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On Iteratable Endofunctors

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Abstract

Completely iterative monads of Elgot et al. are the monads such that every guarded iterative equation has a unique solution. Free completely iterative monads are known to exist on every iteratable endofunctor H, i. e., one with final coalgebras of all functors $H(\_) + X$. We show that conversely, if H generates a free completely iterative monad, then it is iteratable.

Key words: monad, completely iterative, iterable

1 Introduction

There have been various attempts to algebraically capture the concept of computations on data through a program, taking into account that such computations are potentially infinite. During the 1970's the ADJ-group studied continuous algebras, i. e., algebras endowed with a CPO structure. There, an infinite computation is a join of the directed set of its finite approximations, see e. g. [ADJ]. Later, algebras over complete metric spaces were considered, where an infinite computation is a limit of a Cauchy sequence of finite approximations, see e. g. [ARu].

Another approach to infinite computations are iterative algebraic theories, introduced by Calvin C. Elgot in [E]. This notion has been extended to the notion of completely iterative theories by Elgot, Bloom and Tindell, see [EBT]. The latter are algebraic theories (in the sense of Lawvere and Linton [Lin]) that allow for unique solutions of fixed point equations. An important example of a completely iterative theory is the theory of finite and infinite trees over a given signature Σ. In [EBT] it is shown that this is the free completely iterative theory over Σ.

It has recently been discovered by Peter Aczel, Jiří Adámek, Jiří Velebil, and the present author [AAMV], that the above fact is a special case of a much more general categorical result using a coalgebraic approach to infinite

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computation. This coalgebraic approach has also independently been studied by Larry Moss in [M]. Here one considers a category $\mathcal{A}$ with binary coproducts, and an *iteratable* endofunctor H on $\mathcal{A}$, i. e., such that for every object X a final coalgebra

$$TX$$

of $H(\_) + X$ exists. In [AAMV] the notion of a completely iterative monad is introduced. Informally, this is a monad that allows for unique solutions of systems of equations of a certain liberal type. It has been shown that the mapping $X \mapsto TX$ is the object assignment of a completely iterative monad. Moreover, it was proved that this monad T is a free completely iterative monad on H.

In the present paper we investigate the exact relationship between the notion of iteratability and the existence of free completely iterative monads for an endofunctor. The main result of [AAMV] shows that iteratable endofunctors admit free completely iterative monads. Here we prove that no other functors do so. More precisely, if S is a free completely iterative monad over an endofunctor H on $\mathcal{A}$, then H is iteratable, and for all objects X of $\mathcal{A}$, SX is a final coalgebra of $H(\_) + X$.

Before we prove our main result in Section 3 we shall recall the results of [AAMV] and give some motivation for the notion of completely iterative monad in Section 2.

2 Iteratable Endofunctors and Completely Iterative Monads

2.1 A Motivating Example

We take from [AMV] a motivating example for the coalgebraic approach of [AAMV]. Consider the algebra of finite and infinite trees over a given signature Σ. This algebra allows for the unique solution of systems of so-called guarded equations. Let us give the details of this. Denote by

$$T_\Sigma X$$

the algebra of all finite and infinite Σ-labelled trees with variables from X. That is, trees labelled so that a node with $n > 0$ children is labelled by an n-ary operation symbol (an element of Σ_n) and a leaf is labelled by a variable or a constant (an element of $X + \Sigma_0$). The operations on $T_\Sigma X$ are given by tree-tupling. Furthermore, consider a system of equations

$$\begin{aligned} x_0 &\approx t_0(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\ x_1 &\approx t_1(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\ &\vdots \\ x_n &\approx t_n(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\ &\vdots \end{aligned} \tag{1}$$

where t_i are trees with variables from $X = \{x_0, x_1, x_2, \ldots\}$ and parameters from $Y = \{y_0, y_1, y_2, \ldots\}$, i.e.,

$$t_i \in T_\Sigma(X+Y) \quad \text{for } i = 0, 1, 2, \ldots$$

Notice that in a system we denote by $\approx$ formal equations and $=$ is the identity of the two sides. A system is called *guarded* provided that none of the trees t_i is just a variable from X. This condition is enough to force the existence of a unique *solution* of (1), i.e., a unique tuple ${x_i}^\dagger(y_0, y_1, y_2, \ldots)$ of trees in $T_\Sigma Y$ such that the identities

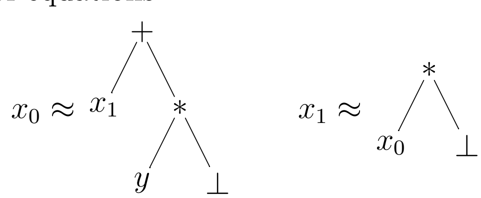
$$\begin{aligned}
{x_0}^\dagger &= t_0({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\
{x_1}^\dagger &= t_1({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\
&\vdots \\
{x_n}^\dagger &= t_n({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\
&\vdots
\end{aligned}$$

hold.

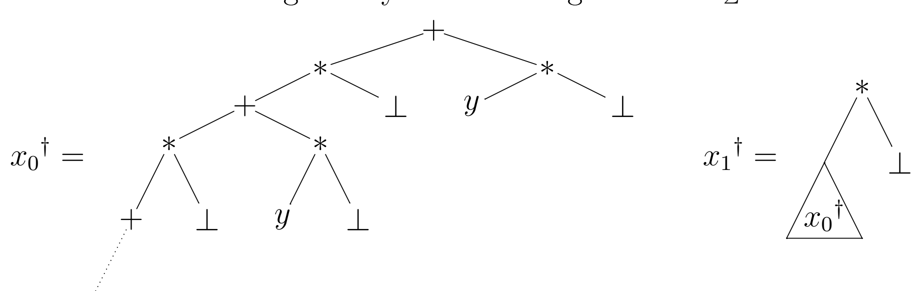
Theorem 2.1 *Every guarded system of equations has a unique solution.*

In fact, this is a special case of a much more general Solution Theorem we mention in Subsection 2.2 below.

