

On the specification of operations on the rational behaviour of systems

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Structural operational semantics can be studied at the general level of distributive laws of syntax over behaviour. This yields specification formats for well-behaved algebraic operations on final coalgebras, which are a domain for the behaviour of all systems of a given type functor. We introduce a format for specification of algebraic operations that restrict to the *rational fixpoint* of a functor, which captures the behaviour of *finite* systems. In other words, we show that rational behaviour is closed under operations specified in our format. As applications we consider operations on regular languages, regular processes and finite weighted transition systems.

1 Introduction

Structural operational semantics (SOS) is a popular and widely used framework for defining operational semantics by means of transition system specifications. Syntactic restrictions on the format of these specifications give rise to algebraic properties of operations on system behaviour [2], e. g., GSOS rules [9] ensure that bisimilarity is a congruence.

The key insight to give a uniform mathematical treatment of various flavours of SOS is that the theory of *coalgebras* provides a common framework for the study of state-based systems and their behaviour. This includes labelled transition systems but also stream automata, (non-)deterministic automata, weighted transition systems and many more. The type of a coalgebra is expressed by an endofunctor F , and a canonical domain for system behaviour is provided by the final F -coalgebra.

Turi and Plotkin [25] show in their seminal paper that the interplay between syntax and behaviour given by transition system specifications can be generalized by *distributive laws* of a functor Σ , representing the syntax, over a functor F , representing the behaviour. They formulate and prove that bisimilarity is a congruence at this level of generality. The final F -coalgebra here plays an important rôle as the denotational model of a transition system specification. In particular, a distributive law induces a canonical Σ -algebra structure on the final coalgebra for F .

But the final F -coalgebra is the domain of the behaviour of *all* F -coalgebras, and often it is interesting to study the behaviour of only *finite-state* systems, such as finite automata or regular processes. In fact, finite-state systems have nice decidability properties and are amenable to automated verification techniques. The *rational fixpoint* of a set functor F is the subcoalgebra of the final coalgebra given by the behaviours of all finite coalgebras [4, 20]. For example, regular languages, rational streams [23], rational formal power series [12] and regular trees for a signature [11] form rational fixpoints of appropriate functors F .

In this paper we investigate *bipointed specifications*, a restricted type of distributive laws which induces operations on the rational fixpoint of a functor F as a restriction of the same operations on the

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final coalgebra. As a result we show that regular system behaviour is closed under operations induced by bipointed specifications. So this yields an easy syntactic criterion to check that regular behaviour is closed under certain algebraic operations. Applications include operations on regular languages and finite automata, such as the well-known shuffle operator, operations on finite weighted transition systems and regular processes.

There is a large body of work on SOS formats and distributive laws (see [17] for a good overview). Bipointed specifications appear (without a name) as an intermediate format between abstract toy SOS [15] and the abstract operational rules of [25]. However, we are not aware of any work on formats for finite coalgebras. The only exception is the work on labelled transition systems by Aceto [1] (see also [2]). When instantiated on coalgebras corresponding to labelled transition systems, bipointed specifications coincide with specifications in the *simple GSOS* format of loc. cit. on finite signatures. Our contribution can thus be seen as a generalization of the simple GSOS format to the realm of distributive laws. In [1, 2] there is also an extension to countable signatures with certain finite dependencies among the operators, and it is proved that the labelled transition system induced by a simple GSOS specification is regular, i. e., for each closed process term P the ensuing transition system defining the operational semantics of P has finitely many states (see [2, Theorem 5.28]). In future work we shall incorporate such a result in our theory.

The outline of this paper is as follows. In the next section we introduce the necessary preliminaries. Then in Section 3 we present our specification format. This induces an algebra on the rational fixpoint, as shown in Section 4. We proceed in Section 5 with several applications of the theory, and we finish in Section 6 with conclusions and suggestions for future work.

2 Preliminaries

We assume that the reader is familiar with basic notions of category theory. With \mathbf{Set} we denote the category of sets and functions. In any category we write products and coproducts with their projections and injections, respectively as $A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$ and $C \xrightarrow{\text{inl}} C + D \xleftarrow{\text{inr}} D$. The corresponding unique induced morphisms are denoted $\langle a, b \rangle : E \rightarrow A \times B$ and $[c, d] : C + D \rightarrow E$.

2.1 Algebras and coalgebras

Let \mathcal{A} be a category and $F : \mathcal{A} \rightarrow \mathcal{A}$ a functor. An F -*algebra* is a pair (A, α) where A is an object of \mathcal{A} called the *carrier* and $\alpha : FA \rightarrow A$ is a morphism called the *structure* of the algebra. Given algebras (A, α) and (B, β) , an *algebra homomorphism* is a map $f : A \rightarrow B$ such that $f \circ \alpha = Ff \circ \beta$. A *signature* is a set Σ of operation symbols with prescribed arity $|\sigma| \in \mathbb{N}$ for each $\sigma \in \Sigma$. This can equivalently be represented as a *polynomial* functor

$$\Sigma X = \prod_{\sigma \in \Sigma} X^{|\sigma|}.$$

(We shall abuse notation and denote by Σ both a signature and its corresponding polynomial functor.) For example, a signature Σ_0 on \mathbf{Set} consisting of a binary operation symbol b and a constant symbol c corresponds to the functor $\Sigma_0 X = X \times X + 1$. A Σ_0 -algebra then is a set A together with an actual binary operation $b_A : A \times A \rightarrow A$ and a constant $c_A \in A$, and algebra homomorphisms are precisely the maps between algebras preserving the binary operation and the constant.

Example 2.1. A *join-semilattice* is a set S with a binary operator $\vee : S \times S \rightarrow S$ called the *join*, and an element $\perp \in S$ (or $\perp : 1 \rightarrow S$) called *bottom*; equivalently, it is an algebra $[\vee, 0] : S \times S + 1 \rightarrow S$. The join

is associative, commutative and idempotent, and the bottom is the identity element with respect to the join. With \mathbf{Jsl} we denote the category of join-semilattices and homomorphisms between them.

An F -coalgebra is a pair (S, f) such that S is an object of \mathcal{A} , called the *carrier*, and $f : S \rightarrow FS$ is an arrow, called the *transition structure* or *dynamics*. For coalgebras (S, f_S) and (T, f_T) , a *coalgebra homomorphism* is a morphism $h : S \rightarrow T$ such that $f_T \circ h = Fh \circ f_S$. If $\mathcal{A} = \mathbf{Set}$ and (S, f_S) and (T, f_T) are coalgebras, then a *bisimulation* is a relation $R \subseteq S \times T$ such that R carries a coalgebra structure f_R and the projection maps $\pi_0 : R \rightarrow S$ and $\pi_1 : R \rightarrow T$ are coalgebra homomorphisms from (R, f_R) to (S, f_S) and (T, f_T) , respectively. We denote by

$$\mathbf{Coalg}(F)$$

the category of F -coalgebras and their homomorphisms. Of special interest are *final coalgebras*, i. e., final objects of categories $\mathbf{Coalg}(F)$, which exist under mild conditions on F . Thus, if a category $\mathbf{Coalg}(F)$ has a final coalgebra $(\nu F, t)$, then there exists, for each F -coalgebra (S, f) a unique coalgebra homomorphism $f^\dagger : S \rightarrow \nu F$. A final coalgebra is determined uniquely up to isomorphism. Moreover, by the famous Lambek Lemma [18], the transition structure $t : \nu F \rightarrow F(\nu F)$ is an isomorphism. The final coalgebra can be thought of as a canonical domain of behaviour of the type of systems corresponding to the functor F . We consider several examples.

