \langle f, g \rangle)^\dagger \\ &= f^\dagger. \end{aligned}$$

And we have

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \blacktriangleright Y \times A & \xrightarrow{\langle h^\dagger, \blacktriangleright Y \times A \rangle} & X \times Y \times (\blacktriangleright Y \times A) & \xrightarrow[p \times (\blacktriangleright Y \times A)]{\circ \text{ by } (\dagger)} & \blacktriangleright (X \times Y) \times \blacktriangleright Y \times A & \xrightarrow{h} & X \times Y \\
 & \searrow \langle f^\dagger, \blacktriangleright Y \times A \rangle & \downarrow \pi_\ell \times \blacktriangleright Y \times A & \circ \text{ by nat. of } p & \downarrow \pi_\ell \times \blacktriangleright Y \times A & \circ & \downarrow \pi_r \\
 & & X \times \blacktriangleright Y \times A & \xrightarrow[p \times \blacktriangleright Y \times A]{} & \blacktriangleright X \times \blacktriangleright Y \times A & \xrightarrow{g} & Y
 \end{array} \\
 \text{(3.6)}
 \end{array}$$

Plugging h into our first computation above we obtain

$$\begin{aligned}
 \pi_r \cdot (\langle f, g \rangle \cdot (\text{can} \times A))^\dagger &= \pi_r \cdot (h^\dagger \cdot (\blacktriangleright \pi_r \times A))^\dagger \\
 &\stackrel{(C)}{=} (\pi_r \cdot h^\dagger)^\dagger \\
 &\stackrel{(3.6)}{=} (g \cdot (p_X \times \blacktriangleright Y \times A) \cdot \langle f^\dagger, \blacktriangleright Y \times A \rangle)^\dagger \\
 &= (g \cdot \langle p_X \cdot f^\dagger, \blacktriangleright Y \times A \rangle)^\dagger \\
 &= e_R^\dagger.
 \end{aligned}$$

Let $z = \langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright \pi_r \times A)$. We saw previously that $(\langle f, g \rangle \cdot (\text{can} \times A))^\dagger = z^{\dagger\dagger}$; thus we have

$$\begin{aligned}
 \pi_\ell \cdot (\langle f, g \rangle \cdot (\text{can} \times A))^\dagger &= \pi_\ell \cdot z^{\dagger\dagger} \\
 &\stackrel{(\dagger)}{=} \pi_\ell \cdot z^\dagger \cdot (p_{X \times Y} \times A) \cdot \langle z^{\dagger\dagger}, A \rangle \\
 &= \pi_\ell \cdot z^\dagger \cdot \langle p_{X \times Y} \cdot z^{\dagger\dagger}, A \rangle \\
 &\stackrel{(P)}{=} \pi_\ell \cdot (z \cdot (\blacktriangleright (X \times Y) \times \langle p_{X \times Y} \cdot z^{\dagger\dagger}, A \rangle))^\dagger \quad (*)
 \end{aligned}$$

Substitute the definition of z and use that

$$\blacktriangleright \pi_r \cdot p_{X \times Y} \cdot z^{\dagger\dagger} = p_Y \cdot \pi_r \cdot z^{\dagger\dagger} = p_Y \cdot \pi_r \cdot (\langle f, g \rangle \cdot (\text{can} \times A))^\dagger = p_Y \cdot e_R^\dagger$$

to obtain that $(*)$ is equal to

$$\begin{aligned}
 &\pi_\ell \cdot (\langle f, g \rangle \cdot (\blacktriangleright \pi_\ell \times \blacktriangleright \pi_r \times A) \cdot (\blacktriangleright (X \times Y) \times \langle p_{X \times Y} \cdot z^{\dagger\dagger}, A \rangle))^\dagger \\
 &= \pi_\ell \cdot (\langle f, g \rangle \cdot (\blacktriangleright X \times \langle p_Y \cdot e_R^\dagger, A \rangle) \cdot (\blacktriangleright \pi_\ell \times A))^\dagger \\
 &\stackrel{(C)}{=} \underbrace{(\pi_\ell \cdot \langle f, g \rangle)}_{=f} \cdot (\blacktriangleright X \times \langle p_Y \cdot e_R^\dagger, A \rangle)^\dagger \\
 &= e_L^\dagger.
 \end{aligned}$$

This completes the proof. \square

Remark 3.11. Notice that the Bekič identity can also be formulated without mentioning e_L . In fact, by the parameter identity we have

$$e_L^\dagger = (A \xrightarrow{\langle p_Y \cdot e_R^\dagger, A \rangle} \blacktriangleright Y \times A \xrightarrow{f^\dagger} X). \quad (3.7)$$

Proposition 3.10 states the Bekič identity can be derived from Conway laws. But it can be also postulated directly as an axiom replacing $(\dagger\dagger)$. This is a guarded counterpart of Proposition 5.3.15 in Bloom and Ésik [14]:

Proposition 3.12. Each guarded fixpoint category $(\mathcal{C}, \blacktriangleright, \dagger)$ satisfying the fixpoint, parameter, composition and Bekič identities is a guarded Conway category.

Proof:

We must derive $(\dagger\dagger)$ from the identities listed in the statement. Given $f : \blacktriangleright X \times \blacktriangleright X \times A \rightarrow X$ we apply the Bekič identity to obtain

$$(\langle f, f \rangle \cdot (\text{can} \times A))^\dagger = \langle e_L^\dagger, e_R^\dagger \rangle,$$

where $e_R = f \cdot \langle p_X \cdot f^\dagger, \blacktriangleright X \times A \rangle$ and $e_L = f \cdot \langle \blacktriangleright X \times \langle p_X \cdot e_R^\dagger, A \rangle \rangle$.
By the fixpoint identity we have $e_R = f^\dagger$. Thus, we obtain

$$\begin{aligned} f^{\dagger\dagger} &= e_R^\dagger \\ &\stackrel{\text{(B}\ddot{\text{c}})}{=} \pi_r \cdot (\langle f, f \rangle \cdot (\text{can} \times A))^\dagger \\ &= \pi_r \cdot (\Delta_X \cdot f \cdot (\text{can} \times A))^\dagger \\ &\stackrel{\text{(C)}}{=} \underbrace{\pi_r \cdot \Delta_X}_{=\text{id}_X} \cdot (f \cdot (\text{can} \times A) \cdot (\blacktriangleright(\Delta_X) \times A))^\dagger \\ &= (f \cdot (\Delta_{\blacktriangleright X} \times A))^\dagger, \end{aligned}$$

where the last equation follows from

$$\text{can} \cdot \blacktriangleright(\Delta_X) = \langle \blacktriangleright\pi_\ell, \blacktriangleright\pi_r \rangle \cdot \blacktriangleright(\Delta_X) = \langle \blacktriangleright(\text{id}_X), \blacktriangleright(\text{id}_X) \rangle = \langle \text{id}_{\blacktriangleright X}, \text{id}_{\blacktriangleright X} \rangle = \Delta_{\blacktriangleright X}.$$

This completes the proof. \square

3.2. Dinaturality

Finally, we discuss a property that is essentially a parametrized version of the composition identity. In fact, Bloom and Ésik [14] use the very name *composition identity* in this context, calling the unguarded counterpart of our earlier (C) *the simplified composition identity* instead. As it turns out, this property and its variants are not easy to understand in the guarded setting, leaving us with Open Problems 3.16 and 3.17. But first, let us state basic notions and facts.

Definition 3.13. We introduce the following possible property of a guarded fixpoint category $(\mathcal{C}, \blacktriangleright, \dagger)$:

Dinaturality (D). For every $f : \blacktriangleright X \times A \rightarrow Y$ and $g : \blacktriangleright Y \times A \rightarrow X$,

$$(\blacktriangleright X \times A \xrightarrow{\langle p_Y \cdot f, \pi_r \rangle} \blacktriangleright Y \times A \xrightarrow{g} X)^\dagger = A \xrightarrow{\langle p_Y \cdot h^\dagger, A \rangle} \blacktriangleright Y \times A \xrightarrow{g} X,$$

where $h = (\blacktriangleright Y \times A \xrightarrow{\langle p_X \cdot g, \pi_r \rangle} \blacktriangleright X \times A \xrightarrow{f} Y)$.

For unguarded fixpoint operators, it is well-known that the four Conway axioms are equivalent to dinaturality, the parameter and double dagger identities (D, P, ††), in other words, dinaturality can replace the fixpoint and composition identities (see, e.g., [30], [14, Ch. 6.2, 6.8–6.9], [31], [20, Ch. 7.1] and references therein). Proposition 3.15 below shows that we can derive dinaturality from the Conway axioms at the price of extra assumptions on \blacktriangleright . However, no extra assumptions are needed for:

Proposition 3.14. Dinaturality holds in each unique guarded fixpoint category $(\mathcal{C}, \blacktriangleright)$.