Example 2.2 Let Σ consist of binary operations $+$ and $*$ and a constant $\perp$. The following system of equations



is guarded. The solution is given by the following trees in $T_\Sigma Y$:



2.2 Substitutions and Solutions Coalgebraically

The coalgebraic approach of [AAMV] and [M] relies on the following observation. To any signature Σ there is an associated polynomial endofunctor $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ defined by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$$

Recall that H_Σ-algebras are just the classical universal Σ-algebras. A final H_Σ-coalgebra is well-known to be the coalgebra $T_\Sigma\emptyset$ of all finite and infinite Σ-labelled trees without variables, see [AK]. Now $H_\Sigma(\_)+X$ is also polynomial (for the signature obtained from Σ by adding a constant symbol for every element of X), thus, $T_\Sigma X$ is a final coalgebra of $H_\Sigma(\_)+X$.

Taking the existence of such a parametrized family of final coalgebras as the primitive notion, one can abstract away from signatures (=polynomial endofunctors) and from the category Set.

Assumption 2.3 *For the rest of this section we assume that $\mathcal{A}$ denotes a category with binary coproducts whose injections are monomorphic, and H is an endofunctor on $\mathcal{A}$.*

Definition 2.4 An endofunctor H of $\mathcal{A}$ is called *iteratable* if for every object X of $\mathcal{A}$ there exists a final coalgebra of $H(\_)+X$.

The following examples of iteratable endofunctors have been taken from [AAMV].

Example 2.5

(i) Accessible (=bounded) endofunctors on Set. An endofunctor is called accessible if it preserves λ-filtered colimits for some infinite cardinal λ. In [AP], it was shown that those are precisely the so-called bounded endofunctors. This example subsumes all the following ones.

(ii) (Generalized) polynomial endofunctors on Set, i. e., H is defined by

$$HZ = \coprod_{i<\lambda} A_i \times Z^i$$

for some cardinal λ; for $\lambda = \omega$ one has a polynomial endofunctor associated to a finitary signature as in 2.1.

(iii) The bounded power set functors defined on objects by

$$\mathcal{P}_\lambda X = \{Y \subseteq X \mid |Y| < \lambda\}$$

for some cardinal λ. Notice that the (unbounded) power set functor $\mathcal{P} : \mathsf{Set} \longrightarrow \mathsf{Set}$ does not allow for a final coalgebra, and hence, it is not iteratable.

Note that the notions of accessibility and iteratability are not equivalent. In fact, there are examples of non-accessible endofunctors that are iteratable (see [AAMV]).

Remark 2.6 If H is an iteratable endofunctor on $\mathcal{A}$ we denote by

$$TX$$

the final coalgebra of $H(\_)+X$. By the Lambek Lemma (see [L]), the structure map of that final coalgebra is an isomorphism, and consequently, TX is a coproduct of HTX and X with injections

$$\begin{aligned} \eta_X : X &\longrightarrow TX && \text{“injection of variables”} \\ \tau_X : HTX &\longrightarrow TX && \text{“TX is an H-algebra”} \end{aligned}$$

The final coalgebras TX have a rich structure. Firstly, the way how substitution works on trees in $T_\Sigma X$ generalizes smoothly to the categorical setting. Recall here that given an interpretation of variables $x \in X$ as trees $s(x)$ over Y, i. e., a function $s : X \longrightarrow T_\Sigma Y$, then the corresponding substitution of trees from $T_\Sigma Y$ into (leaves of) trees of $T_\Sigma X$ is a homomorphism

$$\widehat{s} : T_\Sigma X \longrightarrow T_\Sigma Y$$

of Σ-algebras. Moreover, $\widehat{s}$ is the unique extension of s. This can be generalized to all iteratable endofunctors:

Substitution Theorem 2.7 *For any arrow $s : X \longrightarrow TY$ there exists a unique homomorphism $\widehat{s} : TX \longrightarrow TY$ of H-algebras extending s, i. e., such that $\widehat{s} \cdot \eta_X = s$.*

The proof can be found in [M] or [AAV] (slightly improved in [AAMV]).

Next, one can generalize in a straightforward way the notion of a system of equations. An *equation arrow* is a morphism

$$e : X \longrightarrow T(X + Y)$$

in $\mathcal{A}$. It is called *guarded* if it factors as follows

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & T(X+Y) \\ & \searrow & \uparrow {\scriptstyle [\tau_{X+Y},\, \eta_{X+Y} \cdot \mathsf{inr}]} \\ & & HT(X+Y) + Y\,. \end{array}$$

Notice that for a polynomial endofunctor $H = H_\Sigma$ on Set this is precisely the notion of a guarded system as presented above, since $T(X+Y) = HT(X+Y) + X + Y$. Finally, a *solution* for an equation arrow e is an arrow $e^\dagger : X \longrightarrow TY$ such that the following triangle

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & TY \\ {\scriptstyle e}\downarrow & \nearrow {\scriptstyle \widehat{[e^\dagger, \eta_Y]}} & \\ T(X+Y) & & \end{array}$$

commutes. Again, this corresponds precisely to the notion of solution for systems of equations in case of polynomial endofunctors on Set.

The following result is called Parametric Corecursion in [M] and Solution Theorem in [AAV]; see also an improved version of the proof in [AAMV]:

Solution Theorem 2.8 *Given an iteratable endofunctor H, every guarded equation morphism has a unique solution.*

Remark 2.9 It is an easy consequence of the Substitution Theorem that $(T, \eta, \widehat{(\_)})$ forms a Kleisli triple, i. e., the following three conditions are satisfied

(i) $\widehat{\eta_X} = id_{TX}$ for all objects X,

(ii) $\widehat{s} \cdot \eta_X = s$ for all arrows $s : X \longrightarrow TY$,

(iii) $\widehat{r} \cdot \widehat{s} = \widehat{\widehat{r} \cdot s}$ for any morhisms $s : X \longrightarrow TY$ and $r : Y \longrightarrow TZ$.

