Math. Struct. in Comp. Science (2010), *vol.* 20, *pp.* 419–452.
doi:10.1017/S0960129509990326 First published online 4 February 2010

Iterative reflections of monads

JIŘÍ ADÁMEK[†], STEFAN MILIUS[†] and JIŘÍ VELEBIL[‡§]

[†]*Institute of Theoretical Computer Science,*
Technical University of Braunschweig, Germany
Email: {adamek,milius}@iti.cs.tu-bs.de
[‡]*Faculty of Electrical Engineering, Czech Technical University of Prague,*
Prague, Czech Republic
Email: velebil@math.feld.cvut.cz

Received 16 May 2007; revised 19 December 2008

Iterative monads were introduced by Calvin Elgot in the 1970's and are those ideal monads in which every guarded system of recursive equations has a unique solution. We prove that every ideal monad $\mathbb{M}$ has an iterative reflection, that is, an embedding into an iterative monad with the expected universal property. We also introduce the concept of iterativity for algebras for the monad $\mathbb{M}$, following in the footsteps of Evelyn Nelson and Jerzy Tiuryn, and prove that $\mathbb{M}$ is iterative if and only if all free algebras for $\mathbb{M}$ are iterative algebras.

1. Introduction

At first sight it may seem as if there are few examples in the realm of Calvin Elgot's iterative algebraic theories (Elgot 1975): the free iterative theories (of rational Σ-trees) were described in the work of Elgot and his collaborators, see Elgot *et al.* (1978), together with a treatment of the motivating example of the theory of sequacious functions, but not much else; also, Stephen Bloom and Zoltán Ésik's monograph Bloom and Ésik (1993) did not provide many additional examples. In the current paper we prove that, despite this, iterative theories are in fact abundant: every ideal algebraic theory (for example, semigroups, unary algebras, algebras with a commutative binary operation, and so on) has an iterative reflection. That is, a free 'completion' into an iterative theory.

The concept of an iterative theory is based on the idea that, given a signature Σ, we study systems of recursive equations of the form

$$\begin{aligned} x_1 &\approx t_1(x_1, \dots, x_m, a_1, \dots, a_k) \\ &\vdots \\ x_m &\approx t_m(x_1, \dots, x_m, a_1, \dots, a_k) \end{aligned} \tag{1.1}$$

whose right-hand sides are finite Σ-trees (or Σ-terms) in the given variables x_i and the given parameters $a_1, \dots, a_k$ in a Σ-algebra A. The system (1.1) is *ideal* if none of the trees t_i is either a single variable or a single parameter.

[§] This author acknowledges the support of the Grant Agency of the Czech Republic under Grant Number 201/06/664.

We are interested in *solutions*, which means that to every variable x_i an element $x_i{}^\dagger$ of A is assigned in such a way that the formal equations above become actual identities

$$x_i{}^\dagger = t_i^A(x_1{}^\dagger, \dots, x_m{}^\dagger, a_1, \dots, a_k) \quad i = 1, \dots, m. \tag{1.2}$$

A Σ-algebra A is said to be *iterative*, a concept developed by Evelyn Nelson (Nelson 1983) and Jerzy Tiuryn (Tiuryn 1980), if every ideal system of equations (1.1) has a unique solution. We need to restrict consideration to ideal systems in order to avoid trivial equations such as $x \approx x$; these ideal systems are equivalent to the guarded systems used in Adámek *et al.* (2006) – see Appendix B.

In the present paper we combine this idea of iterativity with that of equational presentation: given a variety, we study algebras in that variety that are iterative. To explain what this means, consider the system (1.1) as a morphism

$$e : X \longrightarrow F_\Sigma(X + A) \quad (X \text{ finite})$$

where X is the set of variables and F_Σ is the (underlying functor of the) monad of free Σ-algebras assigning to every set the set of all finite Σ-terms on it. Now varieties can be expressed by Lawvere's algebraic theories or, equivalently, by finitary monads $\mathbb{M}$ in Set. Recall that a monad $\mathbb{M} = (M, \eta, \mu)$ consists of an endofunctor M, a natural transformation $\mu : MM \longrightarrow M$ (multiplication) that is associative and a natural transformation $\eta : \mathrm{Id} \longrightarrow M$ that is a unit for μ. An example is the monad $\mathbb{F}_\Sigma$ above consisting of F_Σ, the inclusion of variables $\eta_X : X \longrightarrow F_\Sigma X$ and the obvious substitution map $\mu_X : F_\Sigma(F_\Sigma X) \longrightarrow F_\Sigma X$.

We then consider *Eilenberg–Moore algebras* for the monad $\mathbb{M}$, which are those algebras $a : MA \longrightarrow A$ for which $a \cdot \eta_A = \mathrm{id}_A$ and $a \cdot Ma = a \cdot \mu_A$. (For example, Σ-algebras are precisely the Eilenberg–Moore algebras for the monad $\mathbb{F}_\Sigma$.) As a generalisation of the systems (1.1), consider the equation morphisms

$$e : X \longrightarrow M(X + A) \quad (X \text{ finite})$$

in a given Eilenberg–Moore algebra $a : MA \longrightarrow A$.

Calvin Elgot worked with properties of algebraic theories rather than algebras. In order to express the concept of an ideal system of equations, Elgot just considered *ideal algebraic theories*, which are those (Lawvere) theories in which coproduct injections are right cancellative (that is, whenever $u \cdot v$ is a coproduct injection, so is v). In the language of finitary monads, a monad (M, η, μ) is *ideal* if its unit $\eta : \mathrm{Id} \longrightarrow M$ is a coproduct injection of a coproduct

$$M = M' + \mathrm{Id}$$

and its multiplication $\mu : MM \longrightarrow M$ has a domain–codomain restriction $\mu' : M'M \longrightarrow M'$. See Aczel *et al.* (2003) for a proof that this is equivalent to Elgot's concept. Examples of ideal monads include, beside $\mathbb{F}_\Sigma$, the monads of semigroups, or algebras with a commutative binary operation. On the other hand, the monad of groups is not ideal.

Given an ideal monad $\mathbb{M}$, the equation morphism $e : X \longrightarrow M(X + A)$ is said to be *ideal* if it factors through the summand $M'(X + A) \hookrightarrow M(X + A)$. An ideal monad $\mathbb{M}$ is *iterative* if every ideal equation morphism $e : X \longrightarrow M(X + A)$ has a unique *solution*, that is, a unique morphism $e^\dagger : X \longrightarrow MA$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & MA \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_A} \\ M(X+A) & \xrightarrow[M[e^\dagger,\eta_A]]{} & MMA \end{array} \tag{1.3}$$

commutes.

We are going to prove that every ideal monad $\mathbb{M}$ has an iterative reflection, which means an ideal extension

$$\mathbb{M} \hookrightarrow \widehat{\mathbb{M}}$$

with the universal property that every ideal monad morphism from $\mathbb{M}$ to an iterative monad $\mathbb{T}$ can be uniquely extended to an ideal monad morphism from $\widehat{\mathbb{M}}$ to $\mathbb{T}$.

For example, given a finitary endofunctor H, Michael Barr (Barr 1970) proved that H generates a free monad $\mathbb{F} = (F, \eta, \mu)$; moreover, this monad is always ideal due to the canonical isomorphism $F \cong HF + \mathrm{Id}$. An iterative reflection $\widehat{\mathbb{F}}$ is the *rational monad* $\mathbb{R}$ of H that was studied in Adámek *et al.* (2006) and characterised as a free iterative monad on H. In particular, let $H = H_\Sigma$ be the *polynomial endofunctor* of Set for a given signature $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$:

$$H_\Sigma Z = \Sigma_0 + \Sigma_1 \times Z + \Sigma_2 \times Z^2 + \dots \tag{1.4}$$

Then $\mathbb{F}$, denoted by $\mathbb{F}_\Sigma$, is the above monad of finite Σ-trees (or Σ-terms) and $\mathbb{R}$, denoted by $\mathbb{R}_\Sigma$, is Susanna Ginali's monad of rational-trees (Ginali 1979): to every set Z it assigns the algebra $R_\Sigma Z$ of all Σ-trees on Z that are *rational*, that is, have only finitely many subtrees up to isomorphism.

A surprising example is given by the fact that for the ideal monad of semigroups

$$MZ = Z^+ \quad \text{(the free semigroup on } Z\text{)},$$

all infinite 'rational polynomials' collapse to a single absorbing (zero) element. More precisely, an iterative reflection $\widehat{\mathbb{M}}$ is given on objects by

$$\widehat{M}Z = Z^+ + \{0\} \quad \text{(0 absorbing)}.$$

In other words, if Σ_2 is the signature of one binary operation, then, whereas the iterative reflection of $\mathbb{F}$ (the monad of finite binary trees) is the monad $\mathbb{R}$ (of rational binary trees), the associative law makes all the infinite rational trees equal.

In contrast, the commutative law does not collapse anything 'unexpected': here we consider the endofunctor of Set given by

$$HZ = \text{all unordered pairs in } Z,$$

or, equivalently, the monad $\mathbb{F}$ of all finite, binary unordered trees. As proved in Adámek and Milius (2006), the rational monad $\mathbb{R}$ is the monad of all rational, binary unordered trees.

However, commutativity can also be 'devastating', as demonstrated by the following example due to Bruno Courcelle (private communication). Consider the signature of two unary operations a, b. Here $HZ = Z + Z$ and the corresponding free monad is

$$FZ = \{a,b\}^* \times Z.$$

Its iterative reflection is the monad of rational trees, which in the present case has the simple form

$$RZ = \{a,b\}^* \times Z + \{a,b\}^{@}$$

where the right-hand summand is the set of all infinite words on $\{a,b\}$ that are *eventually periodic*:

$$\{a,b\}^{@} = \{uvvv\dots \mid u \in \{a,b\}^* \text{ and } v \in \{a,b\}^{+}\}.$$

Now impose the commutative law:

$$a(b(z)) = b(a(z)).$$

The corresponding ideal monad is

$$MZ = \{a\}^* \times \{b\}^* \times Z.$$

Its iterative reflection is, surprisingly, the collapse of all eventually periodic words to a joint fixed point of a and b:

$$\widehat{M}Z = \{a\}^* \times \{b\}^* \times Z + \{0\} \quad \text{with } a(0) = 0 = b(0).$$

These examples of iterative reflections are based on the concept of an *iterative algebra* for an arbitrary ideal monad $\mathbb{M}$: it is an Eilenberg–Moore algebra $a : MA \longrightarrow A$ such that every ideal equation morphism $e : X \longrightarrow M(X + A)$ has a unique *solution* $e^\dagger : X \longrightarrow A$ *in the algebra*, which means a unique morphism such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle a} \\ M(X+A) & \xrightarrow[M[e^\dagger, A]]{} & MA \end{array}$$

commutes. For the case of a free monad $\mathbb{M} = \mathbb{F}_\Sigma$ on H_Σ this is precisely (1.2) above. We prove that for every ideal monad $\mathbb{M}$:

(a) $\mathbb{M}$ is iterative if and only if every free $\mathbb{M}$-algebra is iterative.

(b) Every object Z generates a free iterative algebra $\widehat{M}Z$ for the monad $\mathbb{M}$.

As a consequence, we obtain a new monad: the monad $\widehat{\mathbb{M}}$ of free iterative algebras for $\mathbb{M}$. The above examples of iterative reflections are based on the following fact:

(c) The monad of free iterative algebras is an iterative reflection of $\mathbb{M}$.

Unfortunately, we have not been able to prove (c) in the same generality as (a) and (b) are proved below. In fact, to date we only have a (rather technically involved) proof of (c) for set-like categories: we need all epimorphisms to split. For this reason, the current paper only gives proofs for (a) and (b), which hold in all extensive, locally finitely presentable categories; the proof of (c) will be given in a subsequent publication – , see Adámek *et al.* (2009b) for an extended abstract.

The result showing that iterative reflections exist is presented in Section 2, with the proof given in Appendix A. Statements (a) and (b) are proved in Section 3. Finally, in Appendix B, we prove that 'ideal' systems of Calvin Elgot's recursive equations are equivalent to the 'guarded' systems we have used previously, for example, in Aczel *et al.* (2003).