Proof:

Given f, g and h as in the definition of dinaturality, we only need to prove that $g \cdot \langle p_Y \cdot h^\dagger, A \rangle : A \rightarrow X$ satisfies the fixpoint identity (2.1) w.r.t. $g \cdot \langle p_Y \cdot f, \pi_r \rangle : \blacktriangleright X \times A \rightarrow X$. Consider the diagram below:

$$\begin{array}{ccccc}
 A & \xrightarrow{\langle p_Y \cdot h^\dagger, A \rangle} & \blacktriangleright Y \times A & \xrightarrow{g} & X \\
 \langle p_Y \cdot h^\dagger, A, A \rangle \downarrow & \searrow \langle p_Y \cdot h^\dagger, A \rangle & \swarrow \langle h^\dagger, A \rangle & \circlearrowleft & \uparrow p_{Y \times A} \\
 \blacktriangleright Y \times A \times A & \xleftarrow{\blacktriangleright Y \times \Delta} & \blacktriangleright Y \times A & \xrightarrow{\langle (*), \circ \rangle} & Y \times A & \xrightarrow{p_{Y \times A}} & \blacktriangleright Y \times A \\
 g \times A \downarrow & & \circlearrowleft & & \swarrow \langle f, \pi_r \rangle & \circlearrowleft & \uparrow \langle p_Y \cdot f, \pi_r \rangle \\
 X \times A & \xrightarrow{p_{X \times A}} & \blacktriangleright X \times A & & \blacktriangleright X \times A & & \blacktriangleright X \times A \\
 & & \circlearrowleft & & \swarrow \langle p_X \cdot g, \pi_r \rangle & &
 \end{array}$$

For (*), recall $h = f \cdot \langle p_X \cdot g, \pi_r \rangle$ and then apply the fixpoint identity. \square

Proposition 3.15. Dinaturality holds in each guarded Conway category $(\mathcal{C}, \blacktriangleright, \dagger)$ such that \blacktriangleright preserves products and is well-pointed (i.e. we have $\blacktriangleright p = p \blacktriangleright$).

Proof:

We prove this property from the fixpoint, composition and Bekič identities. Given $f : \blacktriangleright X \times A \rightarrow Y$ and $g : \blacktriangleright Y \times A \rightarrow X$, let $k = \langle p_Y \cdot f, \pi_r \rangle$. By (C), we have

$$(g \cdot k)^\dagger = (A \xrightarrow{(k \cdot \blacktriangleright g \times A)^\dagger} \blacktriangleright Y \times A \xrightarrow{g} X).$$

Thus, in order to complete the proof it suffices to show that

$$\langle p_Y \cdot h^\dagger, A \rangle = (\blacktriangleright(\blacktriangleright Y \times A) \times A \xrightarrow{\blacktriangleright g \times A} \blacktriangleright X \times A \xrightarrow{k} \blacktriangleright Y \times A)^\dagger.$$

Since \blacktriangleright preserves products, we have $\text{can}^{-1} : \blacktriangleright\blacktriangleright Y \times \blacktriangleright A \rightarrow \blacktriangleright(\blacktriangleright Y \times A)$. Now let

$$\begin{aligned}
 m &= p_Y \cdot f \cdot (\blacktriangleright g \times A) \cdot (\text{can}^{-1} \times A) : \blacktriangleright\blacktriangleright Y \times \blacktriangleright A \times A \rightarrow \blacktriangleright Y, \\
 n &= \pi_r : \blacktriangleright\blacktriangleright Y \times \blacktriangleright A \times A \rightarrow A.
 \end{aligned}$$

Then we clearly have $\langle m, n \rangle \cdot (\text{can} \times A) = k \cdot (\blacktriangleright g \times A)$. Now we apply the Bekič identity to obtain

$$(\langle m, n \rangle \cdot (\text{can} \times A))^\dagger = \langle e_L^\dagger, e_R^\dagger \rangle : A \rightarrow \blacktriangleright Y \times A,$$

where

$$\begin{aligned} e_R &\stackrel{\text{by def}}{=} n \cdot \langle p_{\blacktriangleright Y} \cdot m^\dagger, \blacktriangleright A \times A \rangle = \pi_r : \blacktriangleright A \times A \rightarrow A \\ e_L &\stackrel{\text{by def}}{=} m \cdot (\blacktriangleright\blacktriangleright Y \times \langle p_A \cdot e_R^\dagger, A \rangle) : \blacktriangleright\blacktriangleright Y \times A \rightarrow \blacktriangleright Y. \end{aligned}$$

Using $e_R = \pi_r$ we see that $e_R^\dagger \stackrel{(\dagger)}{=} e_R \cdot (p_A \times A) \cdot \langle e_R^\dagger, A \rangle = \text{id}_A$. So we have

$$e_L = \underbrace{p_Y \cdot f \cdot (\blacktriangleright g \times A) \cdot (\text{can}^{-1} \times A)}_m \cdot (\blacktriangleright\blacktriangleright Y \times \langle p_A, A \rangle).$$

$$\begin{aligned} \text{Thus we obtain } e_L^\dagger &\stackrel{(C)}{=} p_Y \cdot (f \cdot (\blacktriangleright g \times A) \cdot (\text{can}^{-1} \times A) \cdot (\blacktriangleright\blacktriangleright Y \times \langle p_A, A \rangle) \cdot (\blacktriangleright p_Y \times A))^\dagger \\ &= p_Y \cdot (f \cdot \langle p_X \cdot g, \pi_r \rangle)^\dagger, \end{aligned}$$

where the second equation is derived as follows: it is sufficient to prove that the two morphisms inside \dagger after removal of f are equal, and for this one considers the product components of $\blacktriangleright X \times A$ (their codomain) separately. The right-hand component is obviously π_r and the left-hand one follows from

$$p_X \cdot g \stackrel{(\text{nat. of } p)}{=} \blacktriangleright g \cdot p_{\blacktriangleright Y \times A} = \blacktriangleright g \cdot \text{can}^{-1} \cdot (\blacktriangleright p_Y \times p_A),$$

where the second equation is derived using well-pointedness of \blacktriangleright : $\text{can} \cdot p_{\blacktriangleright Y \times A} = p_{\blacktriangleright Y} \times p_A = \blacktriangleright p_Y \times p_A$. \square

Open Problem 3.16. Do (D, P, \dagger) imply the fixpoint and simplified composition identities?

Further inspection reveals a curious asymmetry here. Under the assumption that \blacktriangleright preserves products one can formulate two related versions of dinaturality where the given morphisms only contain one \blacktriangleright . For these properties we use for given $f : \blacktriangleright X \times A \rightarrow Y$ and $g : Y \times A \rightarrow X$ the morphism

$$f \triangleright g = (\blacktriangleright Y \times A \xrightarrow{\blacktriangleright Y \times \langle p_A, A \rangle} \blacktriangleright Y \times \blacktriangleright A \times A \xrightarrow{\text{can}^{-1} \times A} \blacktriangleright (Y \times A) \times A \xrightarrow{\blacktriangleright g \times A} \blacktriangleright X \times A \xrightarrow{f} Y).$$

Property (D₁). Given $f : \blacktriangleright X \times A \rightarrow Y$ and $g : Y \times A \rightarrow X$ we have

$$(\blacktriangleright X \times A \xrightarrow{\langle f, \pi_r \rangle} Y \times A \xrightarrow{g} X)^\dagger = (A \xrightarrow{\langle (f \triangleright g)^\dagger, A \rangle} Y \times A \xrightarrow{g} X).$$

Property (D₂). Given $f : X \times A \rightarrow Y$ and $g : \blacktriangleright Y \times A \rightarrow X$ we have

$$(g \triangleright f)^\dagger = g \cdot \langle p_Y \cdot h^\dagger, A \rangle,$$

where $h = (\blacktriangleright Y \times A \xrightarrow{\langle g, \pi_r \rangle} X \times A \xrightarrow{f} Y)$.

Whenever \blacktriangleright is moreover well-pointed, each of (D_1) and (D_2) implies (D) . One also readily proves, by adapting the proofs for unguarded operators, that (D_1) implies the simplified composition identity (C) and that (D_2) implies the fixpoint identity (\dagger) . Conversely, the Conway axioms imply the first version of dinaturality (D_1) . For the sake of brevity we leave the details to the reader. What defeats us at the moment is:

Open Problem 3.17.

- Do the Conway axioms imply (D_2) ?
- Does (D_1) imply (D_2) ?

4. Guarded trace operators

In the special case of Example 2.4.(1), i.e., \blacktriangleright being the identity functor, it is well-known that a fixpoint operator satisfying the Conway axioms is equivalent to a trace operator w.r.t. the product on \mathcal{C} [17, 20]. In this section we present a similar result for a generalized notion of a guarded trace operator on $(\mathcal{C}, \blacktriangleright)$.

Remark 4.1. Recall that Joyal, Street and Verity [18] introduced the unguarded notion of a trace operator for *(symmetric) monoidal* categories. The applicability to non-cartesian tensor products is in fact one of main reasons of its popularity. Our generalization can also be formulated in the symmetric monoidal setting, see Remark 4.3 below. However, Theorems 4.5 and 4.7, the main results in this section, do not make any use of this added generality. Hence, we keep the Assumption 2.1 like in the remainder of the paper.