Thus setting $\mu_X = \widehat{id_{TX}} : TTX \longrightarrow TX$ we obtain a monad (T, η, μ), and we call it the *completely iterative monad* generated by H.

2.3 The free Completely Iterative Monad

Based on the consideration in the previous section 2.2 it is quite natural to call for any monad (S, η, μ) on $\mathcal{A}$ a morphism

$$e : X \longrightarrow S(X + Y)$$

an *equation arrow*. Recall that for any monad there is an associated Kleisli triple, where for $s : X \longrightarrow SY$ we have $\widehat{s} = \mu_Y \cdot Ss$. Hence, a morphism $e^\dagger : X \longrightarrow SY$ with

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ {\scriptstyle e}\downarrow & \nearrow{\scriptstyle \widehat{[e^\dagger, \eta_Y]}} & \\ S(X+Y) & & \end{array}$$

will be called a *solution*. However, it is in general not obvious how the property of e being guarded is to be expressed for an arbitrary monad.

Elgot, Bloom and Tindell [EBT] use, in their setting of algebraic theories, the notion of an ideal theory introduced by Elgot in [E]. For finitary monads on Set this notion is equivalent to the following notion of ideal monad (see [AAMV] for a simple proof of this fact):

Definition 2.10

(i) Let (S, η, μ) be a monad. A (right) ideal of S is a subfunctor $\sigma : S' \rightarrowtail S$ such that there exists a (necessarily unique) restriction $\mu' : S'S \longrightarrow S'$ of μ, i. e., the following square

$$\begin{array}{ccc} S'S & \xrightarrow{\mu'} & S' \\ {\scriptstyle \sigma S}\downarrow & & \downarrow{\scriptstyle \sigma} \\ SS & \xrightarrow[\mu]{} & S \end{array}$$

commutes.

(ii) A monad together with an ideal of it is called an *idealized monad.* If furthermore we have $S = S' + Id$, i. e., $[\sigma, \eta] : S' + Id \longrightarrow S$ is an isomorphism, then S is called an *ideal monad*.

(iii) An *idealized-monad morphism* between idealized monads S_1 and S_2 with chosen ideals $\sigma_i : S_i' \rightarrowtail S_i$, $i = 1, 2$, is a monad morphism $h : S_1 \longrightarrow S_2$ that preserves the chosen ideals, i. e., there exists a (necessarily unique) natural transformation $h' : S_1' \longrightarrow S_2'$ such that the following square

$$\begin{array}{ccc} S_1' & \xrightarrow{h'} & S_2' \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ S_1 & \xrightarrow[h]{} & S_2 \end{array}$$

commutes.

Example 2.11

(i) Recall that the monad T is a coproduct of HT and Id. Hence the ideal $\tau : HT \rightarrowtail T$, where μ' is given by $H\mu$ makes T into an ideal monad.

(ii) Any monad S has ideals, e. g., the largest one (S itself). If $\mathcal{A}$ has a strict initial object, then the smallest ideal is given by the constant functor on the initial object.

Remark 2.12

(i) Notice that the notion of an ideal of a monad corresponds precisely to the notion of a right ideal for a monoid. Indeed, recall that a right ideal of a monoid M is a subset I of M such that $I \cdot M \subseteq I$. Now a monad is just a monoid in the monoidal category $[\mathcal{A}, \mathcal{A}]$ of endofunctors on $\mathcal{A}$ with tensor product being given by composition of functors.

(ii) It is not difficult to show that the category of ideal monads and ideal monad homomorphisms is a coreflective subcategory of the category of idealized monads with the same morphisms. In fact, if (S, η, μ) is a monad with ideal $\sigma : S' \rightarrowtail S$ the coreflection arrow is given by

$$S' + Id \xrightarrow{[\sigma,\eta]} S.$$

Since this is not needed here, the proof is omitted.

Remark 2.13 Observe that the completely iterative monad T generated by H comes with a natural "embedding of H"

$$\tau^* \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T$$

into it. More generally, we call for any endofunctor H and idealized monad S a natural transformation $\sigma^* : H \longrightarrow S$ *ideal* if it factors through the ideal

$\sigma : S' \rightarrowtail S$ as follows

$$\begin{array}{ccc} H & \xrightarrow{\sigma^*} & S \\ & \searrow^{(\sigma^*)'} & \uparrow \sigma \\ & & S' \end{array}$$

For an idealized monad S we define the notion of a *guarded* equation arrow as a morphism e that factors

$$\begin{array}{ccc} X & \xrightarrow{e} & S(X+Y) \\ & \searrow & \uparrow [\sigma_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \\ & & S'(X+Y)+Y\,. \end{array}$$

Definition 2.14 An idealized monad S is called *completely iterative* if every guarded equation arrow has a unique solution.

Remark 2.15 In [AAMV] a completely iterative monad is required to be ideal. Observe however, that this is an uneccessary restriction. In fact, all of the proofs of [AAMV] use only properties of idealized monads.

The following is the main result of [AAMV].