- Example 2.2.** (1) Coalgebras for the functor $FX = \mathbb{R} \times X$ on \mathbf{Set} , where \mathbb{R} is the set of real numbers, are often called *stream systems* over the reals. The carrier of the final F -coalgebra is the set $\mathbb{R}^\omega = \{\sigma \mid \sigma : \mathbb{N} \rightarrow \mathbb{R}\}$ of all streams (infinite sequences) of elements of \mathbb{R} . The transition structure $\langle o, t \rangle : \mathbb{R}^\omega \rightarrow \mathbb{R} \times \mathbb{R}^\omega$ is defined as $o(\sigma) = \sigma(0)$ and $t(\sigma)(n) = \sigma(n+1)$.
- (2) Deterministic automata with input alphabet A are coalgebras for the functor $FX = 2 \times X^A$, where $2 = \{0, 1\}$. Indeed, to give a coalgebra $f : S \rightarrow 2 \times S^A$ precisely corresponds to giving a set S of states with a map $o : S \rightarrow 2$ (indicating final states) and a map $t : S \rightarrow S^A$, where $t(s)(a)$ is the successor of state s under input a . The final coalgebra is carried by the set of all formal languages $\mathcal{P}(A^*)$ with its coalgebra structure given by $o : \mathcal{P}(A^*) \rightarrow 2$ with $o(L) = 1$ iff L contains the empty word and $t : \mathcal{P}(A^*) \rightarrow \mathcal{P}(A^*)^A$ given by the language derivative $t(L)(a) = \{w \mid aw \in L\}$. For a given automaton (S, f) the unique coalgebra homomorphism maps a state to the language it accepts.
- (3) Labelled transition systems (LTS) with actions from the set A are coalgebras for the functor $FX = \mathcal{P}_f(A \times X)$. Indeed, a coalgebra $f : X \rightarrow \mathcal{P}_f(A \times X)$ corresponds precisely to giving a set X of states and a transition relation $R \subseteq X \times A \times X$ that is *finitely branching*, i. e., for every $x \in X$ there are only finitely many $a \in A$ and $x' \in X$ with $(x, a, x') \in R$. The final coalgebra for F exists and can be thought of as consisting of processes modulo strong bisimilarity of Milner [21]. More precisely, it follows from [4, Proposition 5.16] (cf. also Barr [7]) that the final coalgebra is the coproduct of all countable F -coalgebras modulo the greatest bisimulation.¹
- (4) A very similar example are non-deterministic automata with a finite input alphabet A . They are coalgebras for $FX = 2 \times (\mathcal{P}_f X)^A$. Here the final coalgebra consists of all behaviours modulo bisimilarity of non-deterministic automata; more precisely, νF is the coproduct of all countable F -coalgebras modulo the largest bisimulation as in the previous point. A (necessarily) isomorphic description of νF follows from the description of the final coalgebra for \mathcal{P}_f given by Worrell [26]; (see also [10]): the elements of νF are finitely branching *strongly extensional* trees with edges labelled in A and nodes labelled in 2 . Due to lack of space we omit recalling the definition of a strongly extensional tree and refer the reader to [26, 10] instead.

¹This can be thought of as the coproduct of *all* coalgebras modulo the greatest bisimulation; but this coproduct is a proper class, whence the restriction to countable coalgebras.

- (5) Weighted transition systems (WTS) are labelled transition systems where transitions have weights (modelling multiplicities, costs, probabilities, etc.). We consider WTS's where the weights are elements of a commutative monoid $\mathbb{M} = \langle M, +, 0 \rangle$. In order to define them coalgebraically as done in [16], we first consider the **Set** endofunctor $\mathcal{F}_{\mathbb{M}}$, which acts on a set X and a function $f : X \rightarrow Y$ as

$$\mathcal{F}_{\mathbb{M}}(X) = \{\phi : X \rightarrow M \mid \phi \text{ has finite support}\} \quad \mathcal{F}_{\mathbb{M}}f(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x),$$

where a function $\phi : X \rightarrow M$ has finite support if $\phi(x) \neq 0$ for finitely many $x \in X$. A *weighted transition system* is a coalgebra for the functor $FX = (\mathcal{F}_{\mathbb{M}}X)^A$ for a set of labels A . The final F -coalgebra exists for any monoid \mathbb{M} . Similarly as before, it is the coproduct of all countable F -coalgebras modulo weighted bisimilarity of [16].

- (6) Let $F = \Sigma$ be a polynomial functor on **Set**. The final coalgebra νF is carried by the set of all (finite and infinite) Σ -trees, i. e., rooted and ordered trees labelled in the signature Σ so that inner nodes with n children are labelled by n -ary operation symbols and leaves are labelled by constant symbols. The coalgebra structure of νF is given by the inverse of tree-tupling.

2.2 Locally finitely presentable coalgebras

We are interested in algebraic operations on rational behaviour, i. e., behaviour of *finite* coalgebras (S, f) for a functor F . Anticipating future applications in different categories than **Set**, we present our results for endofunctors on general categories \mathcal{A} in which it makes sense to talk about “finite” objects and the ensuing rational behaviour of “finite” coalgebras. So we work with *locally finitely presentable* categories of Gabriel and Ulmer [13] (see also Adámek and Rosický [5]), and we now briefly recall the basics.

A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called *finitary* if \mathcal{A} has and F preserves filtered colimits. An object X of a category \mathcal{A} is called *finitely presentable* if its hom-functor $\mathcal{A}(X, -)$ is finitary. A category \mathcal{A} is *locally finitely presentable* (lfp) if (a) it is cocomplete, and (b) it has a set of finitely presentable objects such that every object of \mathcal{A} is a filtered colimit of objects from that set.

- Example 2.3.** (1) The category **Set** and the categories of posets and graphs and their morphisms are lfp with finite sets, posets and graphs, respectively, as finitely presentable objects.
- (2) Finitary varieties are categories of algebras for a finitary signature satisfying a set of equations (e. g., groups, monoids, join-semilattices etc.). Such categories are lfp with the finitely presentable objects given by those algebras which can be presented by finitely many generators and relations.
- (3) As a special case consider *locally finite* varieties, which are varieties where the free algebras on finitely many generators are finite (e. g., Jsl, distributive lattices or Boolean algebras). Here the finitely presentable objects are precisely the finite algebras.
- (4) Another special case of point (2) are the categories $\mathbf{Vec}_{\mathbb{F}}$ of vector spaces over a field \mathbb{F} , where the finitely presentable objects are precisely the finite dimensional vector spaces.

Remark 2.4. On the category **Set**, a finitary functor is determined by its behaviour on finite sets. More precisely, a functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ is finitary iff it is *bounded* (see, e. g., Adámek and Trnková [6]), i. e., for every set X and every element $x \in FX$, there is a finite subset $i : Y \hookrightarrow X$ such that $x \in Fi[FY] \subseteq FX$.