Definition 4.2. A (cartesian) *guarded trace operator* on $(\mathcal{C}, \blacktriangleright)$ is a natural family of operations

$$\mathrm{Tr}_{A,B}^X : \mathcal{C}(\blacktriangleright X \times A, X \times B) \rightarrow \mathcal{C}(A, B)$$

subject to the following three conditions:

1. **Vanishing (V1)** For every $f : \blacktriangleright 1 \times A \rightarrow B \cong 1 \times B$ we have

$$\mathrm{Tr}_{A,B}^1(f) = (A \cong 1 \times A \xrightarrow{p_1 \times A} \blacktriangleright 1 \times A \xrightarrow{f} B).$$

- (V2) For every $f : \blacktriangleright X \times \blacktriangleright Y \times A \rightarrow X \times Y \times B$ we have

$$\mathrm{Tr}_{A,B}^Y(\mathrm{Tr}_{\blacktriangleright Y \times A, Y \times B}^X(f)) = \mathrm{Tr}_{A,B}^{X \times Y}(\blacktriangleright(X \times Y) \times A \xrightarrow{\mathrm{can} \times A} \blacktriangleright X \times \blacktriangleright Y \times A \xrightarrow{f} X \times Y \times B).$$

2. **Superposing (S)**. For every $f : \blacktriangleright X \times A \rightarrow X \times B$ we have

$$\mathrm{Tr}_{A \times C, B \times C}^X(f \times C) = \mathrm{Tr}_{A,B}^X(f) \times C.$$

3. **Yanking (Y)**. Consider the canonical isomorphism $c : \blacktriangleright X \times X \rightarrow X \times \blacktriangleright X$. Then we have

$$\mathrm{Tr}_{X, \blacktriangleright X}^X(c) = (X \xrightarrow{p_X} \blacktriangleright X).$$

If Tr is a (cartesian) guarded trace operator on $(\mathcal{C}, \blacktriangleright)$, $(\mathcal{C}, \blacktriangleright, \mathrm{Tr})$ is called a *guarded traced (cartesian) category*.

Of course, when \blacktriangleright is taken to be the identity on \mathcal{C} as in Example 2.4.(1), our notion of guarded trace specializes to the notion of an ordinary trace operator (w.r.t. product) of Joyal, Street and Verity. In addition, the naturality of Tr can equivalently be expressed by three more axioms, just like in the unguarded case:

4. **Left-tightening (Lt).** Given $f : \blacktriangleright X \times A \rightarrow X \times B$ and $g : A' \rightarrow A$ we have

$$\text{Tr}_{A',B}^X(\blacktriangleright X \times A' \xrightarrow{\blacktriangleright X \times g} \blacktriangleright X \times A \xrightarrow{f} X \times B) = (A' \xrightarrow{g} A \xrightarrow{\text{Tr}_{A,B}^X(f)} B).$$

5. **Right-tightening (Rt).** Given $f : \blacktriangleright X \times A \rightarrow X \times B$ and $g : B \rightarrow B'$ we have

$$\text{Tr}_{A,B'}^X(\blacktriangleright X \times A \xrightarrow{f} X \times B \xrightarrow{X \times g} X \times B') = (A \xrightarrow{\text{Tr}_{A,B}^X(f)} B \xrightarrow{g} B').$$

6. **Sliding (Sl).** Given $f : \blacktriangleright X \times A \rightarrow X' \times B$ and $g : X' \rightarrow X$ we have

$$\text{Tr}_{A,B}^X(\blacktriangleright X \times A \xrightarrow{f} X' \times B \xrightarrow{g \times B} X \times B) = \text{Tr}_{A,B}^{X'}(\blacktriangleright X' \times A \xrightarrow{\blacktriangleright g \times A} \blacktriangleright X \times A \xrightarrow{f} X' \times B).$$

Remark 4.3. The generalization for a symmetric monoidal category $(\mathcal{C}, \otimes, I, c)$ equipped with a pointed endofunctor $\blacktriangleright : \mathcal{C} \rightarrow \mathcal{C}$ requires the assumption that \blacktriangleright is *comonoidal*, i.e., equipped with a morphism $m_I : \blacktriangleright I \rightarrow I$ and a natural transformation $m_{X,Y} : \blacktriangleright(X \otimes Y) \rightarrow \blacktriangleright X \otimes \blacktriangleright Y$ satisfying the usual coherence conditions. In fact, in the formulation of Vanishing (V2) we used that in every category the product \times is comonoidal via $m_{X,Y} = \text{can}$.

Construction 4.4. 1. Let $(\mathcal{C}, \blacktriangleright, \text{Tr})$ be a guarded traced category. Define a guarded fixpoint operator $\dagger_{\text{Tr}} : \mathcal{C}(\blacktriangleright X \times A, X) \rightarrow \mathcal{C}(A, X)$ by

$$f^{\dagger_{\text{Tr}}} = \text{Tr}_{A,X}^X(\blacktriangleright X \times A \xrightarrow{\langle f, f \rangle} X \times X) : A \rightarrow X.$$

2. Conversely, suppose $(\mathcal{C}, \blacktriangleright, \dagger)$ is a guarded fixpoint category. Define $\text{Tr}_{\dagger A, B}^X : \mathcal{C}(\blacktriangleright X \times A, X \times B) \rightarrow \mathcal{C}(A, B)$ by setting for every $f : \blacktriangleright X \times A \rightarrow X \times B$

$$\text{Tr}_{\dagger A, B}^X(f) = (A \xrightarrow{\langle (\pi_\ell \cdot f)^\dagger, A \rangle} X \times A \xrightarrow{p_X \times A} \blacktriangleright X \times A \xrightarrow{f} X \times B \xrightarrow{\pi_r} B).$$

Theorem 4.5. 1. Whenever $(\mathcal{C}, \blacktriangleright, \text{Tr})$ is a guarded traced category, $(\mathcal{C}, \blacktriangleright, \dagger_{\text{Tr}})$ is a guarded Conway category. Furthermore, $\text{Tr}_{\dagger_{\text{Tr}}}$ is the original operator Tr .

2. Whenever $(\mathcal{C}, \blacktriangleright, \dagger)$ is a guarded Conway category, $(\mathcal{C}, \blacktriangleright, \text{Tr}_{\dagger})$ is guarded traced. Furthermore, $\dagger_{\text{Tr}_{\dagger}}$ is the original operator \dagger .

The proof details are similar to the unguarded case in Hasegawa [20]. As the derivation of the guarded version of the Bekič identity in Proposition 3.10 has already shown, it is not a completely automatic adaptation. We give a complete proof in Appendix B below.

The process requires some creativity at times.

Hasegawa related uniformity of trace to uniformity of dagger and we can do the same in the guarded setup. Recall that in iteration theories uniformity (called *functorial dagger implication*) plays an important role. On the one hand, this quasiequation implies the so-called *commutative identities*, an infinite set of equational axioms that are added to the Conway axioms in order to yield a complete axiomatization of fixpoint operators in domains. On the other hand, most examples of iteration theories actually satisfy uniformity, and so uniformity gives a convenient sufficient condition to verify that a given Conway theory is actually an iteration theory.

Definition 4.6. A guarded trace operator Tr is called *uniform* if for every $f : \blacktriangleright X \times A \rightarrow X \times B$, $f' : \blacktriangleright X' \times A \rightarrow X' \times B$ and $h : X \rightarrow X'$,

$$\begin{array}{ccc}
 \blacktriangleright X \times A & \xrightarrow{f} & X \times B \\
 \blacktriangleright h \times A \downarrow & \circlearrowleft & \downarrow h \times B \\
 \blacktriangleright X' \times A & \xrightarrow{f'} & X' \times B
 \end{array}
 \implies
 \text{Tr}_{A,B}^X(f) = \text{Tr}_{A,B}^{X'}(f') : A \rightarrow B.$$

A *uniform guarded traced category* is a guarded traced category $(\mathcal{C}, \blacktriangleright, \text{Tr})$ where Tr is uniform.

Theorem 4.7. 1. Whenever $(\mathcal{C}, \blacktriangleright, \text{Tr})$ is a uniform guarded traced category, $(\mathcal{C}, \blacktriangleright, \dagger_{\text{Tr}})$ is a uniform guarded Conway category.

2. Whenever $(\mathcal{C}, \blacktriangleright, \dagger)$ is a uniform guarded Conway category, $(\mathcal{C}, \blacktriangleright, \text{Tr}_{\dagger})$ is a uniform guarded traced category.

The proof is in Appendix C.

Remark 4.8. Actually, Hasegawa proved a slightly stronger statement concerning uniformity than what we stated in Theorem 4.7; he showed that a Conway operator is uniform w.r.t. any fixed morphism $h : X \rightarrow X'$ (i.e. satisfies uniformity just for h) iff the corresponding trace operator is uniform w.r.t. this morphism h . The proof is somewhat more complicated and in our guarded setting we leave this as an exercise to the reader.

