Monoidal Extended Stone Duality

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Abstract. Extensions of Stone-type dualities have a long history in algebraic logic and have also been instrumental for proving results in algebraic language theory. We show how to extend abstract categorical dualities via monoidal adjunctions, subsuming various incarnations of classical extended Stone and Priestley duality as a special case. Guided by these categorical foundations, we investigate residuation algebras, which are algebraic models of language derivatives, and show its subcategory of boolean derivation algebras to be dually equivalent to the category of profinite monoids. We further extend this duality to capture relational morphisms of profinite monoids, which dualize to natural morphisms of residuation algebras.

1 Introduction

Marshall H. Stone's representation theorem for boolean algebras, the foundation for the so called *Stone duality* between boolean algebras and Stone spaces, manifests a tight connection between logic and topology. It has thus become an ubiquitous tool in various areas of theoretical computer science, not only in logic, but also for example in domain theory and automata theory.

From algebraic logic arose the need for extending Stone duality to capture boolean algebras equipped with additional operators (modelling quantifiers or modalities). Originating in Jónsson and Tarski's representation theorem for boolean algebras with operators [20,21], a representation in the spirit of Stone was proven by Halmos [16]; the general categorical picture of the duality of Kripke frames and modal algebras is based on an adjunction between operators and continuous relations developed by Sambin and Vaccaro [31].

In the study of regular languages, the need for extensions of Stone duality was not discovered until this millenium: while Pippenger [26] has already shown that the boolean algebra of regular languages on an alphabet Σ corresponds, under Stone duality, to the Stone space $\widehat{\Sigma^*}$ of profinite words, Gehrke et al. [14] discovered that, under Goldblatt's [15] form of extended Priestley duality, the *residuals* of concatenation product on regular languages correspond to the *multiplication* on the space of profinite words. But while categorical frameworks have identified Stone-type dualities to be one of the cornerstones of algebraic language theory [36,30], the correspondence between residuals and multiplication via extended duality has not yet been placed in the categorical big picture. One reason is that, despite some progress in recent years [5,17], extended (Stone) dualities for (co-)algebras are themselves not fully understood as instances of a crisp categorical idea.

Therefore we introduce as our first main contribution a simple, yet powerful framework to extend any categorical duality $\mathbf{C} \simeq^{\mathrm{op}} \hat{\mathbf{C}}$ via *monoidal adjunctions*: For a given adjunction on $\mathbf{C}$ with a strong monoidal right adjoint U we prove a dual equivalence between the category of U-operators on $\mathbf{C}$ to dual operators in the Kleisli category of the monad on $\hat{\mathbf{C}}$ arising from the dual of the given adjunction. We show how to instantiate the abstract extended duality to Priestley duality, which not only recovers Goldblatt's original duality [15] for distributive lattices with operators but also applies more generally to bialgebraic operators with relational morphisms. Guided by our categorical foundations for extended Stone duality we investigate the correspondence between residuals and multiplication of profinite words in the setting of residuation algebras originally studied by Gehrke [13]. The key observation is that on finite distributive lattices the residuals are equivalent to a *coalgebraic* operator on the lattice, and we show how to lift this correspondence to locally finite structures, i.e. structures built up from finite substructures. By identifying suitable non-full subcategories – boolean derivation algebras and boolean comonoids, respectively – and an appropriate definition of morphism for residuation algebras, we augment Gehrke's characterization of Stone-topological algebras in terms of residuation algebras to a proper categorical duality between the categories of boolean derivation algebras and that of profinite (i.e. Stone-topological) monoids:

$$\mathbf{BDer} \cong \mathbf{BComon} \simeq^{\mathrm{op}} \mathbf{ProfMon}. \tag{1.1}$$

The above duality clarifies the relation between Gehrke's results and the duality by Rhodes and Steinberg [28] between profinite monoids and counital boolean algebras. The extended Stone duality now suggests that the dual equivalence between profinite monoids on the one side and comonoids as well as boolean derivation algebras on the other side extends to a more general duality capturing *relational morphisms* between profinite monoids. To this end, we identify natural notions of relational morphism for residuation algebras and comonoids, and use our abstract extended duality theorem to obtain the dual equivalence

$$\mathbf{RelBDer} \cong \mathbf{RelBComon} \simeq^{\mathrm{op}} \mathbf{RelProfMon}$$

which extends (1.1) to relational morphisms. To our knowledge this is the first duality result for relational morphisms of profinite monoids, which have become an ubiquitous tool in algebraic language theory [25] and semigroup theory [28].

Related Work. Duality for (complete) boolean algebras with operators goes back to Jónsson and Tarski [20,21]. This duality was refined by the topological approach via Stone spaces taken by Halmos [16], which allowed to characterize the relations arising as the duals of operators, namely *boolean relations*. Halmos' duality was extended to distributive lattices with (n-ary) operators by Goldblatt [15] and Cignoli [6]. Kupke et al. [23] recognized that boolean relations elegantly describe descriptive frames as coalgebras for the Vietoris monad on Stone spaces; notions of bisimulation for these coalgebras were investigated by Bezhanishvili et al. [2]. Bosangue et al. [5] introduced a framework for dualities over distributive lattices equipped with a theory of operators for a signature, which are dual to certain coalgebras. Hofmann and Nora [17] have taken a categorical approach to extend natural dualities to algebras for a signature equipped with *unary* operators preserving only some of the operations prescribed by the signature; they relate these to coalgebras for (the underlying functor of) a suitable monad T. In their framework T is a parameter required to satisfy certain conditions for the duality to work, while in our work T is already determined by the adjunction. The recent work by Bezhanishvili et al. [1] clarifies the relation between free constructions on distributive lattices and the different versions of the Vietoris monad to derive several dualities between distributive lattices with different types of operators and their corresponding Priestley relations.

Residuated boolean algebras, i.e. boolean algebras with a residuated operator, were explicitly considered by Jónsson and Tsinakis [22] to highlight the roles of the residuals in relation algebra. Gehrke et al. [14] discovered the connection between the residuals of the concatenation of regular languages and the multiplication on profinite words and investigated applications to automata theory, most notably a duality-theoretic proof of Eilenberg's variety theorem [7]. The duality theory behind the correspondence of residuation algebras and profinite monoids was given via canonical extensions [11,10] and extended Stone duality [13]. However, while these works developed a substantial amount of theory for the correspondence between profinite monoids and residuation algebras, they do not provide a proper categorical duality. The question on when the dual relation to the residuals is functional was posed and answered by Gehrke [13]; Fussner and Palmigiano [9] have shown that functionality of the dual relation is not equationally definable in the language of residuation algebras.

2 Preliminaries

Readers are assumed to be familiar with basic category theory, such as functors, natural transformations, adjunctions and monoidal categories [24]. We briefly recall the foundations of Stone duality [34] and Priestley duality [27]. By the latter we mean the dual equivalence $\mathbf{DL} \simeq^{\mathrm{op}} \mathbf{Priest}$ between the category $\mathbf{DL}$ of bounded distributive lattices and lattice homomorphisms, and the category $\mathbf{Priest}$ of Priestley spaces (ordered compact topological spaces in which for every $x \not\leq y$ there exists a clopen up-set containing x but not y) and continuous monotone maps. The duality sends a distributive lattice D to the space $\mathbf{DL}(D, 2)$ of homomorphisms into the two-element lattice (equivalently prime filters), topologized via pointwise convergence. In the reverse direction, it sends a Priestley space X to the distributive lattice $\mathbf{Priest}(X, 2)$ of continuous maps into the two-element poset $2 = \{0 \leq 1\}$ with discrete topology (equivalently clopen upsets), with the pointwise lattice structure. Priestley duality restricts to Stone duality $\mathbf{BA} \simeq^{\mathrm{op}} \mathbf{Stone}$ between the full subcategories $\mathbf{BA}$ of boolean algebras and

Stone of Stone spaces (discretely ordered Priestley spaces). Moreover, it restricts to Birkhoff duality [3] $\mathbf{DL}_\mathrm{f} \simeq^{\mathrm{op}} \mathbf{Pos}_\mathrm{f}$ between finite distributive lattices and finite posets, sending a finite distributive lattice to its poset of join-irreducibles and a poset to its lattice of upsets. For a comprehensive introduction to ordered structures and their dualities, see the first two chapters of the classic textbook by Johnstone [19].

3 Extending Dualities

We present the first contribution of our paper, a general categorical framework for extending Stone-type dualities via monoidal adjunctions. It serves as the basis for our duality results in the next two sections.

Notation 3.1. (1) For $U\colon \mathbf{C} \to \mathbf{D}$ being right adjoint to $F\colon \mathbf{D} \to \mathbf{C}$ we write

$$F\colon \mathbf{D} \dashv \mathbf{C} : U \quad \text{or simply} \quad F \dashv U.$$

We denote the unit and counit by η and ε and the transposing isomorphisms by

$$(-)^+\colon \mathbf{D}(C, UD) \cong \mathbf{C}(FC, D) : (-)^- \qquad \text{with} \qquad f^+ = \varepsilon \cdot Ff, \quad g^- = Ug \cdot \eta.$$

(2) For dually equivalent categories $\mathbf{C}$ and $\hat{\mathbf{C}}$ we denote the equivalence functors in both directions by $\widehat{(-)}\colon \mathbf{C} \xrightarrow{\simeq} \hat{\mathbf{C}}$ and $\widehat{(-)}\colon \hat{\mathbf{C}} \xrightarrow{\simeq} \mathbf{C}$. Moreover, if if $F\colon \mathbf{C} \to \mathbf{D}$ is a functor and $\hat{\mathbf{D}}$ is dual to $\mathbf{D}$, we denote its dual by $\hat{F} = \widehat{(-)} \circ F \circ \widehat{(-)}\colon \hat{\mathbf{C}} \to \hat{\mathbf{D}}$.

(3) The Kleisli category of a monad (T, η, μ) on $\mathbf{C}$ is denoted by $\mathbf{C}_T$. It has the same objects as $\mathbf{C}$ and $\mathbf{C}_T(X, Y) = \mathbf{C}(X, TY)$ with Kleisli composition $g \circ f = \mu \cdot Tg \cdot f$. A morphism $f\colon C \to TD$ of the Kleisli category is *pure* if $f = \eta \cdot f'$ for some $f'\colon C \to D$ in $\mathbf{C}$. (We omit the components of η and μ.)

Assumptions 3.2. We fix monoidal categories $\mathbf{C}, \mathbf{D}$ with dually equivalent categories $\hat{\mathbf{C}}, \hat{\mathbf{D}}$; we regard $\hat{\mathbf{C}}, \hat{\mathbf{D}}$ as monoidal categories with tensor products $\hat{\otimes}$ dual to the tensor products $\otimes$ of $\mathbf{C}, \mathbf{D}$. Moreover, we fix an adjunction $F\colon \mathbf{D} \dashv \mathbf{C} : U$ with unit $\eta\colon \mathrm{Id} \to UF$ and counit $\varepsilon\colon FU \to \mathrm{Id}$, and assume that U is a *strong monoidal functor* with associated natural isomorphisms $\lambda\colon UX \otimes UY \cong U(X \otimes Y)$ and $\epsilon\colon I_\mathbf{D} \cong UI_\mathbf{C}$. One can extend λ to an isomorphism $\lambda\colon \bigotimes_{i=1}^n UX_i \cong U(\bigotimes_{i=1}^n X_i)$ for every arity n. The dual functor $\hat{U}\colon \hat{\mathbf{C}} \to \hat{\mathbf{D}}$ is a strong monoidal *left* adjoint to $\hat{F}$ and the unit and counit of this dual adjunction are $\hat{\varepsilon}$ and $\hat{\eta}$. We denote the monad induced by the dual adjunction by $T = \hat{F}\hat{U}$ with unit $e = \hat{\varepsilon}\colon \mathrm{Id} \to T$ and multiplication $m = \hat{F}\hat{\eta}\hat{U}\colon TT \to T$.

$$\begin{array}{ccc} \mathbf{D} & \simeq^{\mathrm{op}} & \hat{\mathbf{D}} \\ F \left(\dashv \right) U & & \hat{F} \left(\vdash \right) \hat{U} \\ \mathbf{C} & \simeq^{\mathrm{op}} & \hat{\mathbf{C}} \circlearrowleft T \end{array} \tag{3.1}$$

Remark 3.3. Since $\hat{U}$ is strong monoidal with $\hat{\epsilon}\colon \hat{I}_{\mathbf{D}} \cong \hat{U}\hat{I}_{\mathbf{C}}$ and $\hat{\lambda}\colon \hat{U}X \hat{\otimes} \hat{U}Y \cong \hat{U}(X \hat{\otimes} Y)$ its right adjoint $\hat{F}$ is monoidal (see e.g. [32, p. 17]) with isomorphisms

$$((\hat{\eta} \hat{\otimes} \hat{\eta}) \cdot \hat{\lambda}^{-1})^{-}\colon \hat{F}X \hat{\otimes} \hat{F}Y \to \hat{F}(X \hat{\otimes} Y) \quad \text{and} \quad (\hat{\varepsilon}^{-1})^{-}\colon \hat{I}_{\mathbf{C}} \to \hat{F}\hat{I}_{\mathbf{D}}.$$

This makes $\hat{U} \dashv \hat{F}$ a monoidal adjunction, which then induces a monoidal monad $T = \hat{F}\hat{U}$ on $\hat{\mathbf{C}}$. Let $\delta\colon TX \hat{\otimes} TY \to T(X \hat{\otimes} Y)$ denote the witnessing natural transformation, which also extends to any arity. The tensor product $\hat{\otimes}$ of $\hat{\mathbf{C}}$ lifts to the Kleisli category $\hat{\mathbf{C}}_T$; the lifting sends a pair $(f\colon X \to TY, g\colon X' \to TY')$ of $\hat{\mathbf{C}}_T$-morphisms to the $\hat{\mathbf{C}}_T$-morphism $\delta \cdot (f \hat{\otimes} g)\colon X \hat{\otimes} X' \to TY \hat{\otimes} TY' \to T(Y \hat{\otimes} Y')$. This makes $\hat{\mathbf{C}}_T$ itself a monoidal category [33, Prop. 1.2.2] and the canonical left adjoint $J_T\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}_T$ a strict monoidal functor.

Definition 3.4. Let $G\colon \mathbf{A} \to \mathbf{B}$ be a strict monoidal functor between monoidal categories, and let $m, n \in \mathbb{N}$. An *(m,n)-ary G-operator* consists of an object $A \in \mathbf{A}$ and a morphism $a\colon (GA)^{\otimes m} \to (GA)^{\otimes n}$ of $\mathbf{B}$. An *(m,n)-ary G-operator morphism* from (A, b) to (B, b) is a morphism $h\colon GA \to GB$ of $\mathbf{B}$ such that

$$\begin{array}{ccc} (GA)^{\otimes m} & \xrightarrow{a} & (GA)^{\otimes n} \\ {\scriptstyle h^{\otimes m}}\downarrow & & \downarrow{\scriptstyle h^{\otimes n}} \\ (GB)^{\otimes m} & \xrightarrow{b} & (GB)^{\otimes n} \end{array}$$

commutes. The category of (m,n)-ary G-operators is denoted by $\mathrm{Op}_G^{m,n}(\mathbf{A})$. We call $(m,1)$-ary G-operators *G-algebras* and $(1,n)$-ary G-operators *G-coalgebras.* An operator is *pure* if it is of the form $\lambda^{-1} \cdot Ga' \cdot \lambda$, for λ analogous to Assumptions 3.2, and an operator morphism is *pure* if it is of the form Gh'.

Note that the full subcategory of $\mathbf{B}$ consisting of the objects in the image of G fully embeds into $\mathrm{Op}_G^{1,1}(\mathbf{A})$ via $GA \mapsto (GA, \mathrm{id}_{GA})$.