Theorem 2.16 *For any iteratable endofunctor H, the monad T is the free completely iterative monad on H. More precisely, for all completely iterative monads S and ideal transformations $\lambda : H \longrightarrow S$ there exists a unique idealized-monad morphism $\overline{\lambda} : T \longrightarrow S$ such that $\overline{\lambda} \cdot \tau^* = \lambda$:*

$$\begin{array}{ccc} H & \xrightarrow{\tau^*} & T \\ & \searrow^{\forall \lambda} & \downarrow \exists ! \overline{\lambda} \\ & & S\,. \end{array}$$

Remark 2.17

(i) Since the inclusion of the ideal $\sigma : S' \rightarrowtail S$ is a monomorphism, the last condition is equivalent to stating that

$$\begin{array}{ccc} H & \xrightarrow{H\eta} & HT \\ & \searrow^{\lambda'} & \downarrow \overline{\lambda}' \\ & & S' \end{array}$$

commutes, where $\overline{\lambda}' : HT \longrightarrow S'$ is the restriction of $\overline{\lambda}$ to the ideal of T.

(ii) Categorically, the statement of the theorem says that every iteratable functor H in $[\mathcal{A}, \mathcal{A}]$ has a universal arrow w. r. t. the forgetful functor

$$U : \mathbf{CIM}(\mathcal{A}) \longrightarrow [\mathcal{A}, \mathcal{A}]$$

of the category $\mathbf{CIM}(\mathcal{A})$ of all completely iterative monads and idealized-monad morphisms. Beware! The functor U assigns to every completely iterative monad S its ideal S', not the underlying functor S. This choice of U corresponds to the requirement that $\lambda : H \longrightarrow S$ be an ideal transformation.

The above result states that any iteratable endofunctor admits a free completely iterative monad. However, the obvious question whether these are the only endofunctors with this property remains unanswered in [AAMV]. We will present this answer in the next section.

3 Iteratability is neccessary

We have seen above that any iteratable endofunctor admits a free completely iterative monad. We shall prove in this section that endofunctors that admit a free completely iterative monad are precisely the iteratable ones.

Throughout this section we shall denote by $\mathcal{A}$ a category with binary coproducts such that injections are monomorphic.

Theorem 3.1 *Every endofunctor generating a free completely iterative monad is iteratable.*

Remark 3.2 More detailed, suppose that H is an endofuntor on $\mathcal{A}$ and

$$\sigma^* : H \longrightarrow S$$

is a free completely iterative monad on H (where σ^* is an ideal transformation), then H is iteratable and for all objects X of $\mathcal{A}$, SX is a final coalgebra of $H(\_) + X$.

Before we proceed with the proof of this theorem, let us prove two auxilliary results. First we establish that for any natural transformation $H \longrightarrow S$, where H is any endofunctor and S any monad on $\mathcal{A}$, one can easily obtain an ideal monad $\widetilde{S}$ and an ideal transformation $H \longrightarrow \widetilde{S}$ as follows.

Definition 3.3 Let (S, η, μ) be a monad on $\mathcal{A}$ and let

$$\sigma^* : H \longrightarrow S$$

be a natural transformation from an endofuntor H on $\mathcal{A}$. Define $(\widetilde{S}, \widetilde{\eta}, \widetilde{\mu})$ as follows:

(i) $\widetilde{S} = HS + Id$

(ii) $\widetilde{\eta} \equiv \mathsf{inr} : Id \longrightarrow HS + Id$

(iii) $$\widetilde{\mu} \equiv \widetilde{S}^2 = (HS + Id)^2 = HS(HS + Id) + HS + Id$$
$$\big\downarrow {\scriptstyle HS(\sigma^* S + Id) + HS + Id}$$
$$HS(S^2 + Id) + HS + Id$$
$$\big\downarrow {\scriptstyle HS[\mu,\eta] + HS + Id}$$
$$HS^2 + HS + Id$$
$$\big\downarrow {\scriptstyle [H\mu, \mathsf{inl}] + Id}$$
$$HS + Id = \widetilde{S}$$

Lemma 3.4 *The triple* $(\widetilde{S}, \widetilde{\eta}, \widetilde{\mu})$ *is an ideal monad.*

Proof. Once we have established that $\widetilde{S}$ is a monad, it is obvious that it is ideal: Note that for $\widetilde{S}' = HS$ we have

$$\widetilde{\mu}' \equiv \widetilde{S}'\widetilde{S} = HS(HS + Id) \xrightarrow{HS(\sigma^* S + Id)} HS(S^2 + Id) \xrightarrow{HS[\mu,\eta]} HS^2 \xrightarrow{H\mu} HS = \widetilde{S}'.$$

Hence, it is sufficient to show that $\widetilde{\eta}$ and $\widetilde{\mu}$ satisfy the three axioms of a monad.

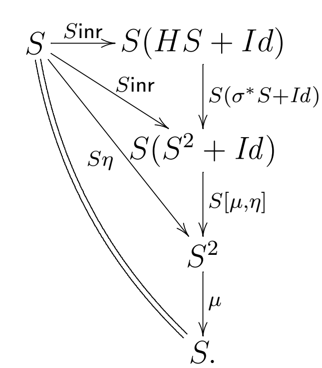
(i) $\widetilde{\mu} \cdot \widetilde{\eta}_{\widetilde{S}} = 1_{\widetilde{S}}$: This is obvious since

$$HS + Id \xrightarrow{\mathsf{inr}} HS(HS + Id) + HS + Id \xrightarrow{\widetilde{\mu}} HS + Id \equiv 1_{HS+Id}.$$

(ii) $\widetilde{\mu} \cdot \widetilde{S}\widetilde{\eta} = 1_{\widetilde{S}}$: Observe that

$$\widetilde{S}\widetilde{\eta} \equiv HS + Id \xrightarrow{HS\mathsf{inr} + \mathsf{inr}} HS(HS + Id) + HS + Id.$$

We compose this with $\widetilde{\mu}$ and consider the components of the coproduct $HS + Id$ separately. On the right-hand component we obviously obtain $\mathsf{inr} : Id \longrightarrow HS + Id$. For the left-hand one we drop H and consider the resulting commutative diagram

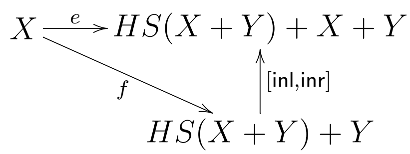


(iii) $\widetilde{\mu} \cdot \widetilde{S}\widetilde{\mu} = \widetilde{\mu} \cdot \widetilde{\mu}\widetilde{S}$: This is a straightforward and not particularly enlightening chase through rather huge diagrams. Since it only involves naturality and the equation $\mu \cdot S\mu = \mu \cdot \mu S$, we leave this as an easy exercise for the Reader.