2. Iterative monads

Assumption 2.1. Throughout this paper we work with finitary monads on an extensive, locally finitely presentable category $\mathscr{A}$.

Recall that a functor is said to be *finitary* if it preserves filtered colimits, and a monad is finitary if its underlying functor is. An object X whose hom-functor $\mathscr{A}(X, -)$ is finitary is said to be *finitely presentable*.

Recall from Carboni *et al.* (1993) that a category is *extensive* if it has universal and disjoint finite coproducts. We will, in particular, use the following facts that hold in extensive categories:

(a) Coproduct injections $\mathsf{inl} : A \longrightarrow A + B$ and $\mathsf{inr} : B \longrightarrow A + B$ are monomorphisms.
(b) For every morphism $f : A \longrightarrow B_1 + B_2$ there exists an essentially unique decomposition $f = f_1 + f_2$ for morphisms $f_i : A_i \longrightarrow B_i$ with $A = A_1 + A_2$.

Finally, recall that a category $\mathscr{A}$ is said to be *locally finitely presentable* in the sense of Peter Gabriel and Friedrich Ulmer (Gabriel and Ulmer 1971), see also Adámek and Rosický (1994), provided that:

(1) $\mathscr{A}$ is cocomplete; and
(2) $\mathscr{A}$ has a set $\mathscr{A}_{\text{fp}}$ of finitely presentable objects whose closure under filtered colimits is all of $\mathscr{A}$.

Examples of extensive, locally finitely presentable categories are sets, posets, graphs and unary algebras. For every extensive, locally finitely presentable category $\mathscr{A}$, all functor categories $[\mathscr{C}, \mathscr{A}]$, with $\mathscr{C}$ small, and the category $\mathsf{FE}(\mathscr{A})$ of all finitary endofunctors of $\mathscr{A}$, are also extensive and locally finitely presentable.

Calvin Elgot's concept of an iterative theory (Elgot 1975) has the following categorical form, as shown in Aczel *et al.* (2003).

Definition 2.2. A monad $\mathbb{M} = (M, \eta, \mu)$ is *ideal* provided that:

(1) its unit $\eta : \mathrm{Id} \longrightarrow M$ is a coproduct injection of a coproduct

$$M = M' + \mathrm{Id} \quad \text{with injections } \sigma : M' \longrightarrow M \text{ and } \eta : \mathrm{Id} \longrightarrow M$$

where M' is a finitary functor (called the *ideal* of $\mathbb{M}$); and

(2) the multiplication μ has a restriction to a natural transformation

$$\mu' : M'M \longrightarrow M' \quad \text{with} \quad \sigma \cdot \mu' = \mu \cdot \sigma M. \tag{2.1}$$

Remark 2.3. Since the category $\mathsf{FE}(\mathscr{A})$ of finitary endofunctors of $\mathscr{A}$ is extensive, the fact that η is a coproduct injection thus determines the 'complementary' coproduct injection $\sigma : M' \longrightarrow M$ uniquely up to natural isomorphism. Also, μ' is determined uniquely by (2.1) since σ is pointwise a monomorphism.

In non-extensive categories, an ideal monad is a structure rather than a property: we have to consider the whole sixtuple $(M, \eta, \mu, M', \sigma, \mu')$. The original definition of ideal theory in Elgot (1975) is different from but equivalent to ours, see Aczel *et al.* (2003, 4.6).

Example 2.4.

(1) The category Set is extensive and locally finitely presentable. A finitary monad $\mathbb{M}$ is a presentation of an equational specification (with MX denoting the free algebra of that specification generated by the set X). To be ideal means that for every term $t(x_1, \ldots, x_n)$ and every substitution y_i/x_i ($i = 1, \ldots, n$), whenever $t(y_1, \ldots, y_n)$ is congruent to some y_i, we have $t(x_1, \ldots, x_n)$ is congruent to some x_j. For example, semigroups, $MX = X^+$, monoids, $MX = X^*$, unary algebras, $MX = \mathbb{N} \times X$, and commutative binary algebras are examples of ideal monads. In contrast, groups do not form an ideal monad: in fact, if we define $M'X = MX \setminus \eta[X]$ for the free-group functor MX, then M' is not a subfunctor of M. Consider $x \neq y$ in X and a map $f : X \longrightarrow Z$ with $f(x) = f(y) = z$. Then the image under Mf for the term $(x \cdot y^{-1}) \cdot x \in M'X$ is $z \in \eta[Z]$.

(2) Let $\mathscr{F}$ denote the category of finite sets and functions. The presheaf category

$$\mathscr{A} = [\mathscr{F}, \mathsf{Set}]$$

can be interpreted as the category of 'sets in context' – see Fiore *et al.* (1999). This is used for the semantics of untyped λ-calculus. The category $\mathscr{A}$ is extensive and locally finitely presentable. The functor

$$HX = X \times X + X^V \quad \text{(for the embedding } V : \mathscr{F} \longrightarrow \mathsf{Set})$$

expressing the algebra of λ-terms as an initial H-monoid defines a free monad that is ideal. An iterative reflection of this monad is, as proved in Adámek *et al.* (2009b), the monad of rational λ-terms used for the semantics of recursive program schemes.

(3) Every finitary endofunctor H of $\mathscr{A}$ generates a free monad $\mathbb{F}$ given on objects Z by

$$FZ = \text{a free } H\text{-algebra on } Z.$$

The Eilenberg–Moore category of $\mathbb{F}$ is the category of H-algebras. This was proved by Michael Barr in Barr (1970).

If $\eta_Z : Z \longrightarrow FZ$ denotes the universal arrow and $\sigma_Z : HFZ \longrightarrow FZ$ the structure of the free H-algebra, then

$$FZ = HFZ + Z \quad \text{with injections } \sigma_Z, \eta_Z. \tag{2.2}$$

Therefore, $\mathbb{F}$ is always an ideal monad with the ideal $F' = HF$. The universal arrow of the free monad is $\kappa = \sigma \cdot H\eta : H \longrightarrow F$.

Definition 2.5. Let $\mathbb{M}$ be an ideal monad.

(1) A (finitary) *equation morphism* is a morphism $e : X \longrightarrow M(X + A)$, where X is a finitely presentable object 'of variables' and A is an arbitrary object 'of parameters'. The equation morphism is said to be *guarded* if it factors through $M'(X + A) + \mathrm{A}$:

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & M(X+A) \\ & \searrow & \uparrow {\scriptstyle [\sigma_{X+A}, \eta_{X+A}\cdot \mathsf{inr}]} \\ & & M'(X+A)+A \end{array}$$

(2) The monad $\mathbb{M}$ is said to be *iterative* if every guarded equation morphism $e : X \longrightarrow M(X + A)$ has a unique *solution*, which means that a unique morphism $e^\dagger : X \longrightarrow MA$ such that

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & MA \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_A} \\ M(X+A) & \xrightarrow[M[e^\dagger, \eta_A]]{} & MMA \end{array} \tag{2.3}$$

commutes.

Remark 2.6.

(1) We could have added the requirement that A be finitely presentable in part (1) of this definition, but it would make no difference, as we prove in Proposition B.1 in Appendix B.

(2) Also, in lieu of guarded equation morphisms, we can work with *ideal* ones, that is, those that factor through $M'(X + A)$:

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & M(X+A) \\ & {\scriptstyle e'}\searrow & \uparrow{\scriptstyle \sigma_{X+A}} \\ & & M'(X+A) \end{array} \tag{2.4}$$

In fact, we prove in Appendix B that every monad that has unique solutions of all ideal equation morphisms is iterative: the assumption that the base category is extensive plays an important role in proving this result.

(3) For ideal equation morphisms, we can fully work with M' in lieu of M – see the following lemma.

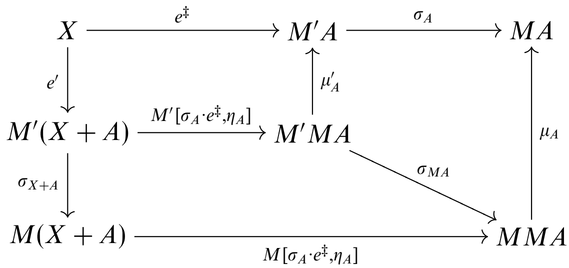
Lemma 2.7. An ideal equation morphism (2.4) has a unique solution if and only if there exists a unique morphism $e^\ddagger : X \longrightarrow M'A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\ddagger \quad} & M'A \\ {\scriptstyle e'}\downarrow & & \uparrow{\scriptstyle \mu'_A} \\ M'(X+A) & \xrightarrow[M'[\sigma_A \cdot e^\ddagger, \eta_A]]{} & M'MA \end{array} \tag{2.5}$$

commutes.

Proof.

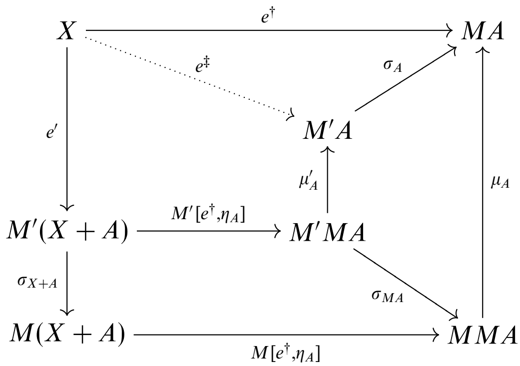
(1) Let $e^{\ddagger}$ exist uniquely. We will obtain a solution $\sigma_A \cdot e^{\ddagger}$ of e: in fact, in the following diagram the upper left-hand part commutes by (2.5), the lower left-hand part commutes by the naturality of σ and the right-hand part commutes by (2.1):



The uniqueness of $e^{\dagger}$ is clear: suppose $e^{\dagger} : X \longrightarrow MA$ is a solution of e and define a morphism $e^{\ddagger} : X \longrightarrow M'A$ to be $e^{\ddagger} = \mu'_A \cdot M'[e^{\dagger}, \eta_A] \cdot e'$. Then we obtain

$$e^{\dagger} = \sigma_A \cdot e^{\ddagger}$$

from the commutativity of the following diagram:



In fact, since all inner parts apart from the upper triangle commute, and the outer part also commutes, we get that the upper triangle must commute too. This implies that the middle left-hand part commutes when extended by the monomorphism σ_A, so (2.5) commutes for $e^{\ddagger}$ as defined above. Since $e^{\ddagger}$ is uniquely determined by hypothesis, the equation $e^{\dagger} = \sigma_A \cdot e^{\ddagger}$ implies that $e^{\dagger}$ is uniquely determined also.

(2) Let $e^{\dagger}$ exist uniquely. Then, arguing as above, we get that

$$e^{\ddagger} = \mu'_A \cdot M'[e^{\dagger}, \eta_A] \cdot e'$$

is the unique morphism such that (2.5) commutes. □

Example 2.8.

(1) The free monad $\mathbb{F}_\Sigma$ on a polynomial functor H_Σ, see (1.4), is given on objects Z by

$$F_\Sigma Z = \text{all finite } \Sigma\text{-trees on } Z.$$

Recall that a Σ-tree on Z is a rooted, ordered tree labelled in $\Sigma + Z$ so that leaves are labelled in $\Sigma_0 + Z$ and nodes with $n > 0$ children are labelled in Σ_n. All trees are considered up to isomorphism throughout this paper.
An equation morphism $e : X \longrightarrow F_\Sigma(X + A)$ with $X = \{x_1, \dots, x_m\}$ can be viewed as a system (1.1) whose right-hand sides are Σ-trees (or Σ-terms) over the given variables x_i and the given parameters $a_1, \dots, a_k$ in A. Such a system is ideal if none of the trees t_i is a single variable or a single parameter. It is guarded if parameters are allowed as a right-hand side, but variables are not.
A solution of (1.1) is a function $e^\dagger : X \longrightarrow F_\Sigma A$ representing a substitution of variables x_i by terms $x_i{}^\dagger$ for $i = 1, \dots, m$ such that the formal equations (1.1) become identities (1.2).
The reason $\mathbb{F}_\Sigma$ is not iterative is that the 'obvious solution' obtained by tree expansions of (1.1) often leads to infinite trees.