Theorem 3.5 (Abstract Extended Duality). *The category of (m,n)-ary U-operators is dually equivalent to the category of (n,m)-ary J_T-operators:*

$$\mathrm{Op}_U^{m,n}(\mathbf{C}) \simeq^{\mathrm{op}} \mathrm{Op}_{J_T}^{n,m}(\hat{\mathbf{C}}).$$

Proof (Sketch). Let $\hat{f}$ be a morphism from an operator $(\hat{A}, \hat{a})$ to an operator $(\hat{B}, \hat{b})$ in $\mathrm{Op}_{J_T}^{n,m}(\hat{\mathbf{C}})$. This means that the following diagram commutes in $\hat{\mathbf{C}}$:

$$\begin{array}{ccccccc} \hat{A}^{\hat{\otimes} n} & \xrightarrow{\hat{a}} & T\hat{A}^{\hat{\otimes} m} & \xrightarrow{T\hat{f}^{\hat{\otimes} m}} & T(T\hat{B})^{\hat{\otimes} m} & \xrightarrow{T\delta} & TT\hat{B}^{\hat{\otimes} m} \\ {\scriptstyle \hat{f}^{\hat{\otimes} n}}\downarrow & & & & & & \downarrow{\scriptstyle \mu} \\ (T\hat{B})^{\hat{\otimes} n} & \xrightarrow{\delta} & T\hat{B}^{\hat{\otimes} n} & \xrightarrow{T\hat{b}} & TT\hat{B}^{\otimes m} & \xrightarrow{\mu} & T\hat{B}^{\otimes m} \end{array}$$

We dualize it by applying the natural isomorphism $\hat{\mathbf{C}}(\hat{X}, \hat{F}\hat{U}\hat{Y}) \cong \mathbf{C}(FUY, X)$. The resulting diagram in $\mathbf{C}$ can then be greatly simplified by a diagram chase and using adjoint transposition $(-)^{-}\colon \mathbf{C}(FUY, X) \cong \mathbf{D}(UY, UX)$; all units and counits vanish either by triangle equalities or the transposition equation $f^{-} = Uf \cdot \eta$. In the final step, the resulting diagram in $\mathbf{D}$ simplifies to a diagram showing that f^{-} is an morphism from (A, a) to (B, b) in $\mathrm{Op}_U^{m,n}(\mathbf{C})$ using that conjugation with λ is an isomorphism. Since all steps of this transformation can be reversed, this establishes the desired equivalence. □

An advantage of extending dualities via adjunctions is that adjunctions compose, making the extensions *modular*: let $\mathbf{E}$ be a monoidal category with monoidal adjunctions $F_1\colon \mathbf{E} \dashv \mathbf{C} : U_1$ and $F_2\colon \mathbf{D} \dashv \mathbf{E} : U_2$ splitting $F \dashv U$, i.e., $F = F_1F_2$ and $U = U_2U_1$ and $\lambda = U_2\lambda_1 \cdot \lambda_2 U_1$. Then the following lifting characterization applies to operators (set $A = B$) as well as operator morphisms (set $m = n = 1$):

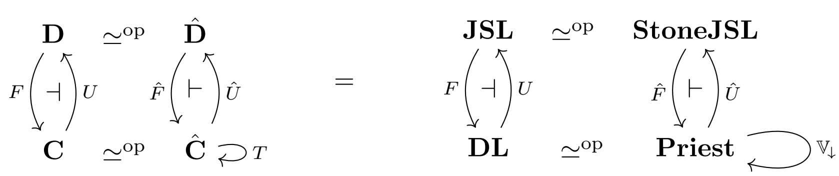
Proposition 3.6. *A morphism $a\colon (UA)^{\otimes m} \to (UB)^{\otimes n}$ in $\mathbf{D}$ lifts to a morphism $b\colon (U_1A)^{\otimes m} \to (U_1B)^{\otimes n}$ with $a = \lambda_2^{-1} \cdot U_2 b \cdot \lambda_2$ iff the dual of a factors through the canonical monad morphism $\hat{F}_1\hat{\varepsilon}_2\hat{U}_1\colon T_1 \to T$, where $T_1 = \hat{F}_1\hat{U}_1$.*

Remark 3.7. (1) A special case of Proposition 3.6 proves that extended Stone duality preserves purity: splitting $F \dashv U$ into $F_1 = \mathrm{Id} \dashv \mathrm{Id} = U_1$ and $F_2 = F \dashv U = U_2$ we see that a U-operator (or operator morphism) a is pure iff its dual f is pure as a Kleisli morphism, i.e. factors through the unit e of T.

(2) The right adjoint U_2 often is faithful and in this case $\hat{F}_1\hat{\varepsilon}_2\hat{U}_1$ is monic, i.e. T_1 is a submonad of T: faithfulness of U_2 is equivalent to ε_2 being epic, hence $\hat{\varepsilon}_2\hat{U}_1$ is mono, and the right adjoint $\hat{F}_1$ preserves monos. In particular, if T is "powerset-like", then $\hat{\mathbf{C}}_T$ is a category of relations, and we think of U-operators (or operator morphisms) of the form $a = \lambda_2^{-1} \cdot U_2 b \cdot \lambda_2$ as dualizing to "more functional" relations. We eloborate this point in Section 4.2.

4 Example: Extended Priestley Duality

As a first application of our adjoint framework, we investigate the classical Priestley duality (Section 2) and derive a generalized version of Goldblatt's duality [15] between distributive lattices with operators and relational Priestley spaces. We instantiate (3.1) to the following categories and functors, which we will subsequently explain in detail:



Categories The upper duality is Hofman-Mislove-Stralka duality [18] between the category of join-semilattices with bottom and the category of Stone semilattices (i.e. topological join-semilattices with bottom whose underlying topological space is a Stone space) and continuous semilattice homomorphisms. The duality maps a join-semilattice J to the Stone semilattice $\mathbf{JSL}(J,2)$ of semilattice homomorphisms into the two-element semilattice, topologized by pointwise convergence. Equivalently, $\mathbf{JSL}(J,2)$ is the space $\mathrm{Idl}(J)$ of *ideals* (downwards closed and upwards directed subsets) of J, ordered by reverse inclusion, with topology generated by the subbasic open sets $\sigma(j) = \{I \in \mathrm{Idl}(J) \mid j \in I\}$ and their complements for $j \in J$. In the other direction, a Stone semilattice X is mapped to its semilattice $\mathbf{StoneJSL}(X,2)$ of clopen ideals, ordered by inclusion.

Functors The functor $U\colon \mathbf{DL} \to \mathbf{JSL}$ is the obvious forgetful functor. Its left adjoint $F\colon \mathbf{JSL} \to \mathbf{DL}$ maps a join-semilattice to the set $\mathcal{U}^{\circ}_{\mathrm{fg}}(J)$ of finitely generated upsets of J ordered by reverse inclusion. The dual right adjoint $\hat{F}$ of the left adjoint F is the forgetful functor mapping a Stone semilattice to its underlying Priestley space. Indeed, as $U2 = 2$ we compute for the underlying Priestley space $|X|$ of a Stone semilattice X that

$$\hat{F}X = \mathbf{DL}(F(\mathbf{StoneJSL}(X,2)),2) \cong |\mathbf{JSL}(\mathbf{StoneJSL}(X,2),U2)| \cong |X|,$$

and this bijection is a homeomorphism. Its left adjoint $\hat{U}\colon \mathbf{Priest} \to \mathbf{StoneJSL}$ maps a Priestley space X to the space

$$\widehat{U\hat{X}} = \mathbf{JSL}(U(\mathbf{Priest}(X,2)),2) \cong \mathrm{Idl}(\mathrm{Cl}_{\uparrow} X) \cong \mathbb{V}_{\downarrow}X$$

of ideals of clopen upsets of X. This space is isomorphic to the *(downset) Vietoris hyperspace* $\mathbb{V}_{\downarrow}X$ of X that has as carrier the set of closed downsets of X. The isomorphism $\mathrm{Idl}(\mathrm{Cl}_{\uparrow} X) \cong \mathbb{V}_{\downarrow}X$ maps an ideal I to the intersection $\bigcap_{U\in I} X \setminus U$; its inverse sends a closed downset C to the ideal $\{U \in \mathrm{Cl}_{\uparrow} X \mid C \subseteq X \setminus U\}$ of complements of the basic clopen downsets that contain it. The topology of pointwise convergence on $\mathbf{JSL}(U(\mathbf{Priest}(X,2)),2)$ translates to the *hit-or-miss topology* on $\mathbb{V}_{\downarrow}X$ generated by the subbasic open sets

$$\{A \subseteq X \text{ closed } \mid A \cap U \neq \emptyset\} \quad \text{for} \quad U \in \mathrm{Cl}_{\uparrow} X$$

and their complements. For a detailed exposition of these results we refer the reader to the recent work by Bezhanishvili et al. [1]; the free join-semilattice structure on $\mathbb{V}_{\downarrow}X$ was already observed by Johnstone [19, Sec. 4.8]. The unit $e\colon X \to \mathbb{V}_{\downarrow}X$ of the Vietoris monad is given by $x \mapsto {\downarrow}x$ and multiplication is given by union [17]. The monad $\mathbb{V}_{\downarrow}$ restricts to the full subcategory **Stone** of Stone spaces. We denote the restriction of this monad simply by $\mathbb{V}$.

Remark 4.1 (Continuous Relations). Continuous maps in **Priest** of the form $\rho\colon X \to \mathbb{V}_{\downarrow}Y$ have a variety of names, we use the term *Priestley relation* as in [6,15] or *Stone relation* if X, Y are Stone spaces. We write $x \mathrel{\rho} y$ for $y \in \rho(x)$, and sometimes identify ρ with a subset of $X \times Y$. Let us note that some authors (e.g. [28]) call a relation $R \subseteq X \times Y$ between topological spaces *continuous* if it is closed as a subspace of $X \times Y$. Every Priestley relation is continuous, but a continuous relation between Priestley spaces is generally not a Priestley relation.

Monoidal Structure The category **JSL** of join-semilattices has a tensor product $\otimes$ with the universal property that it extends join-bilinear maps:

$$\mathrm{Bilin}(J \times J', K) \cong \mathbf{JSL}(J \otimes J', K).$$

Join-bilinear maps $J \times J' \to K$ and their corresponding **JSL**-morphisms $J \otimes J' \to K$ are often tacitly identified. The tensor product $\otimes$ makes **JSL** a monoidal category with unit 2, i.e. $2 \otimes J \cong J$. The tensor product has a representation by the generators $\{j \otimes j' \mid j \in J, j' \in J'\}$ and relations $j \otimes 0 = 0 \otimes j' = 0$ and $(j_1 \vee j_2) \otimes (j_1' \vee j_2') = j_1 \otimes j_1' \vee j_2 \otimes j_1' \vee j_1 \otimes j_2' \vee j_2 \otimes j_2'$. We call elements of the form $j \otimes j'$ *pure tensors*. If D, D' are bounded distributive lattices then so is $UD \otimes UD'$ [8], with meet given on pure tensors as $(d \otimes d') \wedge (e \otimes e') = (d \wedge e) \otimes (d' \wedge e')$. The lattice $UD \otimes UD'$ moreover is the coproduct of D, D' in **DL**: the coproduct injections are $\iota(d) = d \otimes 1'$ and $\iota'(d') = 1 \otimes d'$ for $d \in D, d' \in D'$, and the copairing of lattice homomorphisms $f \colon D \to E, f' \colon D' \to E$ is given by the extension of the join-bilinar map $D \times D' \to E, (d, d') \mapsto f(d) \wedge f(d')$. Taking coproducts yields a monoidal structure on **DL** and since $U(D + D') = UD \otimes UD'$, the functor U is strong monoidal. The monoidal structure on **Priest** is given by binary products, and the natural transformation δ of Remark 3.3 by

$$\delta \colon \mathbb{V}_{\downarrow} X \times \mathbb{V}_{\downarrow} Y \to \mathbb{V}_{\downarrow}(X \times Y), \qquad (C, D) \mapsto C \times D.$$

Spelling out Definition 3.4, the category $\mathrm{Op}^{n,m}_{J_{\mathbb{V}_{\downarrow}}}(\mathbf{Priest})$ is given as follows:

Definition 4.2. A *((n, m)-ary) relational Priestley space* consists of a carrier Priestley space X and a Priestley relation $\rho \colon X^n \to \mathbb{V}_{\downarrow} X^m$. A *relational morphism* from a relational Priestley space (X, ρ) to (X', ρ') is given by a Priestley relation $\beta \colon X \to \mathbb{V}_{\downarrow} Y$ such that, for all $\mathbf{x} \in X^n, \mathbf{y} \in X^m, \mathbf{y}' \in X'^m$,

$$\mathbf{x} \; \rho \; \mathbf{y} \wedge (\forall i \colon y_i \; \beta \; y_i') \Rightarrow \exists \mathbf{x}' \colon (\forall i \colon x_i \; \beta \; x_i') \wedge \mathbf{x}' \; \rho' \; \mathbf{y}',$$

and, for all $\mathbf{x} \in X^n, \mathbf{x}' \in X'^n, \mathbf{y}' \in X'^m$,

$$(\forall i \colon x_i \; \beta \; x_i') \wedge \mathbf{x}' \; \rho' \; \mathbf{y}' \Rightarrow \exists \mathbf{y} \colon \mathbf{x} \; \rho \; \mathbf{y} \wedge (\forall i \colon y_i \; \beta \; y_i').$$

We let $\mathrm{Op}^{n,m}_{J_{\mathbb{V}_{\downarrow}}}(\mathbf{Priest})$ denote the category of (n, m)-ary relational Priestley operators and relational morphisms.

Then Theorem 3.5 instantiates to the following result:

Theorem 4.3 (Extended Priestley duality). *The category of (m, n)-ary U-operators of distributive lattices is dually equivalent to the category of (n, m)-ary relational Priestley spaces and relational morphisms:*

$$\mathrm{Op}^{m,n}_{U}(\mathbf{DL}) \simeq^{\mathrm{op}} \mathrm{Op}^{n,m}_{J_{\mathbb{V}_{\downarrow}}}(\mathbf{Priest}).$$

By taking $n = 1$ and restricting to pure morphisms, we recover Goldblatt's duality [15]. Here, pure relational morphisms are called *bounded morphisms* and n-ary U-algebras $(UD)^{\otimes n} \to UD$ in **JSL** are called n-ary *join-hemimorphisms.*

Corollary 4.4 (Goldblatt, 1989). *The category of distributive lattices with n-ary join-hemimorphisms, and pure morphisms between them, is dually equivalent to the category of $(1, n)$-relational Priestley spaces and bounded morphisms.*

4.1 Deriving Concrete Formulas

We proceed to show how the adjoint framework can be used to methodically derive concrete (i.e. element-based) formulas for the dual join operator of a continuous relation and vice versa. Let us first observe that all involved categories are *order-enriched*, i.e. the homsets are (pointwise) partially ordered; for **JSL** and **DL** this is clear and relations $X \to \mathbb{V}_{\downarrow}Y$ are ordered by inclusion, as usual. Moreover, from the definitions it is clear that the transposing isomorphisms of the adjunction $F \dashv U$ and the duality $\mathbf{DL} \simeq^{\mathrm{op}} \mathbf{Priest}$ are order-isomorphisms.

Second, in **Priest** we can represent an element $\hat{x}$ of a space $\hat{X}$ as a continuous function $1 \to X$ that we also denote by $\hat{x}$; on the lattice side, elements of a join-semilattice J correspond bijectively to **JSL**-morphisms $2 \to J$.

For the rest of the section we fix a U-algebra $h\colon (UX)^{\otimes n} \to UX$ with dual Priestley relation $\rho\colon \hat{X} \to \mathbb{V}_{\downarrow}\hat{X}^n$. We first show how to express ρ in terms of h. Two elements $\hat{x} \in \hat{X}, \hat{\mathbf{x}} \in \hat{X}^n$ are related by ρ (i.e. $\hat{x}\ \rho\ \hat{\mathbf{x}}$) iff the inequality $e(\hat{\mathbf{x}}) = {\downarrow}\hat{\mathbf{x}} \leq \rho(\hat{x})$ holds, equivalently, iff the left diagram below commutes laxly:

$$
\begin{array}{ccccc}
\hat{X} & \xrightarrow{\rho} & \mathbb{V}_{\downarrow}\hat{X}^n & & \\
{\scriptstyle \hat{x}}\uparrow & \leq & \uparrow{\scriptstyle e} & & \\
1 & \xrightarrow{\hat{\mathbf{x}}} & \hat{X}^n & \xleftarrow{\prod_i \hat{x}_i} & 1^n \\
\end{array}
\qquad (\Delta\colon 1 \to 1^n)
\qquad\qquad
\begin{array}{ccc}
UX & \xleftarrow{h} & (UX)^{\otimes n} \\
{\scriptstyle Ux}\downarrow & \leq & \downarrow{\scriptstyle \bigotimes_i Ux_i} \\
U2 & \xleftarrow{\nabla} & (U2)^{\otimes n}
\end{array}
$$

The duals of $\hat{x}, \hat{x}_i$ are **DL** morphisms $x, x_i\colon X \to 2$. Under duality and transposition the left diagram corresponds to the right diagram where ∇ is the codiagonal given by n-fold conjunction, i.e. it sends $\bigotimes_{i=1}^n x_i$ to $\bigwedge_{i=1}^n x_i$. Writing $F_z = z^{-1}(1)$ for the prime filter corresponding to a morphism $z \in \mathbf{DL}(X,2)$ the right diagram yields Goldblatt's formula [15, p. 186] for the dual Priestley relation of an algebra h: we have $\hat{x}\ \rho\ \hat{\mathbf{x}}$ iff $h[\prod_i F_{x_i}] \subseteq F_x$.