Example 2.5. We list some examples of finitary functors.

- (1) The *finite* powerset functor \mathcal{P}_f is finitary, whereas the ordinary powerset functor \mathcal{P} is not.
- (2) The functor $FX = X^A$ is finitary if and only if A is a finite set.

- (3) More generally, the class of finitary set functors contains all constant functors and the identity functor, and it is closed under finite products, arbitrary coproducts and composition. Thus, a polynomial functor Σ is finitary iff every operation symbol of the corresponding signature has finite arity (but there may be infinitely many operations).
- (4) The functors $\mathcal{F}_{\mathbb{M}}$ are finitary for every monoid \mathbb{M} .
- (5) The functor $FX = \mathbb{R} \times X$ is finitary both on **Set** and on $\mathbf{Vec}_{\mathbb{R}}$.

Assumption 2.6. Throughout the rest of this paper we assume, unless stated otherwise, that \mathcal{A} is a locally finitely presentable category and $F : \mathcal{A} \rightarrow \mathcal{A}$ is a finitary functor. So F has a final coalgebra $t : \nu F \rightarrow F(\nu F)$ (see Makkai and Paré [19]).

For a functor F on an lfp category \mathcal{A} the notion of a “finite” coalgebra is captured by a coalgebra having a finitely presentable carrier. We denote by

$$\mathbf{Coalg}_f(F)$$

the full subcategory of F -coalgebras $f : S \rightarrow FS$ with S finitely presentable. In order to talk about the behaviour of finite coalgebras in this setting we would like to consider a coalgebra that is final among all coalgebras in $\mathbf{Coalg}_f(F)$. However, $\mathbf{Coalg}_f(F)$ does not have a final object in general, and so we consider the larger category of locally finitely presentable coalgebras in which the desired final object exists.

An F -coalgebra (S, f) is called *locally finitely presentable* if the canonical forgetful functor

$$\mathbf{Coalg}_f(F)/(S, f) \rightarrow \mathcal{A}/S$$

is cofinal [10, 20]. In lieu of going into the details of this definition we recall the following result, which gives a structure theoretic characterisation of locally finitely presentable coalgebras that we will use later:

Theorem 2.7 ([20]). *A coalgebra is locally finitely presentable iff it is a filtered colimit of a diagram of coalgebras from $\mathbf{Coalg}_f(F)$, i. e., a colimit of a diagram of the form $\mathcal{D} \rightarrow \mathbf{Coalg}_f(F) \hookrightarrow \mathbf{Coalg}(F)$.*

Example 2.8. We recall from [20, 10] more concrete descriptions of locally finitely presentable coalgebras in some categories of interest.

- (1) A coalgebra for a functor on **Set** is locally finitely presentable iff it is *locally finite*, i. e., every finite subset of its carrier is contained in a finite subcoalgebra.
- (2) Similarly, for a functor on a locally finite variety a coalgebra is locally finitely presentable iff every finite subalgebra of its carrier is contained in a finite subcoalgebra.
- (3) A coalgebra (S, f) for a functor on $\mathbf{Vec}_{\mathbb{F}}$ is locally finitely presentable if and only if every finite dimensional subspace of its carrier S is contained in a subcoalgebra (S', f') of (S, f) whose carrier S' is finite dimensional.

2.3 The rational fixpoint

The final F -coalgebra is thought to capture the behaviour of all systems of type F . The behaviour of all “finite” systems is captured by the so-called *rational fixpoint*. We now recall its definition and key properties as well as some illustrative examples from [4, 20, 10].

First it is easy to see that the category $\mathbf{Coalg}_f(F)$ is closed under finite colimits, so the embedding

$$E : \mathbf{Coalg}_f(F) \hookrightarrow \mathbf{Coalg}(F) \tag{2.1}$$

is an (essentially small) filtered diagram. We define a coalgebra

$$r : \varrho F \rightarrow F(\varrho F)$$

to be the colimit of E , i.e., $(\varrho F, r) = \text{colim } E$. This coalgebra is a fixpoint of F [4], and it is characterized by a universal property both as a coalgebra and as an algebra. This is the content of the following theorem. Statement 3 in the theorem below mentions iterative algebras for F . We do not recall that concept as it is not needed in the present paper; we refer the interested reader to [4].

Theorem 2.9. *Let $(\varrho F, r)$ be as above. Then*

1. $(\varrho F, r)$ is a fixpoint of F , i.e., r is an isomorphism, and
2. $(\varrho F, r)$ is the final locally finitely presentable F -coalgebra, and finally
3. $(\varrho F, r^{-1})$ is the initial iterative F -algebra.

Remark 2.10. For $\mathcal{A} = \text{Set}$ the rational fixpoint ϱF is the union of all images $f^\dagger[S] \subseteq \nu F$, where $f : S \rightarrow FS$ ranges over the *finite* F -coalgebras and $f^\dagger : S \rightarrow \nu F$ is the unique coalgebra homomorphism (see [4, Proposition 4.6 and Remark 4.3]). So, in particular, we see that ϱF is a subcoalgebra of νF .

For endofunctors on different categories than Set , this need not be the case as shown in [10, Example 3.15]. However, for functors preserving monomorphisms on categories of vector spaces over a field and on locally finite varieties such as Jsl the rational fixpoint always is a subcoalgebra of νF (see [10, Proposition 3.12]).

Example 2.11. For each of the functors in Example 2.2 we now mention the rational fixpoints. For more examples see [4, 10].

- (1) For the functor $FX = \mathbb{R} \times X$ on Set whose final coalgebra is carried by the set of all streams over \mathbb{R} , the rational fixpoint consists of all streams that are *eventually periodic*, i.e., of the form $\sigma = vw\text{www}\dots$ for words $v \in \mathbb{R}^*$ and $w \in \mathbb{R}^+$. If we consider the similar functor $FV = \mathbb{R} \times V$ on the category of vector spaces over \mathbb{R} , the rational fixpoint consists precisely of all *rational streams* (see, e. g., Rutten [23]).
- (2) Recall that deterministic automata are modeled by the functor $FX = 2 \times X^A$ on Set . The carrier of the rational fixpoint of F is the set of all languages accepted by *finite* automata, viz. the set of all *regular* languages. If we define F instead on the category Jsl of join-semilattices, its rational fixpoint is still given by all regular languages, this time with the join-semilattice structure given by union and \emptyset .
- (3) For $FX = \mathcal{P}_f(A \times X)$ on Set we saw in Example 2.2(3) that the coalgebras are labelled transition systems and νF consists of processes (modulo strong bisimilarity). In this case the rational fixpoint contains all *finite-state* processes (modulo bisimilarity); more precisely, ϱF is the coproduct of all *finite* F -coalgebras modulo the largest bisimulation—this follows from the construction of ϱF as the colimit of the diagram in (2.1).
- (4) Similarly, for $FX = 2 \times (\mathcal{P}_f X)^A$ on Set , ϱF can be described as the coproduct of all finite F -coalgebras modulo the largest bisimulation. A different (isomorphic) description is that ϱF consists of all *rational* finitely branching strongly extensional trees with edges labelled in A and nodes labelled in 2 , where a tree is rational if it has (up to isomorphism) only a finite number of subtrees.
- (5) For the functor $FX = (\mathcal{F}_{\mathbb{M}} X)^A$ of weighted transition systems the rational fixpoint is obtained as the coproduct of all finite WTS's modulo weighted bisimilarity.
- (6) Let $F = \Sigma$ be a polynomial functor on Set , where the final coalgebra is carried by all Σ -trees. Then the rational fixpoint is given by all *regular* Σ -trees (see Courcelle [11]), i. e., all those Σ -trees having (up to isomorphism) only finitely many different subtrees; this description of regular trees is due to Ginali [14].