Finally, let us note that the bijective correspondence between guarded Conway operators and guarded trace operators established in Theorem 4.5 yields an isomorphism of the (2-)categories of (small) guarded Conway categories and guarded traced (cartesian) categories. The corresponding notions of morphisms are, of course, as expected:

Definition 4.9. 1. $F : (\mathcal{C}, \blacktriangleright^{\mathcal{C}}, \dagger) \rightarrow (\mathcal{D}, \blacktriangleright^{\mathcal{D}}, \ddagger)$ is a morphism of guarded Conway categories whenever $F : \mathcal{C} \rightarrow \mathcal{D}$ is a finite-product-preserving functor satisfying

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\blacktriangleright^{\mathcal{C}}} & \mathcal{C} \\ F \downarrow & \circlearrowleft & \downarrow F \\ \mathcal{D} & \xrightarrow{\blacktriangleright^{\mathcal{D}}} & \mathcal{D} \end{array} \quad \text{and} \quad p_{FX}^{\mathcal{D}} = F(p_X^{\mathcal{C}}) : FX \rightarrow \blacktriangleright^{\mathcal{D}}FX = F(\blacktriangleright^{\mathcal{C}}X), \quad (4.1)$$

and preserving dagger, i.e., for every $f : \blacktriangleright X \times A \rightarrow X$ we have

$$F(f^\dagger) = (\blacktriangleright^{\mathcal{D}}FX \times FA \cong F(\blacktriangleright^{\mathcal{C}}X \times A) \xrightarrow{Ff} FX)^\ddagger.$$

2. A morphism $F : (\mathcal{C}, \blacktriangleright^{\mathcal{C}}, \text{Tr}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \blacktriangleright^{\mathcal{D}}, \text{Tr}_{\mathcal{D}})$ of guarded traced categories is a finite-product-preserving $F : \mathcal{C} \rightarrow \mathcal{D}$ satisfying (4.1) above and preserving the trace operation: for every $f : \blacktriangleright^{\mathcal{C}}X \times A \rightarrow X \times B$ in \mathcal{C} we have

$$F(\text{Tr}_{\mathcal{C}}^X_{A,B}(f)) = \text{Tr}_{\mathcal{D}}^{FX}_{FA,FB}(\blacktriangleright^{\mathcal{D}}FX \times FA \cong F(\blacktriangleright^{\mathcal{C}}X \times A) \xrightarrow{Ff} F(X \times B) \cong FX \times FB).$$

Corollary 4.10. The (2-)categories of guarded Conway categories and of guarded traced (cartesian) categories are isomorphic.

The proof is in Appendix D.

5. Conclusions and future work

We have made the first steps in the study of equational properties of guarded fixpoint operators popular in the recent literature [3, 4, 5, 6, 9, 7, 8, 9, 11, 12]. We began with an extensive list of examples, including some not discussed so far as instances of delay endofunctors—e.g., Example 2.4.(6) or completely iterative theories in Section 2.2. Furthermore, we formulated the four Conway axioms and the uniformity property in analogy to their unguarded counterparts and we showed their soundness w.r.t. the models discussed in Section 2. In particular, Theorem 3.4 proved that our axioms hold in all categories with a unique guarded dagger. In Theorem 4.5, we have a generalization of a result by Hasegawa for ordinary fixpoint operators: we proved that to give a (uniform) guarded fixpoint operator satisfying the Conway axioms is equivalent to giving a (uniform) guarded trace operator on the same category.

Our paper can be considered as a work in progress report. The long-term goal is to arrive at completeness results similar to the ones for iteration theories. We do not claim that the axioms we presented are complete. In the unguarded setting, completeness is obtained by adding to the Conway axioms an infinite set of equational axioms called the *commutative identities* [14, 15]. We did not consider those here, but we considered the quasi-equational property of uniformity which implies the commutative identities and is satisfied in most models of interest. Only further research can show whether this property can ensure completeness in the guarded setup or one needs to postulate stronger ones.

Let us recall Open Problem 3.8 regarding soundness of $(\dagger\dagger)$ in the general setting of Crole and Pitts [21, 16] and intriguingly complex status of guarded dinaturality leading to Open Problems 3.16 and 3.17.

It would also be interesting to study further examples of guarded traced monoidal categories which are not ordinary traced monoidal categories and which do not arise from guarded Conway categories. We have obtained some such examples but more work is needed to develop a full-blown theory. We postpone a detailed discussion to future work.

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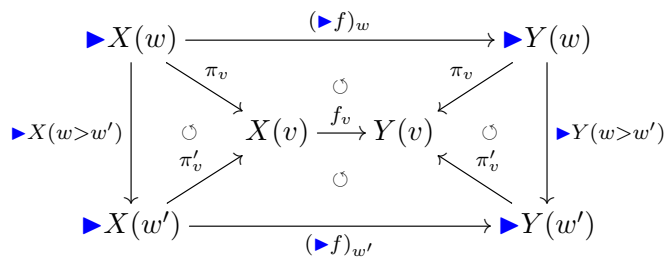
A. Details for example 2.4.(5)

First observe that $\blacktriangleright X$ is clearly a presheaf: for every $w' \leq w$ there exists a canonical morphism $\blacktriangleright X(w) = \lim_{v < w} X(v) \rightarrow \lim_{v < w'} X(v) = \blacktriangleright X(w')$ induced by the universal property of the limit in the codomain; the functoriality easily follows from the uniqueness.

Next we define \blacktriangleright on a morphism $f : X \rightarrow Y$ componentwise: for every w , $(\blacktriangleright f)_w$ is the unique morphism such that the following equations hold:

$$\pi_v \cdot (\blacktriangleright f)_w = f_v \cdot \pi_v, \quad (v < w),$$

where $\pi_v : X(w) = \lim_{v < w} X(v) \rightarrow X(v)$ denotes the limit projection. To see that $(\blacktriangleright f)_w$ is natural in w it suffices to show that for any $w > w'$ the corresponding naturality square commutes when extended by the projection $\pi'_v : \blacktriangleright Y(w') \rightarrow Y(v)$ for every $v < w'$:



A routine calculation then shows that $\blacktriangleright : \mathcal{C} \rightarrow \mathcal{C}$ is functorial.

The point $p : \text{Id} \rightarrow \blacktriangleright$ is given componentwise as the unique morphism $(p_X)_w : X(w) \rightarrow \blacktriangleright X(w)$ such that $\pi_v \cdot (p_X)_w = X(w > v)$ for all $v < w$. Two easy routine calculations using the definitions of $\blacktriangleright X$ and \blacktriangleright on morphisms, respectively, show that each component p_X is natural in w and that p is natural in X .

Let us now turn to the guarded fixpoint operator \dagger . We first prove simultaneously that each f_w^\dagger is well-defined and that f^\dagger is a morphism of \mathcal{C} , i.e., f_w^\dagger is natural in w . This is done by induction on (W, \leq) (note that we do not have to distinguish the base case and induction step here). Fix any $w \in W$ and assume that f_v^\dagger is well-defined for any $v < w$ and that the naturality condition $f_{v'}^\dagger \cdot Y(w > v') = X(w > v') \cdot f_v^\dagger$ holds for any $v' < v$ which are smaller than w . (Note that for a minimal $w \in W$ this holds trivially.) The latter naturality condition implies the cone property for $f_v^\dagger \cdot Y(w > v)$ inducing $k : Y(w) \rightarrow \blacktriangleright X(w)$ so that f_w^\dagger is well-defined. We proceed to showing the naturality condition for any $w > w'$ using the following diagram (with k' induced by the cone $f_{v'}^\dagger \cdot Y(w' > v)$):

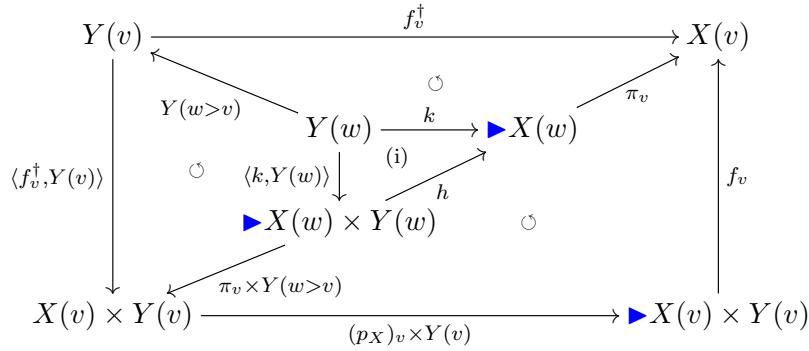
$$\begin{array}{ccccc}
 & & f_w^\dagger & & \\
 & & \circlearrowleft & & \\
 Y(w) & \xrightarrow{\langle k, Y(w) \rangle} & \blacktriangleright X(w) \times Y(w) & \xrightarrow{f_w} & X(w) \\
 \downarrow Y(w > w') & \langle (*), \circlearrowleft \rangle & \downarrow \blacktriangleright X(w > w') \times Y(w > w') & \circlearrowleft & \downarrow X(w > w') \\
 Y(w') & \xrightarrow{\langle k', Y(w') \rangle} & \blacktriangleright X(w') \times Y(w') & \xrightarrow{f_{w'}} & X(w') \\
 & & \circlearrowright & & \\
 & & f_{w'}^\dagger & &
 \end{array}$$

(Note that $\langle (*), \circlearrowleft \rangle$ indicates that the right-hand product component of that part obviously commutes and the left-hand part, called $(*)$ is considered further.) Part $(*)$ is seen commutative by extending it with the limit projection $\pi_v : Y(w') \rightarrow Y(v)$ for every $v < w'$ and performing a routine calculation. (Note again that this covers the cases where w or w' are minimal and consequently k and k' , respectively, are the unique morphisms with codomain 1.)