To express h in terms of ρ, it suffices to describe $h(\mathbf{x})$ for a pure tensor $\mathbf{x} \in (UX)^{\otimes n}$ by the universal property of the tensor product. We factor $\mathbf{x} = \bigotimes_i x_i \cdot \nabla^{-1}\colon U2 \cong (U2)^{\otimes n} \to (UX)^{\otimes n}$ to see that the element $h(\mathbf{x})$ corresponds to the following morphism representing an element of the join-semilattice UX:

$$h \cdot \bigotimes_i x_i \cdot \nabla^{-1}\colon U2 \cong (U2)^{\otimes n} \to (UX)^{\otimes n} \to UX.$$

Its dual is the characteristic function

$$\hat{X} \xrightarrow{\rho} \mathbb{V}_{\downarrow}\hat{X}^n \xrightarrow{\mathbb{V}_{\downarrow}(\prod_i C_i)} \mathbb{V}_{\downarrow}(\mathbb{V}_{\downarrow}1)^n \xrightarrow{\mathbb{V}_{\downarrow}\delta} \mathbb{V}_{\downarrow}\mathbb{V}_{\downarrow}1^n \xrightarrow{\bigcup} \mathbb{V}_{\downarrow}1^n \xrightarrow{\mathbb{V}_{\downarrow}\Delta^{-1}} \mathbb{V}_{\downarrow}1 = 2,$$

where $C_i = \widehat{x_i^+}$ is the clopen upset of $\hat{X}$ dual to

$$x_i \in \mathbf{JSL}(U2, UX) \cong \mathbf{DL}(FU2, X) \cong \mathbf{Priest}(\hat{X}, \mathbb{V}_{\downarrow}1) \cong \mathbf{Priest}(\hat{X}, 2).$$

This shows that $h(\mathbf{x}) \in X \cong \mathrm{Cl}_{\uparrow}\hat{X}$ corresponds to the clopen upset

$$h(\mathbf{x}) = \{a \in \hat{X} \mid \exists (b_1, \dots, b_n) \in \rho(a)\colon \forall i\colon b_i \in C_i = \widehat{x_i^+}\} \in \mathrm{Cl}_{\uparrow}(\hat{X}),$$

which is Goldblatt's formula [15, p. 184] for the dual algebra of a relation ρ.

4.2 Partial Functions and Total Relations

As a further application of the adjoint framework we characterize those operators whose dual Priestley relation is a partial function or a total relation, respectively. We achieve this by considering two splittings of the adjunction $F\colon \mathbf{JSL} \dashv \mathbf{DL} : U$ (Proposition 3.6 and Remark 3.7). The tensor on all categories considered is the tensor product of their underlying join-semilattices.

First split the adjunction into $Q\colon \mathbf{DL}_0 \dashv \mathbf{DL} : P$ and $Q'\colon \mathbf{JSL} \dashv \mathbf{DL}_0 : P'$, where $\mathbf{DL}_0$ is the category of distributive lattices that are only bounded from below, and P, P' are forgetful functors. The left adjoint Q add s a fresh top element to a lattice in $\mathbf{DL}_0$. The dual submonad $\hat{Q}\hat{P} \hookrightarrow \mathbb{V}_{\downarrow}$ on **Priest** is given by

$$\hat{Q}\hat{P}\hat{D} \cong \widehat{QPD} \cong \mathbf{DL}(QPD, 2) \cong \mathbf{DL}_0(PD, P2).$$

Every $f \in \mathbf{DL}_0(PD, P2)$ either satisfies $f(1) = 1$, in which case $f \in \hat{D}$ is prime, or $f(1) = 0$ but then $f = 0!$ is the constant zero map. In the pointwise ordering of $\mathbf{DL}_0(PD, P2)$ the morphism $0!$ is clearly the bottom element. Hence, the monad $\hat{Q}\hat{P}$ just freely adds a bottom element. In particular, the dual category of $\mathbf{DL}_0$ is readily seen to be equivalent to $\mathbf{Priest}_0$, the category of Priestley spaces with a bottom element, and bottom-preserving continuous monotone maps. A continuous relation $\rho\colon X \to \hat{Q}\hat{P}\hat{X}$ is thus simply a *partial continuous function.*

Another splitting of the adjunction $F \dashv U$ is given by $L\colon \mathbf{JSL}_1 \dashv \mathbf{DL} : R$ and $L'\colon \mathbf{JSL} \dashv \mathbf{JSL}_1 : R'$ and $Q'\colon \mathbf{JSL} \dashv \mathbf{DL}_0 : P'$ where $\mathbf{JSL}_1$ is the category of join-semilattices with both a bottom and top element (which are preserved by homomorphisms). The right adjoints R, R' are forgetful functors. The left adjoint L maps $J \in \mathbf{JSL}_1$ to the distributive lattice $\mathcal{U}^{\circ}_{\mathrm{fg}+}$ of *non-empty* finitely generated upsets of J, ordered by reverse inclusion. The submonad $\hat{L}\hat{R} \hookrightarrow \mathbb{V}_{\downarrow}$ thus maps a Priestley space $\hat{D}$ to

$$\hat{L}\hat{R}\hat{D} \cong \mathbf{DL}(LRD, 2) \cong \mathbf{JSL}_1(RD, R2) \cong \mathbb{V}^{+}_{\downarrow}\hat{D},$$

where $\mathbb{V}^{+}_{\downarrow}$ is the submonad of $\mathbb{V}_{\downarrow}$ taking non-empty closed downsets. Coalgebras for $\mathbb{V}^{+}_{\downarrow}$ are *total Priestley relations.* Hence, we recover from Proposition 3.6 the following result (the unary case is well-known, see e.g. [17, Lemma 4.6]):

Corollary 4.5. *The dual Priestley relation of a U-operator (operator morphism, respectively) is a partial function iff the operator (operator morphism, respectively) preserves non-empty meets, and total iff it preserves $\top$.*

5 Residuation Algebras

We are now ready to prove the main result of this paper: a fully fledged categorical duality between profinite monoids and a subcategory of residuation algebras we call *boolean derivation algebras*, refining Gehrke's correspondence [12,13] between profinite monoids and residuation algebras. Our result is obtained by combining two ingredients: our framework for extended Stone duality from the previous sections and an isomorphism between residuation algebras and certain lattice coalgebras. The latter is first established for finite algebras via an operator on complete lattices we call *tensor implication*; extending it to locally finite algebras (Definition 5.22) then yields the desired duality with the category of profinite monoids. To this end we introduce the notion of residuation morphism (Definition 5.8). In addition, we also consider *relational* morphisms and obtain a dual equivalence between the categories of profinite monoids and that of boolean derivation algebras, both equipped with (Stone) relational morphisms.

5.1 The Tensor Product of Distributive Lattices Revisited

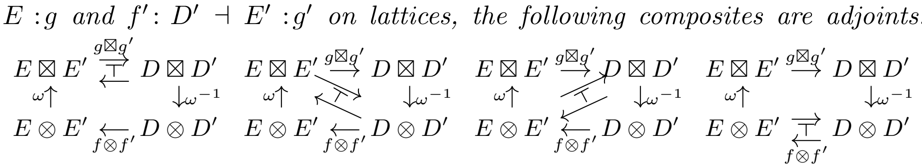
Notation 5.1. By a *lattice* we always mean a bounded and distributive lattice, i.e. an object of **DL**. We often write de for $d \wedge e$. The dual lattice of D is denoted D°. The category of meet-semilattices (with a top element) is denoted **MSL**. Analogous to **JSL** it has a tensor product $M \boxtimes M'$ and is dual to the category of Stone meet-semilattices [18]. From now on we denote the forgetful functors from **DL** to **JSL** and **MSL** by $U_\vee$ and $U_\wedge$, respectively. Sometimes we omit the forgetful functors $U_\wedge$ and $U_\vee$ for notational brevity and just write the respective tensor products of the underlying semilattices as $D \otimes D'$ and $D \boxtimes D'$.

Remark 5.2. The monad induced by the dual of $F_\wedge \dashv U_\wedge$ sends X to its hyperspace $\mathbb{V}_\uparrow X$ of closed *upsets* [1]. The comonads of the adjunctions $F_\wedge \dashv U_\wedge$ and $F_\vee \dashv U_\vee$ are not isomorphic but *conjugate*: $F_\wedge U_\wedge \cong (F_\vee U_\vee(-)^\circ)^\circ$. Their restrictions to the category of boolean algebras are isomorphic since their dual monads satisfy $\mathbb{V}_\downarrow = \mathbb{V} = \mathbb{V}_\uparrow$ as the order on their dual Priestley space is discrete. On the category of finite Priestley spaces, which are simply posets, the Vietoris monad $\mathbb{V}_\downarrow$ is simply the downset monad, which restricts to the finite powerset monad on the category of finite sets (discrete finite posets).

Remark 5.3 (Adjunctions on Lattices). By the adjoint functor theorem [24, Thm. V.6.1] a monotone function $f\colon D \to D'$ between complete lattices preserves all joins iff it has a right adjoint $f_*\colon D' \to D$, which is then given by $f_*(d') = \bigvee_{f(d) \leq d'} d$; dually, it preserves all meets iff it has a left adjoint $f^*\colon D' \to D$, given by $f^*(d') = \bigwedge_{d' \leq f(d)} d$. Finite lattices are complete, so every lattice homomorphism f between finite lattices has a left and a right adjoint. The join-irreducibles $\mathcal{J}D$ of a finite lattice D are precisely those elements $p \in D$ whose characteristic function $\chi_p\colon D \to 2$ (mapping $x \in D$ to 1 iff $p \leq x$) is a lattice morphism. The left adjoint of χ_p, also denoted $p\colon 2 \to D$, maps $1 \mapsto p$.

Lemma 5.4. (1) *The join- and meet-semilattice tensor products of distributive lattices D, E are isomorphic, that is, there is an isomorphism $\omega\colon D \otimes E \cong D \boxtimes E$.*

(2) *Adjunctions on lattices "compose horizontally": Given adjunctions $f\colon D \dashv E : g$ and $f'\colon D' \dashv E' : g'$ on lattices, the following composites are adjoints:*



Construction 5.5. For every finite lattice D the map $x \otimes (-): D \to D \otimes D$ preserves all joins, so it admits a right adjoint $x \multimap (-): U_\wedge(D \otimes D) \to U_\wedge D$ which we call *tensor implication*. By Remark 5.3, it is given by $x \multimap T = \bigvee_{x \otimes y \leq T} y$. Analogously, we let $(-) \circ\!\!- x$ denote the right adjoint of $(-) \otimes x$.

Definition 5.6. A *(boolean) residuation algebra* consists of a (boolean) lattice $R \in \mathbf{DL}$ equipped with $\mathbf{MSL}$-morphisms $\backslash : R^\circ \boxtimes R \to R$ and $/ : R \boxtimes R^\circ \to R$, the *left* and *right residual*, satisfying the *residuation property*: $b \leq a \backslash c \iff a \leq c \,/\, b$. We call R *associative* if it satisfies $x \backslash (z \,/\, y) = (x \backslash z) \,/\, y$ for all $x, y, z \in R$. A join-irreducible element $e \in \mathcal{J}R$ is a *unit* if it satisfies $e \backslash z = z = z \,/\, e$.

Residuals may be thought of algebraic generalizations of language derivatives, but as the following examples indicate they are not limited to this interpretation.

Examples 5.7. (1) Every distributive Heyting algebra is an associative residuation algebra with residuals $a \backslash c = a \to c$ and $c \,/\, b = b \to c$.

(2) Every boolean algebra B is a non-associative residuation algebra with $x \backslash 1 = 1$ and $x \backslash z = \neg x$ for $z \neq 1$, but for $|B| > 1$ it does not have a unit.

(3) For a binary algebra $m: X \times X \to X$ on a Stone space X, the dual boolean algebra $\hat{X}$ of clopens forms a residuation algebra: given clopens $A, C \subseteq X$, put

$$A \backslash C = \{x \in X \mid \forall (a \in A): m(a, x) \in C\},$$
$$C \,/\, A = \{x \in X \mid \forall (b \in B): m(x, b) \in C\}.$$

(4) The regular languages $\mathrm{Reg}\,\Sigma$ over a finite alphabet Σ form an associative boolean residuation algebra with residuals given by (extended) left and right *derivatives*: $K \backslash L = \{v \in \Sigma^* \mid Kv \subseteq L\}$ and $L \,/\, K = \{v \in \Sigma^* \mid vK \subseteq L\}$. The singleton empty word $\{\varepsilon\}$ is a unit. This example is a special case of item (3) obtained by taking the Stone algebra given by the *free profinite monoid* $\widehat{\Sigma^*}$.

We now introduce the notion of a *residuation morphism* between residuation algebras and also its *relational* generalization.

Definition 5.8. (1) A lattice morphism $f: R \to S$ between unital residuation algebras is a *(pure) residuation morphism* if it satisfies the conditions

$$f(x \backslash z) \leq f(x) \backslash f(z) \tag{Forth}$$
$$\forall (y, z) \in S \times R: \exists (x_{y,z} \in R): y \leq f(x_{y,z}) \quad \wedge \quad y \backslash f(z) = f(x_{y,z} \backslash z) \tag{Back}$$
$$\forall x: e \leq x \Leftrightarrow e' \leq f(x) \tag{Unit}$$

The morphism f is *open* if, additionally, it has a left adjoint. The category of unital residuation algebras with residuation morphisms is denoted $\mathbf{Res}$.

(2) A *(lax) relational residuation morphism* from a unital residuation algebra R to a unital residuation algebra S is a morphism $\rho \in \mathbf{JSL}_1(R, S)$ satisfying

$$\rho(x \backslash z) \leq \rho(x) \backslash \rho(z) \quad \text{and} \quad e' \leq \rho(e).$$

Unital residuation algebras with relational residuation morphisms form a category **RelRes**.

We use the convention that for a subcategory **C** of **Res** we denote the full subcategory of **C** with boolean carriers by **BC**, and analogously for **RelRes**.

Remark 5.9. Let us provide some intuition behind the choices made in Definition 5.8. Recall that a *relational monoid morphism* from a finite monoid M to N is a total relation $\rho\colon M \to \mathcal{P}N$ such that $\rho(x)\rho(y) \subseteq \rho(xy)$ and $1_N \in \rho(1_M)$.

(1) The notion of residuation morphism is derived from a result by Gehrke [13, Theorem 3.19], where it is shown to capture precisely the conditions satisfied by the duals of morphisms of binary Stone algebras.

(2) We speak about *relational* morphisms of residuation algebras since for finite algebras these will dualize precisely to *relational morphisms* of finite monoids, which model inverses of surjective monoids homomorphisms [28, p. 38]: on finite monoids the inverse relation $e^{-1}\colon N \to \mathcal{P}M$ of a surjective homomorphism $e\colon M \twoheadrightarrow N$ is the *right adjoint* $e \dashv e^{-1}$ in the category of $\mathcal{P}$-coalgebras (relations), i.e. as relations they satisfy $\mathrm{id} \leq e^{-1} \cdot e$ and $e \cdot e^{-1} \leq 1$. Under duality the composition is reversed, so e^{-1} dualizes to a *left adjoint* $\widehat{e^{-1}} \dashv \hat{e}$. As left adjoints between finite lattices are precisely the join-preserving functions this suggests the choice that relational morphisms of residuation algebras preserve finite joins (and not necessarily meets). Surjectivity of e is equivalent to totality of e^{-1}, which by Corollary 4.5 is equivalent to $\widehat{e^{-1}}$ preserving the top element.

(3) This is also the reasoning behind the naming for *open* residuation morphisms: if $e\colon M \twoheadrightarrow N$ is a continuous surjection between *profinite* monoids (that is, topological monoids in **Stone**), then $e^{-1}\colon N \to \mathbb{V}M$ is continuous precisely iff e is an open map.

For open residuation morphisms the conditions (Back) and (Forth) can be combined into a much simpler condition. Over finite residuation algebras this is particularly convenient since every residuation morphism is open.

Lemma 5.10. *Let R, S be residuation algebras. A lattice morphism $f\colon R \to S$ is an open residuation morphism iff $f^*(e') = e$ and it satisfies the condition*

$$y \setminus f(z) = f(f^*(y) \setminus z). \tag{Open}$$

Example 5.11. Let Σ, Δ be finite alphabets. Every substitution $f_0\colon \Sigma \to \Delta^*$ can be extended to a monoid homomorphism $f\colon \Sigma^* \to \Delta^*$, and for regular languages $L \in \mathrm{Reg}\,\Sigma$ and $K \in \mathrm{Reg}\,\Delta$ both $f[L]$ and $f^{-1}[K]$ are also regular. Then $f^{-1}\colon \mathrm{Reg}\,\Delta \to \mathrm{Reg}\,\Sigma$ is an open residuation morphism. Indeed, its left adjoint is $f[-]$, and we have $f[\{\varepsilon\}] = \{f(\varepsilon)\} = \{\varepsilon\}$ and

$$K \setminus f^{-1}[L] = \{w \mid Kw \subseteq f^{-1}[L]\} = \{w \mid f[K]f(w) \subseteq L\} = f^{-1}(f[K] \setminus L).$$

5.2 Finite Residuation Algebras

Construction 5.12. In a finite residuation algebra R the partially applied residuals $(x \setminus -), (- \,/\, y)$ have left adjoints $\mu(x, -) \dashv (x \setminus -), \mu(-, y) \dashv (- \,/$

y) that can be combined, by the universal property of $\otimes$, into a $U_\vee$-algebra $\mu\colon U_\vee R \otimes U_\vee R \to U_\vee R$ called *multiplication*. Every algebra $U_\vee D \otimes U_\vee D \to U_\vee D$ on a finite lattice D has a right adjoint $\gamma\colon U_\wedge D \to U_\wedge(D \otimes D)$ that can, by using the isomorphism ω from Lemma 5.4, be extended to a $U_\wedge$-coalgebra

$$\hat{\gamma} = U_\wedge \omega \cdot \gamma\colon U_\wedge D \to U_\wedge(D \otimes D) \cong U_\wedge(D \boxtimes D) = U_\wedge D \boxtimes U_\wedge D.$$

Since γ and $\hat{\gamma}$ are essentially the same function (differing only by the isomorphism ω) we refer to both as *comultiplication* or *coalgebra structure*. Conversely, we obtain a $U_\vee$-algebra from a comultiplication $\gamma\colon U_\wedge D \to U_\wedge(D \otimes D)$ by taking its left adjoint. In summary, each of $/$, $\backslash$, μ, γ determine each other uniquely:

$$x \le z \;/\; y \iff y \le x \setminus z \iff \mu(x \otimes y) \le z \iff x \otimes y \le \gamma(z),$$

Lemma 5.13. *In a finite residuation algebra R the residuals can be expressed via comultiplication γ and tensor implication as $x \setminus z = x \multimap \gamma(z)$ and $z \;/\; y = \gamma(z) \mathbin{\circ\!-} y$. Conversely, the comultiplication can be expressed via residuals as*

$$\gamma(z) = \bigvee\nolimits_{x \in R} x \otimes (x \setminus z) = \bigvee\nolimits_{p \in \mathcal{J}R} p \otimes (p \setminus z).$$

First we investigate when the comultiplication is a *pure*, i.e. lifts to a lattice morphism $R \to R + R$.