(2) By dropping the finiteness requirement of the previous example, we can define a monad $\mathbb{T}_\Sigma$ of Σ-trees with

$$T_\Sigma Z = \text{all } \Sigma\text{-trees on } Z.$$

This is indeed a monad, as observed by Eric Badouel (Badouel 1989). However, this is not a finitary monad. A finitary submonad $\mathbb{R}_\Sigma$ of $\mathbb{T}_\Sigma$ is given by

$$R_\Sigma Z = \text{all rational } \Sigma\text{-trees on } Z,$$

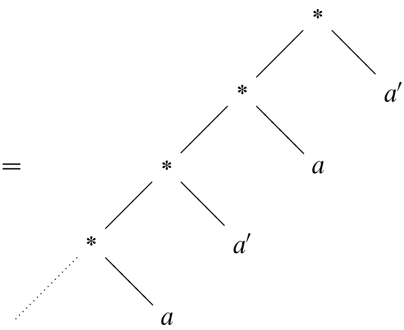
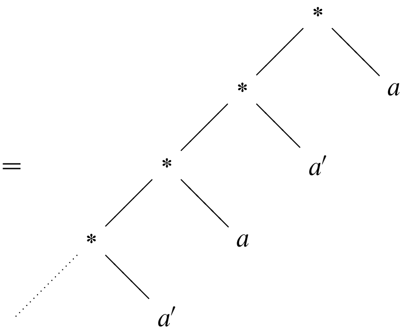
where a tree is *rational* if it only has finitely many subtrees (up to isomorphism) – see Ginali (1979).
The monad $\mathbb{R}_\Sigma$ is ideal since, analogously to (2.2) above,

$$R_\Sigma = H_\Sigma R_\Sigma + \mathrm{Id}.$$

Moreover, $\mathbb{R}_\Sigma$ is iterative: given an equation system (1.1) where t_i are now rational trees, and assuming guardedness (no t_i is a single variable), we have an obvious tree expansion s_i of the variable x_i for $i = 1, \dots, m$, so s_i is a rational tree also. This is the unique solution in $R_\Sigma Z$.
For example, if Σ_2 denotes the signature of one binary operation $*$, the equation system

$$x_1 \approx x_2 * a \qquad x_2 \approx x_1 * a' \tag{2.6}$$

has the unique solution given by the rational trees

$s_1 =$  and $s_2 =$ 

(3) Let H be the endofunctor of Set whose algebras are commutative binary algebras:

$$HZ = \text{all unordered pairs in } Z.$$

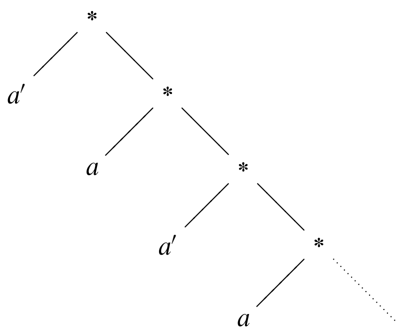
A free monad $\mathbb{F}$ on H can be described by

$$FZ = \text{all finite unordered binary trees on } Z$$

(where an unordered tree has no order on the two children of a given node). Adámek and Milius (2006) described a free iterative monad $\mathbb{R}$ on H: it is given on objects by

$$RZ = \text{all rational unordered binary trees on } Z.$$

For example, the solution of (2.6) is given by the above trees s_1 and s_2, which are unordered, so s_1 can also be represented by



Definition 2.9. Let $\mathbb{M} = (M, \eta, \mu)$ and $\overline{\mathbb{M}} = (\overline{M}, \overline{\eta}, \overline{\mu})$ be ideal monads with

$$M = M' + \mathrm{Id} \quad \text{and} \quad \overline{M} = \overline{M'} + \mathrm{Id}.$$

A monad morphism $h : \mathbb{M} \longrightarrow \overline{\mathbb{M}}$ is said to be *ideal* if it has the form $h = h' + \mathrm{id}$ for a (uniquely determined) natural transformation $h' : M' \longrightarrow \overline{M'}$.

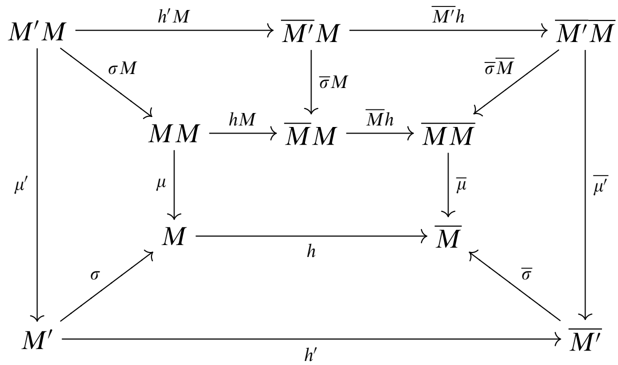
Example 2.10. All the 'usual' monad morphisms are ideal. For example, given a natural transformation between endofunctors, the unique extension to free monads (see Example 2.4(3)) is ideal.

On the other hand, for the monad X^* of free monoids, the monad (endo-)morphism given by $h_X : X^* \longrightarrow X^*$ defined by $h_X(\varepsilon) = \varepsilon$ and $h_X(x_1 \dots x_n) = x_1$ is not ideal.

Remark 2.11. It follows that h' in Definition 2.9 fulfills

$$h' \cdot \mu' = \overline{\mu'} \cdot \overline{M'}h \cdot h'M.$$

In fact, we will now prove that the following diagram commutes:



The middle square commutes since h is a monad morphism, the left- and right-hand parts commute by (2.1), the upper left-hand and the lower part commute by the definition of an ideal monad morphism, and the remaining upper right-hand part trivially commutes. So, since $\overline{\sigma}$ is componentwise a monomorphism (see Assumption 2.1(a)), the outer square commutes as desired.

Remark 2.12. We are going to prove that every ideal monad $\mathbb{M}$ has an iterative reflection, that is, an ideal monad morphism $\mathbb{M} \longrightarrow \widehat{\mathbb{M}}$ to an iterative monad $\widehat{\mathbb{M}}$ with the universal property that every ideal monad morphism from $\mathbb{M}$ to an iterative monad has a unique ideal extension to $\widehat{\mathbb{M}}$.

Notation 2.13. We use

$$\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$$

to denote the category of all ideal monads on $\mathscr{A}$ and ideal monad morphisms, and

$$\mathsf{IFM}_{\mathrm{id}}(\mathscr{A})$$

to denote its full subcategory of iterative monads.

Theorem 2.14. Every ideal monad has an iterative reflection. That is, the full embedding $\mathsf{IFM}_{\mathrm{id}}(\mathscr{A}) \longrightarrow \mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ has a left adjoint.

Remark 2.15. The full proof of the theorem is postponed to Appendix A. For the category Set (and any other where all epimorphisms split) there is a shorter proof based on an idea suggested by one of the referees.

Proof for the case in which epimorphisms split in $\mathscr{A}$.

(a) The category $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ is complete, and iterative monads are closed under limits in it – see (b2) in Appendix A.

(b) The theorem can then be proved using the Adjont Functor Theorem: for every ideal monad $\mathbb{M} = (M, \eta^M, \mu^M)$ we need a set of ideal monad morphisms $h : \mathbb{M} \longrightarrow \mathbb{S}$, with $\mathbb{S}$ iterative, through which all other morphisms factorise. To find such a set, let $\mathbb{R} = (R, \eta^R, \mu^R)$ be a free iterative monad on the functor M with the universal arrow $i : M \longrightarrow R$. Since M is finitary, $\mathbb{R}$ exists (Adámek *et al.* 2006).
Every ideal morphism $h : \mathbb{M} \longrightarrow \mathbb{S}$ with $\mathbb{S}$ iterative extends to a unique ideal monad morphism $\overline{h} : \mathbb{R} \longrightarrow \mathbb{S}$ with $\overline{h}{\cdot}i = h$ and we can factorise $\overline{h}_A = m_A {\cdot} h^*_A$ as an epimorphism $h^*_A : RA \longrightarrow S^*A$ followed by a (strong) monomorphism $m_A : S^*A \longrightarrow SA$. It is well known that S^* carries a unique structure of a monad $\mathbb{S}^*$ such that $h^* : \mathbb{R} \longrightarrow \mathbb{S}^*$ and $m : \mathbb{S}^* \longrightarrow \mathbb{S}$ are monad morphisms. It is also easy to verify that $\mathbb{S}^*$ is ideal, as are h^* and m. Thus, to conclude the proof, we only need to verify below that $\mathbb{S}^*$ is an iterative monad, since then all the morphisms h^* form the desired solution set (because, due to the finitarity of M, they are determined by their components h^*_A with A finitely presentable, thus, they essentially form a set).

(c) Every ideal equation morphism $e : X \longrightarrow S^*(X + Y)$ with X finitely presentable has a solution. To prove this, choose a splitting

$$r : S^*(X + Y) \longrightarrow R(X + Y)$$

of the epimorphism h^*_{X+Y}, that is $h^*_{X+Y} \cdot r = \mathrm{id}$. It is easy to see that the morphism $f = r \cdot e : X \longrightarrow R(X + Y)$ is ideal. Thus, it has a solution $e^\dagger : X \longrightarrow RY$ with respect to $\mathbb{R}$. A simple computation then shows that $h^*_Y \cdot e^\dagger : X \longrightarrow S^*Y$ is a solution with respect to $\mathbb{S}^*$.

(d) The uniqueness of solutions $e^\dagger$ of e with respect to $\mathbb{S}$ follows from the fact that $m_{X+Y} \cdot e : X \longrightarrow S(X + Y)$ is ideal with respect to $\mathbb{S}$. Then a simple computation shows that $m_Y \cdot e^\dagger : X \longrightarrow SY$ is a solution with respect to $\mathbb{S}$. Since m_Y is monomorphic, $e^\dagger$ is unique. □

3. Iterative algebras

It is often easier and more intuitive to work with the concept of iterativity for algebras rather than monads. In the classical case of Σ-algebras in Set, this has already been observed by Evelyn Nelson (Nelson 1983) and Jerzy Tiuryn (Tiuryn 1980). We prove that a monad is iterative if and only if each free Eilenberg–Moore algebra for that monad is iterative.

Recall that an Eilenberg–Moore algebra for a monad $\mathbb{M} = (M, \eta, \mu)$ is an M-algebra $a : MA \longrightarrow A$ satisfying

$$a \cdot \eta_A = \mathrm{id}_A \quad \text{and} \quad a \cdot \mu_A = a \cdot Ma.$$

We use $\mathscr{A}^{\mathbb{M}}$ to denote the category of all Eilenberg–Moore algebras and homomorphisms.

Definition 3.1. Let $\mathbb{M}$ be an ideal monad. An $\mathbb{M}$-algebra $a : MA \longrightarrow A$ is said to be *iterative* if for every guarded equation morphism $e : X \longrightarrow M(X + A)$ there exists a unique *solution in* A, which means a unique morphism $e^\dagger : X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle a} \\ M(X+A) & \xrightarrow[M[e^\dagger,A]]{} & MA \end{array}$$

commutes.

Remark 3.2. We can use 'ideal' instead of 'guarded' in the above definition – see Proposition B.1 in Appendix B.

Example 3.3. If $\mathbb{M}$ is a free monad on a finitary endofunctor H, see Example 2.4(3), we can work with H-algebras $a : HA \longrightarrow A$ instead of monadic $\mathbb{M}$-algebras. We proved in Adámek *et al.* (2006) that an H-algebra is iterative if and only if for every *flat equation morphism*, that is, a morphism $e : X \longrightarrow HX + A$ with X finitely presentable, there exists a unique *solution* defined by the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [a,A]} \\ HX+A & \xrightarrow[He^\dagger+A]{} & HA+A \end{array}$$

(1) For $H = H_\Sigma$ and $\mathscr{A} = \mathsf{Set}$, the above concept is the concept of an iterative Σ-algebra A given in the Introduction: in fact, every ideal system (1.1) has a unique solution if and only if every flat system (one, where the right-hand sides t_i are either elements of A or flat terms $\sigma(x_{i_1}, \dots, x_{i_k})$ for a k-ary operation σ) has a unique solution. See Nelson (1983).
(2) In particular, a unary algebra $a : A \longrightarrow A$ is iterative if and only if its operation a has a fixed point, which is the unique cycle of a (Adámek *et al.* 2006).
(3) Analogously, for algebras on two unary operations, a and b, an algebra is iterative if and only if for every non-empty word on $\{a, b\}$ the corresponding derived operation has a unique fixed point (Adámek *et al.* 2006).