3 Bipointed specifications

We still assume that $F : \mathcal{A} \rightarrow \mathcal{A}$ is a finitary endofunctor on the lfp category \mathcal{A} .

Definition 3.1. Let $\Sigma : \mathcal{A} \rightarrow \mathcal{A}$ be a functor. We call a natural transformation

$$\lambda : \Sigma(F \times Id) \Rightarrow F(\Sigma + Id)$$

a *bipointed specification*.

While this is a rather abstract and seemingly unusable specification format, by considering a specific functor F one can often devise more concrete formats. We discuss several examples in Section 5. For now let us consider the definition of a parallel operator on transition systems, to give a basic example of a bipointed specification. Klin [15, §5.2] presents a similar example and notices that it gives rise to a bipointed specification.

Example 3.2. Recall that the functor corresponding to transition systems is $FX = \mathcal{P}_f(A \times X)$ on **Set** and that we think of the elements of νF as processes.

We would like to define a parallel operator on processes, which can be defined in standard SOS as follows:

$$\frac{s \xrightarrow{a} s'}{s||t \xrightarrow{a} s'||t} \qquad \frac{t \xrightarrow{a} t'}{s||t \xrightarrow{a} s||t'}$$

Intuitively this means that whenever s can make an a -transition to some state s' , then $s||t$ can make an a -transition to $s'||t$, and similarly for t . Since we are interested in a single binary operator, the corresponding signature is $\Sigma X = X \times X$. Thus, the bipointed specification $\lambda : \Sigma(F \times Id) \Rightarrow F(\Sigma + Id)$ is given by the following family of maps:

$$\lambda_X : (\mathcal{P}_f(A \times X) \times X) \times (\mathcal{P}_f(A \times X) \times X) \rightarrow \mathcal{P}_f(A \times (X \times X + X)).$$

Now a for a 4-tuple (S, s, T, t) in the domain of λ_X , S and T are the sets of outgoing transitions of s and t , respectively. Moreover, an element $(a, (u, v))$ in the codomain of λ_X corresponds to an a -transition to the state $u||v$. Thus, we may define λ_X as

$$\lambda_X(S, s, T, t) = \{(a, (s', t)) \mid (a, s') \in S\} \cup \{(a, (s, t')) \mid (a, t') \in T\}.$$

It has been shown by Turi and Plotkin [25] and Bartels [8] that natural transformations as in the previous definition and more general ones (see Klin [17] for an overview) induce algebraic structures on the final coalgebra νF . We recall how this construction works for our bipointed specifications. To this end let $\lambda : \Sigma(F \times Id) \Rightarrow F(\Sigma + Id)$ be a bipointed specification. We define a functor $\Phi : \text{Coalg}(F) \rightarrow \text{Coalg}(F)$ as follows:

$$\begin{aligned} \Phi(S, f) &= \left(\Sigma S + S \xrightarrow{\Sigma(f, id) + f} \Sigma(FS \times S) + FS \xrightarrow{[\lambda_S, \text{FInr}]} F(\Sigma S + S) \right), \\ \Phi h &= \Sigma h + h, \quad \text{for any coalgebra homomorphism } h : (S, f) \rightarrow (T, g). \end{aligned} \tag{3.1}$$

In order for Φ to be well-defined Φh must be a coalgebra homomorphism, which indeed follows from naturality of λ and functoriality of Σ . We do not spell out the details, but refer the interested reader to [8, 17]. Observe that Φ is a *lifting* of $\Sigma + Id$ to $\text{Coalg}(F)$, i. e., for the forgetful functor $U : \text{Coalg}(F) \rightarrow \text{Set}$ we have $(\Sigma + Id) \cdot U = U \cdot \Phi$.

Now if we apply Φ to the final coalgebra $(\nu F, t)$ we obtain the following:

$$\Sigma(\nu F) + \nu F \xrightarrow{\Sigma\langle t, id \rangle + t} \Sigma(F(\nu F) \times \nu F) + F(\nu F) \xrightarrow{[\lambda_{\nu F}, \text{Finr}]} F(\Sigma(\nu F) + \nu F).$$

By finality, there is a unique coalgebra homomorphism from $\Phi(\nu F, t)$ to $(\nu F, t)$, and it is easy to prove that its right-hand component is the identity on νF ; so the homomorphism has the form

$$[\alpha, id] : \Sigma(\nu F) + \nu F \rightarrow \nu F.$$

Thus, we obtain a unique Σ -algebra $\alpha : \Sigma\nu F \rightarrow \nu F$ making the diagram below commute:

$$\begin{array}{ccc} \Sigma(\nu F) & \xrightarrow{\Sigma\langle t, id \rangle} & \Sigma(F(\nu F) \times \nu F) & \xrightarrow{\lambda_{\nu F}} & F(\Sigma(\nu F) + \nu F) \\ \alpha \downarrow & & & & \downarrow F[\alpha, id] \\ \nu F & \xrightarrow{t} & F\nu F & & \end{array} \quad (3.2)$$

In concrete instances, α provides the denotational semantics of the algebraic operations as specified by λ , taking as arguments elements of the final coalgebra. Returning to the above Example 3.2, for two processes s and t , $\alpha(s, t)$ is indeed the parallel composition $s \parallel t$.

Remark 3.3. The original abstract GSOS format considered by Turi and Plotkin is given by natural transformations of the form

$$\lambda : \Sigma(F \times Id) \Rightarrow FT_{\Sigma}$$

where T_{Σ} is the free monad on Σ ; for a polynomial functor Σ on Set , $T_{\Sigma}X$ is the set of all terms of operations in Σ over variables of X . This is more general than the bipointed specifications of Definition 3.1. However, we will be interested in operations on the *rational* fixpoint. And in general, operations on νF defined by the above format need not restrict to ϱF as demonstrated by the following example.