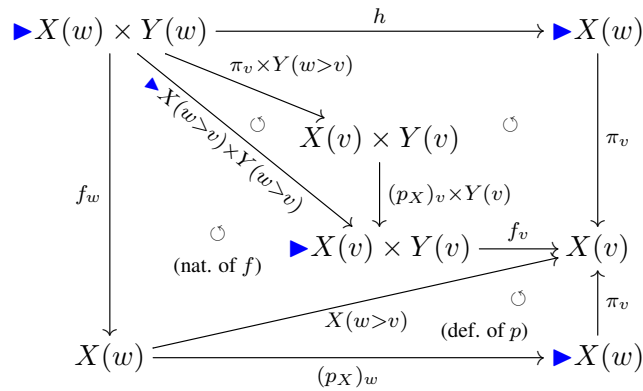
We are ready to verify the commutativity of (2.1). This is done componentwise by induction on (W, \leq) . Assume that for a given w all components of (2.1) at $v < w$ commute. Then we obtain for the w -component of f^\dagger the following diagram (where k is again induced by the cone $f_v^\dagger \cdot Y(w > v)$ and h by the cone $f_v \cdot ((p_X)_v \times Y(v)) \cdot (\pi_v \times Y(w > v)) : \blacktriangleright X(w) \times Y(w) \rightarrow X(v)$ for all $v < w$):

$$\begin{array}{ccccc}
 & & f_w^\dagger & & \\
 & & \circlearrowleft & & \\
 Y(w) & \xrightarrow{\langle k, Y(w) \rangle} & \blacktriangleright X(w) \times Y(w) & \xrightarrow{f_w} & X(w) \\
 \downarrow \langle k, Y(w), Y(w) \rangle & \langle (i), \circlearrowleft \rangle & \nearrow h \times Y(w) & \circlearrowleft & \downarrow f_w \\
 \blacktriangleright X(w) \times Y(w) \times Y(w) & & & & \\
 \downarrow f_w \times Y(w) & & \langle (ii), \circlearrowleft \rangle & & \\
 X(w) \times Y(w) & \xrightarrow{(p_X)_w \times Y(w)} & \blacktriangleright X(w) \times Y(w) & &
 \end{array}$$

Note that we are done if w is minimal since $\blacktriangleright X(w) = 1$ is the terminal object. Otherwise for part (i) we extend with the limit projection π_v for every $v < w$ to obtain the following diagram (its outside commutes by the induction hypothesis, hence, so does part (i) extended by π_v):



For part (ii) observe first that $(p_X)_v \cdot \pi_v = \blacktriangleright X(w > v)$; indeed, this follows by routine calculation extending both sides by the limit projection $\pi_u : \blacktriangleright X(v) \rightarrow X(u)$ for every $u < v$. Now we obtain the commutativity of part (ii) by extending it with every limit projection π_v :



It remains to prove that f^\dagger is unique such that (2.1) commutes. Suppose that $s : Y \rightarrow X$ is such that $s = f \cdot (p_X \times Y) \cdot \langle s, Y \rangle$. Then we prove that $f^\dagger = s$ componentwise by induction on (W, \leq) . Assume that $s_v = f_v^\dagger$ holds for all $v < w$. This implies that k is the morphism induced by the cone $s_v \cdot Y(w > v) = X(w > v) \cdot s_w : Y(w) \rightarrow X(v)$. Hence, for all $v < w$ we have

$$\pi_v \cdot k = X(w > v) \cdot s_w = \pi_v \cdot (p_X)_w \cdot s_w$$

from which we conclude that $k = (p_X)_w \cdot s_w$. (In the special case where w is minimal this equation holds since it is an equation between morphisms with codomain $\blacktriangleright X(w) = 1$.)

Thus, we obtain

$$\begin{aligned}
f_w^\dagger &= f_w \cdot \langle k, Y(w) \rangle && \text{(def. of } f_w^\dagger) \\
&= f_w \cdot \langle (p_X)_w \cdot s_w, Y(w) \rangle && \text{(since } k = (p_X)_w \cdot s_w) \\
&= s_w && \text{(since } s = f \cdot (p_X \times Y) \cdot \langle s, Y \rangle).
\end{aligned}$$

This completes the proof.

B. Proof of theorem 4.5

The proof of Theorem 4.5 proceeds in three steps:

1. We show that \dagger_{Tr} defined in Construction 4.4.1 is a guarded trace operator.
2. We show that Tr_\dagger defined in Construction 4.4.2 satisfies the Conway axioms.
3. We show that the two constructions are mutually inverse, i.e. $\dagger_{\text{Tr}_\dagger} = \dagger$ and $\text{Tr}_{\dagger_{\text{Tr}}} = \text{Tr}$.

In the first two sections we shall drop the subscripts and only write Tr and \dagger in lieu of Tr_\dagger and \dagger_{Tr} , respectively. The proof is an adaptation of Hasegawa's proof for ordinary traces and fixpoint operators in [20].

B.1. From trace to dagger

We will now prove that the \dagger -operation defined in Construction 4.4.1 satisfies the Conway axioms. But before we need an analogue of the fixpoint identity for traces:

Lemma B.1. For every $f : \blacktriangleright X \times A \rightarrow X \times B$ we form

$$h = \text{Tr}_{A,X}^X(\blacktriangleright X \times A \xrightarrow{f} X \times B \xrightarrow{\pi_\ell} X \xrightarrow{\Delta} X \times X).$$

Then we have

$$\text{Tr}_{A,B}^X(f) = (A \xrightarrow{\langle h, A \rangle} X \times A \xrightarrow{p_X \times A} \blacktriangleright X \times A \xrightarrow{f} X \times B \xrightarrow{\pi_r} B).$$

Proof:

Let $c : \blacktriangleright X \times X \rightarrow X \times \blacktriangleright X$ denote the canonical isomorphism swapping components. Observe that we have

$$c \cdot (p_X \times X) \cdot \Delta_X = c \cdot \langle p_X, X \rangle = \langle X, p_X \rangle = (X \times p_X) \cdot \Delta_X \quad (\text{B.1})$$

and

$$(X \times c) \cdot (c \times X) \cdot (\blacktriangleright X \times \Delta_X) = (\Delta_X \times \blacktriangleright X) \cdot c. \quad (\text{B.2})$$

Now we compute

$$\begin{aligned}
p_X \cdot h &= p_X \cdot \text{Tr}_{A,X}^X(\Delta_X \cdot \pi_\ell \cdot f) && \text{by def. of } h \\
&\stackrel{(Rt)}{=} \text{Tr}_{A,\blacktriangleright X}^X((X \times p_X) \cdot \Delta_X \cdot \pi_\ell \cdot f) \\
&\stackrel{(B.1)}{=} \text{Tr}_{A,\blacktriangleright X}^X(c \cdot (p_X \times X) \cdot \Delta_X \cdot \pi_\ell \cdot f) \\
&\stackrel{(Y)}{=} \text{Tr}_{A,\blacktriangleright X}^X(c \cdot (\text{Tr}_{X,\blacktriangleright X}^X(c) \times X) \cdot \Delta_X \cdot \pi_\ell \cdot f) \\
&\stackrel{(S)}{=} \text{Tr}_{A,\blacktriangleright X}^X(c \cdot \text{Tr}_{X \times X,\blacktriangleright X \times X}^X(c \times X) \cdot \Delta_X \cdot \pi_\ell \cdot f) \\
&\stackrel{(Lt)}{=} \text{Tr}_{A,\blacktriangleright X}^X(c \cdot \text{Tr}_{\blacktriangleright X \times A,\blacktriangleright X \times X}^X((c \times X) \cdot (\blacktriangleright X \times (\Delta_X \cdot \pi_\ell \cdot f)))) \\
&\stackrel{(Rt)}{=} \text{Tr}_{A,\blacktriangleright X}^X(\text{Tr}_{\blacktriangleright X \times A,X \times \blacktriangleright X}^X((X \times c) \cdot (c \times X) \cdot (\blacktriangleright X \times (\Delta_X \cdot \pi_\ell \cdot f)))) \\
&\stackrel{(B.2)}{=} \text{Tr}_{A,\blacktriangleright X}^X(\text{Tr}_{\blacktriangleright X \times A,X \times \blacktriangleright X}^X((\Delta_X \times \blacktriangleright X) \cdot c \cdot (\blacktriangleright X \times (\pi_\ell \cdot f)))) \\
&\stackrel{(V2)}{=} \text{Tr}_{A,\blacktriangleright X}^{X \times X}((\Delta_X \times \blacktriangleright X) \cdot c \cdot (\blacktriangleright X \times (\pi_\ell \cdot f)) \cdot (\text{can} \times A)) \\
&\stackrel{(Sl)}{=} \text{Tr}_{A,\blacktriangleright X}^X(c \cdot (\blacktriangleright X \times (\pi_\ell \cdot f)) \cdot (\text{can} \times A) \cdot (\blacktriangleright (\Delta_X) \times A)) \\
&= \text{Tr}_{A,\blacktriangleright X}^X(c \cdot (\blacktriangleright X \times (\pi_\ell \cdot f)) \cdot (\Delta_{\blacktriangleright X} \times A))
\end{aligned}$$