Example 3.4 (Bruno Courcelle, private communication). Consider the case of two unary operations a and b that commute:

$$a \cdot b = b \cdot a.$$

The corresponding free-algebra monad $\mathbb{M}$ on Set can be described on objects Z by

$$MZ = \{a\}^* \times \{b\}^* \times Z.$$

Every iterative $\mathbb{M}$-algebra (A, a, b) has a joint fixed point of a and b. In fact, the recursive equation $x \approx a(x)$ for a fixed point of a is represented by the ideal equation morphism

$$e : \{x\} \longrightarrow \{a\}^* \times \{b\}^* \times (\{x\} + A), \quad e(x) = (a, \varepsilon, x).$$

Its solution $e^\dagger : \{x\} \longrightarrow A$ is an element $t = e^\dagger(x)$, which is also a fixed point of b. To verify this, all we need to show is that $b(t)$ is a fixed point of a, and then $t = b(t)$ follows from the uniqueness of $e^\dagger$. We have

$$b(t) = b(a(t)) = a(b(t)).$$

Consequently, t is the unique fixed point of each $a^n \cdot b^k$. This shows that commutativity of unary operations trivialises iterativity.

Example 3.5. In contrast, commutativity of one binary operation $*$, that is, the law

$$x * y = y * x,$$

does not make iterativity trivial. This follows from Example 3.3: take the functor of unordered pairs as H and use Example 2.8(3).

Example 3.6. Unfortunately, the associativity of a binary operation

$$x * (y * z) = (x * y) * z$$

does trivialise iterativity. Here we take the monad

$$MZ = Z^+ \quad \text{(free semigroup on } Z\text{)}.$$

For every semigroup A that is iterative with respect to $\mathbb{M}$, there exists an absorbing element $0 \in A$:

$$0 * x = 0 = x * 0 \quad \text{for all } x.$$

In fact, since A is an iterative semigroup, the unique idempotent 0 (the unique solution of $x \approx x * x$) fulfills

$$0 * s = 0 \quad \text{for all } s \in A.$$

To prove this, using $\bar{s}$ to denote the unique solution of $x \approx x * s$, we have $\bar{s} * \bar{s}$ is also a solution:

$$(\bar{s} * \bar{s}) * s = \bar{s} * (\bar{s} * s) = \bar{s} * \bar{s}.$$

Since solutions are unique, $\bar{s} * \bar{s} = \bar{s}$; and since idempotents are unique, $\bar{s} = 0$. Analogously,

$$s * 0 = 0 \quad \text{for all } s \in A.$$

Remark 3.7. Notice that the existence of a unique idempotent element that is absorbing is a necessary but not a sufficient condition for iterativity of semigroups. In fact, consider the semigroup of all 2 by 2 matrices with entries from $\mathbb{N}$ whose determinant is 0, with matrix multiplication as the operation. The zero matrix is the unique idempotent in this semigroup, and it is absorbing. However, the formal equation $x \approx a * x * b$ has for

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

two different solutions:

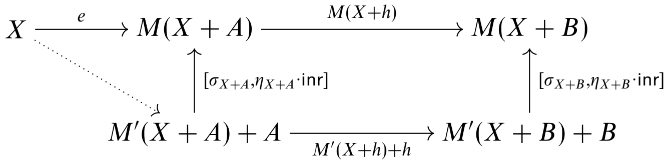
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The situation is simpler for a single binary associative and commutative operation. Evelyn Nelson (Nelson 1983) observed that among commutative semigroups, the iterative ones are precisely those with a unique idempotent element that is, moreover, absorbing.

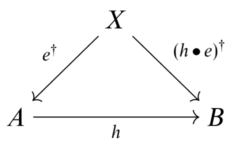
Convention 3.8. Given an equation morphism $e : X \longrightarrow M(X + A)$, then for every morphism $h : A \longrightarrow B$, we obtain a new equation morphism:

$$h \bullet e \equiv X \xrightarrow{e} M(X + A) \xrightarrow{M(X+h)} M(X + B).$$

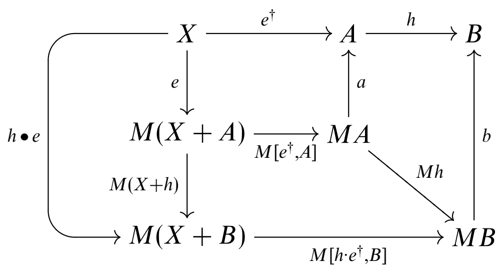
If e is guarded, so is $h \bullet e$. In fact, in the following diagram, the left-hand triangle commutes, since e is guarded, and the right-hand square commutes by the naturality of σ and η:



Proposition 3.9 (Solution-preserving morphisms = homomorphisms). Given iterative algebras (A, a) and (B, b), a morphism $h : A \longrightarrow B$ of $\mathscr{A}$ is a homomorphism if and only if for every guarded equation morphism $e : X \longrightarrow M(X + A)$ the composite $h \cdot e^\dagger$ is a solution of $h \bullet e$ in B:



Proof. If h is a homomorphism, we see that $h \cdot e^\dagger$ solves $h \bullet e$:



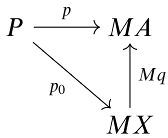
The uniqueness of solutions means the desired triangle commutes.

Conversely, assuming that h preserves solutions, we prove that h is a homomorphism.

Recall that M is a finitary functor and, since the base category is locally finitely presentable, A is a filtered colimit of the comma-catgegory $\mathscr{A}_{\text{fp}}/A$ of all morphisms $q : X \longrightarrow A$ where X is a finitely presentable object of $\mathscr{A}$. It then follows that MA is a filtered colimit with the colimit cocone $Mq : MX \longrightarrow MA$. Therefore, for every morphism

$$p : P \longrightarrow MA$$

in $\mathscr{A}_{\mathrm{fp}}/MA$, there exists a factorisation through Mq for some $q : X \longrightarrow A$ in $\mathscr{A}_{\mathrm{fp}}/A$:



(3.1)

To prove that h is a homomorphism, that is, $h \cdot a = b \cdot Mh$, it is sufficient to verify that for every p in $\mathscr{A}_{\mathrm{fp}}/MA$, we have

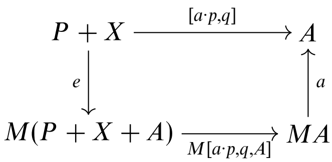
$$h \cdot a \cdot p = b \cdot Mh \cdot p. \tag{3.2}$$

Define an equation morphism $e : P + X \longrightarrow M(P + X + A)$ to have components

$$e \cdot \mathsf{inr} \equiv X \xrightarrow{\;q\;} A \xrightarrow{\;\mathsf{inr}\;} P + X + A \xrightarrow{\;\eta_{P+X+A}\;} M(P + X + A)$$

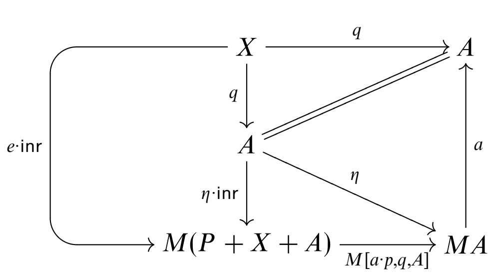
$$e \cdot \mathsf{inl} \equiv P \xrightarrow{\;p_0\;} MX \xrightarrow{\;[\sigma,\eta]^{-1}\;} M'X + X \xrightarrow{\;M'\mathsf{inr}+q\;} M'(P + X) + A \xrightarrow{\;[\sigma\cdot M'\mathsf{inl},\eta\cdot\mathsf{inr}]\;} M(P + X + A),$$

which is obviously guarded. We prove that $e^{\dagger} = [a \cdot p, q]$, in other words, that the square



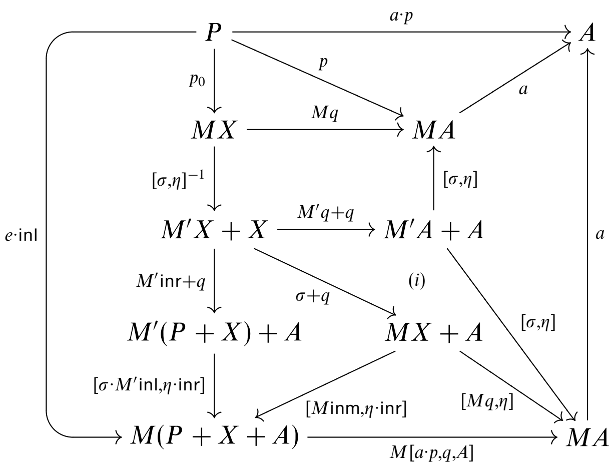
(3.3)

commutes. To see this, consider the components of $P + X$ separately. The right-hand component of diagram (3.3) is the outer shape of the diagram



Its lower triangle commutes because of the naturality of η, the right-hand triangle is the unit law for the algebra a, and the remaining two parts are obvious.

For the left-hand component of diagram (3.3), we obtain a commutative diagram:



The left-hand part is the definition of e, and the right-hand and uppermost parts are obvious. The upper middle triangle is (3.1). Part (i), the square above it, and the two triangles below it commute because of the naturality of σ and η, respectively. This proves that diagram (3.3) commutes.

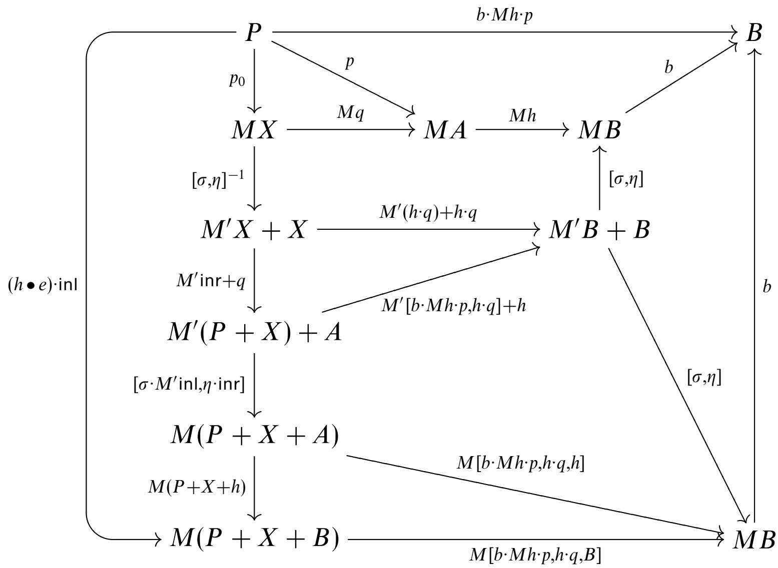
Since h preserves solutions, we have

$$(h \bullet e)^{\dagger} = h \cdot e^{\dagger} = h \cdot [a \cdot p, q] = [h \cdot a \cdot p, h \cdot q]\,. \tag{3.4}$$

Equation (3.2) follows from (3.4) and the equation

$$(h \bullet e)^{\dagger} = [b \cdot Mh \cdot p, h \cdot q] \,:\, P + X \longrightarrow MA, \tag{3.5}$$

which we prove by verifying that $[h \cdot a \cdot p, h \cdot q]$ is a solution of $h \bullet e$. To do this, consider the components of $P + X$ separately again. From equation (3.4), the right-hand component, $h \cdot q$, is indeed the right-hand component of $(h \bullet e)^{\dagger}$. For the left-hand component $b \cdot Mh \cdot p$, it is sufficient to verify the commutativity of the diagram



The left-hand part commutes from the definition of e and $h \bullet e$. The right-most and upper right-hand parts are trivial. The upper left-hand triangle is (3.1). The square below it commutes by the naturality of σ and η, as does the big middle part. The remaining two triangles are obvious, which proves equation (3.2). □

Remark 3.10.