Example 3.4. Recall from Example 2.11(1) the functor $FX = \mathbb{R} \times X$ whose coalgebras are stream systems. A unary operation p on the final coalgebra $\nu F = \mathbb{R}^{\omega}$ of all real streams is specified by the following behavioural differential equations:

$$p(\sigma)(0) = \sigma(0) + 1 \quad p(\sigma)' = p(p(\sigma)'),$$

where $\sigma' = (\sigma(1), \sigma(2), \sigma(3), \dots)$ denotes the tail of the stream σ . Let $\Sigma X = X$ be the polynomial functor for the signature with one unary operation symbol p . Then the above behavioural differential equations give rise to the natural transformation

$$\ell_X : \Sigma FX = \mathbb{R} \times X \rightarrow \mathbb{R} \times T_{\Sigma}X = FT_{\Sigma}X \quad (r, x) \mapsto (r + 1, p(p(x))),$$

and we get an abstract GSOS rule as follows: $\lambda = (\Sigma(F \times Id) \xrightarrow{\Sigma\pi_0} \Sigma F \xrightarrow{\ell} FT_{\Sigma})$, where $\pi_0 : F \times Id \Rightarrow F$ denotes the left-hand product projection. It is easy to see that the ensuing operation $p : \nu F \rightarrow \nu F$ satisfies

$$(0, 0, 0, \dots) \xrightarrow{p} (1, 2, 4, 8, \dots, 2^n, \dots).$$

Clearly, the rational fixpoint ϱF , which consists of eventually periodic streams, is not closed under the operation p .

Even operations defined using bipointed specifications will not restrict to ϱF in general, when we simultaneously specify infinitely many operations that depend on one another.

Example 3.5. For $FX = \mathbb{R} \times X$ on \mathbf{Set} with $\nu F = \mathbb{R}^\omega$ we define infinitely many unary operations u_n , $n \in \mathbb{N}$, by the following behavioural differential equations:

$$u_n(\sigma)(0) = n \quad u_n(\sigma)' = u_{n+1}(\sigma').$$

Let $\Sigma X = \mathbb{N} \times X$ be the polynomial functor corresponding to the signature with the unary operation symbols u_n , $n \in \mathbb{N}$. Then the above behavioral differential equations give rise to the natural transformation

$$\ell : \Sigma FX = \mathbb{N} \times \mathbb{R} \times X \rightarrow \mathbb{R} \times \mathbb{N} \times X = F\Sigma X \quad (n, r, x) \mapsto (n, n+1, x),$$

and we get a bipointed specification as follows: $\lambda = (\Sigma(F \times Id) \xrightarrow{\Sigma\pi_0} \Sigma F \xrightarrow{\ell} F\Sigma \xrightarrow{F\text{inl}} F(\Sigma + Id))$. The ensuing operations $u_n : \nu F \rightarrow \nu F$ satisfy $(0, 0, 0, \dots) \xrightarrow{u_n} (n, n+1, n+2, n+3, \dots)$. So the rational fixpoint ϱF is not closed under these operations.

4 Algebras on the rational fixpoint

In this section we show how a bipointed specification defines an algebraic structure $\beta : \Sigma(\varrho F) \rightarrow \varrho F$ on the rational fixpoint similar to the structure $\alpha : \Sigma(\nu F) \rightarrow \nu F$ in (3.2). We will also see that the new structure β on ϱF is a “restriction” of α ; more precisely the unique coalgebra homomorphism $(\varrho F, r) \rightarrow (\nu F, t)$ is also a Σ -algebra homomorphism. In order to proceed we make

Assumption 4.1. We still assume that F is a finitary functor on the lfp category \mathcal{A} . We now assume also that $\Sigma : \mathcal{A} \rightarrow \mathcal{A}$ is a *strongly finitary* functor, i. e., Σ is finitary and it preserves finitely presentable objects. We also assume that $\lambda : \Sigma(F \times Id) \rightarrow F(\Sigma + Id)$ is a bipointed specification. We still write Φ for the functor in (3.1), which lifts $\Sigma + Id$ to $\mathbf{Coalg}(F)$.

Example 4.2. The notion of strongly finitary functor is taken from [3] and we discuss some examples below.

- (1) The class of strongly finitary functors on \mathbf{Set} contains the identity functor, all constant functors on finite sets, the finite power-set functor \mathcal{P}_f , and it is closed under finite products, finite coproducts and composition.
- (2) From the previous point we see that a polynomial functor Σ on \mathbf{Set} is strongly finitary iff the corresponding signature has finitely many operation symbols of finite arity.
- (3) The functor $FX = 2 \times X^A$ is strongly finitary iff A is a finite set.
- (4) The type functor $FX = \mathbb{R} \times X$ of stream systems as coalgebras is finitary but not strongly so. However, if we consider F as a functor on $\mathbf{Vec}_{\mathbb{R}}$, then it is strongly finitary; in fact, for every finite dimensional real vector space X , $\mathbb{R} \times X$ is finite dimensional, too.

First we need the following lemma which states that Φ is a finitary functor that restricts to the subcategory of coalgebras with a finitely presentable carrier.

Lemma 4.3. *The lifting Φ (a) is finitary and (b) restricts to $\mathbf{Coalg}_f(F)$.*

Proof. Ad (a). By assumption, Σ is a finitary functor, and so $\Sigma + Id : \mathcal{A} \rightarrow \mathcal{A}$ is clearly finitary, too. Since the forgetful functor $U : \mathbf{Coalg}(F) \rightarrow \mathcal{A}$ creates all colimits, it follows that Φ is finitary since $(\Sigma + Id) \cdot U = U \cdot \Phi$.

Ad (b). Let (S, f) be an object of $\mathbf{Coalg}_f(F)$. Then S is finitely presentable, and, since Σ is strongly finitary, ΣS is also finitely presentable. Finally, since finitely presentable objects are clearly closed under finite colimits, $\Sigma S + S$ is finitely presentable, too. Thus, $\Phi(S, f)$ is an object of $\mathbf{Coalg}_f(F)$. \square

Now in order to use the universal property of ϱF we prove that the lifting Φ applied to it is locally finitely presentable:

Lemma 4.4. *The coalgebra $\Phi(\varrho F, r)$ is locally finitely presentable.*

Proof. Since $(\varrho F, r) = \text{colim } E$ (see (2.1)) and Φ is finitary (Lemma 4.3(a)), $\Phi(\varrho F, r)$ can be obtained as the filtered colimit of the diagram $\text{Coalg}_f(F) \xleftarrow{E} \text{Coalg}(F) \xrightarrow{\Phi} \text{Coalg}(F)$. By Lemma 4.3(b), this is a diagram of coalgebras from $\text{Coalg}_f(F)$. Therefore, by Theorem 2.7, $\Phi(\varrho F, r)$ is a locally finitely presentable coalgebra. \square

From the above lemma, by the universal property of the rational fixpoint we obtain

Corollary 4.5. *There exists a unique algebra structure $\beta : \Sigma(\varrho F) \rightarrow \varrho F$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \Sigma(\varrho F) & \xrightarrow{\Sigma\langle r, id \rangle} & \Sigma(F(\varrho F) \times \varrho F) & \xrightarrow{\lambda_{\varrho F}} & F(\Sigma(\varrho F) + \varrho F) \\ \beta \downarrow & & & & \downarrow F[\beta, id] \\ \varrho F & \xrightarrow{\quad r \quad} & & & F(\varrho F) \end{array}$$

Indeed, by Lemma 4.4 and the finality of ϱF as a locally finitely presentable coalgebra there is a unique coalgebra homomorphism from $\Phi(\varrho F, r)$ to $(\varrho F, r)$, and it is again easy to show that its right-hand coproduct component must be the identity, and so its left-hand component is the desired Σ -algebra structure β .