Lemma 5.14. *For a finite residuation algebra R, the following are equivalent:*

(1) *The comultiplication is pure, i.e., $\gamma(0) = 0$ and $\gamma(x \vee y) = \gamma(x) \vee \gamma(y)$.*

(2) *For all $p \in \mathcal{J}R$ we have $p \setminus 0 = 0 = 0 \;/\; p$, and the following equations hold:*

$$p \setminus (x \vee y) = p \setminus x \vee p \setminus y \qquad \textit{and} \qquad (x \vee y) \;/\; p = x \;/\; p \vee y \;/\; p.$$

(3) *For all $x, y \in R$: $\mu(x \otimes y) = 0 \iff x = 0 \vee y = 0$, and $\mu[\mathcal{J}(R + R)] \subseteq \mathcal{J}R$.*

Next we inspect how structural identities like (co-)associativity or unitality translate to the other operations. Note that while the statements are to be expected, the proof is non-trivial due to the complication introduced by the seemingly innocent isomorphism $\omega\colon R \otimes R \cong R \boxtimes R$. Recall that a coalgebra $c\colon U_\wedge R \to U_\wedge R \boxtimes U_\wedge R$ is *coassociative* if $(c \boxtimes \mathrm{id}) \cdot c = (\mathrm{id} \boxtimes c) \cdot c$ and *counital* if it is equipped with a *counit* $\varepsilon \in \mathbf{DL}(R, 2)$ such that $(\varepsilon \boxtimes \mathrm{id}) \cdot c = \mathrm{id} = (\mathrm{id} \boxtimes \varepsilon) \cdot c$.

Lemma 5.15. *The following are equivalent for a finite residuation algebra R:*

(1) *The comultiplication on R is coassociative and has a counit.*

(2) *The residuals are associative and R has a unit.*

(3) *The multiplication μ is associative and has a unit, i.e. a join-irreducible $e \in \mathcal{J}R$ satisfying $\mu(e \otimes -) = \mathrm{id} = \mu(- \otimes e)$.*

These lemmas suggest the following definitions.

Definition 5.16. (1) A finite residuation algebra R is *pure* if it satisfies the equivalent conditions of Lemma 5.14.

(2) A finite residuation algebra R is a *finite derivation algebra* if it is pure, associative and has a unit. The respective full subcategories of $\mathbf{Res}_{\mathrm{f}}$ and $\mathbf{RelRes}_{\mathrm{f}}$ are denoted by $\mathbf{Der}_{\mathrm{f}}$ and $\mathbf{RelDer}_{\mathrm{f}}$.

(3) A (not necessarily finite) $U_\wedge$-coalgebra $\hat{\gamma}\colon U_\wedge C \to U_\wedge C \boxtimes U_\wedge C$ is a $U_\wedge$-*comonoid* if its coassociative and counital. It is a *comonoid* if $\hat{\gamma}$ is pure.

In order to extend the correspondence of (finite) residuation algebras and $U_\wedge$-coalgebras to a categorical equivalence we introduce appropriate morphisms.

Definition 5.17. (1) A *pure morphism* from a counital $U_\wedge$-coalgebra $(C, \hat{\gamma}, \epsilon)$ to $(C', \hat{\gamma}', \epsilon')$ is a lattice morphism $f\colon C \to D$ satisfying $(f \boxtimes f) \cdot \hat{\gamma} = \hat{\gamma}' \cdot f$ and $\epsilon = \epsilon' \cdot f$.

$$\begin{array}{ccc} U_\wedge C & \xrightarrow{U_\wedge f} & U_\wedge C' \\ \downarrow{\scriptstyle \hat{\gamma}} & & \downarrow{\scriptstyle \hat{\gamma}'} \\ U_\wedge C \boxtimes U_\wedge C & \xrightarrow{U_\wedge f \boxtimes U_\wedge f} & U_\wedge C' \boxtimes U_\wedge C' \end{array} \qquad \begin{array}{ccc} U_\wedge C & \xrightarrow{U_\wedge f} & U_\wedge C' \\ & {\scriptstyle U_\wedge \epsilon}\searrow & \downarrow{\scriptstyle U_\wedge \epsilon'} \\ & & U_\wedge 2 \end{array}$$

The category of counital $U_\wedge$-coalgebras with pure morphisms is denoted by $\mathbf{Coalg}(U_\wedge)$ and its full subcatgegory of $U_\wedge$-comonoids by $\mathbf{Comon}(U_\wedge)$, again with the full subcategory $\mathbf{Comon}$ of comonoids.

(2) Let C and C' be comonoids. A *(lax) relational morphism* from C to C' is a morphism $\rho \in \mathbf{JSL}_1(C, C')$ satisfying $(\rho \otimes \rho) \cdot \gamma \leq \gamma' \cdot \rho$ and $\epsilon \leq \epsilon' \cdot \rho$, i.e. the following diagrams in $\mathbf{JSL}$ commute laxly:

$$\begin{array}{ccc} U_\vee C & \xrightarrow{\rho} & U_\vee C' \\ \downarrow{\scriptstyle U_\vee \gamma} & \leq & \downarrow{\scriptstyle U_\vee \gamma'} \\ U_\vee C \otimes U_\vee C & \xrightarrow{\rho \otimes \rho} & U_\vee C' \otimes U_\vee C' \end{array} \qquad \begin{array}{ccc} U_\vee C & \xrightarrow{\rho} & U_\vee C' \\ & {\scriptstyle U_\vee \epsilon}\searrow \;\leq & \downarrow{\scriptstyle U_\vee \epsilon'} \\ & & U_\vee 2 \end{array}$$

Comonoids with relational morphisms form a category $\mathbf{RelComon}$.

Theorem 5.18. *The following categories are isomorphic:*

$$\mathbf{Coalg}_{\mathrm{f}}(U_\wedge) \cong \mathbf{Res}_{\mathrm{f}}, \qquad \mathbf{Comon}_{\mathrm{f}} \cong \mathbf{Der}_{\mathrm{f}} \quad \textit{and} \quad \mathbf{RelComon}_{\mathrm{f}} \cong \mathbf{RelDer}_{\mathrm{f}}.$$

Proof (Sketch). On objects the isomorphism swaps between residuals and comultiplication; the residual unit is left adjoint of the counit. The first isomorphism restricts to the second by Lemmas 5.14 and 5.15. On morphisms one proves that a lattice morphism $f\colon C \to C'$ is a pure coalgebra morphism iff it is an (open) residuation morphism, and if C and C' are comonoids, then $\rho \in \mathbf{JSL}_1(C, C')$ is a relational comonoid morphism iff it is a relational residuation morphism.

From Theorem 5.18 we obtain the following dual characterization of finite monoids; an analogous result can be stated for finite ordered monoids.

Theorem 5.19. (1) *The category of finite monoids is dually equivalent to the category of finite boolean derivation algebras (or finite boolean comonoids):*

$$\mathbf{Mon}_{\mathrm{f}} \simeq^{\mathrm{op}} \mathbf{BComon}_{\mathrm{f}} \cong \mathbf{BDer}_{\mathrm{f}}.$$

(2) *The category of finite monoids with relational morphisms is dually equivalent to the category of finite boolean derivation algebras (or finite boolean comonoids) with relational morphisms.*

$$\mathbf{RelMon}_\mathrm{f} \simeq^{\mathrm{op}} \mathbf{RelBComon}_\mathrm{f} \cong \mathbf{RelBDer}_\mathrm{f}.$$

Proof. The first statement is a trivial extension of Theorem 5.18 by (finite) Stone duality since finite monoids dualize to finite boolean comonoids. For item (2) note that a relational monoid morphism $(M, \cdot_M, 1_M) \to (N, \cdot_N, 1_N)$ is a total relation $\rho\colon M \to \mathcal{P}N$ making the following diagrams commute laxly:

$$\begin{array}{ccccc} M \times M & & \xrightarrow{\quad \cdot_M \quad} & & M \\ \downarrow{\scriptstyle \rho\times\rho} & & \leq & & \downarrow{\scriptstyle \rho} \\ \mathcal{P}N \times \mathcal{P}N & \xrightarrow{\delta} & \mathcal{P}(N \times N) & \xrightarrow{\mathcal{P}(\cdot_N)} & \mathcal{P}N \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{1_M} & M \\ \downarrow{\scriptstyle 1_N} & \leq & \downarrow{\scriptstyle \rho} \\ N & \xrightarrow{\eta} & \mathcal{P}N \end{array}$$

If we view N as a finite discrete Priestley space, then $\mathcal{P}N = \mathbb{V}N$, thus under extended duality this yields precisely a relational morphism $\hat{\rho}^- \in \mathbf{JSL}_1(\hat{N}, \hat{M})$ of finite boolean comonoids, or equivalently, a relational residuation morphism.

5.3 Locally Finite Residuation Algebras

The main complication in the generalization from finite to infinite structures comes from the reliance on adjoints, as these may not exist anymore on infinite lattices. The prime example of a residuation algebra in automata theory suggests a *local* translation between residuals and comultiplication:

Example 5.20. It is well-known that the boolean algebra $\mathrm{Reg}\,\Sigma$ of regular languages dualizes under Stone duality to the *free profinite monoid* $\widehat{\Sigma^*}$ (see Pippenger [26]). The multiplication $\mu\colon \widehat{\Sigma^*} \times \widehat{\Sigma^*} \to \widehat{\Sigma^*}$ of profinite words dualizes under Stone duality to a comultiplication $\mu^{-1}\colon \mathrm{Reg}\,\Sigma \to \mathrm{Reg}\,\Sigma + \mathrm{Reg}\,\Sigma$ on regular languages defined on $L \in \mathrm{Reg}\,\Sigma$ by

$$\mu^{-1}(L) = \bigvee\nolimits_{[v]\in \mathrm{Syn}_L} [v] \otimes [v] \setminus L. \tag{5.1}$$

Here Syn_L is the *syntactic monoid* of L, whose elements are the equivalence classes of the equivalence relation on Σ^* defined by $v \equiv_L w$ iff v, w belong to the same residuals $K \setminus L \mathbin{/} M$. Gehrke [12, Thm. 15] has shown that, under Stone duality, Syn_L dualizes to the *residuation ideal* generated by $L \in \mathrm{Reg}\,\Sigma$.

Definition 5.21. A *residuation ideal* of a residuation algebra R is a sublattice $I \hookrightarrow R$ such that for all $z \in I$ and $x \in R$ one has $x \setminus z, z \mathbin{/} x \in R$. We denote the residuation ideal generated by a subset $X \subseteq R$ by $\backslash X /$.

Residuation ideals were used by Gehrke [13] to characterize quotients of Priestley topological algebras. Note that in the formula (5.1) for the comultiplication on regular languages it is crucial that the residuation ideal $\backslash\{L\}/$ generated by a single regular language L is *finite*, as otherwise the join might not exist. This leads to the following restriction.

Definition 5.22. (1) A residuation algebra R is *locally finite* if every finite subset of R is contained in a finite residuation ideal of R.

(2) A $U_\wedge$-coalgebra C is *locally finite* if every finite subset of C is contained in a finite subcoalgebra of C.

Note that not every residuation algebra is locally finite, consider for example an infinite boolean algebra in Example 5.7(2).

Proposition 5.23. (1) *Every locally finite residuation algebra R yields a locally finite $U_\wedge$-coalgebra $\gamma_\backslash \colon U_\wedge R \to U_\wedge(R \otimes R)$ with comultiplication given by*

$$\gamma_\backslash(z) = (\iota_A \otimes \iota_A)(\gamma_A(z)) = \bigvee\nolimits_{x \in A} \iota_A(x) \otimes \iota_A(x \setminus z) = \bigvee\nolimits_{p \in \mathcal{J}A} \iota_A(p) \otimes \iota_A(p \setminus z)$$

for any finite residuation ideal $\iota_A \colon A \hookrightarrow R$ containing z (here γ_A is the comultiplication on A as in Construction 5.12).

(2) *Every locally finite $U_\wedge$-coalgebra (C, γ) yields any locally finite residuation algebra with the left residual given by $x \setminus_\gamma z = \iota_A(x \setminus_A z) = \iota_A(x \multimap \gamma(z))$ for any finite subcoalgebra $\iota_A \colon A \hookrightarrow C$ containing x, z (here $\setminus_A$ is the residual on A as given by Construction 5.12). The residual has a canonical presentation as $x \setminus_\gamma z = \iota_z(\iota_z^*(x) \setminus z)$, where $\iota_z \colon \langle z \rangle \to C$ is the smallest (finite) subcoalgebra containing z. The right residual is defined analogously.*

(3) *These translations are mutually inverse.*

Proposition 5.23 shows that every locally finite residuation algebra carries a unique $U_\wedge$-coalgebra structure and vice versa. We may thus translate at will between the residuals and comultiplication as in the finite case and omit the subscripts. We extend Lemmas 5.14 and 5.15 to locally finite structures:

Lemma 5.24. *Let R be a locally finite residuation algebra.*

(1) *Finite residuation ideals correspond to finite subcoalgebras.*

(2) *The residuals are associative iff the comultiplication is coassociative.*

(3) *The residuals have a unit iff the comultiplication is counital.*

(4) *The comultiplication is pure iff every finite residuation ideal is pure (see Definition 5.16).*

Definition 5.25. A residuation algebra R is a *derivation algebra* if it is locally finite, associative, unital and every finite residuation ideal I is pure. The ensuing full subcategories of **Res** and **RelRes** are denoted **Der** and **RelDer**.

Theorem 5.26. (1) *The category of locally finite residuation algebras and residuation morphisms is isomorphic to the category of locally finite unital $U_\wedge$-coalgebras and pure coalgebra morphisms.*

(2) *The isomorphism restricts to the full subcategories of derivation algebras and locally finite comonoids.*

(3) *The categories of derivation algebras and relational residuation morphisms and locally finite comonoids with relational morphisms are isomorphic.*

Combining this characterization with our approach to extended Stone duality we finally arrive at our main result. We define a *Stone relational morphism* from a profinite monoid X to a profinite monoid Y to be a Stone relation $\rho\colon X \to \mathbb{V}Y$ such that $\rho(x)\rho(x') \subseteq \rho(xx')$ and $1_N \in \rho(1_M)$.

Theorem 5.27. (1) *The category of boolean derivation algebras is dually equivalent to the category of profinite monoids:*

$$\mathbf{BDer} \cong \mathbf{BComon} \simeq^{\mathrm{op}} \mathbf{ProfMon}.$$

(2) *The category of boolean derivation algebras and relational residuation morphisms is dually equivalent to the category of profinite monoids and Stone relational morphisms:*

$$\mathbf{RelBDer} \cong \mathbf{RelBComon} \simeq^{\mathrm{op}} \mathbf{RelProfMon}.$$

Remark 5.28. (1) By extended Priestley duality, Theorem 5.27 extends to all derivation algebras and *Priestley* monoids with Priestley relational morphisms.

(2) All results of Section 5 hold analogously for the extension of the "discrete" duality between posets (or sets) and algebraic completely distributive lattices (or CABAs) along the free-forgetful adjunction between completely distributive lattices and *complete join-semilattices.* This yields a duality between the category of all monoids and completely atomic boolean residuation algebras with open residuation morphisms that can further be extended to relational morphisms.

6 Conclusion and Future Work

We have presented an abstract approach to extending Stone-type dualities based on adjunctions between monoidal categories and instantiated it to recover and generalize extended Priestley duality. Guided by these foundations we have investigated residuation and derivation algebras and proved a duality between the latter and profinite monoids, Moreover, we have extended this duality to relational morphisms.

Our next goal is to apply our abstract duality framework beyond classical Stone and Priestley dualities. Specifically, we aim to develop an extended duality theory for the recently developed *nominal* Stone duality [4], which would allow to generalize our present results on residuation algebras to the nominal setting and uncover new results about data languages.