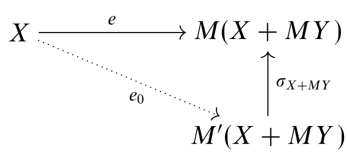
(1) The above proposition shows that the 'correct' concept of morphism for iterative algebras is the ordinary homomorphism.

(2) The relationship between iterativity of algebras and that of monads has the expected form, as shown by the following theorem.

Theorem 3.11. An ideal monad is iterative if and only if every free Eilenberg–Moore algebra is iterative.

Proof.

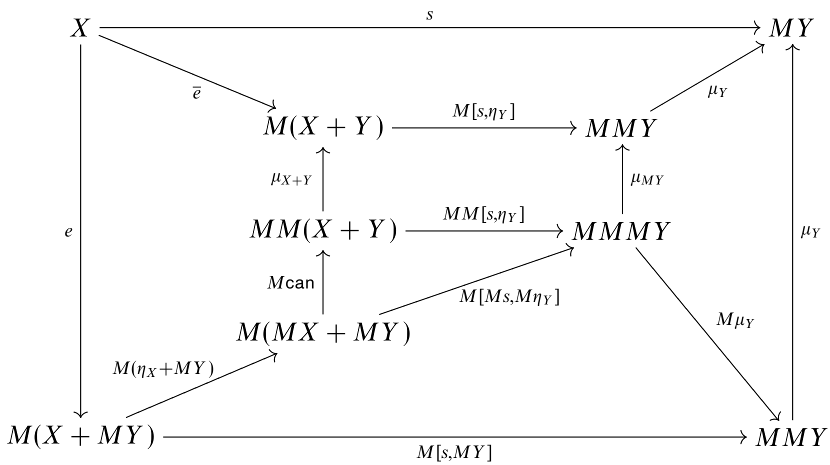
(1) Let $\mathbb{M}$ be an iterative monad. We wish to prove the iterativity of (MY, μ_Y) for every object Y. Let



be an ideal equation morphism. We form the equation morphism $\overline{e} : X \longrightarrow M(X + Y)$ as follows:

$$X \xrightarrow{e} M(X + MY) \xrightarrow{M(\eta_X + MY)} M(MX + MY) \xrightarrow{M\mathsf{can}} MM(X + Y) \xrightarrow{\mu_{X+Y}} M(X + Y).$$

It is easy to prove that $\overline{e}$ is ideal using the fact that e is. It is thus sufficient to prove that solutions of e in the algebra (MY, μ_Y) are precisely the solutions of $\overline{e}$ with respect to the monad $\mathbb{M}$. To this end, consider the diagram

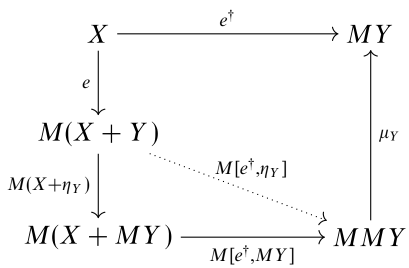


Suppose s is a solution of $\overline{e}$. Then the upper part of the above diagram commutes, and all the other inner parts commute also: the left-hand part commutes by the definition of $\overline{e}$; the right-hand part commutes by the monad laws of the monad $\mathbb{M}$; the middle square commutes by the naturality of μ; the triangle below it trivially commutes; and, finally, to see the commutativity of the lower part, remove M and consider the two components of $X + MY$ separately. So the outer square commutes, which means that s solves e in the free algebra (MY, μ_Y).

Conversely, if s is a solution of e, the outer square commutes, and since all the inner parts except for the upper square commute, the upper square also commutes. So s is a solution of $e^\dagger$.

Thus, since $\overline{e}$ has a unique solution, we conclude that e does too.

(2) Let all algebras (MY, μ_Y) be iterative. For every ideal equation morphism $e : X \longrightarrow M(X + Y)$ with respect to M, the equation morphism $\eta_Y \bullet e$ (see Convention 3.8) has a unique solution $e^\dagger$ in the free algebra (MY, μ_Y):



Since $M[e^\dagger, \eta_Y] = M[e^\dagger, MY] \cdot M(X + \eta_Y)$, this shows precisely that $\mathbb{M}$ is an iterative monad. □

Proposition 3.12. A limit or a filtered colimit of iterative algebras in $\mathscr{A}^{\mathbb{M}}$ is iterative.

The proof is completely analogous to the proof of Adámek *et al.* (2006, Proposition 2.15), so we omit it here.

Corollary 3.13. Every algebra has an iterative reflection (that is, the full subcategory of $\mathscr{A}^{\mathbb{M}}$ formed by iterative algebras is reflective). In particular, every object Y of $\mathscr{A}$ generates a free iterative algebra

$$\widehat{M}Y\,,$$

which is a reflection of the free algebra MY.

Proof. The statement follows from Adámek and Rosický (1994, 2.48 and 2.78). □

Example 3.14.

(1) For the monad $\mathbb{F}_\Sigma$ of finite Σ-trees, in other words, for classical Σ-algebras, a free iterative algebra on A is the algebra of rational Σ-trees on A (Nelson 1983).
In particular, if

$$MA = \text{finite binary trees on } A$$

(the case of a single binary operation), we have a free iterative algebra on A given by

$$\widehat{M}A = \text{rational binary trees on } A.$$

(2) Analogously, for a single commutative binary operation, a free iterative algebra is the algebra of all rational binary unordered trees of Example 2.8(3).

(3) For a single associative binary operation (semigroups),

$$MA = A^+,$$

we have a free iterative algebra on A given by

$$\widehat{M}A = A^+ + \{0\}, \quad 0 \text{ absorbing}.$$

In fact, in view of Example 3.6, all we have to prove is that the algebra $A^+ + \{0\}$ is iterative. Let

$$e : X \longrightarrow (X + A)^+$$

be an ideal equation morphism. One solution is the following function $e^\dagger : X \longrightarrow A^+ + \{0\}$:

(a) For all variables x_0 in $X_0 = e^{-1}(A^+)$, put $e^\dagger(x_0) = e(x_0)$. For all variables x_1 in

$$X_1 = e^{-1}(X_0 + A)^+,$$

let $e^\dagger(x_1)$ be the word obtained from $e(x_1)$ by substituting every variable $y \in X_0$ with $e^\dagger(y)$ above, and analogously for all X_n ($n \in \mathbb{N}$), where

$$X_{n+1} = e^{-1}(X_n + A)^+.$$

(b) For all variables x in $X \setminus \bigcup_{n\in\mathbb{N}} X_n$, put $e^\dagger(x) = 0$.

To verify uniqueness, let $f : X \longrightarrow A^+ + \{0\}$ be a solution of e. It is easy to see by induction on n that $f(x)$ is equal to $e^\dagger(x)$ for all $x \in X_n$. It then remains to prove that

$$f(x_0) = 0 \quad \text{for all } x_0 \in X \setminus \textstyle\bigcup_{n\in\mathbb{N}} X_n.$$

In order to show a contradiction, we assume, to the contrary, that there exists $x_0 \in X \setminus \bigcup_{n\in\mathbb{N}} X_n$ with $f(x_0) \in A^+$.

The word $e(x_0)$ in $(X + A)^+$ contains at least one variable $x_1 \in X \setminus \bigcup_{n\in\mathbb{N}} X_n$ (in fact, if all variables in $e(x_0)$ lay in X_n, we would have $x_0 \in X_{n+1}$). Since $f(x_0) \in A^+$, it is clear that $f(x_1) \neq 0$. Consequently, $f(x_1) \in A^+$ and the length of the word $f(x_0)$ is bigger than that of $f(x_1)$: recall that $e(x_0) \neq x_1$ since e is ideal, and the word $f(x_1)$ is a subword of $f(x_0)$. Analogously, for x_1, the word $e(x_1)$ contains $x_2 \in X \setminus \bigcup_{n\in\mathbb{N}} X_n$, and the length of $f(x_1)$ is bigger than that of $f(x_2)$, and so on. Since X is a finite set, we obtain a cycle, which contradicts the above growing length of words.

(4) For the monad

$$MA = \{a\}^* \times \{b\}^* \times A$$

of two commuting unary operations, the free iterative algebras are given by

$$\widehat{M}A = \{a\}^* \times \{b\}^* \times A + \{0\} \quad \text{with } a(0) = 0 = b(0).$$

In fact, in view of Example 3.4, all we need to prove is that the algebra $\widehat{M}A$ is iterative. The argument is entirely analogous to that of part (3). Let

$$e : X \longrightarrow \{a\}^* \times \{b\}^* \times (X + A) + \{0\}$$

be an ideal equation morphism. One solution is the function

$$e^\dagger : X \longrightarrow \{a\}^* \times \{b\}^* \times A + \{0\}$$

defined by:

(a) For all variables x_0 in $X_0 = e^{-1}(\{a\}^* \times \{b\}^* \times A)$, put $e^\dagger(x_0) = e(x_0)$, and for all variables in

$$X_1 = e^{-1}(\{a\}^* \times \{b\}^* \times (X_0 + A)),$$

let $e^\dagger(x_1)$ be the element of $\{a\}^* \times \{b\}^* \times A$ obtained from $e(x_1)$ by substituting the variable $x_0 \in X_0$ (if any) by the already given $e^\dagger(x_0)$, and so on.

(b) For all variables x in $X \setminus \bigcup_{n\in\mathbb{N}} X_n$, put $e^\dagger(x) = 0$.

The verification of uniqueness is analogous to that given for part (3).

4. Conclusions and future research

The aim of this paper has been to prove that all ideal algebraic theories in Set can be freely completed to iterative theories of Calvin Elgot, and, more generally, to show that given an extensive, locally finitely presentable category, all ideal finitary monads on it have iterative reflections.

We have also extended the result of Evelyn Nelson (Nelson 1983) and Jerzy Tiuryn (Tiuryn 1980) characterising iterativity of theories by means of the iterativity of algebras for theories by showing that what they achieved for the classical Σ-algebras in fact holds for algebras for an arbitrary ideal monad.

There is an obvious missing step connecting our two results: the proof that for every ideal algebraic theory $\mathbb{M}$, the theory of free iterative $\mathbb{M}$-algebras is an iterative reflection of $\mathbb{M}$. We have not presented such a proof here because we do not yet know one that would work in the present generality. However, we do have a proof that the above result holds for ideal algebraic theories in Set, see Adámek *et al.* (2009b).

We have given several examples of ideal monads $\mathbb{M}$ (for example, the finite-list monad or the monad of two commuting unary operations) for which the iterative reflection is trivial: one just adds a single element. How one might be able to see directly that an ideal monad has a trivial iterative reflection remains an open problem.

Appendix A. Proof of the Reflection theorem (Theorem 2.14)

Notation A.1.

(1) We use

$$\mathsf{FE}(\mathscr{A})$$

to denote the category of all finitary endofunctors of $\mathscr{A}$ and

$$V : \mathsf{FM}_{\mathrm{id}}(\mathscr{A}) \longrightarrow \mathsf{FE}(\mathscr{A})$$

to denote the functor assigning the endofunctor M' to an ideal monad $M = M' + \mathrm{Id}$.

(2) For two natural transformations $m : M_1 \longrightarrow M_2$ and $n : N_1 \longrightarrow N_2$, we write $m * n : M_1N_1 \longrightarrow M_2N_2$ for the parallel composition defined by

$$m * n = mN_2 \cdot M_1n = M_2n \cdot mN_1.$$

Recall that for natural transformations m, n, p, q with appropriate domains and codomains, we have the *interchange law*

$$(m * n) \cdot (p * q) = (m \cdot p) * (n \cdot p). \tag{A.1}$$

Proof of Theorem 2.14. We will prove:

(a) $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ is a locally finitely presentable category.

(b) $\mathsf{IFM}_{\mathrm{id}}(\mathscr{A})$ is closed in it under limits and filtered colimits. The theorem then follows from the Reflection Theorem of Adámek and Rosický (1994).

Proof of part (a). This part of the proof is divided into showing that V:

(a1) has a left adjoint;

(a2) creates coequalisers of V-split pairs;

(a3) is finitary.

Beck's Theorem, see Mac Lane (1998), then tells us that V is monadic. Since $\mathsf{FE}(\mathscr{A})$ is locally finitely presentable by Adámek and Rosický (1994, 1.45 and 1.46), we conclude by Adámek and Rosický (1994, 2.78) that $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ is also.