Proposition 4.6. *Let $h : (\varrho F, r) \rightarrow (\nu F, t)$ be the unique F -coalgebra homomorphism. Then h is also a Σ -algebra homomorphism from $(\varrho F, \beta)$ to $(\nu F, \alpha)$.*

Proof. We are to prove the equation $h \cdot \beta = \alpha \cdot \Sigma h$. This is equivalent to proving

$$[h \cdot \beta, h] = [\alpha \cdot \Sigma h, h] : \Sigma(\varrho F) + \varrho F \rightarrow \nu F,$$

which is established by proving that both sides form coalgebra homomorphisms from $\Phi(\varrho F, r)$ to $(\nu F, t)$. Indeed, they are both compositions of two coalgebra homomorphisms:

$$[h \cdot \beta, h] = (\Phi(\varrho F, r) \xrightarrow{[\beta, id]} (\varrho F, r) \xrightarrow{h} (\nu F, t)), \quad [\alpha \cdot \Sigma h, h] = (\Phi(\varrho F, r) \xrightarrow{\Phi h} \Phi(\nu F, t) \xrightarrow{[\alpha, id]} (\nu F, t)). \quad \square$$

As a consequence we obtain the following closure property of ϱF : Suppose that h in the previous proposition is a monomorphism (cf. Remark 2.10). Then $(\varrho F, r)$ is a subcoalgebra of $(\nu F, t)$ and $(\varrho F, \beta)$ is a subalgebra of $(\nu F, \alpha)$ via h .

Remark 4.7. Notice that the results of this section are easily seen to generalize from bipointed specifications to the more general *coGSOS laws*, i. e., natural transformations of the form

$$\lambda : \Sigma C_F \rightarrow F(\Sigma + Id),$$

where C_F denotes the cofree comonad on F (see, e. g., [17]). (Observe that the cofree comonad on F is given objectwise by assigning to an object X of \mathcal{A} the final coalgebra $\nu(F(-) \times X)$.) This is formally dual to the abstract GSOS format we recalled in Remark 3.3. *coGSOS laws* allow to specify important operations not captured by bipointed specifications, e. g., the tail operation $\sigma \mapsto \sigma'$ on streams. And in the case of transition system specifications (i. e., where $FX = \mathcal{P}_f(A \times X)$) it is well-known that specifications in the so-called safe ntree format are instances of *coGSOS laws* (see [25]), but it is not known whether every *coGSOS law* arises from a safe ntree specification. We defer a thorough treatment of *coGSOS laws* to future work.

5 Applications

In this section we consider algebraic operations defined on the rational fixpoint for several concrete types of systems, as applications of Corollary 4.5 and Proposition 4.6. We discuss concrete SOS formats corresponding to bipointed specifications. There are many such concrete specification formats for similar distributive laws studied in the literature [17], and we can only cover a few examples here. For most of these formats it is easy to obtain a restriction to bipointed specifications, so that our results apply and the obtained specifications define operations which restrict to the rational fixpoint. Throughout this section we assume that Σ is a signature represented as a strongly finitary polynomial functor on \mathbf{Set} . To the best of our knowledge, all the results we present in the corollaries in this section are new.

Streams. Consider the \mathbf{Set} functor $FX = \mathbb{R} \times X$ of streams over the reals. A bipointed specification then is a natural transformation λ with components

$$\lambda_X : \Sigma(\mathbb{R} \times X \times X) \Rightarrow \mathbb{R} \times (\Sigma X + X). \quad (5.1)$$

We recall from [17] that these natural transformations can be expressed in a more convenient SOS format as follows. A *bipointed stream SOS rule* for an operator f in Σ of arity n is a rule

$$\frac{x_1 \xrightarrow{r_1} x'_1 \quad \dots \quad x_n \xrightarrow{r_n} x'_n}{f(x_1, \dots, x_n) \xrightarrow{r} t}$$

where $x_1, \dots, x_n, x'_1, \dots, x'_n$ is a collection of pairwise distinct variables, which we call V . Further, t is a variable in V or a term of the form $g(y_1, \dots, y_m)$ where g is an m -ary operation symbol of Σ , and $y_i \in V$ for all $1 \leq i \leq m$, and finally $r, r_1, \dots, r_n \in \mathbb{R}$. We say the above rule is *triggered* by the n -tuple (r_1, \dots, r_n) . A *bipointed stream SOS specification* for the strongly finitary signature Σ then is a collection of bipointed stream SOS rules for Σ such that for each operator f in Σ and for each sequence of real numbers r_1, \dots, r_n , there exists precisely one rule for f triggered by (r_1, \dots, r_n) . Bipointed stream SOS specifications are in one-to-one correspondence with natural transformations of the above type (5.1). Therefore, by Proposition 4.6 we have

Corollary 5.1. *The operations defined by a bipointed stream SOS specification on the final coalgebra of the \mathbf{Set} functor $FX = \mathbb{R} \times X$ restrict to the rational fixpoint of F , i.e., the coalgebra of eventually periodic streams.*

As an example consider the well-known *zip* (or *merge*) operation, which takes two streams and returns a new stream which alternates between the two given arguments. The standard definition of *zip* can be given as a bipointed stream SOS rule:

$$\frac{\sigma \xrightarrow{r_1} \sigma' \quad \tau \xrightarrow{r_2} \tau'}{\text{zip}(\sigma, \tau) \xrightarrow{r_1} \text{zip}(\tau, \sigma')}$$

A direct consequence of the above corollary is the basic insight that for any two streams σ and τ which are eventually periodic, $\text{zip}(\sigma, \tau)$ is again eventually periodic.

Remark 5.2. (1) Another way of specifying operations on streams is using *behavioural differential equations* [22] (cf. Example 3.4). In fact the above bipointed stream specifications also correspond precisely to behavioural differential equations in which each of the derivatives is restricted to be either a variable or a single operator applied to variables (precisely as t in the definition of bipointed stream SOS rules). Thus, such differential equations define operations which restrict to eventually periodic streams as well.

- (2) If we consider $FX = \mathbb{R} \times X$ as a functor on $\mathbf{Vec}_{\mathbb{R}}$ then bipointed specifications are natural transformations λ where Σ is a functor on $\mathbf{Vec}_{\mathbb{R}}$ and where the components λ_X in (5.1) are linear maps. By Proposition 4.6 we obtain that operations defined by a bipointed specification on νF , the final coalgebra of all streams, restrict to the rational fixpoint ρF formed by all rational streams. An example of such an operation is the above specification of *zip*. Consequently, we obtain that rational streams are closed under *zip*.