□

Lemma 3.5 *If S in Definition 3.3 above is a completely iterative monad and $\sigma^* : H \longrightarrow S$ is an ideal transformation, then $\widetilde{S}$ is completely iterative, too.*

Proof. We have to show that for each guarded equation morphism $e : X \longrightarrow \widetilde{S}(X+Y)$ with a factorization



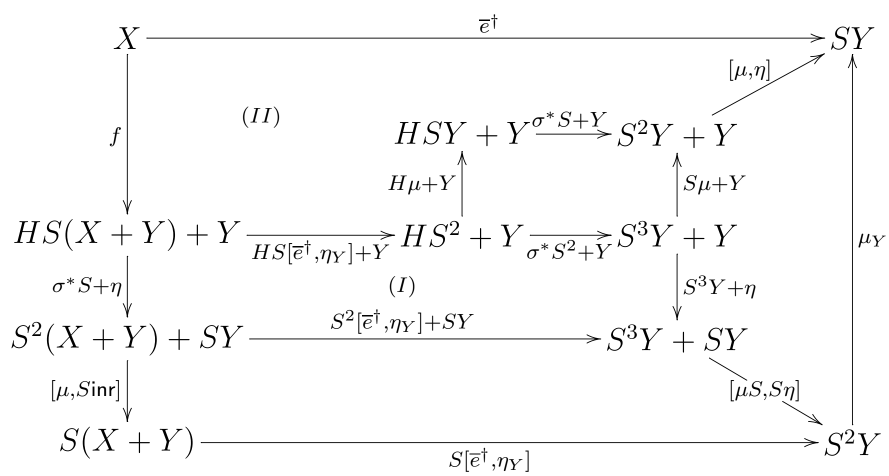
we have a unique solution $e^\dagger : X \longrightarrow \widetilde{S}Y$. We define a guarded equation arrow $\overline{e} : X \longrightarrow S(X+Y)$ as follows

$$\overline{e} \equiv X \xrightarrow{f} HS(X+Y)+Y \xrightarrow{\sigma^* S+\eta} S^2(X+Y)+SY \xrightarrow{[\mu, S\mathsf{inr}]} S(X+Y).$$

In order to see that $\overline{e}$ is indeed guarded, use that S is an idealized monad and that σ^* is an ideal transformation.

We solve $\overline{e}$ to obtain a unique arrow $\overline{e}^\dagger : X \longrightarrow SY$ such that the outer shape of the following diagram

(2)

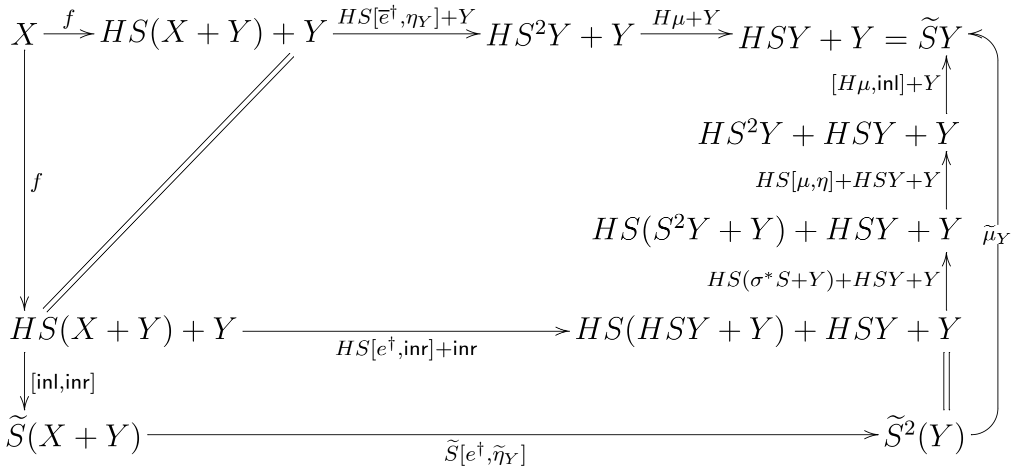


commutes. To see that square (I) commutes consider the components of the coproduct $HS(X+Y)+Y$ separately. The left-hand component commutes by naturality of σ^*, whereas the right-hand one obviously does. Hence, region (II) commutes since all other parts of the above Diagram (2) clearly do.

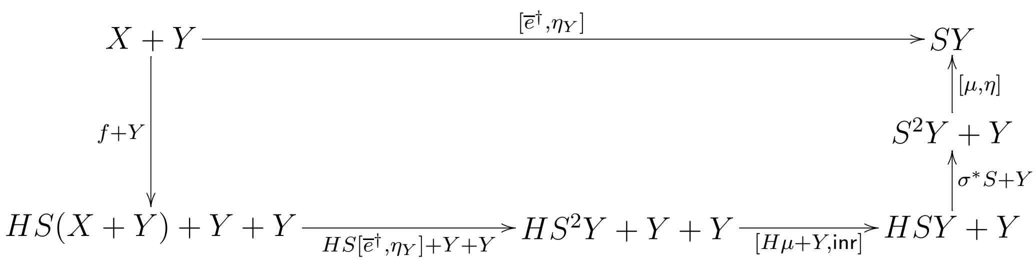
We define

$$e^\dagger \equiv X \xrightarrow{f} HS(X+Y)+Y \xrightarrow{HS[\overline{e}^\dagger,\eta_Y]+Y} HS^2Y+Y \xrightarrow{H\mu+Y} HSY+Y = \widetilde{S}Y,$$

and check that this yields a solution for e. Indeed, consider the following diagram:



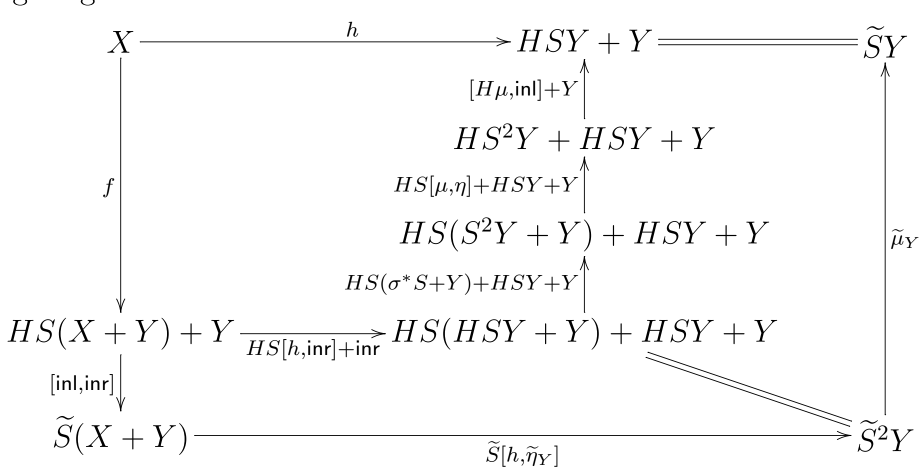
It obviously commutes, except perhaps the upper middle part. We consider its components separately. The right-hand component is the identity on Y. For the left-hand one notice that the last arrow is $H\mu$ on both paths. We show that the rest is already commutative, in fact, even if we drop HS. That is, we consider the resulting diagram:



This is obviously commutative. Indeed, the right-hand component is η_Y and the left-hand one is region (II) of diagram (2). This concludes the proof of the existence of a solution for e.

As for the unicity of solutions, consider any $h : X \longrightarrow \widetilde{S}Y$ such that the following diagram

(3)



commutes.

Below we will show that

$$X \xrightarrow{h} HSY + Y \xrightarrow{\sigma^* S + Y} S^2Y + Y \xrightarrow{[\mu,\eta]} SY \tag{4}$$

solves $\overline{e}$. But then it is not difficult to show that $e^\dagger = h$. In fact, we start with the definition of the solution $e^\dagger$

$$e^\dagger = (H\mu_Y + Y) \cdot (HS[\overline{e}^\dagger, \eta_Y] + Y) \cdot f,$$

then substitute (4) for $\overline{e}^\dagger$ to obtain

$$(H\mu_Y + Y) \cdot (HS\,[[\mu_Y, \eta_Y] \cdot (\sigma^*_{SY} + Y) \cdot h, \eta_Y] + Y) \cdot f, \tag{5}$$

and finally, we use the equation

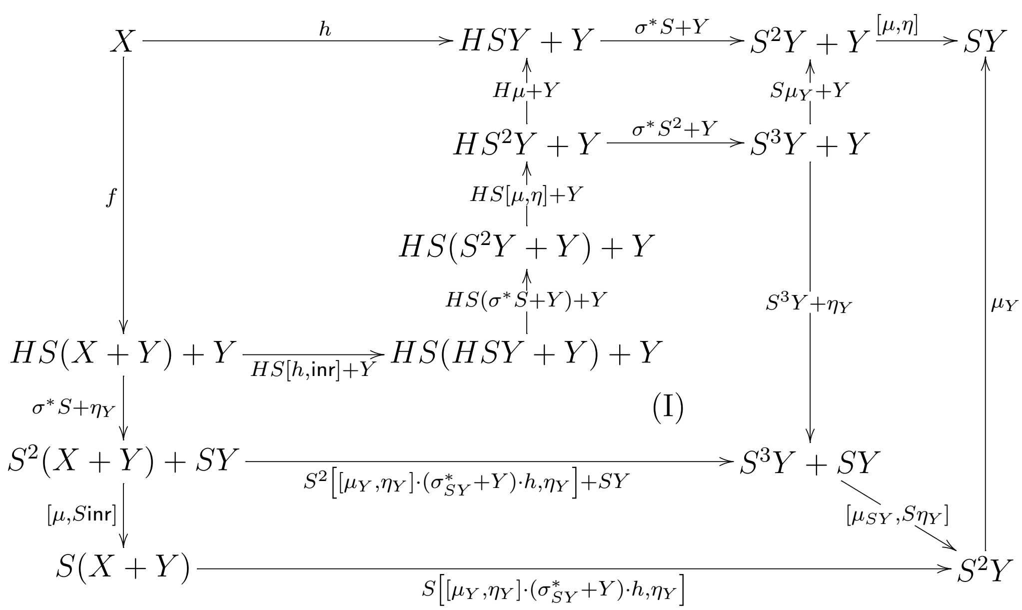
$$\eta_Y = [\mu_Y, \eta_Y] \cdot (\sigma^*_{SY} + Y) \cdot \mathsf{inr}. \tag{6}$$

in order to see that (5) is the same as

$$(H\mu_Y + Y) \cdot (HS[\mu_Y, \eta_Y] + Y) \cdot (HS(\sigma^*_{SY} + Y) + Y) \cdot (HS[h, \mathsf{inr}] + Y) \cdot f,$$

which according to the upper left-hand part of Diagram (3) is just h.