A conceptually rather different dual characterization of the category of profinite monoids and continuous monoid morphisms in terms of semi-Galois categories has been provided by Uramoto [35]. Extending this result to relational morphisms, similar to our Theorem 5.27, is another interesting point for future work.

Finally, by extending the duality on objects we note that by dualizing a *non-pure* lattice comonoid one obtains a *relational* profinite monoid. General relational monoids are equivalent to power quantales, which recognize precisely *context-free* languages [29]; this leads to the question of the expressivity of relational (pro-)finite monoids.

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A Appendix

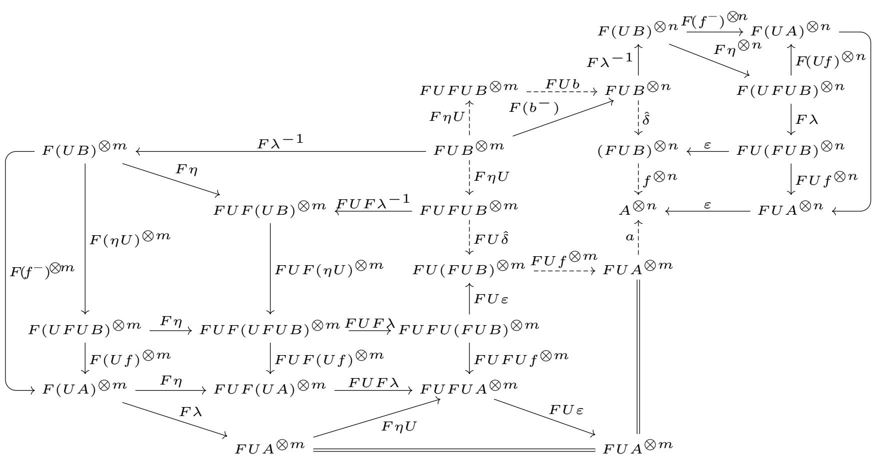
This appendix provides full proofs of all results and technical statements omitted for space reasons.

Proof of Theorem 3.5

Let $\hat{a}\colon \hat{A}^{\hat{\otimes} n} \to T\hat{A}^{\hat{\otimes} m}$ and $\hat{b}\colon \hat{B}^{\hat{\otimes} n} \to T\hat{B}^{\hat{\otimes} m}$ be objects of $\mathrm{Op}^{n,m}_{J_T}(\hat{\mathbf{C}})$, and let $\hat{f}\colon (\hat{A},\hat{a}) \to (\hat{B},\hat{b})$ be a morphism of operators. Unfolding the definitions of composition and monoidal structure in the Kleisli category, this means $\hat{f}$ is a morphism $\hat{A} \to T\hat{B} = \hat{F}\hat{U}\hat{B}$ such that the following diagram in $\hat{\mathbf{C}}$ commutes:

$$\begin{array}{ccccccc}
\hat{A}^{\hat{\otimes} n} & \xrightarrow{\hat{a}} & T\hat{A}^{\hat{\otimes} m} & \xrightarrow{T\hat{f}^{\hat{\otimes} m}} & T(T\hat{B})^{\hat{\otimes} m} & \xrightarrow{T\delta} & TT\hat{B}^{\hat{\otimes} m} \\
\big\downarrow{\scriptstyle \hat{f}^{\hat{\otimes} n}} & & & & & & \big\downarrow{\scriptstyle \mu} \\
(T\hat{B})^{\hat{\otimes} n} & \xrightarrow{\delta} & T\hat{B}^{\hat{\otimes} n} & \xrightarrow{T\hat{b}} & TT\hat{B}^{\hat{\otimes} m} & \xrightarrow{\mu} & T\hat{B}^{\hat{\otimes} m}
\end{array}$$

Its dual diagram is the dashed center square of the following commuting diagram in $\mathbf{C}$. (Note that we extend δ to arbitrary tensor powers, and also that the choice of the order in which we compose λ, λ^{-1} and the associators is irrelevant by the *Coherence Theorem* [24] for monoidal categories.)



Since the counit $\varepsilon = \mathrm{id}^+$ is the adjoint transpose of the identity and transposition is natural, transposing the outer paths of the above diagram yield the inner square of the following commutative diagram in $\mathbf{D}$:

$$\begin{array}{ccccccccccc}
(UB)^{\otimes m} & \xrightarrow{\lambda} & UB^{\otimes m} & \xrightarrow{b^-} & UB^{\otimes n} & \xrightarrow{\lambda^{-1}} & (UB)^{\otimes n} & \xrightarrow{(f^-)^{\otimes n}} & (UA)^{\otimes n} & & \\
& \searrow{\scriptstyle =} & \big\downarrow{\scriptstyle \lambda^{-1}} & & & & & & \big\downarrow{\scriptstyle \lambda} & \searrow{\scriptstyle =} & \\
& & (UB)^{\otimes m} & \xrightarrow{(f^-)^{\otimes m}} & (UA)^{\otimes m} & \xrightarrow{\lambda} & UA^{\otimes m} & \xrightarrow{a^-} & UA^{\otimes n} & \xrightarrow{\lambda^{-1}} & U(A^{\otimes n})
\end{array}$$

By pre- and postcomposition of this square with λ and λ^{-1}, respectively, and replacing a^- and b^- by their respective conjugates $\alpha = \lambda \cdot a^- \cdot \lambda^{-1}$ and $\beta = \lambda \cdot b^- \cdot \lambda^{-1}$ this diagram simply becomes the square

$$\begin{array}{ccc} (UB)^{\otimes m} & \xrightarrow{\beta} & (UB)^{\otimes n} \\ {\scriptstyle (f^-)^{\otimes m}}\downarrow & & \downarrow{\scriptstyle (f^-)^{\otimes n}} \\ (UA)^{\otimes m} & \xrightarrow{\alpha} & (UA)^{\otimes n} \end{array}$$

in $\mathbf{D}$, which is a homomorphism diagram of (m,n)-ary U-operators. All steps of this transformation are reversible. We have thus shown that the functor sending an operator $(\hat{A}, \hat{a})$ in $\mathrm{Op}^{n,m}_{J_T}(\hat{\mathbf{C}})$ to the operator $(A, \lambda \cdot a^- \cdot \lambda^{-1})$ in $\mathrm{Op}^{m,n}_U(\mathbf{C})$, and an operator morphism $\hat{f}$ to f^-, defines an equivalence of categories. ◻

Proof of Proposition 3.6

We denote the transposition of $F_1 \dashv U_1$ by $(-)^\$ \colon \mathbf{C}(F_1X, Y) \cong \mathbf{D}(X, U_1Y)$. Let $\hat{h} \colon \hat{B}^{\hat{\otimes} n} \to T\hat{A}^{\hat{\otimes} m} = \hat{F}_1\hat{F}_2\hat{U}_2\hat{U}_1\hat{A}^{\otimes m}$ be the dual of $a \colon (U_2U_1A)^{\otimes m} \to (U_2U_1B)^{\otimes n}$ under the abstract extended duality. This is precisely the case iff the outer path of the following diagram commutes:

$$\begin{array}{ccc} U_2U_1A^{\otimes m} & \xrightarrow{h^-} & U_2U_1B^{\otimes n} \\ {\scriptstyle U_2\lambda_1}\uparrow & & \downarrow{\scriptstyle U_2\lambda_1^{-1}} \\ U_2(U_1A^{\otimes m}) & & U_2(U_1B^{\otimes n}) \\ {\scriptstyle \lambda_2}\uparrow & & \downarrow{\scriptstyle \lambda_2^{-1}} \\ (U_2U_1A)^{\otimes m} & \xrightarrow{a} & (U_2U_1B)^{\otimes n} \end{array} \qquad (\text{outer arrows } \lambda, \ \lambda^{-1})$$

Let $\hat{g} \colon B^{\otimes n} \to T_1A^{\otimes m} = \hat{F}_1\hat{U}_1A^{\otimes m}$ be a morphism in $\hat{\mathbf{C}}$ with $\hat{h} = \hat{F}_1\hat{\varepsilon}_2\hat{U}_1 \cdot \hat{g}$. This holds iff

$$h = g \cdot F_1\varepsilon_2U_1 \iff h^\$ = g^\$ \cdot \varepsilon_2U_1 \iff h^- = U_2g^\$$$

since the counit ε_2 vanishes under the second transposition. The dual b of $\hat{g}$ under the extended duality along the adjunction $F_1 \dashv U_1$ makes the upper square of the following diagram commute

$$\begin{array}{ccc} U_2U_1A^{\otimes m} & \xrightarrow{U_2g^\$ = h^-} & U_2U_1B^{\otimes n} \\ {\scriptstyle U_2\lambda_1}\uparrow & & \downarrow{\scriptstyle U_2\lambda_1^{-1}} \\ U_2(U_1A^{\otimes m}) & \overset{U_2b}{\dashrightarrow} & U_2(U_1B)^{\otimes n} \\ {\scriptstyle \lambda_2}\uparrow & & \downarrow{\scriptstyle \lambda_2^{-1}} \\ (U_2U_1A^{\otimes m}) & \xrightarrow{a} & (U_2U_1)B^{\otimes n} \end{array} \qquad (\text{outer arrows } \lambda, \ \lambda^{-1})$$

whence the whole diagram commutes. But this is equivalent to a admitting the lifting b with $a = \lambda_2^{-1} \cdot U_2 b \cdot \lambda_2$. $\square$

Details for Section 4

Lemma A.1. *The map δ witnessing that $\mathbb{V}_\downarrow$ is a monoidal monad is given by*

$$\delta\colon \mathbb{V}_\downarrow X \times \mathbb{V}_\downarrow Y \to \mathbb{V}_\downarrow (X \times Y), \qquad (C, D) \mapsto C \times D.$$

Proof. Let $C \in \mathbb{V}_\downarrow X, D \in \mathbb{V}_\downarrow Y$ be closed downsets. We first represent them by their respective ideals I_C, I_D of $\mathrm{Cl}_\uparrow X$, which are equivalently **JSL**-morphisms

$$c\colon U \,\mathrm{Cl}_\uparrow X \to U2, d\colon U \,\mathrm{Cl}_\uparrow X \to U2.$$

The dual of the distributive law δ is

$$\hat{\delta}\colon \mathbf{DL}(FU \,\mathrm{Cl}_\uparrow X + FU \,\mathrm{Cl}_\uparrow X, 2) \to \mathbf{DL}(FU(\mathrm{Cl}_\uparrow X + \mathrm{Cl}_\uparrow X), 2)$$

mapping $[c^+, d^+]$ to the transpose of

$$\begin{aligned} U[c^+, d^+] \cdot (\eta \otimes \eta)\colon & U(\mathrm{Cl}_\uparrow X + \mathrm{Cl}_\uparrow X) \\ &= U(\mathrm{Cl}_\uparrow X) \otimes U(\mathrm{Cl}_\uparrow X) \to UFU \,\mathrm{Cl}_\uparrow X \otimes UFU \,\mathrm{Cl}_\uparrow X \\ &= U(FU \,\mathrm{Cl}_\uparrow X + FU \,\mathrm{Cl}_\uparrow X) \to U2. \end{aligned}$$

The latter map sends a pure tensor $A \otimes B \in U(\mathrm{Cl}_\uparrow X + \mathrm{Cl}_\uparrow X)$ to its "product" $c(A) \wedge d(B)$. Therefore the closed set $\delta(C, D)$ corresponding to $U[c^+, d^+] \cdot (\eta \otimes \eta)$ contains a pair of elements x, y iff $x \in C$ and $y \in D$. $\square$

Details for Lemma 5.4

For the proofs, we need an extended version of Lemma 5.4

Lemma A.2. (1) *The join- and meet-semilattice tensor products of distributive lattices D, E yield isomorphic lattices.*

$$U_\vee D \otimes U_\vee E \cong U_\wedge D \boxtimes U_\wedge E.$$

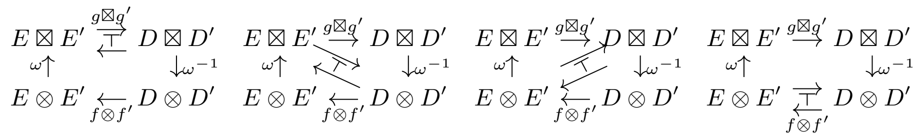
More precisely, the unique lattice morphism $\omega\colon U_\vee D \otimes U_\vee E \to U_\wedge D \boxtimes U_\wedge E$ commuting with the coproduct injections is an isomorphism. It acts on pure tensors as $d \otimes e \mapsto d \boxtimes 0 \wedge 0 \boxtimes e$, and on general elements ω is given by

$$\bigvee\nolimits_{i \in I} d_i \otimes e_i \mapsto \bigwedge\nolimits_{A \in \mathcal{P} I} (\bigvee\nolimits_{i \in A} d_i) \boxtimes (\bigvee\nolimits_{i \notin A} e_i)$$

with inverse ω^{-1} given by

$$\bigwedge\nolimits_{i \in I} d_i \boxtimes e_i \mapsto \bigvee\nolimits_{A \in \mathcal{P} I} (\bigwedge\nolimits_{i \in A} d_i) \otimes (\bigwedge\nolimits_{i \notin A} e_i).$$

(2) *Adjunctions on lattices "compose horizontally": Given adjunctions* $f\colon D \dashv E : g$ *and* $f'\colon D' \dashv E' : g$ *on lattices we get adjunctions:*



If the right adjoints g, g' *preserve finite joins this simplifies to*

$$f \otimes f' \dashv g \otimes g' = \omega^{-1}(g \boxtimes g')\omega$$

and dually $\omega(f \otimes f')\omega^{-1} = f \boxtimes f'$ *if both* f, f' *preserve finite meets.*

Proof. (1) By order-duality, the tensor product $U_\wedge D \boxtimes U_\wedge E$ in the category of meet-semilattices also gives a representation of the coproduct $D + E$ in **DL**, its inclusions $\hat\imath_1, \hat\imath_2$ map $d \in D, e \in E$ to $\hat\imath_1(d) = d \boxtimes 0$ and $\hat\imath_2(e) = 0 \boxtimes e$, respectively. The canonical isomorphism ω is the coparing $\omega = [\hat\imath_1, \hat\imath_2]$ of the inclusions of the meet-semilattice tensor product. Therefore on pure tensors ω maps $d \otimes e \mapsto d \boxtimes 0 \wedge 0 \boxtimes e$, which extends to general elements of $D \otimes E$ via distributivity as

$$\begin{aligned}
\omega(\bigvee\nolimits_{i\in I} d_i \otimes e_i) &= \bigvee\nolimits_{i\in I} \omega(d_i \otimes e_i)\\
&= \bigvee\nolimits_{i\in I} d_i \boxtimes 0 \wedge 0 \boxtimes e_i\\
&= \bigwedge\nolimits_{A\in\mathcal{P}I} \bigvee\nolimits_{i\in A} d_i \boxtimes 0 \vee \bigvee\nolimits_{i\notin A} 0 \boxtimes e_i\\
&= \bigwedge\nolimits_{A\in\mathcal{P}I} (\bigvee\nolimits_{i\in A} d_i) \boxtimes 0 \vee 0 \boxtimes (\bigvee\nolimits_{i\notin A} e_i)\\
&= \bigwedge\nolimits_{A\in\mathcal{P}I} (\bigvee\nolimits_{i\in A} d_i) \boxtimes (\bigvee\nolimits_{i\notin A} e_i),
\end{aligned}$$

where we in the last two steps that $(a \boxtimes b) \vee (c \boxtimes d) = (a \vee c) \boxtimes (b \vee d)$ holds in $D \boxtimes E$.

(2) It suffices to prove that only one of the squares is an adjunction, since one gets all others by suitable composition with ω.