Labelled transition systems. Recall from Example 2.2(3) that labelled transition systems are coalgebras for the functor $FX = \mathcal{P}_f(A \times X)$ on \mathbf{Set} . In this case a bipointed specification for a strongly finitary signature Σ is a natural transformation with components

$$\lambda_X : \Sigma(\mathcal{P}_f(A \times X) \times X) \Rightarrow \mathcal{P}_f(A \times (\Sigma X + X)). \quad (5.2)$$

This corresponds to a restricted “flat” version of the well-known GSOS format [9], where on the right-hand side of the transition in the conclusions of a rule there may only be a variable or single operation symbol applied to variables in lieu of an arbitrary term. For a strongly finitary signature, this is precisely the simple GSOS format of [2]. Indeed, following the presentation in [17], we define a *bipointed LTS SOS rule* for an operator f in Σ of arity n as

$$\frac{\{x_{i_j} \xrightarrow{a_j} y_j\}_{j=1..m} \quad \{x_{i_k} \xrightarrow{b_k} \cdot\}_{k=1..l}}{f(x_1, \dots, x_n) \xrightarrow{c} t} \quad (5.3)$$

where m is the number of positive premises and l is the number of negative premises. The variables $x_1, \dots, x_n, y_1, \dots, y_m$ are again pairwise distinct; let V denote the set of these variables. Then t is either a variable in V or a flat term $g(z_1, \dots, z_p)$, where g is an p -ary operation symbol in Σ and $z_1, \dots, z_p \in V$. Finally $a_1, \dots, a_m, b_1, \dots, b_l, c \in A$ are labels. The above rule is *triggered* by an n -tuple (E_1, \dots, E_n) , where each $E_i \subseteq A$, if for each $i = 1..n$ we have $a_j \in E_{i_j}$ for all $j = 1..m$ and $b_k \notin E_{i_k}$ for all $k = 1..l$. A *bipointed LTS SOS specification* then is a collection of rules of the above type such that for each operator f in Σ , each $c \in A$ and each n -tuple $\bar{E} = (E_1, \dots, E_n)$ of sets of labels, there are finitely many rules for f with c as the conclusion label that are triggered by \bar{E} . Bipointed specifications for labelled transition systems (5.2) are in one-to-one correspondence with bipointed LTS SOS specifications. So by Proposition 4.6 we have

Corollary 5.3. *The operations defined by a bipointed LTS SOS specification on the final coalgebra of the Set functor $FX = \mathcal{P}_f(A \times X)$ restrict to the rational fixpoint of F , i.e., the coalgebra of all finite labelled transition systems modulo the largest bisimulation.*

As an example we recall the semantics of the operators of Milner’s CCS [21], which forms a bipointed LTS SOS specification:

$$\frac{}{a.P \xrightarrow{a} P} \quad \frac{P_1 \xrightarrow{a} P'_1}{P_1 + P_2 \xrightarrow{a} P'_1} \quad \frac{P_2 \xrightarrow{a} P'_2}{P_1 + P_2 \xrightarrow{a} P'_2} \quad \frac{P \xrightarrow{a} P'}{P \setminus L \xrightarrow{a} P' \setminus L} \quad (a, \bar{a} \notin L)$$

$$\frac{P_1 \xrightarrow{a} P'_1}{P_1 \parallel P_2 \xrightarrow{a} P'_1 \parallel P_2} \quad \frac{P_2 \xrightarrow{a} P'_2}{P_1 \parallel P_2 \xrightarrow{a} P_1 \parallel P'_2} \quad \frac{P_1 \xrightarrow{a} P'_1 \quad P_2 \xrightarrow{\bar{a}} P'_2}{P_1 \parallel P_2 \xrightarrow{\tau} P'_1 \parallel P'_2} \quad \frac{P \xrightarrow{a} P'}{P[\rho] \xrightarrow{\rho(a)} P'[\rho]}$$

Note that in order for the signature corresponding to these operations to be strongly finitary, the set of actions A must be finite. Then, by the above Corollary 5.3, finite-state processes are closed under all of the above operations.

Remark 5.4. Aceto [1] proved (see [2, Theorem 5.28]) that for a simple GSOS specification the induced transition system on the process terms is regular, i. e., for every closed process term P the transition system giving P its operational semantics has finitely many states. Note that this result is not a direct consequence of our results in Section 4. In fact, the transition systems induced by a (simple) GSOS specification is (generalized by) the *operational model* of Turi and Plotkin [25] for the corresponding abstract GSOS specification; this operational model is the initial Σ -algebra $\mu\Sigma$ equipped with the F -coalgebra structure induced by the abstract GSOS specification. The corresponding generalization of Aceto's result then states that for a bipointed specification λ the induced F -coalgebra on $\mu\Sigma$ is locally finitely presentable. We shall state and prove this result in future work.

Non-deterministic automata. Recall from Example 2.2(4) that non-deterministic automata are coalgebras for the **Set** functor $FX = 2 \times (\mathcal{P}_f X)^A$. Bipointed specifications for this functor instantiate to natural transformations with components

$$\lambda_X : \Sigma(2 \times \mathcal{P}_f(X)^A \times X) \Rightarrow 2 \times \mathcal{P}_f(\Sigma X + X)^A. \quad (5.4)$$

We are not aware of an existing SOS format for non-deterministic automata corresponding precisely to these natural transformations, which we call *bipointed NDA specifications*. However, it is not hard to devise a format based on the above LTS SOS specifications, such that each specification gives rise to a bipointed NDA specification, but not necessarily vice versa, i. e., an incomplete format. Define an *output rule* for an operator f in Σ of arity n as

$$\frac{\{x_{i_j} \downarrow\}_{j=1..k}}{f(x_1, \dots, x_n) \downarrow} \quad (5.5)$$

where $k \leq n$. The above output rule is *triggered* by an n -tuple $(o_1, \dots, o_n) \in 2^n$ provided that for all j , $o_{i_j} = 1$ iff $x_{i_j} \downarrow$ is in the premise of the rule. Intuitively, such a rule specifies that $f(x_1, \dots, x_n) \downarrow$, meaning that $f(x_1, \dots, x_n)$ is a final state, whenever each of its arguments x_{i_j} are final, and all of the other arguments are not final. Notice that one way to extend this format would be to make the transitions also depend on the output of the arguments; for technical convenience and lack of space we do not discuss such extensions here. A *bipointed NDA SOS specification* is a bipointed LTS SOS specification together with a collection of output rules such that for each operator f and for each n -tuple $\bar{o} = (o_1, \dots, o_n) \in 2^n$, there is at most one output rule triggered by f and \bar{o} . Any bipointed NDA SOS specification is easily seen to give rise to a bipointed NDA specification (5.4). By Proposition 4.6 we now have

Corollary 5.5. *The operations defined by a bipointed NDA (SOS) specification on the final coalgebra of the Set functor $FX = 2 \times \mathcal{P}_f(X)^A$, where A is a finite set, restrict to the rational fixpoint of F .*

Besides inducing an algebra structure $\alpha : \Sigma(\nu F) \rightarrow \nu F$ that restricts to ϱF , a bipointed specification as in (5.4) also induces an algebra on formal languages, i. e., $\tilde{\alpha} : \Sigma(\nu G) \rightarrow \nu G$ for $GX = 2 \times X^A$ on **Set**. To see this recall from Examples 2.2(4) and 2.11(4) the descriptions of νF (and ϱF) as (rational) strongly extensional trees. Now consider the following map $s : \nu G \rightarrow \nu F$: it takes a formal language L and first interprets its characteristic map $A^* \rightarrow 2$ as a complete ordered $|A|$ -ary tree t_L with nodes labelled in 2 ; the strongly extensional tree $s(L)$ is then obtained by forgetting the order on the children of every node of t_L and labelling the outgoing edges of every node with the corresponding letter from A . So $s(L)$ has the same shape as t_L , and every node of $s(L)$ has for every $a \in A$ precisely one a -labelled edge to a successor node. Secondly, let $q : \nu F \rightarrow \nu G$ be the map that assigns to every strongly extensional tree t in νF its corresponding formal language of all words given by paths from the root of t to a node labelled by 1. Clearly, we have $q \cdot s = id_{\nu G}$. Now define

$$\tilde{\alpha} = (\Sigma(\nu G) \xrightarrow{\Sigma s} \Sigma(\nu F) \xrightarrow{\alpha} \nu F \xrightarrow{q} \nu G).$$