(a1) *V has a left adjoint.*

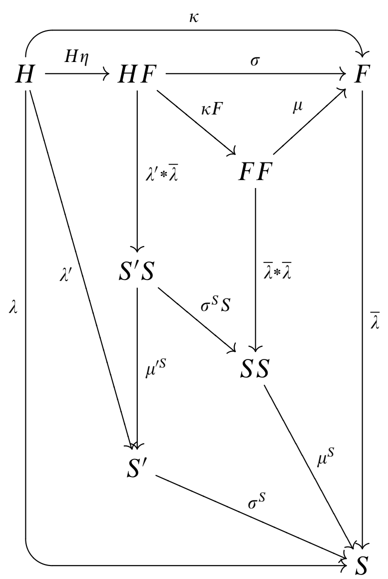
This follows from Barr's result, see Example 2.8(3), that if we are given an ideal monad $\mathbb{S}$ with $S = S' + \mathrm{Id}$ and a natural transformation $\lambda : H \longrightarrow S$ that factors through the coproduct injection $\sigma^S : S' \longrightarrow S$ as shown by

$$\lambda \equiv H \xrightarrow{\lambda'} S' \xrightarrow{\sigma^S} S, \tag{A.2}$$

then we have the unique monad morphism $\overline{\lambda} : \mathbb{F} \longrightarrow \mathbb{S}$ with

$$\lambda = \overline{\lambda} \cdot \kappa \,. \tag{A.3}$$

We will show that the diagram

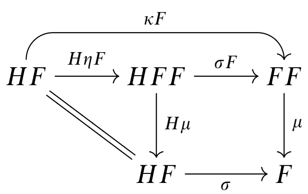


(A.4)

commutes, where $\kappa = \sigma \cdot H\eta : H \longrightarrow F$ denotes the universal natural transformation of the free monad $\mathbb{F}$ on H, see Example 2.8(3).

In fact, the outer part commutes by definition of $\overline{\lambda}$, the upper part is the definition of κ, and the lower left-hand part commutes by (A.2). The right-hand square commutes since $\overline{\lambda}$ is a monad morphism. The lower part commutes since $\mathbb{S}$ is an ideal monad. The upper middle square commutes because of the interchange law (A.1) and equations (A.3) and (A.2). For the upper left-hand triangle, we use the interchange law again, together with the equations $\overline{\lambda} \cdot \eta = \eta^S$ and $\mu'^S \cdot S'\eta^S = \mathrm{id}_{S'}$.

Finally, we prove that the upper right-hand triangle commutes. To do this, consider the diagram



The upper part commutes by the definition of κ, and the left-hand triangle by the unit laws of the monad $\mathbb{F}$. The right-hand square commutes since the monad multiplication μ arises componentwise as an H-algebra homomorphism from the universal property of the free H-algebra FFX, for any object X (*cf.* Example 2.8(3)).

All we need to prove now is that $\overline{\lambda}$ is an ideal monad morphism. In fact, the natural transformation

$$\overline{\lambda'} \equiv HF \xrightarrow{\lambda' * \overline{\lambda}} S'S \xrightarrow{\mu'^S} S'$$

is the desired restriction of $\overline{\lambda}$ to the ideals of F (which is HF, see Example 2.4(3)) and S, which follows from the commutativity of diagram (A.4).

(a2) *V creates coequalisers of V-split pairs.*
Let

$$S \underset{g}{\overset{f}{\rightrightarrows}} T$$

be a pair of ideal monad morphisms that is V-split. This means that we have a diagram

$$\begin{array}{ccccc} & \xrightarrow{f'} & & \xrightarrow{c'} & \\ S' & \xrightarrow{g'} & T' & \xleftarrow{s'} & C' \\ & \xleftarrow{t'} & & & \end{array}$$

in $\mathsf{FE}(\mathscr{A})$ such that $c'f' = c'g'$, $c's' = \mathrm{id}_{C'}$, $s'c' = g't'$ and $f't' = \mathrm{id}_{T'}$. We apply the endofunctor $(-) + \mathrm{Id}$ to this diagram and obtain the split coequaliser

$$\begin{array}{ccccc} & \xrightarrow{f} & & \xrightarrow{c} & \\ S & \xrightarrow{g} & T & \xleftarrow{s} & C \\ & \xleftarrow{t} & & & \end{array}$$

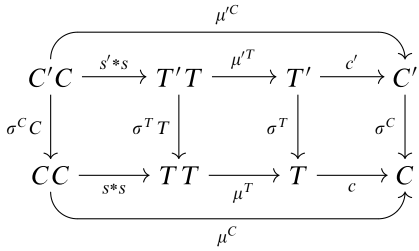
where $C = C' + \mathrm{Id}$, $c = c' + \mathrm{id}$, $s = s' + \mathrm{id}$ and $t = t' + \mathrm{id}$.
The functor $U : \mathsf{FM}(\mathscr{A}) \longrightarrow \mathsf{FE}(\mathscr{A})$ assigning to every finitary monad (M, η, μ) the functor M is monadic. This follows from Barr (1970) by an easy application of Beck's Theorem (Mac Lane 1998). Thus, there exists a unique structure of a finitary monad on C such that $c : T \longrightarrow C$ is a coequaliser of f and g in $\mathsf{FM}(\mathscr{A})$.
We now only need to prove that C is an ideal monad and that c is a coequaliser of f and g in $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$. The latter is clear: in fact, we observe that for any ideal monad D and an ideal monad morphism $d : T \longrightarrow D$ with $df = dg$, because $cs = \mathrm{id}$, the unique induced monad morphism $h : C \longrightarrow D$ with $hc = d$ is given by $h = ds$. So, since $d = d' + \mathrm{id}$ is an ideal monad morphism, we have $h = d's' + \mathrm{id}$. Therefore, h is a morphism of $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$. To prove that C is an ideal monad, we again use $cs = \mathrm{id}$ to conclude that the multiplication $\mu^C : CC \longrightarrow C$ is obtained as follows:

$$\mu^C \equiv CC \xrightarrow{s*s} TT \xrightarrow{\mu^T} T \xrightarrow{c} C.$$

Now consider

$$\mu'^C \equiv C'C \xrightarrow{s'*s} T'T \xrightarrow{\mu'^T} T' \xrightarrow{c'} C'.$$

This is the required restriction of μ^C, that is, the diagram



commutes: in fact, the right-hand square commutes since $c = c' + \mathrm{id}$, the middle one does since T is an ideal monad, and for the left-hand square, we use the interchange law (A.1) and the fact that $s = s' + \mathrm{id}$.

(a3) *V is finitary.*
In fact, the above forgetful functor $U : \mathsf{FM}(\mathscr{A}) \longrightarrow \mathsf{FE}(\mathscr{A})$ is clearly finitary. Moreover, $\mathsf{FE}(\mathscr{A})$ is closed under colimits in the functor category $[\mathscr{A}, \mathscr{A}]$, so filtered colimits are formed pointwise in $\mathsf{FM}(\mathscr{A})$. Since (filtered) colimits commute with finite coproducts, we conclude that filtered colimits are formed pointwise in $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ also: given a filtered diagram $D : \mathscr{T} \longrightarrow \mathsf{FM}_{\mathrm{id}}(\mathscr{A})$, it has the form $D = D' + \mathrm{Id}$ and its poinwise colimit $\operatorname{colim} D$ has the form $\operatorname{colim} D' + \mathrm{Id}$. Therefore, $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ is closed under filtered colimits in $\mathsf{FM}(\mathscr{A})$ and $V : \mathsf{FM}_{\mathrm{id}}(\mathscr{A}) \longrightarrow \mathsf{FE}(\mathscr{A})$ preserves filtered colimits.

Proof of part (*b*). We proceed in two steps:

(b1) *$\mathsf{IFM}_{\mathrm{id}}(\mathscr{A})$ is closed under filtered colimits in $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$.*
Let $D : \mathscr{T} \longrightarrow \mathsf{IFM}_{\mathrm{id}}(\mathscr{A})$ be a filtered diagram with objects $Dt = \mathbb{M}_t = (M_t, \eta_t, \mu_t)$ where $M_t = M'_t + \mathrm{Id}$. Let

$$m_t : \mathbb{M}_t \longrightarrow \mathbb{M} \quad \text{with } \mathbb{M} = (M, \eta, \mu)$$

be a colimit cocone in $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ with $M = M' + \mathrm{Id}$ and $m_t = m'_t + \mathrm{id}$. Recall from (a3) that $M' = \operatorname{colim} M'_t$, so $M'Y = \operatorname{colim} M'_t Y$ for finitely presentable objects Y.
We will prove that $\mathbb{M}$ is iterative using the formulation of Lemma 2.7. Let a morphism

$$e' : X \longrightarrow M'(X + A) \quad \text{with } X \text{ finitely presentable}$$

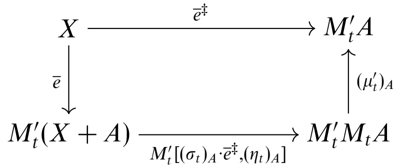
be given. We prove that there exists a unique morphism $e^\ddagger$ such that diagram (2.5) commutes. Since X is finitely presentable and e' is a morphism into a filtered colimit $M'(X + A) = \operatorname{colim} M'_t(X + A)$, e' factors through one of the colimit morphisms, $(m'_t)_{X+A} : M'_t(X + A) \longrightarrow M'(X + A)$. More precisely, there exists a morphism

$$\overline{e} : X \longrightarrow M'_t(X + A) \quad \text{with } e' = (m'_t)_{X+A} \cdot \overline{e}.$$

Since $\mathbb{M}_t$ is an iterative monad, we have a unique morphism

$$\overline{e}^\ddagger : X \longrightarrow M'_t A$$

such that the square

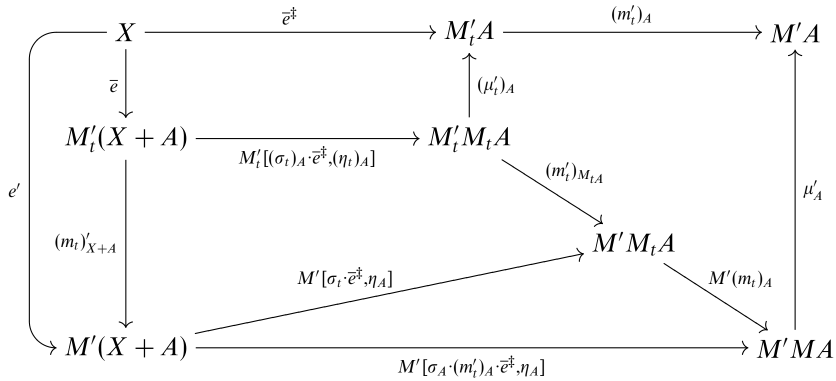


(A.5)

commutes. It follows that e' has a solution with respect to $\mathbb{M}$: the morphism

$$e^{\ddagger} \equiv X \xrightarrow{\overline{e}^{\ddagger}} M'_tA \xrightarrow{(m'_t)_A} M'A$$

makes the diagram



(A.6)

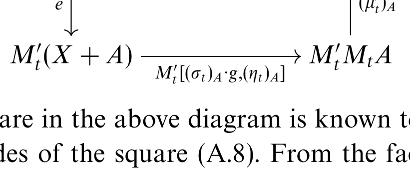
commute – the upper left-hand square is (A.5); the middle part is the naturality of m'_t; the lower triangle follows from m_t being an ideal monad morphism ($m_t \cdot \eta_t = \eta$ and $m_t \cdot \sigma_t = \sigma \cdot m'_t$) and the right-hand square commutes by Remark 2.11.
To prove uniqueness, let a morphism

$$e^{\ddagger} : X \longrightarrow M'A \quad \text{with } e^{\ddagger} = (\mu')_A \cdot M'[\sigma_A \cdot e^{\ddagger}, \eta_A] \cdot e' \tag{A.7}$$

be given. Since $M'A = \operatorname{colim} M'_tA$ is a filtered colimit and X is finitely presentable, we know that $e^{\ddagger}$ factors through one of the colimit maps $(m'_t)_A$ – and without loss of generality, we can assume that this index t is the same as the one we used above. Given