Using this we can finish the proof:

$$\begin{aligned}
&\pi_r \cdot f \cdot \langle p_X \cdot h, A \rangle \\
&= \pi_r \cdot f \cdot ((p_X \cdot h) \times A) \cdot \Delta_A \\
&\stackrel{(S)}{=} \pi_r \cdot f \cdot \text{Tr}_{A \times A,\blacktriangleright X \times A}^X((c \times A) \cdot (\blacktriangleright X \times (\pi_\ell \cdot f) \times A) \cdot (\Delta_{\blacktriangleright X} \times A \times A)) \cdot \Delta_A \\
&\stackrel{(Lt)}{=} \pi_r \cdot f \cdot \text{Tr}_{A,\blacktriangleright X \times A}^X((c \times A) \cdot (\blacktriangleright X \times (\pi_\ell \cdot f) \times A) \cdot \underbrace{(\Delta_{\blacktriangleright X} \times A \times A) \cdot (\blacktriangleright X \times \Delta_A)}_{=\Delta_{\blacktriangleright X} \times \Delta_A})) \\
&\stackrel{(Rt)}{=} \text{Tr}_{A,B}^X((X \times (\pi_r \cdot f)) \cdot (c \times A) \cdot (\blacktriangleright X \times (\pi_\ell \cdot f) \times A) \cdot (\Delta_{\blacktriangleright X} \times \Delta_A)) \\
&= \text{Tr}_{A,B}^X(\langle \pi_\ell \cdot f, \pi_r \cdot f \rangle) \\
&= \text{Tr}_{A,B}^X(f),
\end{aligned}$$

which completes the proof of the lemma. \square

We now verify the Conway axioms for the \dagger -operation from Construction 4.4.1.

(1) Fixpoint identity. Given $f : \blacktriangleright X \times A \rightarrow X$ we apply Lemma B.1 to $\langle f, f \rangle$; then

$$h = \text{Tr}_{A,X}^X(\Delta_X \cdot \pi_\ell \cdot \langle f, f \rangle) = \text{Tr}_{A,X}^X(\langle f, f \rangle) = f^\dagger$$

and therefore we have

$$\begin{aligned}
f^\dagger &= \text{Tr}_{A,X}^X \langle f, f \rangle && \text{by def. of } \dagger \\
&= \pi_r \cdot \langle f, f \rangle \cdot (p_X \times A) \cdot \langle h, A \rangle && \text{by Lemma B.1} \\
&= f \cdot (p_X \times A) \cdot \langle f^\dagger, A \rangle && \text{since } h = f^\dagger.
\end{aligned}$$

(2) Parameter identity. Let $f : \blacktriangleright X \times A \rightarrow X$ and $h : A' \rightarrow A$. Then we have

$$\begin{aligned}
 (f \cdot (\blacktriangleright X \times h))^\dagger &= \text{Tr}_{A',X}^X(\langle f \cdot (\blacktriangleright X \times h), f \cdot (\blacktriangleright X \times h) \rangle) && \text{by def. of } \dagger \\
 &= \text{Tr}_{A',X}^X(\langle f, f \rangle \cdot (\blacktriangleright X \times h)) \\
 &\stackrel{(Lt)}{=} \text{Tr}_{A',X}^X(\langle f, f \rangle) \cdot h \\
 &= f^\dagger \cdot h && \text{by def. of } \dagger.
 \end{aligned}$$

(3) Composition identity. Given $f : \blacktriangleright X \times A \rightarrow Y$ and $g : Y \rightarrow X$ we compute

$$\begin{aligned}
 (g \cdot f)^\dagger &= \text{Tr}_{A,X}^X(\langle g \cdot f, g \cdot f \rangle) && \text{by def. of } \dagger \\
 &= \text{Tr}_{A,X}^X((X \times g) \cdot \langle f, f \rangle) \\
 &\stackrel{(Rt)}{=} g \cdot \text{Tr}_{A,Y}^X(\langle f, f \rangle) \\
 &= g \cdot \text{Tr}_{A,Y}^X((g \times X) \cdot \langle f, f \rangle) \\
 &\stackrel{(Sl)}{=} g \cdot \text{Tr}_{A,Y}^Y(\langle f, f \rangle \cdot (\blacktriangleright g \times A)) \\
 &= g \cdot \text{Tr}_{A,Y}^Y(\langle f \cdot (\blacktriangleright g \times A), f \cdot (\blacktriangleright g \times A) \rangle) \\
 &= g \cdot (f \cdot (\blacktriangleright g \times A))^\dagger && \text{by def. of } \dagger.
 \end{aligned}$$

(4) Double dagger identity. Given $f : \blacktriangleright X \times \blacktriangleright X \times A \rightarrow X$ we have

$$\begin{aligned}
 f^{\dagger\dagger} &= \text{Tr}_{A,X}^X(\langle f^\dagger, f^\dagger \rangle) && \text{by def. of } \dagger \\
 &= \text{Tr}_{A,X}^X(\Delta_X \cdot \text{Tr}_{\blacktriangleright X \times A, X}^X(\langle f, f \rangle)) && \text{by def. of } \dagger \\
 &\stackrel{(Rt)}{=} \text{Tr}_{A,X}^X(\text{Tr}_{\blacktriangleright X \times A, X \times X}^X((X \times \Delta_X) \cdot \langle f, f \rangle)) \\
 &\stackrel{(V2)}{=} \text{Tr}_{A,X}^{X \times X}(\langle f, f, f \rangle \cdot (\text{can} \times A)) \\
 &= \text{Tr}_{A,X}^{X \times X}((\Delta_X \times X) \cdot \langle f, f \rangle \cdot (\text{can} \times A)) \\
 &\stackrel{(Sl)}{=} \text{Tr}_{A,X}^X(\langle f, f \rangle \cdot \underbrace{(\text{can} \times A) \cdot (\blacktriangleright (\Delta_X) \times A)}_{=\Delta_{\blacktriangleright X \times A}}) \\
 &= \text{Tr}_{A,X}^X(\langle f \cdot (\Delta_{\blacktriangleright X} \times A), f \cdot (\Delta_{\blacktriangleright X} \times A) \rangle) \\
 &= (f \cdot (\Delta_{\blacktriangleright X} \times A))^\dagger && \text{by def. of } \dagger.
 \end{aligned}$$

B.2. From dagger to trace

We prove that the operation Tr defined in Construction 4.4.2 satisfies all the axioms of a guarded trace operator. Again we start with a technical lemma.

Lemma B.2. Let $f : \blacktriangleright X \times A \rightarrow X \times B$ and define

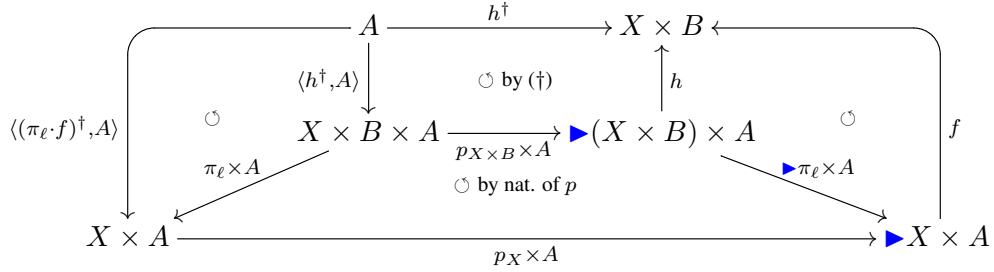
$$h = (\blacktriangleright(X \times B) \times A \xrightarrow{\blacktriangleright \pi_\ell \times A} \blacktriangleright X \times A \xrightarrow{f} X \times B).$$

Then we have

$$\text{Tr}_{A,B}^X(f) = (A \xrightarrow{h^\dagger} X \times B \xrightarrow{\pi_r} B).$$

Proof:

Notice first that by the simplified composition identity we have $\pi_\ell \cdot h^\dagger = (\pi_\ell \cdot f)^\dagger$. This implies that



The result follows by postcomposing with π_r ; by the definition of Tr we have

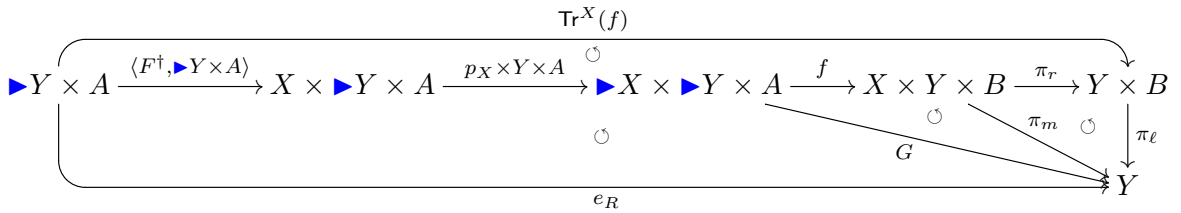
$$\text{Tr}_{A,B}^X(f) = \pi_r \cdot f \cdot (p_X \times A) \cdot \langle (\pi_\ell \cdot f)^\dagger, A \rangle = \pi_r \cdot h^\dagger.$$

□

We now verify the properties of a guarded trace for Tr .