We choose the third, i.e. we show that there is an adjunction

$$(f \otimes f') \cdot \omega^{-1}\colon D \boxtimes D' \dashv E \otimes E' : (g \boxtimes g') \cdot \omega.$$

by verifying the unit and counit inequalities

$$\mathrm{id} \le (g \boxtimes g')\omega(f \otimes f')\omega^{-1} \qquad \text{and} \qquad (f \otimes f')\omega^{-1}(g \boxtimes g')\omega \le \mathrm{id}.$$

We only prove the counit inequality; the proof of the unit inequality is dual. Recall that the right adjoint g preserves meets. Therefore, for every $x_A = \bigvee_{i\in A} x_i$

we have:

$$
\begin{aligned}
& (f \otimes f')\omega^{-1}(g \boxtimes g')\omega(\textstyle\bigvee_i x_i \otimes y_i) \\
= \;& (f \otimes f')\omega^{-1}(g \boxtimes g')(\textstyle\bigwedge_{A \in \mathcal{P}I} x_A \boxtimes y_{A^c}) && \text{def. } \omega \\
= \;& (f \otimes f')\omega^{-1}(\textstyle\bigwedge_{A \in \mathcal{P}I} g(x_A) \boxtimes g'(y_{A^c})) && \text{def. } g \boxtimes g' \\
= \;& (f \otimes f')(\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} (\bigwedge_{A \in B} g(x_A)) \otimes (\bigwedge_{A \in B^c} g'(y_{A^c}))) && \text{def. } \omega^{-1} \\
= \;& (f \otimes f')(\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} g(\bigwedge_{A \in B} x_A) \otimes g'(\bigwedge_{A \in B^c} y_{A^c})) && g \text{ pres. meets} \\
= \;& (\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} fg(\bigwedge_{A \in B} x_A) \otimes f'g'(\bigwedge_{A \in B^c} y_{A^c})) && \text{def. } f \otimes f' \\
\leq \;& (\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} \mathrm{id}(\bigwedge_{A \in B} x_A) \otimes \mathrm{id}(\bigwedge_{A \in B^c} y_{A^c})) && \text{counits } f \dashv g, f' \dashv g' \\
= \;& \omega^{-1}\omega(\textstyle\bigvee_i x_i \otimes y_i) = \bigvee_i x_i \otimes y_i.
\end{aligned}
$$

If g, g' preserve finite joins, then $g \otimes g'$ is defined (otherwise it would not be!) and it is clear that $f \otimes f' \dashv g \otimes g'$. By uniqueness of adjoints this implies $g \otimes g' = \omega^{-1}(g \boxtimes g')\omega$. □

Details for Construction 5.5

Proposition A.3. *Let D be a finite distributive lattice.*

(1) *The function*

$$x \otimes (-) \colon D \to D \otimes D, \qquad y \mapsto x \otimes y$$

has a right adjoint

$$x \multimap (-) \colon D \otimes D \to D, \qquad T \mapsto x \multimap T = \bigvee_{x \otimes y \leq T} y$$

called tensor implication. *If $p \in \mathcal{J}D$ is join-irreducible then $p \multimap (-)$ is the lattice homomorphism*

$$\lambda \cdot (\chi_p + \mathrm{id}) \colon D + D \to 2 + D \cong D, \qquad \bigvee_{i \in I} p_i \otimes q_i \mapsto \bigvee_{p \leq p_i} q_i.$$

(2) *It can be extended to a function $(-) \multimap (-) \colon D^\circ \boxtimes (D \otimes D) \to D$.*

(3) *Every adjunction $l \dashv r$ between finite distributive lattices satisfies*

$$x \multimap \omega^{-1}(r \boxtimes r)\omega(T) = r(l(x) \multimap T)$$

as well as

$$l(x \multimap T) \leq l(x) \multimap (l \otimes l)(T) \qquad \text{and} \qquad r(x \multimap T) \leq r(x) \multimap \omega^{-1}(r \boxtimes r)\omega(T),$$

where the latter equation is an equality if r is order-reflecting.

Proof. (1) The function $x \otimes (-)$ preserves finite joins by definition, so its right adjoint $x \multimap (-)$ exists and is given by $T \mapsto \bigvee_{x\otimes y \le T} y$. We can express $x \otimes (-)$ as

$$x \otimes (-) = \lambda^{-1} \cdot (x \otimes \mathrm{id})\colon D \cong 2 \otimes D \to D \otimes D,$$

so it has by Lemma A.2(2) the right adjoint $\lambda \cdot \omega^{-1} \cdot (\chi_x \boxtimes \mathrm{id}) \cdot \omega$. If $x = p \in \mathcal{J}D$ is join-irreducible then χ_p preserves joins, whence the right adjoint simplifies to

$$\lambda^{-1} \cdot (\chi_p \otimes \mathrm{id})\colon D \otimes D \to 2 \otimes D \cong 2, \quad \bigvee_i p_i \otimes q_i \mapsto \bigvee_{p \le p_i} q_i$$

which is a lattice morphism.

(2) This is an instance of an *adjunction with a parameter* (cf. [24, Chapter IV.7]), and it is easy to verify that $(-) \multimap T$ sends joins (meets in D°) to meets in D.

(3) Let $T \in D \otimes D$ and $x \in E$, then for all $y \in E$ we have

$$\begin{aligned} y \le x \multimap \omega^{-1}(r \boxtimes r)\omega(T) &\iff x \otimes y \le \omega^{-1}(r \boxtimes r)\omega(T) \\ &\iff l(x) \otimes l(y) \le T \\ &\iff l(y) \le l(x) \multimap T \\ &\iff y \le r(l(x) \multimap T), \end{aligned}$$

so the first statement follows. With this we compute

$$x \multimap T \le x \multimap \omega^{-1}(r \boxtimes r)\omega(l \otimes l)(T) = r(l(x) \multimap (l \otimes l)(T)),$$

which by adjunction is equivalent to

$$l(x \multimap T) \le l(x) \multimap (l \otimes l)(T).$$

Similarly,

$$r(x \multimap T) \le r(l(r(x)) \multimap T) = r(x) \multimap \omega^{-1}(r \boxtimes r)\omega(T),$$

and the first step is an equality if $l \cdot r = \mathrm{id}$, which is equivalent to r being order-reflecting. ◻

Proof of Lemma 5.10

Let $f\colon R \to S$ be a lattice morphism with left adjoint $f^*\colon S \to R$.

We first show that if f satisfies (Open) then it is an open residuation morphism. The (Forth) condition follows from the counit $f^*(f(x)) \le x$ and contravariance:

$$f(x \setminus z) \le f(f^*(f(x)) \setminus z) = f(x) \setminus f(z).$$

The (Back) condition is satisfied since one can choose for every $y \in S$ the element $x_{y,z} = f^*(y) \in R$ *independently* of $z \in R$. By the unit of the adjunction it satisfies $y \le f(f^*(y)) = f(x_{y,z})$, and thus via (Open)

$$y \setminus f(z) = f(f^*(y) \setminus z) = f(x_{y,z} \setminus z).$$

For the other direction we prove that every open residuation morphism satisfies the condition (Open). Let $(y, z) \in S \times R$, then by the (Back) condition there exists $x_{y,z} \in R$ with $y \leq f(x_{y,z})$ and $y \setminus f(z) = f(x_{y,z} \setminus z)$. This implies $f^*(y) \leq x_{y,z}$ and whence via (Back) also

$$y \setminus f(z) = f(x_{y,z} \setminus z) \leq f(f^*(y) \setminus z).$$

On the other hand the adjunction unit $y \leq f(f^*(y))$ and (Forth) combine to

$$f(f^*(y) \setminus z) \leq f(f^*(y)) \setminus f(z) \leq y \setminus f(z).$$

This proves that f indeed satisfies (Open).

For the respective unitality conditions we have by $f^* \dashv f$ that

$$\forall x\colon e \leq x \Leftrightarrow e' \leq f(x) \Leftrightarrow f^*(e') \leq x$$

which is equivalent to $e = f^*(e')$. □

Proof of Lemma 5.13

The identity $x \setminus z = x \multimap \gamma(z)$ holds because, for every $y \in R$,

$$y \leq x \setminus z \iff \mu(x \otimes y) \leq z \iff x \otimes y \leq \gamma(z) \iff y \leq x \multimap \gamma(z).$$

The proof of $z \mathbin{/} y = \gamma(z) \multimapinv y$ is analogous. Finally, we have

$$\begin{aligned}
\gamma(z) &= \bigvee\{T = \bigvee\nolimits_i x_i \otimes y_i \mid \mu(T) \leq z\} && \text{formula right adjoint}\\
&= \bigvee\{x \otimes y \mid \mu(x \otimes y) \leq z\} && \mu \text{ preserves joins}\\
&= \bigvee\{x \otimes y \mid y \leq x \setminus z\} && \mu(x \otimes -) \dashv (x \setminus -)\\
&= \bigvee\{x \otimes x \setminus z\} && \text{simplification}\\
(&= \bigvee\{z \mathbin{/} x \otimes x\} && \text{symmetry}).
\end{aligned}$$

It is clear that $\bigvee_{p \in \mathcal{J}R} p \otimes p \setminus z \leq \bigvee_{x \in R} x \otimes x \setminus z$, for the reverse inclusion we compute

$$x \otimes x \setminus z = (\bigvee\nolimits_{p \leq x} p) \otimes x \setminus z = \bigvee\nolimits_{p \leq x} p \otimes x \setminus z \leq \bigvee\nolimits_{p \leq x} p \otimes p \setminus z,$$

where p ranges over $\mathcal{J}R$ and we use contravariance of $(- \setminus z)$ in the last step. □

Proof of Lemma 5.14

We first prove the equivalence of (1) and (3). First, we have

$$\begin{aligned}
\gamma(0) = 0 &\iff \gamma(0) \leq 0\\
&\iff \forall T\colon T \leq \gamma(0) \Rightarrow T \leq 0\\
&\iff \forall T = \bigvee\nolimits_i x_i \otimes y_i\colon \mu(T) = \bigvee\nolimits_i \mu(x_i \otimes y_i) \leq 0 \Longrightarrow \forall i\colon x_i \otimes y_i \leq 0\\
&\iff \forall x, y\colon \mu(x \otimes y) = 0 \Rightarrow x \otimes y = 0\\
&\iff \forall x, y\colon \mu(x \otimes y) = 0 \Leftrightarrow x = 0 \lor y = 0.
\end{aligned}$$

Moreover, note that the join-irreducibles of $R \otimes R$ are given by pure tensors of $p \otimes q$ join-irreducibles $p, q \in \mathcal{J}R$, and that in a distributive lattice every join-irreducible element j is join-prime (i.e. $j \leq x \vee y$ implies $j \leq x$ or $j \leq y$), and vice versa. Consequently,

$$\begin{aligned}
&\forall x, y : \gamma(x \vee y) = \gamma(x) \vee \gamma(y) \\
\iff &\forall x, y : \forall a, b \in \mathcal{J}R : a \otimes b \leq \gamma(x \vee y) \Rightarrow a \otimes b \leq \gamma(x) \vee \gamma(y) \\
\iff &\forall x, y : \forall a, b \in \mathcal{J}R : a \otimes b \leq \gamma(x \vee y) \Rightarrow [a \otimes b \leq \gamma(x) \text{ or } a \otimes b \leq \gamma(y)] \\
\iff &\forall x, y : \forall a, b \in \mathcal{J}R : \mu(a \otimes b) \leq x \vee y \Rightarrow [\mu(a \otimes b) \leq x \text{ or } \mu(a \otimes b) \leq y] \\
\iff &\forall x, y : \forall a \otimes b \in \mathcal{J}[R \otimes R] : \mu(a \otimes b) \in \mathcal{J}R.
\end{aligned}$$

For the equivalence (1) $\iff$ (2) we combine Lemma 5.13 with the preservation properties of $x \multimap (-)$ from Proposition A.3(1): If γ preserves finite joins, then so does $(p \setminus -) = p \multimap \gamma(-)$ for $p \in \mathcal{J}R$, and if every $(p \setminus -)$ preserves finite joins then so does $\gamma = \bigvee_{p \in \mathcal{J}R} p \otimes p \setminus (-)$. □

Proof of Lemma 5.15

For the proof we only have to do "adjunctional calculus": The equivalence (3) $\iff$ (2) follows from uniqueness of adjoints: We write associativity of μ as

$$\forall x, y \colon \mu(- \otimes y) \cdot \mu(x \otimes -) = \mu(x \otimes -) \cdot \mu(- \otimes y)$$

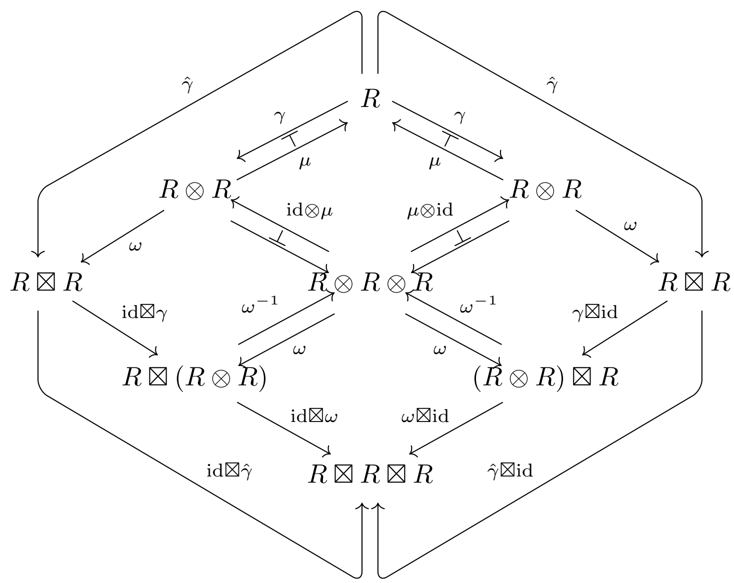
and associativity of the residuals as

$$\forall x, y \colon (x \setminus -) \cdot (- \mathbin{/} y) = (- \mathbin{/} y) \cdot (x \setminus -).$$

Since the respective left and right sides of the equalities are adjoint, and adjoints are unique, it is clear that one of the equations hold iff the other one does. The equivalence of the (co-)unit properties is analogous.

The equivalence (1) $\iff$ (3) works similar, but we have to be careful since μ and $\hat{\gamma}$ are technically not adjoint, only up to the isomorphism $\omega \colon R \otimes R \to R \boxtimes R$.

By Lemma 5.4(2) we have the following diagram of adjunctions:



The left and right diamonds come from the horizontal composition of adjunctions under the respective tensor products. The bottom diamond is easily seen to commute. So if μ is associative, then the top inner diamond commutes, and so by uniqueness the outer big diamond commutes by uniqueness of adjoints, proving $\hat{\gamma}$ coassociative. Dually, if $\hat{\gamma}$ is coassociative then μ is associative. The unit of μ is the left adjoint of $\gamma = \omega^{-1} \cdot \hat{\gamma}$.

Proof of Theorem 5.18

The correspondence on operators follows from the following proposition.

Proposition A.4. *Let R and R' be finite residuation algebras with unit.*

(1) *A lattice morphism $f\colon R \to R'$ is a pure morphism of coalgebras iff it is a residuation morphism.*

(2) *If R and R' are comonoids, then $\rho \in \mathbf{JSL}_1(R, R')$ is a relational comonoid morphism iff it is a (unital) relational residuation morphism.*

Proof. (1) First, let f be a pure coalgebra morphism. Then

$$\begin{aligned} x \setminus f(z) &= x \multimap \gamma' f(z) \\ &= x \multimap \omega^{-1}\hat{\gamma}' f(z) \\ &= x \multimap \omega^{-1}(f \boxtimes f)\omega\gamma(z) && f \text{ coalgebra morphism} \\ &= f(f^*(x) \multimap \gamma(z)) && \text{Proposition A.3(3)} \\ &= f(f^*(x) \setminus z), \end{aligned}$$

which by Lemma 5.10 shows that f is an (open) residuation morphism. Conversely, if f is an (open) residuation morphism, then for every z we compute

$$\begin{aligned}\gamma' f(z) &= \bigvee_{x'\in R'} x' \otimes x' \setminus f(z) \\ &= \bigvee_{x'\in R'} x' \otimes f(f^*(x') \setminus z) \\ &\leq \bigvee_{x'\in R'} ff^*(x') \otimes f(f^*(x') \setminus z) \\ &= (f \otimes f)(\bigvee_{x'\in R'} f^*(x') \otimes f^*(x') \setminus z) \\ &\leq (f \otimes f)(\bigvee_{x\in R} x \otimes x \setminus z) \\ &= (f \otimes f)\gamma(z),\end{aligned}$$

and (order-isomorphic) postcomposition with ω gives

$$\hat{\gamma}' f = \omega\gamma' f \leq \omega(f \otimes f)\gamma = (f \boxtimes f)\omega\gamma = (f \boxtimes f)\hat{\gamma}.$$

Conversely,

$$\begin{aligned}(f \otimes f)\gamma(z) &= \bigvee_{x\in R} f(x) \otimes f(x \setminus z) \\ &\leq \bigvee_{x\in R} f(x) \otimes f(f^* f(x) \setminus z) && \text{counit + contravariance} \\ &= \bigvee_{x\in R} f(x) \otimes f(x) \setminus f(z) && f \text{ open res. hom.} \\ &\leq \bigvee_{x'\in R'} x' \otimes x' \setminus f(z) = \gamma'(f(z))\end{aligned}$$

and postcomposition with ω again yields $\hat{\gamma}' f \geq (f \boxtimes f)\hat{\gamma}$. Hence we shown that $\hat{\gamma}' f = (f \boxtimes f)\hat{\gamma}$, i.e. f is a pure coalgebra morphism. Moreover, it is clear that the counit condition is equivalent to the unit conditions since $e \dashv \epsilon$ and $f^*(e) \dashv \epsilon' \cdot f$ and since adjoints are unique one of the equations holds iff the other one does.