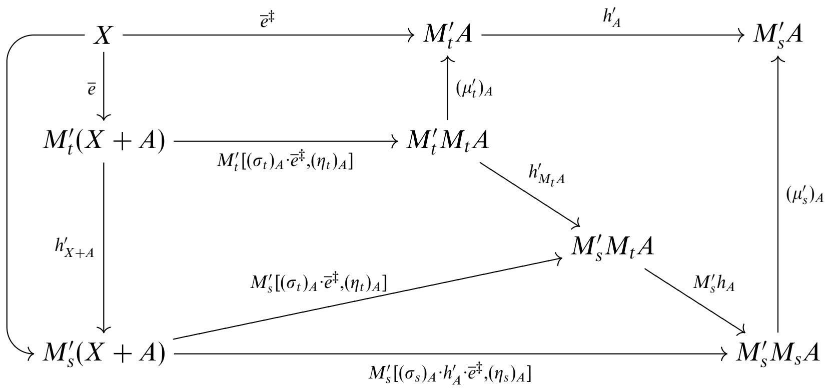
$$e^{\ddagger} = (m'_t)_A \cdot g \quad \text{for } g : X \longrightarrow M'_tA,$$

consider the above diagram (A.6) in which all occurences of $\overline{e}^{\ddagger}$ are substituted by g. Then all inner parts commute as before, with the exception of the upper square



(A.8)

Now, since the outer square in the above diagram is known to commute, we conclude that $(m_t)'_A$ merges the sides of the square (A.8). From the fact that $(m'_t)_A$ is a colimit map of the filtered colimit $M'A = \mathrm{colim}(M'_tA)$ and that X is finitely presentable, we deduce that the two sides of (A.8) are also merged by some connecting morphism $h'_A : M'_tA \longrightarrow M'_sA$ of the diagram. But h'_A also merges the two sides of (A.5) because of the commutative diagram



We conclude that $h'_A \cdot g = h'_A \cdot \overline{e}^\ddagger$.
Consequently, $e^\ddagger$ is determined uniquely, since h' is a connecting morphism, we get $m'_t = m'_s \cdot h'$, and thus

$$e^\ddagger = (m'_t)_A \cdot g = (m'_s)_A \cdot h'_A \cdot g = (m'_s)_A \cdot h'_A \cdot \overline{e}^\ddagger = (m'_t)_A \cdot \overline{e}^\ddagger.$$

(b2) $\mathsf{IFM}_{\mathrm{id}}(\mathscr{A})$ *is closed under limits in* $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$.
First recall from part (a) that $V : \mathsf{FM}_{\mathrm{id}}(\mathscr{A}) \longrightarrow \mathsf{FE}(\mathscr{A})$ preserves limits. The category $\mathsf{FE}(\mathscr{A})$ is coreflective in the functor category $[\mathscr{A},\mathscr{A}]$: given a functor H, its finitary coreflection is obtained from its domain restriction H_0 to the full subcategory $\mathscr{A}_{\mathrm{fp}}$ of all finitely presentable objects by a left Kan extension of H_0 along the inclusion functor $\mathscr{A}_{\mathrm{fp}} \hookrightarrow \mathscr{A}$. Consequently, limits in $\mathsf{FE}(\mathscr{A})$ are computed pointwise when evaluation takes place at a finitely presentable object. We conclude that for a diagram

$$D : \mathscr{T} \longrightarrow \mathsf{IFM}_{\mathrm{id}}(\mathscr{A})$$

with objects $Dt = \mathbb{M}_t$ (notation as in (b1)) and its limit cone $m_t : \mathbb{M} \longrightarrow \mathbb{M}_t$ in $\mathsf{FM}_{\mathrm{id}}(\mathscr{A})$ where $\mathbb{M} = (M, \eta, \mu)$ and $m_t = m'_t + \mathrm{id}$, we have

$$M'Y = \lim M'_tY \quad \text{for finitely presentable } Y.$$

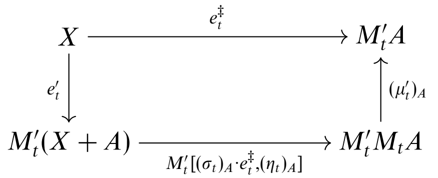
We now prove that $\mathbb{M}$ is iterative by applying the condition of Lemma 2.7.
Thus, let a morphism

$$e' : X \longrightarrow M'(X+A) \quad \text{with } X \text{ finitely presentable}$$

be given. For every t, the morphism

$$e'_t \equiv X \xrightarrow{\;e'\;} M'(X+A) \xrightarrow{\;(m_t)'_{X+A}\;} M'_t(X+A)$$

allows for a unique $e_t^\ddagger : X \longrightarrow M'_t A$ with a commutative square

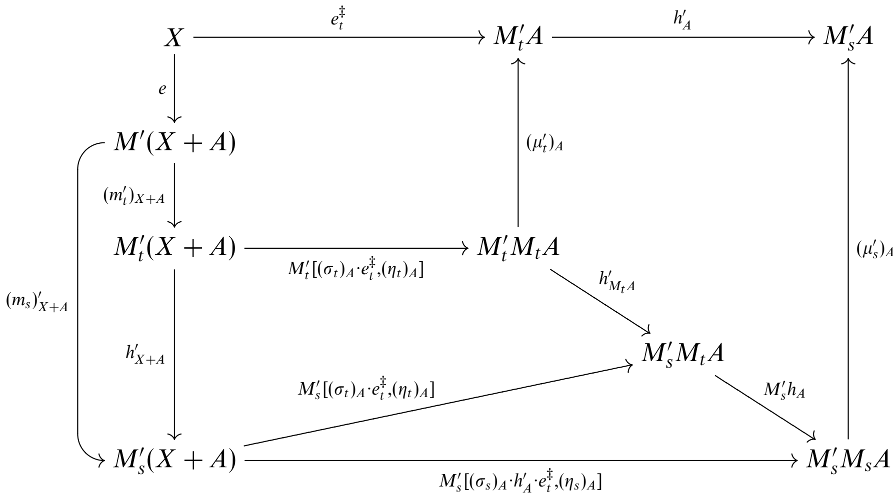


(A.9)

The cone of all $e_t^\ddagger$ is compatible – if $h : \mathbb{M}_t \longrightarrow \mathbb{M}_s$ is a connecting morphism of the diagram D with $h = h' + \mathrm{id}$, we are going to prove that

$$e_s^\ddagger = h'_A \cdot e_t^\ddagger : X \longrightarrow M'_s A$$

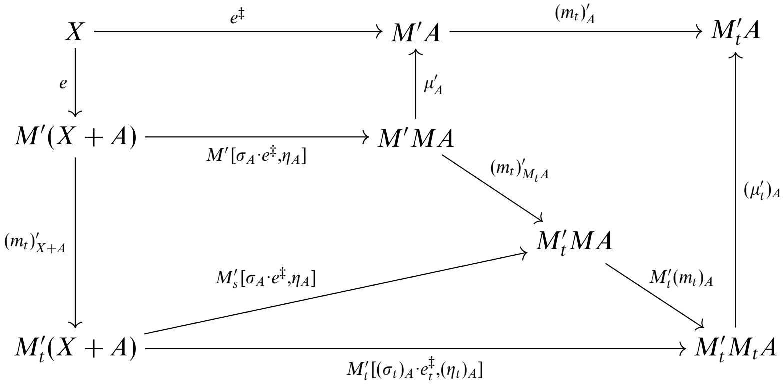
by verifying that the right-hand side has the property (2.5) characterising $e_s^\ddagger$:



In fact, the lower triangle commutes: delete M'_s and consider the components of $X + A$ separately. The right-hand part follows from $h \cdot \eta_t = \eta_s$, and the left-hand part follows from $h \cdot \sigma_t = \sigma_s \cdot h'$ (recall that $h : \mathbb{M}_t \longrightarrow \mathbb{M}_s$ is an ideal monad morphism). All the other parts are obvious (in fact, we can use similar arguments to those we used in the verification of the commutativity of (A.6)). We know that $M'A = \lim M'_t A$ and can define $e^\ddagger : X \longrightarrow M'A$ by

$$(m_t)'_A \cdot e^\ddagger = e_t^\ddagger \quad \text{for all } t \text{ in } \mathscr{T}. \tag{A.10}$$

We now verify that $e^\ddagger$ has the desired property that the upper left-hand square in the diagram



commutes. In fact, the whole diagram commutes: its outer square commutes because of (A.9) and (A.10), and, apart from the (desired) upper square, the inner parts also clearly commute. The cone of all

$$(m'_t)_A : M'A \longrightarrow M'_tA$$

is a limit cone, and thus a monocone. Consequently, the desired square also commutes. To prove the uniqueness of $e^\ddagger$, suppose that in the above diagram we know that the upper left-hand square commutes for some morphism $e^\ddagger : X \longrightarrow M'A$. Then the whole diagram commutes, which shows that $(m'_t)_A \cdot e^\ddagger : X \longrightarrow M'_tA$ has the property characterising $e_t^\ddagger$. This proves that

$$(m'_t)_A \cdot e^\ddagger = e_t^\ddagger \quad \text{for all } t \text{ in } \mathscr{T},$$

and this determines $e^\ddagger$, since $(m'_t)_A$ is a monocone. □

Appendix B. Ideal and guarded equation morphisms

In this appendix we clarify the relationship between ideal equation morphisms, as used by Calvin Elgot, and guarded ones – see Remark 2.6. We still assume that the base category $\mathscr{A}$ is locally finitely presentable and extensive.

Proposition B.1. For every ideal monad $\mathbb{M}$ the following conditions are equivalent:

(1) all guarded equation morphisms $e : Z \longrightarrow M(Z+A)$, with A arbitrary, have unique solutions;
(2) all ideal equation morphisms $e : Z \longrightarrow M(Z+A)$, with A finitely presentable, have unique solutions.

Proof. Assuming (2), we prove (1). Let

$$\begin{array}{ccc} Z & \xrightarrow{\;e\;} & M(Z+A) \\ & \searrow^{e_0} & \uparrow{\scriptstyle [\sigma_{Z+A},\eta_{Z+A}\cdot \mathsf{inr}]} \\ & & M'(Z+A)+A \end{array}$$

be a guarded equation morphism. Since the base category is extensive, we have a decomposition

$$Z = X + Y \quad \text{and} \quad e_0 = f_0 + g_0$$

for some morphisms $f_0 : X \longrightarrow M'(Z+A)$ and $g_0 : Y \longrightarrow A$.