Observe that s maps a regular language to a regular tree in νF , and q maps a regular tree in νF to a regular language. Thus, s and q restrict to the corresponding rational fixpoints and we have

Corollary 5.6. *The set of regular languages over a finite alphabet A is closed under any operation defined in a bipointed NDA (SOS) specification.*

More precisely, the above algebra structure $\tilde{\alpha} : \Sigma(\nu G) \rightarrow \nu G$ restricts to an algebra structure $\tilde{\beta} : \Sigma(\varrho G) \rightarrow \varrho G$ on the rational fixpoint (i. e., on regular languages) with $\tilde{\beta} = q' \cdot \beta \cdot s'$, where q' , β and s' are the restrictions of q , α and s , respectively, to the rational fixpoints ϱF and ϱG .

Given two words w and v , the *shuffle* of w and v , denoted $w \bowtie v$, is the set of words obtained by arbitrary interleavings of w and v [24]. For example, $ab \bowtie c = \{abc, acb, cab\}$. The shuffle of two languages L_1 and L_2 is the pointwise extension: $L_1 \bowtie L_2 = \bigcup_{w \in L_1, v \in L_2} w \bowtie v$. The shuffle operator can be defined in terms of a bipointed NDA SOS specification as follows:

$$\frac{s \xrightarrow{a} s'}{s \bowtie t \xrightarrow{a} s' \bowtie t} \quad \frac{t \xrightarrow{a} t'}{s \bowtie t \xrightarrow{a} s \bowtie t'} \quad \frac{s \downarrow \quad t \downarrow}{(s \bowtie t) \downarrow}$$

By Corollary 5.5, this operation restricts to the rational fixpoint of non-deterministic automata, and by Corollary 5.6 we obtain the fact that regular languages are closed under shuffle.

The *perfect shuffle* of two words w and v of the same length is defined as the alternation between the two words, reminiscent of the *zip* operation on streams discussed above [24]. The operation assigning to two formal languages the language of all perfect shuffles of their words can also easily be defined as a bipointed specification; in fact it can be defined using a bipointed specification w.r.t. the type functor G of deterministic automata.

Weighted transition systems. Recall from Example 2.2(5) that weighted transition systems are coalgebras for the functor $FX = (\mathcal{F}_{\mathbb{M}}X)^A$ on \mathbf{Set} ; here, we assume A to be finite. In this case a bipointed specification is a natural transformation with components

$$\lambda_X : \Sigma((\mathcal{F}_{\mathbb{M}}X)^A \times X) \Rightarrow (\mathcal{F}_{\mathbb{M}}(\Sigma X + X))^A. \quad (5.6)$$

We call these natural transformations *bipointed WTS specifications*. A general GSOS format for weighted transition systems is given in [16]. We restrict it to bipointed specifications as follows. A *bipointed WTS SOS rule* for an operator f in Σ of arity n is defined as

$$\frac{\{x_{i_j} \xrightarrow{a_j, u_j} y_j\}_{j=1..m} \quad \{x_i \xrightarrow{a} w_{a,i}\}_{a \in D_i, i=1..n}}{f(x_1, \dots, x_n) \xrightarrow{c, \beta(u_1, \dots, u_m)} t} \quad (5.7)$$

where m is the number of weighted transitions in the premise. The variables $x_1, \dots, x_n, y_1, \dots, y_m$ are again pairwise distinct; let V be the set consisting of these variables. Then t is either a variable in V or a flat term $g(z_1, \dots, z_p)$, where g is an p -ary operation symbol in Σ and $z_1, \dots, z_p \in V$. Further $D_i \subseteq A$ is a subset of labels for which the *total weight* of the outgoing transitions from x_i is specified by $w_{a,i}$. Finally $a_1, \dots, a_m, c \in A$ are labels, u_1, \dots, u_m are weight variables, and $\beta : \mathbb{M}^m \rightarrow \mathbb{M}$ is a multi-additive function. A bipointed WTS SOS specification then is a collection of rules of the above type such that only finitely many rules share the same operator f in the source, the same label c in the conclusion, and the same partial function from $\{1, \dots, n\} \times A$ to \mathbb{M} arising from their sets of total weight premises [16]. Each bipointed WTS SOS specification induces a distributive law as in (5.6) (but the converse does not hold, see [16]). So by Proposition 4.6 we have

Corollary 5.7. *The operations defined by a bipointed WTS (SOS) specification on the final coalgebra of the Set functor $FX = (\mathcal{F}_{\mathbb{M}}X)^A$ where A is a finite set, restrict to the rational fixpoint of F , i.e., the coalgebra of all finite weighted transition systems modulo weighted bisimilarity.*

All of the examples of operations on WTS's from [16] are bipointed specifications, from which it follows that the rational fixpoint is closed under those operations. We recall here the priority operator. To this end we consider the weights to be in $\mathbb{R}^{+\infty}$, which is the set consisting of all positive reals augmented with infinity (denoted ∞). By taking *minimum* as the sum operation, this forms a monoid with ∞ as the unit. The unary operation ∂_{ab} is defined by the rules

$$\frac{x \xRightarrow{a} w \quad x \xRightarrow{b} v \quad x \xrightarrow{a,u} x'}{\partial_{ab}(x) \xrightarrow{a,u} \partial_{ab}(x')} \quad \frac{x \xRightarrow{a} v \quad x \xRightarrow{b} w \quad x \xrightarrow{b,u} x'}{\partial_{ab}(x) \xrightarrow{b,u} \partial_{ab}(x')}$$

for all $w \leq v \in \mathbb{R}^{+\infty}$. The operator ∂_{ab} preserves only the a -transitions if the minimum weight of all a -transitions is less than or equal to the minimum of all outgoing b -transitions, and vice versa.

6 Conclusions and future work

In this paper we have presented a general categorical framework for the specification of algebraic operations on regular behaviour based on distributive laws. The theory we have presented works not only in **Set** but also in many other categories including vector spaces and other algebraic categories. In this paper we have instantiated the general theory to several concrete specification formats in **Set**. It remains an interesting challenge to study concrete formats for distributive laws on other categories, not only for our bipointed specifications but also for distributive laws corresponding to GSOS. For example, working out a format for the functor $FX = 2 \times X^A$ on the category of join-semilattices will give a more direct way to define operations like the shuffle product of formal languages which cannot be captured by a bipointed specification for F on **Set**. Finally, it is interesting to study extensions of the format introduced in this paper. We already mentioned the coGSOS format, and we will investigate this more thoroughly in the future. One would also hope for formats covering all the standard operations on formal languages such as the Kleene star which, presently, does not arise as an application of our theory. Since checking if a specification gives rise to operations under which regular behaviour is closed is in general undecidable, a complete format cannot exist [2].

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