(2) If $\rho\colon R \to R'$ is a relational morphism of pure coalgebras, then by Proposition A.3(3)

$$\rho(x \setminus z) = \rho(x \multimap \gamma(z)) \leq \rho(x) \multimap (\rho \otimes \rho)\gamma(z) \leq \rho(x) \multimap \gamma'(\rho(z)) = \rho(x) \setminus \rho(z).$$

Conversely, if $\rho\colon R \to R'$ is a relational morphism of residuation algebras, then

$$(\rho \otimes \rho)\gamma(z) = \bigvee_{x\in R} \rho(x) \otimes \rho(x \setminus z) \leq \bigvee_{x\in R} \rho(x) \otimes \rho(x) \setminus \rho(z) \leq \gamma'(\rho(z)).$$

For the respective counits we compute

$$\begin{aligned} & \epsilon \leq \epsilon' \cdot \rho \\ \iff \quad & \forall x\colon \epsilon(x) \leq \epsilon'(\rho(x)) \\ \iff \quad & \forall x, y\colon y \leq \epsilon(x) \Rightarrow y \leq \epsilon'(\rho(x)) \\ \iff \quad & \forall x, y\colon e(y) \leq x \Rightarrow e'(y) \leq \rho(x) && e \dashv \epsilon, e' \dashv \epsilon' \\ \iff \quad & \forall x\colon e \leq x \Rightarrow e' \leq \rho(x) && y \in \{0, e\} \text{ and } e(0) = e'(0) = 0 \\ \iff \quad & e' \leq \rho(e),\end{aligned}$$

where in the last step we set $x = e$ for the downwards implication we use monotonicity: if $e' \leq \rho(e)$ and $e \leq x$ then $e' \leq \rho(e) \leq \rho(x)$.

Details for Example 5.20

By duality, $\mathrm{Reg}\,\Sigma$ is the filtered colimit of its finite sub-coalgebras $\mathcal{P}M \hookrightarrow \mathrm{Reg}\,\Sigma$ dual to monoid quotients $\widehat{\Sigma^*} \twoheadrightarrow M$, we can apply the coalgebra structure to a language $L \in \mathrm{Reg}\,\Sigma$ in its syntactic monoid and then embed into regular languages via the preimage of its syntactic morphism $h\colon \Sigma^* \twoheadrightarrow M$:

$$\begin{array}{ccccc}
\mathrm{Cl}(\widehat{\Sigma^*}) & \xrightarrow{\mu^{-1}} & \mathrm{Cl}(\widehat{\Sigma^*} \times \widehat{\Sigma^*}) & \xrightarrow{\cong} & \mathrm{Cl}(\widehat{\Sigma^*}) + \mathrm{Cl}(\widehat{\Sigma^*}) \\
{\scriptstyle h^{-1}}\big\uparrow & & {\scriptstyle (h\times h)^{-1}}\big\uparrow & & {\scriptstyle h^{-1}+h^{-1}}\big\uparrow \\
\mathcal{P}M & \xrightarrow{\mu^{-1}} & \mathcal{P}(M \times M) & \xrightarrow{\cong} & \mathcal{P}M + \mathcal{P}M
\end{array}$$

If we denote by $[w]$ the syntactic equivalence class of w with respect to L we thus can compute

$$\begin{aligned}
\mu^{-1}(L) &= \mu^{-1}(h^{-1}(h[L])) && L \text{ recognized by } M \\
&= (h \times h)^{-1}(\mu^{-1}(h[L])) && h^{-1} \text{ coalgebra hom.} \\
&= (h \times h)^{-1}(\{(m,n) \mid mn \in h[L]\}) \\
&= (h \times h)^{-1}(\bigcup_{mn \in h[L]} \{(m,n)\}) \\
&\mapsto (h^{-1} + h^{-1})(\bigvee\nolimits_{mn \in h[L]} \{m\} \otimes \{n\}) && \mathcal{P}(M \times M) \cong \mathcal{P}M + \mathcal{P}M \\
&= \bigvee\nolimits_{mn \in h[L]} h^{-1}[\{m\}] \otimes h^{-1}[\{n\}] \\
&= \bigvee\nolimits_{h(v)h(w) \in h[L]} h^{-1}h[v] \otimes h^{-1}h[w] && h \text{ surj.} \\
&= \bigvee\nolimits_{vw \in L} [v] \otimes [w] && \text{syntactic equivalence classes} \\
&= \bigvee\nolimits_{v \in \Sigma^*} \bigvee\nolimits_{w \in v^{-1}L} [v] \otimes [w] && \text{definition of } v^{-1}L \\
&= \bigvee\nolimits_{v \in \Sigma^*} [v] \otimes (\bigvee\nolimits_{w \in v^{-1}L} [w]) && \text{fin. many equivalence classes} \\
&= \bigvee\nolimits_{v \in \Sigma^*} [v] \otimes v^{-1}L, \\
&= \bigvee\nolimits_{[v] \in \mathrm{Syn}_L} [v] \otimes [v] \setminus L.
\end{aligned}$$

In the second to last step we use that $v^{-1}L = \bigcup_{w \in v^{-1}L} [w]$, which holds by the definition of syntactic equivalence.

Proof of Proposition 5.23

(1a) We first show that the formula for $\gamma_{\setminus}(z)$ is well-defined, i.e. it does not depend on the residuation ideal containing z. First, let $\iota_K \cdot \iota \colon I \hookrightarrow K \hookrightarrow R$ be finite residuation ideals containing z. Since $I \subseteq K$ it is clear that

$$\begin{aligned}(\iota_I \otimes \iota_I)(\gamma(z)) &= (\iota_I \otimes \iota_I)(\bigvee\nolimits_{x\in I} x \otimes x \setminus z)\\ &\le (\iota_K \otimes \iota_K)(\bigvee\nolimits_{x\in K} x \otimes x \setminus z)\\ &= (\iota_K \otimes \iota_K)(\gamma(\iota(z))).\end{aligned}$$

For the other direction note that for every $q \in \mathcal{J}(K)$ we find $p \in \mathcal{J}I$ with $q \le \iota(p)$ since $q \le 1 = \iota(1) = \iota(\bigvee_{p\in\mathcal{J}I} p) = \bigvee_{p\in\mathcal{J}I} \iota(p)$ and q is join-irreducible. We so compute

$$\begin{aligned}(\iota_K \otimes \iota_K)(\gamma(\iota(z))) &= \bigvee\nolimits_{q\in\mathcal{J}K} \iota_K(q) \otimes \iota_K(q \setminus \iota(z))\\ &= \bigvee\nolimits_{p\in\mathcal{J}I} \bigvee\nolimits_{q\le\iota(p)} \iota_K(q) \otimes \iota_K(q \setminus \iota(z))\\ &= \bigvee\nolimits_{p\in\mathcal{J}I} \bigvee\nolimits_{q\le\iota(p)} \iota_K(q) \otimes \iota_I(\iota^*(q) \setminus z) && (*)\\ &= \bigvee\nolimits_{p\in\mathcal{J}I} \bigvee\nolimits_{q\le\iota(p)} \iota_I\iota^*(q) \otimes \iota_I(\iota^*(q) \setminus z) && (**)\\ &\le \bigvee\nolimits_{p\in\mathcal{J}I} \iota_I(p) \otimes \iota_I(p \setminus z) && (***)\\ &= (\iota_I \otimes \iota_I)(\gamma(z)).\end{aligned}$$

For step $(*)$ we use the for $p \in \mathcal{J}I, q \in \mathcal{J}K$ with $q \le \iota(p)$ the following holds:

$$\begin{aligned}\iota_K(q \setminus \iota(z)) &= \iota_K(q) \setminus \iota_K(\iota(z))\\ &= \iota_K(q) \setminus \iota_I(z)\\ &= \iota_I(\iota_I^*(\iota_K(q)) \setminus z) && \iota_I \text{ finite (open) residuation morphism}\\ &= \iota_I(\iota^*(\iota_K^*(\iota_K(q))) \setminus z) && \iota_I = \iota_K\iota\\ &= \iota_I(\iota^*(q) \setminus z) && \iota_K \text{ embedding.}\end{aligned}$$

For step $(**)$ we use that $q \le \iota(\iota^*(q))$ to get

$$\iota_K(q) \le \iota_K(\iota(\iota^*(q))) = \iota_I(\iota^*(q)).$$

For step $(***)$ observe that $\iota^*(q) \in \mathcal{J}(I)$ for $q \in \mathcal{J}(K)$ (indeed, $\iota^*(q) \le x \vee y$ in I implies $q \le \iota(x) \vee \iota(y)$ in K, hence $q \le \iota(x)$ or $q \le \iota(y)$, hence $\iota^*(q) \le x$ or $\iota^*(q) \le y$. In particular, each $\iota_I\iota^*(q) \otimes \iota_I(\iota^*(q) \setminus z)$ where $q \in \mathcal{J}(K)$ is equal to $\iota_I(p) \otimes \iota_I(p) \setminus z)$ for some $p \in \mathcal{J}(I)$, which proves $(***)$.

Now, if I, I' are finite residuation ideals containing z they are both contained in a finite residuation ideal $\iota\colon I \hookrightarrow K \hookleftarrow I' : \iota'$, using that R is locally finite, and we have

$$
\begin{aligned}
(\iota_I \otimes \iota_I)(\gamma(z)) &= (\iota_K \otimes \iota_K)(\iota \otimes \iota)(\gamma(z)) \\
&= (\iota_K \otimes \iota_K)(\gamma(\iota(z))) && \iota \text{ coalgebra morphism} \\
&= (\iota_K \otimes \iota_K)(\gamma(\iota'(z))) && \iota, \iota' \text{ subcoalgebras} \\
&= (\iota_{I'} \otimes \iota_{I'})(\gamma(z)) && \text{backwards.}
\end{aligned}
$$

This shows that the mapping

$$\gamma_\backslash \colon |R| \mapsto |R \otimes R|, \qquad z \mapsto (\iota_I \otimes \iota_I)(\gamma(z))$$

does not depend on the choice of I.

(1b) We show that the mapping indeed yields a $U_\wedge$-coalgebra , i.e. preserves all finite meets. Let $F \subseteq R$ be a finite subset. By local finiteness we find a residuation ideal I containing F. Now we simply use that both the comultiplication on I and $\iota_I \otimes \iota_I$ preserve finite meets:

$$\gamma_\backslash(\bigwedge_{x\in F} x) = (\iota_I \otimes \iota_I)(\gamma(\bigwedge_{x\in F} x)) = \bigwedge_{x\in F} (\iota_I \otimes \iota_I)(\gamma(x)) = \bigwedge_{x\in F} \gamma_\backslash(x).$$

(1c) The coalgebra is easily seen to be locally finite, since for every finite subset $X \subseteq R$ we find a finite residuation ideal I containing X, and the corresponding coalgebra structure on I is per definition a subcoalgebra of $(R, \gamma_\backslash)$.

(2a) Again we first show that for finite subcoalgebras A, A' of R containing both x, z we have

$$\iota_A(x \setminus z) = \iota_{A'}(x \setminus z).$$

First, let $\iota_B \cdot \iota \colon A \hookrightarrow B \hookrightarrow R$ be finite subcoalgebras. Then

$$
\begin{aligned}
\iota_A(x \setminus z) &= \iota_A(x \multimap \gamma(z)) \\
&= \iota_B(\iota(x \multimap \gamma(z))) \\
&= \iota_B(\iota(x) \multimap (\iota \otimes \iota)(\gamma(z))) && \text{embeddings preserve } \multimap \\
&= \iota_B(\iota(x) \multimap \gamma(\iota(z))) \\
&= \iota_B(\iota(x) \setminus \iota(z)).
\end{aligned}
$$

From this it follows that

$$\iota_B(x \setminus \iota(z)) = \iota_B(x \multimap (\iota \otimes \iota)(\gamma(z))) = \iota_B(\iota(\iota^*(x) \multimap \gamma(z))) = \iota_A(\iota^*(x) \setminus z). \quad \text{(A.1)}$$

In particular, for $x \setminus_\gamma z$ no matter what subalgebra A one chooses it certainly contains the subalgebra generated by z, i.e. $\iota \colon \langle z \rangle \hookrightarrow A$ and we hence obtain the canonical presentation

$$x \setminus_\gamma z = \iota_A(x \setminus z) = \iota_A(x \setminus \iota(z)) = \iota_{\langle z \rangle}(\iota^*(x) \setminus z).$$

For general finite subcoalgebras $A, A' \hookrightarrow R$ containing x, z we find an upper bound $\iota \colon A \hookrightarrow B \hookleftarrow A' : \iota'$ and compute

$$\iota_A(x \setminus z) = \iota_B(\iota(x) \setminus \iota(z)) = \iota_B(\iota'(x) \setminus \iota'(z)) = \iota_{A'}(x \setminus z).$$

(2b) The proof that the residuals preserve finite meets in the covariant component is analogous to the comultiplication.

(2c) The residuation structure is locally finite since we have the canonical representation $x \setminus_\gamma z = \iota_z(\iota_z^*(x) \setminus z)$ and $\langle z \rangle$ is finite.

(2d) It remains to verify the residuation property:

$$\begin{aligned}
y \leq x \setminus_\gamma z &\iff y \leq \iota_z(\iota_z^*(x) \setminus_z z) && \text{definition } \setminus_\gamma \\
&\iff y \leq \iota_z(\iota_z^* x \multimap \gamma z) && \text{definition } \setminus_z \\
&\iff \iota_z^* y \leq \iota_z^* x \multimap \gamma z && \iota_z^* \dashv \iota_z \\
&\iff \iota_z^* x \otimes \iota_z^* y \leq \gamma z && \iota_z^* x \otimes (-) \dashv \iota_z^* x \multimap (-) \\
&\iff \iota_z^* x \leq \gamma z \multimapinv \iota_z^* y && (-) \otimes \iota_z^* y \dashv (-) \multimapinv \iota_z^* y \\
&\iff x \leq \iota_z(\gamma z \multimapinv \iota_z^* y) && \iota_z^* \dashv \iota_z \\
&\iff x \leq \iota_z(z /_z \iota_z^* y) && \text{definition } /_z \\
&\iff x \leq z /_\gamma y && \text{definition } /_\gamma
\end{aligned}$$

(3) The translations are inverse since they are liftings of the translations between the operators on the finite substructures: To show that $\gamma_{\setminus_\gamma} = \gamma$ note that for $z \in R$ the subcoalgebra $\iota_z \colon \langle z \rangle \hookrightarrow R$ generated by z is a residuation ideal of $\setminus_\gamma$:

$$x \setminus_\gamma z = \iota_z(\iota_z^*(x) \multimap \gamma(z)) \in \langle z \rangle.$$

We can therefore choose it as a residuation ideal containing z in the definition of $\gamma_{\setminus_\gamma}$ to get

$$\gamma_{\setminus_\gamma}(z) = (\iota_z \otimes \iota_z)(\gamma_{\setminus_\gamma}(z)) = (\iota_z \otimes \iota_z)(\gamma(z)) = \gamma(z).$$

An analogous argument proves $\setminus_{\gamma_\setminus} = \setminus$. □

Proof of Lemma 5.24

(1) If $\iota_I \colon I \hookrightarrow R$ is a finite residuation ideal then by definition its comultiplication makes I a subcoalgebra of (R, γ). On the other hand, if $\iota_A \colon A \hookrightarrow R$ is a finite subcoalgebra then ι_A by definition is a residuation embedding since $x \setminus z = \iota_A(x \setminus z)$. To show that A is a residuation ideal, let $x \in R$ and $z \in A$. There exists a finite subcoalgebra B containing x, z with $\iota \colon A \hookrightarrow B$. By (A.1) we then have

$$x \setminus z = \iota_B(x \setminus z) = \iota_A(\iota^*(x) \setminus z) \in A.$$

(2) First, let γ be the coassociative comultiplication and let $x, y, z \in R$. By local finiteness they are contained in a finite coassociative subcoalgebra $\iota_A \colon A \hookrightarrow R$. Then by Lemma 5.24(1) ι_A is an associative finite residuation ideal of R, whence

$$x \setminus (z \mathbin{/} y) = \iota_A(x \setminus (z \mathbin{/} y)) = \iota_A((x \setminus z) \mathbin{/} y) = (x \setminus z) \mathbin{/} y.$$

The other direction works analogous: if the residuals are associative and $z \in R$, then it is contained in a finite associative residuation ideal $\iota_I \colon I \hookrightarrow R$. Whence I

is a finite associative subcoalgebra of R and whence

$$\begin{aligned}(\hat{\gamma} \boxtimes \mathrm{id})(\hat{\gamma}(\iota_I(z))) &= (\iota_I \boxtimes \iota_I \boxtimes \iota_I)((\hat{\gamma} \boxtimes \mathrm{id})(\hat{\gamma}(z))) \\ &= (\iota_I \boxtimes \iota_I \boxtimes \iota_I)((\mathrm{id} \boxtimes \hat{\gamma})(\hat{\gamma}(z))) \\ &= (\mathrm{id} \boxtimes \hat{\gamma})(\hat{\gamma}(\iota_I(z)))\end{aligned}$$

(3) Let $\epsilon\colon R \to 2$ be a counit for the comultiplication $\hat{\gamma} = \omega \cdot \gamma\colon U_\wedge R \to U_\wedge(R \otimes R) \cong U_\wedge R \boxtimes U_\wedge R$ with right adjoint $e \in R$. Note that $\epsilon \cdot \iota_A$ is a counit for every subcoalgebra $\iota_A\colon A \hookrightarrow R$: Let $z \in A$, then

$$\iota_A(((\epsilon \cdot \iota_A) \boxtimes \mathrm{id}_A)(\hat{\gamma}(z))) = (\epsilon \boxtimes \mathrm{id})(\iota_A \boxtimes \iota_A)(\hat{\gamma}(z)) = (\epsilon \boxtimes \mathrm{id})(\hat{\gamma}(\iota_A(z))) = \iota_A(z).$$

Whence its left adjoint $e_A = \iota_A^*(e) \in A$ is a unit for the finite residuation ideal A. Thus for all $z \in R$ we have $e \setminus z = \iota_z(\iota_z^*(e) \setminus z) = \iota_z(e_z \setminus z) = \iota_z(z) = z$, so e is a unit for the residuals. Conversely, let $e \in R$ be a unit for the residuals with right adjoint $\epsilon\colon R \to 2$. For every residuation ideal $\iota_I\colon I \hookrightarrow R$ the element $\iota_I^*(e) \in I$ is the unit of I: The embedding trivially is an open residuation morphism and whence $\iota_I(\iota_I^*(e) \setminus z) = e \setminus \iota_I(z) = \iota_I(z)$ for every $z \in I$. So the subcoalgebra structure I has the counit $\epsilon \cdot \iota_I$, and whence for every $z \in R$ contained we have

$$\begin{aligned}(\epsilon \boxtimes \mathrm{id})(\hat{\gamma}(z)) &= (\epsilon \boxtimes \mathrm{id})(\hat{\gamma}(\iota_I(z))) \\ &= (\epsilon \boxtimes \mathrm{id})((\iota_I \boxtimes \iota_I)(\hat{\gamma}(z))) \\ &= \iota_I((\epsilon\iota_I \boxtimes \mathrm{id})(\hat{\gamma}(z))) \\ &= \iota_I(z) = z.\end{aligned}$$

This shows that ϵ is a counit for $\hat{\gamma}$. By definition ϵ is coprime iff e is a prime element of R.