(a) Assume that A is a finitely presentable object. The ideal equation morphism

$$f \equiv X \xrightarrow{f_0} M'(Z+A) = M'(X+Y+A) \xrightarrow{M'(X+[g_0,A])} M'(X+A) \xrightarrow{\sigma_{X+A}} M(X+A)$$

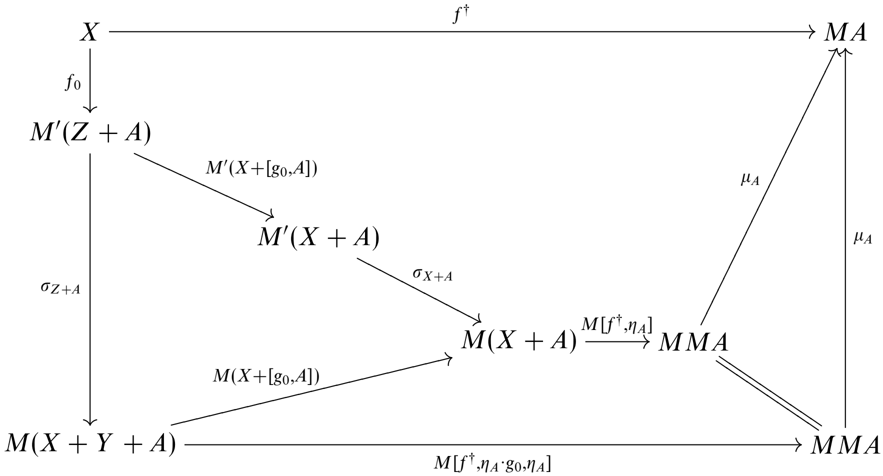
has a unique solution $f^\dagger : X \longrightarrow MA$. We prove now that the morphism

$$e^\dagger = [f^\dagger, \eta_A \cdot g_0] : Z \longrightarrow MA$$

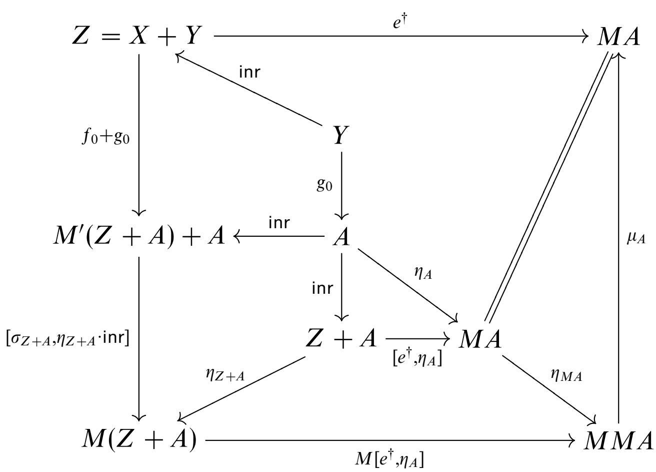
is a solution of e. To this end, consider the diagram

$$\begin{array}{ccc} X+Y=Z & \xrightarrow{[f^\dagger,\eta_A\cdot g_0]} & MA \\ \downarrow{\scriptstyle f_0+g_0} & & \\ M'(Z+A)+A & & \uparrow{\scriptstyle \mu_A} \\ \downarrow{\scriptstyle [\sigma_{Z+A},\eta_{Z+A}\cdot \mathsf{inr}]} & & \\ M(Z+A)=M(X+Y+A) & \xrightarrow[M[f^\dagger,\eta_A\cdot g_0,\eta_A]]{} & MMA \end{array}$$

In fact, we will analyse the two components of $X+Y$ separately. The right-hand component clearly yields $\eta_A \cdot g_0$ on both sides (since $\mu_A \cdot M\eta_A = \mathrm{id}$). The left-hand component commutes because in the diagram

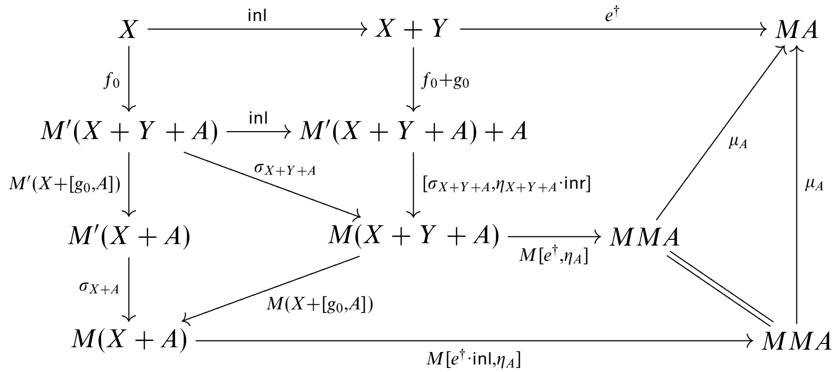


the upper part commutes by the definition of $f^\dagger$, the left-hand triangle commutes by the naturality of σ, the right-hand triangle is obvious and the lower square trivially commutes: delete M and consider the components of $X + Y + A$ separately.
To prove uniqueness, let $e^\dagger : X + Y \longrightarrow MA$ be a solution of e. We will verify $e^\dagger = [f^\dagger, \eta_A \cdot g_0]$ componentwise. For the right-hand component, consider the diagram



The outer square commutes because $e^\dagger$ is a solution of e, and since all other inner parts clearly commute, so does the upper right-hand part, as desired.

For the left-hand component it suffices to prove that $e^\dagger \cdot \mathsf{inl}$ is a solution of f:



In the above diagram, all inner parts commute: the upper left-hand square and the triangle below it trivially commute; the lower left-hand triangle commutes by the naturality of σ; for the lower square, delete M and consider the components separately using the already established equation $e^\dagger \cdot \mathsf{inr} = \eta_A \cdot g_0$; and for the upper right-hand part, recall that $e = [\sigma_{Z+A}, \eta_{Z+A} \cdot \mathsf{inr}] \cdot e_0$ and $e_0 = f_0 + g_0$.

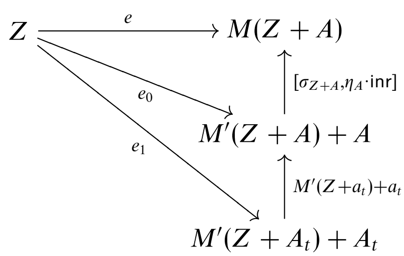
(b) Let A be arbitrary. We can express A as a filtered colimit

$$A = \operatorname*{colim}_{t \in T} A_t \quad \text{with a colimit cocone } a_t : A_t \longrightarrow A,\ t \in T,$$

of finitely presentable objects A_t. Then $Z + A$ is a filtered colimit of $Z + A_t$, and since the functor M' is finitary, we conclude that

$$M'(Z + A) + A = \operatorname{colim} M'(Z + A_t) + A_t \quad \text{with a colimit cocone } M'(Z + a_t) + a_t.$$

Consequently, the morphism $e_0 : Z \longrightarrow M'(Z + A) + A$ factors through one of the colimit morphisms:

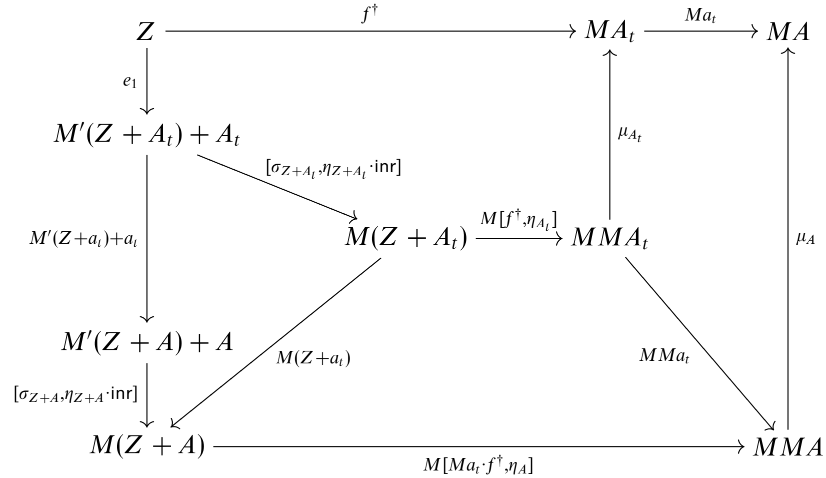


The equation morphism

$$f \equiv Z \xrightarrow{e_1} M'(Z + A_t) + A_t \xrightarrow{[\sigma_{Z+A_t}, \eta_{A_t} \cdot \mathsf{inr}]} M(Z + A_t)$$

is guarded, so, by part (a), it has a unique solution $f^\dagger : Z \longrightarrow MA_t$.

To verify that $Ma_t \cdot f^\dagger$ is a solution of e, consider the diagram



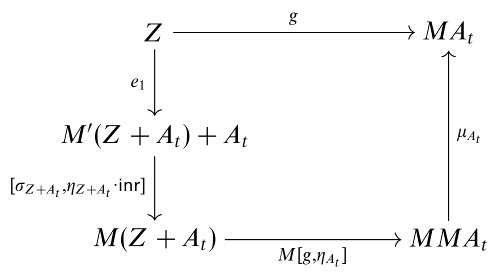
(B.11)

The upper left-hand part commutes because $f^\dagger$ solves f; the lower left-hand triangle commutes by the naturality of σ and η; and the right-hand part commutes by the naturality of μ. Finally, to see that the lower part is commutative, remove M and consider the coproduct components separately: it is obvious that the left-hand one commutes and the right-hand one commutes by the naturality of η.

To prove the uniqueness of solutions, let $e^\dagger : Z \longrightarrow MA$ be a solution of e. Since $MA = \operatorname{colim} MA_t$, the morphism $e^\dagger$ factors through the colimit map Ma_t for some index t – without loss of generality, we can assume that this is the same index as we used above. We thus have

$$e^\dagger = Ma_t \cdot g \quad \text{for some } g : Z \longrightarrow MA_t.$$

In the above diagram, substitute g for $f^\dagger$: all parts of the resulting diagram are commutative apart from, possibly, the upper square



and since the outer part of (B.11) now commutes, we conclude that Ma_t merges the two sides of the last square. The rest of the proof is completely analogous to the end of the proof of (b1) in Theorem 2.14: we find a connecting morphism $h : A_t \longrightarrow A_s$ such that Mh merges the two sides of the above square and conclude that

$$e^\dagger = Ma_s \cdot Mh \cdot f^\dagger = Ma_t \cdot f^\dagger. \qquad \square$$

Remark B.2. Proposition B.1 also holds 'locally': for a given algebra (A, a) and an ideal monad $\mathbb{M}$, the following conditions are equivalent:

(1) The algebra (A, a) is iterative.

(2) Every ideal equation morphism $e : X \longrightarrow M(X + A)$ has a unique solution in A.

The proof is similar to that of B.1.

Acknowledgements

We are grateful to the referees for their suggestions, which have improved our presentation. In particular, one of the referees suggested a simplification to the proof of Theorem 2.14 – see the remark following that theorem.

References

Aczel, P., Adámek, J., Milius, S. and Velebil, J. (2003) Infinite trees and completely iterative theories: a coalgebraic view. *Theoretical Computer Science* **300** 1–45.

Adámek, J. and Milius, S. (2006) Terminal coalgebras and free iterative theories. *Inform. and Comput.* **204** 1139–1172.

Adámek, J., Milius, S. and Velebil, J. (2006) Iterative algebras at work. *Mathematical Structures in Computer Science* **16** (6) 1085–1131.

Adámek, J., Milius, S. and Velebil, J. (2009a) Semantics of higher-order recursion schemes. Proceedings CALCO 2009. *Springer-Verlag Lecture Notes in Computer Science* **5728** 49–63.

Adámek, J., Milius, S. and Velebil, J. (2009b) A description of iterative reflections of monads. Extended abstract in Proc. FOSSACS 2009. *Springer-Verlag Lecture Notes in Computer Science* **5504** 152–166.

Adámek, J. and Rosický, J. (1994) *Locally presentable and accessible categories*, Cambridge University Press.

Badouel, E. (1989) Terms and infinite trees as monads over a signature. *Springer-Verlag Lecture Notes in Computer Science* **351** 89–103.

Barr, M. (1970) Coequalizers and free triples. *Math. Z.* **116** 307–322.

Bloom, S. and Ésik, Z. (1993) *Iteration theories: the equational logic of iteration processes*, EATCS Monographs on Theoretical Computer Science.

Carboni, A., Lack, S. and Walters, R. F. C. (1993) Introduction to extensive and distributive categories. *J. Pure Appl. Algebra* **84** 145–158.

Elgot, C. C. (1975) Monadic computation and iterative algebraic theories. In: Rose, H. E. and Shepherdson, J. C. (eds.) *Logic Colloquium '73*, North-Holland.

Elgot, C. C., Bloom, S. and Tindell, R. (1978) On the algebraic structure of rooted trees. *J. Comput. System Sci.* **16** 361–399.

Fiore, M., Plotkin, G. and Turi, D. (1999) Abstract syntax and variable binding. *Proc. Logic in Computer Science 1999*, IEEE Press 193–202.

Gabriel, P. and Ulmer, F. (1971) Lokal präsentierbare Kategorien. *Springer-Verlag Lecture Notes in Mathematics* **221**.

Ginali, S. (1979) Regular Trees and the Free Iterative Theory. *J. Comput. System Sci.* **18** 228–242.

MacLane, S. (1998) *Categories for the working mathematician*, 2nd edition, Springer-Verlag.

Nelson, E. (1983) Iterative algebras. *Theoretical Computer Science* **25** 67–94.

Tiuryn, J. (1980) Unique fixed points vs. least fixed points. *Theoretical Computer Science* **12** 229–254.