Let us complete our proof by showing that the arrow (4) solves $\overline{e}$. In fact, the following diagram



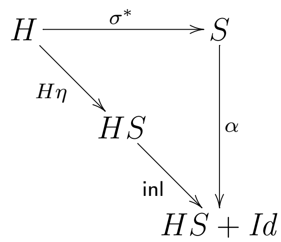
commutes. The upper left-hand square is just the upper left-hand square of Diagram (3). For the inner part (I), consider the components of the coproduct $HS(X+Y)+Y$ separately. The right-hand components obviously commute, for the left-hand ones use naturality of σ^* and Equation (6). All other parts clearly commute. □

Proof of Theorem 3.1. Suppose that (S, η, μ) is a free completely iterative monad on H, i. e., there exists a universal ideal transformation

$$\sigma^* : H \longrightarrow S.$$

By Lemma 3.4, $\widetilde{S} = HS + Id$ is an ideal monad, and by Lemma 3.5 it is completely iterative. Then by the universal property we have a unique idealized-monad morphism $\alpha : S \longrightarrow HS + Id$ such that the following diagram

(7)



commutes.

Note that for all objects Y of $\mathcal{A}$ the arrows

$$\alpha_Y : SY \longrightarrow HSY + Y$$

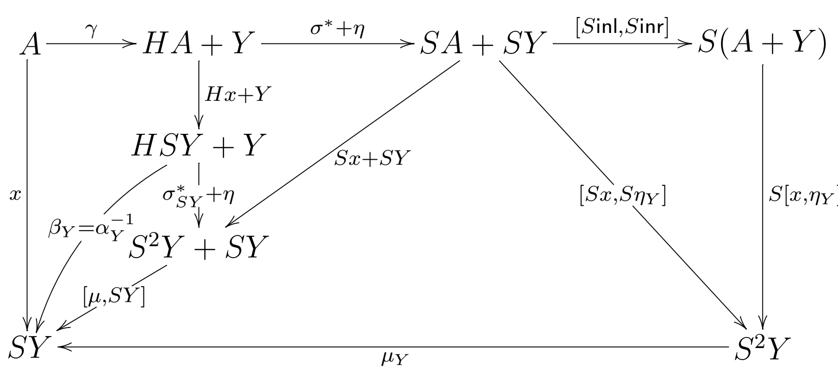
define a coalgebra structure for $H(\_) + Y$ on SY. We shall establish below that α is an isomorphism with an inverse given by the natural transformation

$$\beta \equiv HS + Id \xrightarrow{\sigma^* S + Y} S^2 + Id \xrightarrow{[\mu,\eta]} S.$$

In order to establish that (SY, α_Y) is a final coalgebra suppose that $\gamma : A \longrightarrow HA + Y$ is any coalgebra of $H(\_) + Y$. Then

$$\overline{\gamma} \equiv A \xrightarrow{\gamma} HA + Y \xrightarrow{\sigma^* + \eta} SA + SY \xrightarrow{[S\mathsf{inl}, S\mathsf{inr}]} S(A + Y)$$

is a guarded equation arrow (since σ^* is ideal, i. e., it factors through $\sigma : S' \longrightarrow S$) whose solution $\overline{\gamma}^\dagger : A \longrightarrow SY$ yields the desired unique homomorphism of coalgebras. Indeed, consider the following diagram:

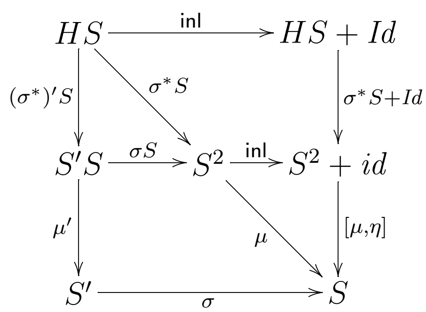


Suppose we put $\overline{\gamma}^\dagger$ in place of x in the diagram. Then the outer square commutes, and we conclude that the upper left-hand part commutes, since all other parts obviously do. This shows that $\overline{\gamma}^\dagger$ is a coalgebra homomorphism.

Conversely, put any coalgebra homomorphism $h : (A, \gamma) \longrightarrow (SY, \alpha_Y)$ in place of x. Then the upper left-hand part commutes, and therefore the whole diagram does. But then $h = \overline{\gamma}^\dagger$, by the uniqueness of solutions. This concludes the proof.

Finally, we show that β is the inverse of α.

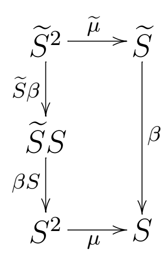
(i) $\beta \cdot \alpha = 1_S$: We will first show that $\beta : HS + Id \longrightarrow S$ is an idealized-monad morphism. In fact, once we know it is a monad morphism, it is easily established that it is ideal. To see this, consider the following commutative diagram:



Let us show that β is a monad homomorphism. We clearly have

$$
\begin{aligned}
\beta \cdot \widetilde{\eta} &= [\mu, \eta] \cdot (\sigma S + Id) \cdot \mathsf{inr} \\
&= [\mu, \eta] \cdot \mathsf{inr} \\
&= \eta.
\end{aligned}
$$