(1) Vanishing (V1). For any $f : \blacktriangleright 1 \times A \rightarrow B$ the definition of $\text{Tr}_{A,B}^1(f)$ yields $f \cdot (p_1 \times A)$; for if we consider B as the product $1 \times B$ we see that both $\pi_\ell \cdot f : \blacktriangleright 1 \times A \rightarrow 1$ and its dagger $(\pi_\ell \cdot f)^\dagger : A \rightarrow 1$ are unique morphisms, which implies that $\langle (\pi_\ell \cdot f)^\dagger, A \rangle : A \rightarrow 1 \times A$ is the canonical isomorphism $A \cong 1 \times A$, and $\pi_r : 1 \times B \rightarrow B$ is the canonical isomorphism $1 \times B \cong B$.

(2) Vanishing (V2). Given $f : \blacktriangleright X \times \blacktriangleright Y \times A \rightarrow X \times Y \times B$ we form $F = \pi_\ell \cdot f : \blacktriangleright X \times \blacktriangleright Y \times A \rightarrow X$ and $G = \pi_m \cdot f : \blacktriangleright X \times \blacktriangleright Y \times A \rightarrow Y$, where $\pi_m : X \times Y \times B \rightarrow B$ denotes the middle product projection. Then by the Bekič identity (see Proposition 3.10) we have $\langle (F, G) \cdot (\text{can} \times A) \rangle^\dagger = \langle e_L^\dagger, e_R^\dagger \rangle$ for appropriate $e_L : \blacktriangleright X \times A \rightarrow X$ and $e_R : \blacktriangleright Y \times A \rightarrow Y$. From the following diagram we see that $\pi_\ell \cdot \text{Tr}_{\blacktriangleright Y \times A, Y \times B}^X(f) = e_R$:



By the naturality of p we have

$$p_X \times p_Y = (X \times Y \xrightarrow{p_{X \times Y}} \blacktriangleright (X \times Y) \xrightarrow{\text{can}} \blacktriangleright X \times \blacktriangleright Y). \quad (\text{B.3})$$

Now we obtain

$$\begin{aligned}
\mathrm{Tr}_{A,B}^Y(\mathrm{Tr}_{\blacktriangleright Y \times A, Y \times B}^X(f)) &= \pi_r \cdot \mathrm{Tr}^X(f) \cdot (p_Y \times A) \cdot \langle e_R^\dagger, A \rangle \\
&= \pi_r \cdot f \cdot (p_X \times \blacktriangleright Y \times A) \cdot \underbrace{\langle F^\dagger, \blacktriangleright Y \times A \rangle \cdot (p_Y \times A) \cdot \langle e_R^\dagger, A \rangle}_{=} \\
&= \pi_r \cdot f \cdot (p_X \times \blacktriangleright Y \times A) \cdot \langle F^\dagger \cdot \langle p_Y \cdot e_R^\dagger, A \rangle, p_Y \cdot e_R^\dagger, A \rangle \\
&\stackrel{(3.7)}{=} \pi_r \cdot f \cdot (p_X \times \blacktriangleright Y \times A) \cdot \langle e_L^\dagger, p_Y \cdot e_R^\dagger, A \rangle \\
&= \pi_r \cdot f \cdot (p_X \times p_Y \times A) \cdot \langle e_L^\dagger, e_R^\dagger, A \rangle \\
&\stackrel{(B.3)}{=} \pi_r \cdot f \cdot (\mathrm{can} \times A) \cdot (p_{X \times Y} \times A) \cdot \langle e_L^\dagger, e_R^\dagger, A \rangle
\end{aligned}$$

That this is $\mathrm{Tr}_{A,B}^{X \times Y}(f \cdot (\mathrm{can} \times A))$ now follows from the definition of Tr , the fact that $\langle e_L^\dagger, e_R^\dagger \rangle = (\langle F, G \rangle \cdot (\mathrm{can} \times A))^\dagger$ holds by the Bekič identity and since $\langle F, G \rangle = \pi'_\ell \cdot f$ where $\pi'_\ell : X \times Y \times A \rightarrow X \times Y$.

(3) Superposing. Let $f : \blacktriangleright X \times A \rightarrow X \times B$ and denote by $\pi'_r : X \times B \times C \rightarrow B \times C$ and $\pi'_\ell : X \times B \times C \rightarrow X$ the projections. Notice first that we have

$$\begin{aligned}
(\pi'_\ell \cdot (f \times C))^\dagger &= (\blacktriangleright X \times A \times C \xrightarrow{\blacktriangleright X \times \pi_\ell} \blacktriangleright X \times A \xrightarrow{f} X \times B \xrightarrow{\pi_\ell} X)^\dagger \\
&\stackrel{(P)}{=} (A \times C \xrightarrow{\pi_\ell} A \xrightarrow{(\pi_\ell \cdot f)^\dagger} X).
\end{aligned}$$

Using this we obtain

$$\begin{aligned}
\mathrm{Tr}_{A \times C, B \times C}^X(f \times C) &\stackrel{\mathrm{def}}{=} \pi'_r \cdot (f \times C) \cdot (p_X \times A \times C) \cdot \underbrace{\langle (\pi'_\ell \cdot (f \times C))^\dagger, A \times C \rangle}_{= \langle (\pi_\ell \cdot f)^\dagger, A \rangle \times C} \\
&= (\pi_r \cdot f \cdot (p_X \times A) \cdot \langle (\pi_\ell \cdot f)^\dagger, A \rangle) \times C \\
&\stackrel{\mathrm{def}}{=} \mathrm{Tr}_{A,B}^X(f) \times C.
\end{aligned}$$

(4) Yanking. Consider $c : \blacktriangleright X \times X \rightarrow X \times \blacktriangleright X$. Then by definition we have

$$\mathrm{Tr}_{X, \blacktriangleright X}^X(c) = \pi_r \cdot c \cdot (p_X \times X) \cdot \langle (\pi_\ell \cdot c)^\dagger, X \rangle.$$

Thus, we are done if we show that $(\pi_\ell \cdot c)^\dagger$ is the identity on X , which easily follows from the fixpoint identity:

$$(\pi_\ell \cdot c)^\dagger \stackrel{(\dagger)}{=} \pi_\ell \cdot c \cdot (p_X \times X) \cdot \langle (\pi_\ell \cdot c)^\dagger, X \rangle = \pi_r \cdot (p_X \times X) \cdot \langle (\pi_\ell \cdot c)^\dagger, X \rangle = X.$$

(5) Left tightening. Let $f : \blacktriangleright X \times A \rightarrow X \times B$ and $g : A' \rightarrow A$. By the parameter identity we have

$$(\pi_\ell \cdot f \cdot (\blacktriangleright X \times g))^\dagger = (\pi_\ell \cdot f)^\dagger \cdot g. \quad (\text{B.4})$$

Then we have

$$\begin{aligned}
\mathrm{Tr}_{A',B}^X(f \cdot (\blacktriangleright X \times g)) &\stackrel{\mathrm{def}}{=} \pi_r \cdot f \cdot (\blacktriangleright X \times g) \cdot (p_X \times A') \cdot \langle (\pi_\ell \cdot f \cdot (\blacktriangleright X \times g))^\dagger, A' \rangle \\
&\stackrel{(B.4)}{=} \pi_r \cdot f \cdot (\blacktriangleright X \times g) \cdot (p_X \times A') \cdot \langle (\pi_\ell \cdot f)^\dagger \cdot g, A' \rangle \\
&= \pi_r \cdot f \cdot (p_X \times A) \cdot \langle (\pi_\ell \cdot f)^\dagger, A \rangle \cdot g \\
&\stackrel{\mathrm{def}}{=} \mathrm{Tr}_{A,B}^X(f) \cdot g.
\end{aligned}$$

(6) Right tightening. Let $f : \blacktriangleright X \times A \rightarrow X \times B$ and $g : B \rightarrow B'$. We compute

$$\begin{aligned} \text{Tr}_{A,B'}^X((X \times g) \cdot f) &\stackrel{\text{def}}{=} \underbrace{\pi_r \cdot (X \times g)}_{=g \cdot \pi_r} \cdot f \cdot (p_X \times A) \cdot \langle \underbrace{(\pi_\ell \cdot (X \times g) \cdot f)^\dagger}_{=\pi_\ell}, A \rangle \\ &\stackrel{\text{def}}{=} g \cdot \text{Tr}_{A,B}^X(f). \end{aligned}$$

(7) Sliding. Let $f : \blacktriangleright X \times A \rightarrow X' \times B$ and $g : X' \rightarrow X$. Notice first that we have

$$(\pi_\ell \cdot (g \times B) \cdot f)^\dagger = (g \cdot (\pi_\ell \cdot f))^\dagger \stackrel{(C)}{=} g \cdot (\pi_\ell \cdot f \cdot (\blacktriangleright g \times A))^\dagger. \quad (\text{B.5})$$