(4) If the comultiplication is pure, associative and has a counit, then this holds for every finite subcoalgebra. Every finite residuation ideal I of R is a finite pure subcoalgebra, which therefore is a derivation algebra. On the other hand, if every finite residuation ideal is a derivation algebra, then we only have to show that the comultiplication preserves finite joins, since it clearly is coassociative and has a counit. But this is clear since the join of finitely many elements is taken in a finite subcoalgebra, which thus is a derviation algebra and hence preserves finite joins. □

Proof of Lemma A.5

Gehrke [13, Proposition 3.15] has identified a condition on general residuation algebras which is related to the previous lemma in the following way.

Lemma A.5. *If C is a locally finite residuation algebra where every residuation finite ideal is pure then it is join-preserving at primes:*

$$\forall F \in \mathbf{DL}(C,2)\colon \forall(a \in F), \forall(b, c \in C)\colon \exists a' \in F\colon a \setminus (b \vee c) \leq (a' \setminus b) \vee (a' \setminus c).$$

Proof. If F is a prime filter on C and $a \in F$, then we get by local finiteness for all $b, c \in C$ a finite *pure* residuation ideal $\iota\colon I \hookrightarrow C$ containing a, b, c. In this residuation ideal we have $a = \bigvee K$ for join-irreducibles $K \subseteq \mathcal{J}I$. Since F is prime some $a' \in K$ lies in F and satisfies

$$\begin{aligned} a \setminus (b \vee c) &= \iota(a \setminus (b \vee c)) \\ &= \iota((\bigvee K) \setminus (b \vee c)) \\ &\leq \iota(a' \setminus (b \vee c)) \\ &= \iota(a' \setminus b \vee a' \setminus c) && I \text{ is pure} \\ &= \iota(a' \setminus b) \vee \iota(a' \setminus c) \\ &= a' \setminus b \vee a' \setminus c. \end{aligned}$$

This shows that the residuals are join-preserving at primes. □

Proof of Theorem 5.26

The bijection on objects is given by Lemma 5.24, so we just have to prove it on morphisms, which is the content of the following

Proposition A.6. *Let R, R' be locally finite residuation algebras with units.*

(1) *A lattice morphism $f \in \mathbf{DL}(R, R')$ is a residuation morphism iff it is a counital morphism of $U_\wedge$-coalgebras.*

(2) *If R, R' are comonoids a finite join-preserving function $\rho \in \mathbf{JSL}(R, R')$ is a relational residuation morphism iff it is a relational comonoid morphism.*

Proof. (1) First, let $f\colon R \to R'$ be a residuation morphism. For $z \in R$ we choose a finite residuation ideal $I' \hookrightarrow R'$ containing $f(z)$. Since f is a residuation morphism we have by the (Back) condition that for every $y \in I'$ there exists some $x_{y,z}$ with $y \leq f(x_{y,z})$ and $y \setminus f(z) = f(x_{y,z} \setminus z)$. We now choose a finite ideal $I \hookrightarrow R$ containing z and all $x_{y,z}$ for $y \in I'$. We therefore have

$$\begin{aligned} \gamma_\setminus(f(z)) &= \bigvee\nolimits_{y \in I'} y \otimes y \setminus f(z) \\ &\leq \bigvee\nolimits_{y \in I', y \leq f(x_{y,z})} f(x_{y,z}) \otimes f(x_{y,z} \setminus z) \\ &= (f \otimes f)(\bigvee\nolimits_{y \in I', y \leq f(x_{y,z})} x_{y,z} \otimes x_{y,z} \setminus z) \\ &\leq (f \otimes f)(\bigvee\nolimits_{x \in I} x \otimes x \setminus z) \\ &= (f \otimes f)(\gamma_\setminus(z)). \end{aligned}$$

For the reverse direction choose a finite residuation ideal $J \hookrightarrow R'$ containing $f(z)$ and all $f(x), x \in I$, and use the (Forth) condition:

$$\begin{aligned}(f \otimes f)(\gamma_\backslash(z)) &= \bigvee_{x \in I} f(x) \otimes f(x \setminus z) \\ &\leq \bigvee_{x \in I} f(x) \otimes f(x) \setminus f(z) \\ &\leq \bigvee_{y \in J} y \otimes y \setminus f(z) \\ &= \gamma_\backslash(f(z)),\end{aligned}$$

This proves that f is a morphism of $U_\wedge$-coalgebras.

Conversely, let $f\colon R \to R'$ be a morphism of $U_\wedge$-coalgebras. For every $z \in R$ the morphism f restricts to the finite subcoalgebras generated by $z, f(z)$ as $f_z\colon \langle z \rangle \to \langle f(z) \rangle$. If we denote the respective inclusions by $\iota_z\colon \langle z \rangle \hookrightarrow R$ and $\iota_{f(z)}\colon \langle f(z) \rangle \hookrightarrow R'$ then this is equivalent to saying that $f \cdot \iota_z = \iota_{f(z)} \cdot f_z$. From the unit of ι_z we thus get $f \leq f \cdot \iota_z \cdot \iota_z^* = \iota_{f(z)} \cdot f_z \cdot \iota_z^*$, and so by adjunction

$$\iota_{f(z)}^* \cdot f \leq f_z \cdot \iota_z^*. \tag{A.2}$$

This gives the (Forth) condition by

$$\begin{aligned}f(x \setminus_\gamma z) &= f(\iota_z(\iota_z^*(x) \multimap \gamma(z))) && \text{def. } \setminus_\gamma \\ &= \iota_{f(z)}(f_z(\iota_z^*(x) \multimap \gamma(z))) && f_z \text{ restriction} \\ &\leq \iota_{f(z)}(f_z(\iota_z^*(x)) \multimap (f_z \otimes f_z)\gamma(z))) && \multimap \text{ Proposition A.3(3)} \\ &= \iota_{(z)}(f_z(\iota_z^*(x)) \multimap \gamma(f_z(z))) && f_z \text{ coalgbra morphism} \\ &\leq \iota_{f(z)}(\iota_{f(z)}^*(f(x)) \multimap \gamma(f_z(z))) && \text{(A.2) + contravariance} \\ &\leq \iota_{f(z)}(\iota_{f(z)}^*(f(x)) \multimap \gamma(f(z))) && f_z(z) = f(z) \\ &= f(x) \setminus_\gamma f(z).\end{aligned}$$

To verify the (Back) condition, let $y \in R', z \in R$ and put

$$x_{y,z} = \iota_z(f_z^*(\iota_{f(z)}^*(y))).$$

Then

$$y \leq \iota_{f(z)}\iota_{f(z)}^*(y) \leq \iota_{f(z)}(f_z(f_z^*(\iota_{f(z)}^*(y)))) = f(\iota_z(f_z^*(\iota_{f(z)}^*(y)))) = f(x_{y,z})$$

and

$$\begin{aligned}y \setminus_\gamma f(z) &= \iota_{f(z)}(\iota_{f(z)}^*(y) \multimap \gamma(f_z(z))) && \text{def. } \setminus_\gamma \\ &= \iota_{f(z)}(\iota_{f(z)}^*(y) \multimap (f_z \otimes f_z)(\gamma(z))) && f_z \text{ coalgebra morphism} \\ &= \iota_{f(z)}(f_z(f_z^*(\iota_{f(z)}^*(y)) \multimap \gamma(z))) && \text{Proposition A.3(3)} \\ &= f(\iota_z(f_z^*(\iota_{f(z)}^*) \multimap \gamma(z))) && \iota_{f(z)} \cdot f_z = f \cdot \iota_z \\ &= f(\iota_z(\iota_z^*(\iota_z(f_z^*(\iota_{f(z)}^*(y)))) \multimap \gamma(z))) && \iota_z^* \iota_z = \text{id} \\ &= f_z(x_{y,z} \setminus_\gamma z) && \text{def. } x_{y,z}.\end{aligned}$$

For the (co-)unitality conditions we split the pointwise equality $\forall x\colon \epsilon'(f(x)) = \epsilon(x)$ into $\epsilon'(f(x)) \le \epsilon(x)$ and $\epsilon(x) \le \epsilon'(f(x))$. These are equivalent to $e' \le f(x) \Rightarrow e \le x$ and $e \le x \Rightarrow e' \le f(x)$, respectively, combining to $\forall x\colon e' \le f(x) \Leftrightarrow e \le x$.

(2) Now let R, R' be comonoids and let $f \in \mathbf{JSL}(R, R')$ be a relational comonoid morphism. For $x, z \in R$ we choose a finite subcomonoid $\iota\colon I \hookrightarrow R$ that contains x, z and a finite subcomonoid $\iota'\colon I' \hookrightarrow R'$ (with structure γ') containing $\rho[I]$. Then ρ restricts to a relational morphism $\rho\colon I \to I'$ of finite comonoids. By Theorem 5.18 it is a relational morphism of the equivalent finite residuation algebra, so it satisfies $\rho(a \setminus_{\gamma'} b) \le \rho(a) \setminus_{\gamma'} \rho(b)$ for all $a, b \in I$. We therefore get

$$\rho(x \setminus_{\gamma} z) = \rho(\iota(x \setminus_{\gamma'} z)) = \iota'(\rho(x \setminus_{\gamma'} z)) \le \iota'(\rho(x) \setminus_{\gamma'} \rho(z)) = \rho(x) \setminus_{\gamma} (z).$$

which proves that f is a relational residuation morphism. To verify $e' \le \rho(e)$ we again choose finite subcoalgebras A, A' with $e \in A$ and $\rho[A] \cup \{e'\} \subseteq A'$. Since ρ is counital it satisfies $\epsilon \le \epsilon' \cdot \rho$ and therefore also $\epsilon \cdot \iota_A \le \epsilon' \cdot \rho \cdot \iota_A = \epsilon' \cdot \iota_{A'} \cdot \rho$. As $\epsilon \cdot \iota_A$ and $\epsilon' \cdot \iota_{A'}$ are the counits for A and A', respectively, its restriction $\rho\colon A \to A'$ is thus also counital and whence satisfies $\iota_A^*(e) \le \rho(\iota_{A'}^*(e'))$ for the corresponding units of the residuals on A, A'. But $e \in A$ and $e' \in A'$, so this equation simplifies to the desired $e \le \rho(e')$.

Conversely, if f is a relational residuation morphism choose for $z \in R$ a finite residuation ideal $\iota\colon I \hookrightarrow R$ containing z and a finite residuation ideal $\iota'\colon I' \hookrightarrow R$ containing $\rho[I]$. Then we have

$$\begin{aligned}(\rho \otimes \rho)(\gamma(z)) &= (\rho \otimes \rho)(\gamma(\iota(z))) \\ &= (\rho \otimes \rho)(\iota \otimes \iota)(\gamma(z)) \\ &= (\iota' \otimes \iota')(\rho \otimes \rho)(\gamma(z)) \\ &\le (\iota' \otimes \iota')\gamma(\rho(z)) \\ &= \gamma(\rho(z)),\end{aligned}$$

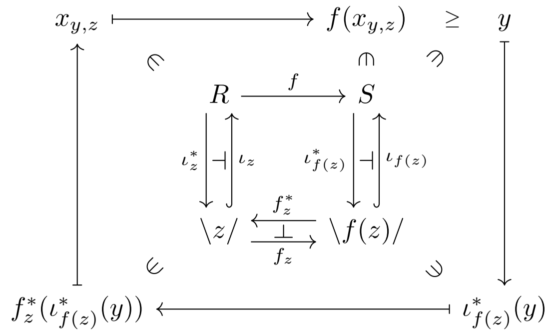
so ρ is a relational residuation morphism. To show that $\epsilon \le \epsilon' \cdot \rho$, take $z \in R$ with ideals I, I' chosen as before. Recall that $\iota^*(e)$ is a unit of the residuation ideal I and $\epsilon \cdot \iota$ is a counit for the corresponding subcoalgebra. Since ρ is unital it satisfies

$$e' \le \rho(e) \le \rho(\iota(\iota^*(e))) = \iota'(\rho(\iota^*(e)))$$

which is equivalent to $\iota'^*(e') \le \rho(\iota^*(e))$. Since $\rho\colon I \to I'$ is a relational residuation morphism it is relational morphism of the coalgebra structures on I, I' and we thus get

$$\epsilon(z) = \epsilon(\iota(z)) \le \epsilon'(\iota'(\rho(z))) = \epsilon'(\rho(\iota(z))) = \epsilon'(\rho(z)). \qquad \square$$

Remark A.7. We note that the proof of Proposition A.6 gives an alternative formulation of the (Back) condition for locally finite residuation algebras as $y \setminus f(z) = f((\iota_z \cdot f_z^* \cdot \iota_{f(z)}^*)(y) \setminus z)$. Here one "chooses locally" the existentially quantified $x_{y,z}$ via the local left adjoint f_z^*.



Compare this with open residuation morphisms, where the existence of a global left adjoint f^* allows one to choose $x_{y,z} = f^*(y)$ *independently of* z.

Proof of Theorem 5.27

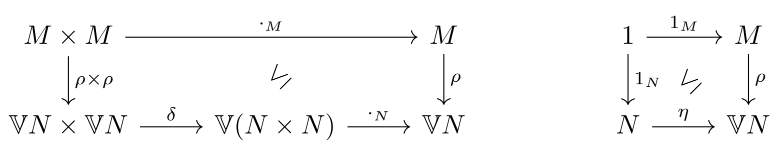
To extend Theorem 5.26 by (extended) Stone duality we need the following characterization. We write $\mathrm{Ind}(\mathbf{C})$ and $\mathrm{Pro}(\mathbf{C})$ for the free completion of a category $\mathbf{C}$ under directed colimits and codirected limits, respectively.

Lemma A.8. *All pure boolean comonoids are locally finite.*

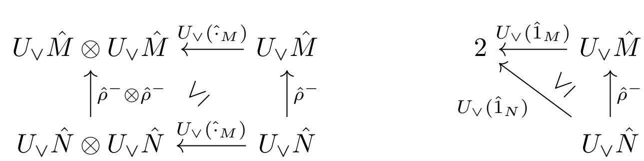
Proof. Every Stone monoid is profinite (see e.g. [19, Chapter VI, Example 2.9]), i.e. one has $\mathrm{Pro}(\mathbf{Mon}_\mathrm{f}) \simeq \mathbf{ProfMon} \simeq \mathbf{StoneMon}$, the category of Stone-topological monoids and continuous monoid morphisms. Under Stone duality this statement dualizes to $\mathrm{Ind}(\mathbf{Comon}_\mathrm{f}) \simeq \mathbf{Comon}$, which tells us that all boolean comonoids are locally finite. □

(1) All Stone monoids are profinite, so by Stone duality the category of profinite monoids is dual to the category of boolean comonoids. We have just seen that all boolean comonoids are locally finite, and locally finite pure boolean comonoids are isomorphic, as a category, to boolean derivation algebras by Theorem 5.26(2).

(2) A Stone relational morphism from M to N is precisely a Stone relation $\rho\colon M \to \mathbb{V}N$ such that the following diagrams laxly commute.



Recall that $\mathbb{V} \cong \hat{F}_\vee \hat{U}_\vee$ for $U_\vee\colon \mathbf{BA} \to \mathbf{JSL}$, so under extended duality this dualizes precisely to a relational morphism of pure boolean comonoids:



□

Together with Theorem 5.26(3) this yields the desired duality.