Hence, it suffices to prove that the following square



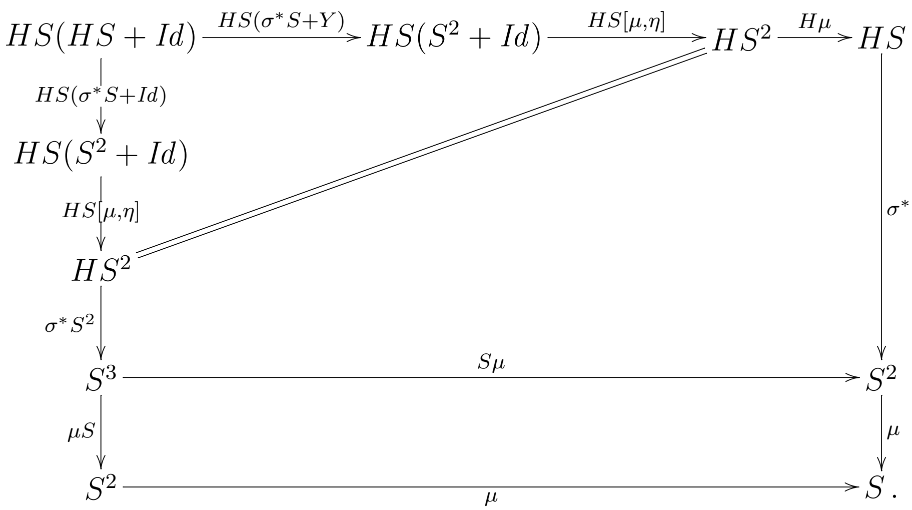
is commutative. We apply the definition of $(\widetilde{S}, \widetilde{\eta}, \widetilde{\mu})$ and consider the components of the coproduct

$$\widetilde{S}^2 = (HS + Id)^2 = HS(HS + Id) + HS + Id$$

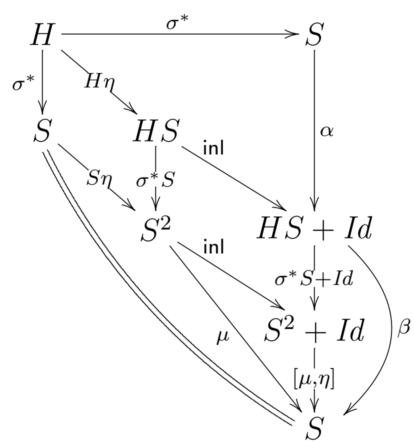
separately. For the right-hand component $HS + Id$ we obtain

$$
\begin{aligned}
\beta \cdot \widetilde{\mu} \cdot \mathsf{inr} &= \beta && (\mathsf{inr} = \widetilde{\eta}\widetilde{S}) \\
&= \mu \cdot \eta S \cdot \beta && (\mu \cdot \eta S = 1_S) \\
&= \mu \cdot \beta S \cdot \widetilde{\eta} S \cdot \beta && (\text{since } \beta \cdot \widetilde{\eta} = \eta) \\
&= \mu \cdot \beta S \cdot \widetilde{S}\beta \cdot \mathsf{inr} && (\text{naturality of } \widetilde{\eta}).
\end{aligned}
$$

For the left-hand component we obtain the following commutative diagram



Now $\beta \cdot \alpha$ is an idealized-monad morphism such that $\beta \cdot \alpha \cdot \sigma^* = \sigma^*$. In fact, the following diagram



commutes. Therefore, by the freeness of S on H, we have $\beta \cdot \alpha = 1_S$, as desired.

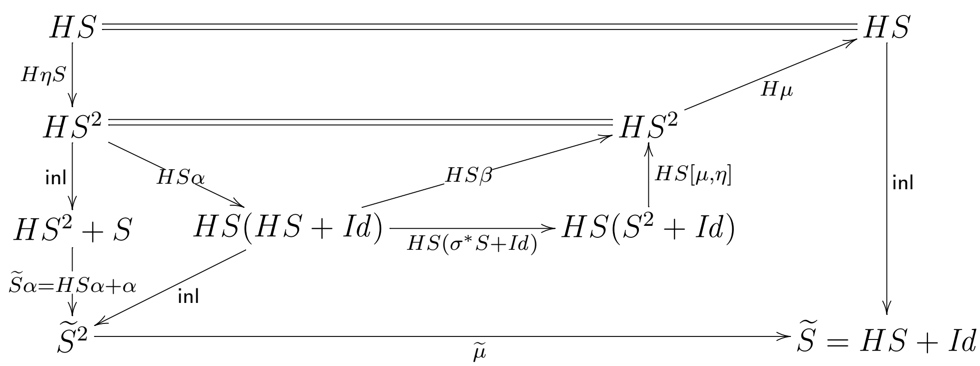
(ii) $\alpha \cdot \beta = 1_{HS+Id}$: We check this on the components of $HS + Id$. For the right-hand component we obtain

$$
\begin{aligned}
\alpha \cdot \beta \cdot \mathsf{inr} &= \alpha \cdot [\mu, \eta] \cdot (\sigma^* S + Id) \cdot \mathsf{inr} && \text{(definition of } \beta\text{)} \\
&= \alpha \cdot \eta \\
&= \mathsf{inr} && (\alpha \text{ is a monad morphism}),
\end{aligned}
$$

For the left-hand component we have

$$\begin{aligned}
\alpha \cdot \beta \cdot \mathsf{inl} &= \alpha \cdot [\mu, \eta] \cdot (\sigma^* S + Id) \cdot \mathsf{inl} && \text{(definition of } \beta) \\
&= \alpha \cdot \mu \cdot \sigma^* S \\
&= \widetilde{\mu} \cdot \widetilde{S}\alpha \cdot \alpha S \cdot \sigma^* S && (\alpha \text{ is a monad morphism}) \\
&= \widetilde{\mu} \cdot \widetilde{S}\alpha \cdot \mathsf{inl} \cdot H\eta S && (\alpha \cdot \sigma^* = \mathsf{inl} \cdot H\eta, \text{see (7)}).
\end{aligned}$$

We analyze the last expression further and obtain the following commutative diagram:



Note that the inner triangle commutes since $\beta \cdot \alpha = 1_S$, and the other parts obviously commute. Thus we have shown that

$$\alpha \cdot \beta = [\mathsf{inl}, \mathsf{inr}] = 1_{HS+Id}$$

as required.

□

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