Then we obtain

$$\begin{aligned} \text{Tr}_{A,B}^X((g \times B) \cdot f) &\stackrel{\text{def}}{=} \underbrace{\pi_r \cdot (g \times B)}_{=\pi_r} \cdot f \cdot (p_X \times A) \cdot \langle (\pi_\ell \cdot (g \times B) \cdot f)^\dagger, A \rangle \\ &\stackrel{(B.5)}{=} \pi_r \cdot f \cdot (p_X \times A) \cdot \langle g \cdot (\pi_\ell \cdot f \cdot (\blacktriangleright g \times A))^\dagger, A \rangle \\ &= \pi_r \cdot f \cdot \underbrace{(p_X \times A) \cdot (g \times A)}_{=(\blacktriangleright g \times A) \cdot (p_{X'} \times A)} \cdot \langle (\pi_\ell \cdot f \cdot (\blacktriangleright g \times A))^\dagger, A \rangle \\ &\stackrel{\text{def}}{=} \text{Tr}_{A,B}^{X'}(f \cdot (\blacktriangleright g \times A)). \end{aligned}$$

B.3. Mutual inverses

From dagger to trace and back. We show that $\dagger \text{Tr}_\dagger = \dagger$. For any $f : \blacktriangleright X \times A \rightarrow X$ we compute

$$\dagger \text{Tr}_\dagger(f) \stackrel{\text{def}}{=} (\text{Tr}_\dagger)_{A,X}^X(\langle f, f \rangle) \stackrel{\text{def}}{=} \underbrace{\pi_r \cdot \langle f, f \rangle}_{=f} \cdot (p_X \times A) \cdot \langle \underbrace{(\pi_\ell \cdot \langle f, f \rangle)^\dagger}_{=\langle f^\dagger, A \rangle}, A \rangle \stackrel{(\dagger)}{=} f^\dagger.$$

From trace to dagger and back. We show that $\text{Tr}_{\dagger \text{Tr}} = \text{Tr}$. Let $f : \blacktriangleright X \times A \rightarrow X \times B$. Then we have by the definition of $\dagger \text{Tr}$

$$(\pi_\ell \cdot f)^\dagger \text{Tr} \stackrel{\text{def}}{=} \text{Tr}_{A,X}^X(\langle \pi_\ell \cdot f, \pi_\ell \cdot f \rangle) = \text{Tr}_{A,X}^X(\Delta_X \cdot \pi_\ell \cdot f) =: h. \quad (\text{B.6})$$

Using Lemma B.1, this allows us to conclude

$$(\text{Tr}_{\dagger \text{Tr}})_{A,B}^X(f) \stackrel{\text{def}}{=} \pi_r \cdot f \cdot (p_X \times A) \cdot \underbrace{\langle (\pi_\ell \cdot f)^\dagger \text{Tr}, A \rangle}_{=(h,A)} \stackrel{\text{Lem. B.1}}{=} \text{Tr}_{A,B}^X(f).$$

C. Proof of theorem 4.7

1. From trace to dagger. Let f, f' and h form the commutative square on the left below:

$$\begin{array}{ccc} \blacktriangleright X \times A & \xrightarrow{f} & X \\ \blacktriangleright h \times A \downarrow & \circlearrowleft & \downarrow h \\ \blacktriangleright X' \times A & \xrightarrow{f'} & X' \end{array} \qquad \begin{array}{ccccc} \blacktriangleright X \times A & \xrightarrow{\langle f, f \rangle} & X \times X & \xrightarrow{X \times h} & X \times X' \\ \blacktriangleright h \times A \downarrow & & \circlearrowleft & \searrow h \times h & \downarrow h \times X' \\ \blacktriangleright X' \times A & \xrightarrow{\langle f', f' \rangle} & X' \times X' & & X' \times X' \end{array}$$

Then the diagram on the right above commutes, too, and thus, by uniformity of Tr we have

$$\text{Tr}_{A,X'}^X((X \times h) \cdot \langle f, f \rangle) = \text{Tr}_{A,X'}^{X'} \langle f', f' \rangle. \quad (\text{C.1})$$

Thus, we obtain: $h \cdot f^\dagger \stackrel{\text{def}}{=} h \cdot \text{Tr}_{A,X}^X \langle f, f \rangle \stackrel{(\text{Rt})}{=} \text{Tr}_{A,X'}^X((X \times h) \cdot \langle f, f \rangle) \stackrel{(\text{C.1})}{=} \text{Tr}_{A,X'}^{X'} \langle f', f' \rangle \stackrel{\text{def}}{=} (f')^\dagger$.

2. From dagger to trace. Let f , f' and h form the commutative square on the left below:

$$\begin{array}{ccc} \blacktriangleright X \times A & \xrightarrow{f} & X \times B \\ \blacktriangleright h \times A \downarrow & \circlearrowleft & \downarrow h \times B \\ \blacktriangleright X' \times A & \xrightarrow{f'} & X' \times B \end{array} \quad \begin{array}{ccccc} \blacktriangleright (X \times B) \times A & \xrightarrow{\blacktriangleright \pi_\ell \times A} & \blacktriangleright X \times A & \xrightarrow{f} & X \times B \\ \blacktriangleright (h \times B) \times A \downarrow & \circlearrowleft & \blacktriangleright h \times A \downarrow & \circlearrowleft & \downarrow h \times B \\ \blacktriangleright (X' \times B) \times A & \xrightarrow{\blacktriangleright \pi_\ell \times A} & \blacktriangleright X' \times A & \xrightarrow{f'} & X' \times B \end{array}$$

Then the diagram on the right above commutes, too, and thus, by uniformity of \dagger we have

$$(h \times B) \cdot (f \cdot (\blacktriangleright \pi_\ell \times A))^\dagger = (f' \cdot (\blacktriangleright \pi_\ell \times A))^\dagger \quad (\text{C.2})$$

Using Lemma B.2 we now compute:

$$\begin{aligned} \text{Tr}_{A,B}^{X'}(f') &\stackrel{\text{Lem. B.2}}{=} \pi_r \cdot (f' \cdot (\blacktriangleright \pi_\ell \times A))^\dagger \\ &\stackrel{(\text{C.2})}{=} \underbrace{\pi_r \cdot (h \times B) \cdot (f \cdot (\blacktriangleright \pi_\ell \times A))^\dagger}_{=\pi_r} \\ &\stackrel{\text{Lem. B.2}}{=} \text{Tr}_{A,B}^X(f). \end{aligned}$$

D. Proof of corollary 4.10

1. Let $F : (\mathcal{C}, \blacktriangleright^{\mathcal{C}}, \text{Tr}_{\mathcal{C}}) \rightarrow (\mathcal{D}, \blacktriangleright^{\mathcal{D}}, \text{Tr}_{\mathcal{D}})$ be a morphism of guarded traced categories. We show that F preserves \dagger_{Tr} as defined in Construction 4.4.1. Let $f : \blacktriangleright X \times A \rightarrow X$ in \mathcal{C} . Then we have (dropping subscripts of \dagger and Tr)

$$\begin{aligned} F(f^\dagger) &= F(\text{Tr}^X \langle f, f \rangle) \quad (\text{by definition of } \dagger) \\ &= \text{Tr}^{FX} (F \langle f, f \rangle) \quad (F \text{ trace preserving}) \\ &= \text{Tr}^{FX} \langle Ff, Ff \rangle \quad (F \text{ finite product preserving}) \\ &= (Ff)^\dagger \quad (\text{by definition of } \dagger). \end{aligned}$$

2. Let $F : (\mathcal{C}, \blacktriangleright^{\mathcal{C}}, \dagger) \rightarrow (\mathcal{D}, \blacktriangleright^{\mathcal{D}}, \dagger)$ be a morphism of guarded Conway categories. We show that F preserves Tr_\dagger as defined in Construction 4.4.2. Let $f : \blacktriangleright^{\mathcal{C}} X \times A \rightarrow X \times B$ in \mathcal{C} . Then we have

(again we drop all subscripts of Tr and \dagger)

$$\begin{aligned}
F(\text{Tr}^X(f)) &= F(\pi_r \cdot (f \cdot (\blacktriangleright^{\mathcal{C}} \pi_\ell \times A))^\dagger) && \text{(by Lemma B.2)} \\
&= F\pi_r \cdot (Ff \cdot F(\blacktriangleright^{\mathcal{C}} \pi_\ell \times A))^\ddagger && (F \text{ dagger preserving}) \\
&= \pi_r \cdot (Ff \cdot (F(\blacktriangleright^{\mathcal{C}} \pi_\ell) \times FA))^\ddagger && (F \text{ finite product preserving}) \\
&= \pi_r \cdot (Ff \cdot (\underbrace{\blacktriangleright^{\mathcal{D}} F\pi_\ell}_{=\pi_\ell} \times FA))^\ddagger && \text{(by (4.1))} \\
&= \text{Tr}^{FX}(Ff) && \text{(by Lemma B.2).}
\end{aligned}$$

This completes the proof.