# How Iterative Reflections of Monads are Constructed

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**Abstract**

Every ideal monad $\mathbb{M}$ on the category of sets is known to have a reflection $\widehat{\mathbb{M}}$ in the category of all iterative monads of Elgot. Here we describe the iterative reflection $\widehat{\mathbb{M}}$ as the monad of free iterative Eilenberg-Moore algebras for $\mathbb{M}$. This yields numerous concrete examples: if $\mathbb{M}$ is the free-semigroup monad, then $\widehat{\mathbb{M}}$ is obtained by adding a single absorbing element; if $\mathbb{M}$ is the monad of finite trees then $\widehat{\mathbb{M}}$ is the monad of rational trees, etc.

*Keywords:* monad, iterative theory, recursive equations, equational laws

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In mathematics you don't understand things.
You just get used to them.

John von Neumann (1903–1957)

## 1. Introduction

The semantics of recursive definitions is a topic at the heart of theoretical computer science. Iterative theories of Calvin Elgot are a well-established formalism in which recursive equation systems can be solved. So far, iterative theories were considered over arbitrary signatures [1, 2] or arbitrary endofunctors [3] but without studying the effect of equational laws on given operations. For example, Elgot et al. described in [2] the free iterative theory on a signature $\Sigma$ as the theory $\Sigma$ of all rational $\Sigma$-trees (that is, $\Sigma$-trees with only finitely many subtrees up to isomorphism). The free iterative theory can be thought of as the closure of the theory formed by all $\Sigma$-terms under unique solutions of recursive equations. In our present paper we attend to the influence that equations have on iteration. This topic is relevant, e. g., for process algebra where processes are defined recursively and operations on processes typically satisfy equational laws such as associativity, commutativity or idempotency. Let us consider the simple case of one binary operation: by the above result the free iterative theory is the theory of rational binary trees. What happens if the operation is required to be commutative? The answer is simple: the free iterative theory consists of all

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rational non-ordered binary trees. This has been known before since commutativity can be expressed by working with algebras for an endofunctor $H$, and in that case the free iterative theory was described in [4] as follows: one applies the given equations to rational $\Sigma$-trees possibly infinitely often. Next question: what happens if the given binary operation is required to be associative? That is, the theory we start with is the theory of finite lists. It follows from our results in the present paper that the free iterative theory is the extension of the finite-list theory by just a single absorbing element. (Informally, for every infinite binary tree one gets, by applying the associative law infinitely often, the complete binary tree viz. the unique solution of $x \approx x \cdot x$.) The same answer is true for a commutative and associative binary operation, in other words, for the finite-bag theory. Last question: what about an idempotent binary operation? We cannot provide an answer to this question because the corresponding theory is not ideal, see below, and one can only form iterative reflections for ideal theories—in fact, the question makes no sense for general theories.

We are going to work with finitary monads $\mathbb{M}$ rather than equational theories—recall that the underlying functor $M$ of the monad assigns to every set $X$ the free algebra $MX$ on $X$ for the given equational theory, and that the inclusion of generators forms a natural transformation $\eta : \mathrm{Id} \longrightarrow M$. Elgot [1] called $\mathbb{M}$ *ideal* if $M$ is a coproduct of Id and a subfunctor $M'$ such that $\eta$ is the right-hand coproduct injection, and the monad multiplication $\mu : MM \longrightarrow M$ has a restriction to $\mu' : M'M \longrightarrow M'$; in the language of theories that means that the presentation of $M$ by operations modulo equations is such that the property of a term not being equivalent to a variable is preserved by substitution. Commutativity and associativity of a binary operation are examples of equational specifications yielding ideal monads, idempotency is not.

We already know that every ideal monad has an iterative reflection; this states in category-theoretic terms that unique solutions of guarded recursive equations can be added freely to the given monad. This was proved in [5] under much less restrictive side conditions than those required below. However, a concrete description of the iterative reflection was missing: we proved that for a given ideal monad $\mathbb{M}$ every object $X$ generates a free iterative algebra $\widehat{MX}$, and thus, we obtain a new monad $\widehat{\mathbb{M}}$. Here we prove that $\widehat{\mathbb{M}}$ is iterative and that it is the desired iterative reflection of $\mathbb{M}$.

Although the statement *"the iterative reflection is the monad of free iterative algebras"* may sound almost tautological, we have not found an easy proof. In fact, the proof presented in our paper is not only technically involved, it also requires at one point that every strong epimorphism is split—this forces us to restrict our attention essentially to monads in the category of sets. In contrast, the existence of iterative reflections was proved for ideal monads in all extensive locally finitely presentable categories (see [5]).

*Related Work*

This paper is an extension of the paper [6] presented at the conference FoSSaCS 2009. Most of the proofs there were omitted or just sketched, and we also present here additional examples of iterative reflections. One example in [6], 2.8(1), was incorrect and we present a correction (see Example 2.11(3)–(4) below).

*Acknowledgement*

We are grateful to Bruno Courcelle for a discussion of iterative reflections and for providing us with Example 2.10(2) below.

## 2. Ideal and Iterative Monads

In this section we recall the concepts introduced by Calvin Elgot [1] in the language of monads in lieu of algebraic theories used originally. Throughout this section $\mathbb{M} = (M, \eta, \mu)$ denotes a finitary monad on a category $\mathcal{A}$ (that is, $M$ preserves filtered colimits). Recall that given another monad $\overline{\mathbb{M}} = (\overline{M}, \overline{\eta}, \overline{\mu})$, a *monad morphism* is a natural transformation $h : M \longrightarrow \overline{M}$ such that $h \cdot \eta = \overline{\eta}$ and $h \cdot \mu = \overline{\mu} \cdot (h * h)$.

**Assumption 2.1.** Throughout the paper we assume that the base category $\mathcal{A}$ is locally finitely presentable, extensive, and has split strong epimorphisms. More detailed, we assume that

(1) $\mathcal{A}$ has colimits,

(2) for every strong epimorphism $e : X \longrightarrow Y$ there exists $m : Y \longrightarrow X$ with $e \cdot m = \mathrm{id}$ (where "strong" means the diagonal fill-in property w. r. t. all monomorphisms),

(3) $\mathcal{A}$ has a set $\mathcal{A}_{\mathrm{fp}}$ of finitely presentable objects $\mathcal{A}$ (i. e., such that the hom-functor is finitary) whose closure under filtered colimits is all of $\mathcal{A}$, and

(4) finite coproducts are universal and disjoint, see [7].

For example, the categories of sets and many-sorted sets satisfy the above assumptions, with $\mathcal{A}_{\mathrm{fp}}$ formed by all finite sets.

**Notation 2.2.** The category of algebras for the monad $\mathbb{M}$ is denoted by $\mathcal{A}^{\mathbb{M}}$. Recall that its objects are algebras $a : MA \longrightarrow A$ for the functor $M$ such that

$$a \cdot \eta_A = \mathrm{id} \qquad \text{and} \qquad a \cdot Ma = a \cdot \mu_A. \tag{2.1}$$

The morphisms of $\mathcal{A}^{\mathbb{M}}$ are the usual $M$-algebra homomorphisms, i. e., $h$ is a homomorphism from an algebra $a : MA \longrightarrow A$ to $b : MB \longrightarrow B$ if $h \cdot a = b \cdot Mh$.

**Definition 2.3.** (C. Elgot [1]) An *ideal monad* consists of a finitary monad $\mathbb{M} = (M, \eta, \mu)$, a finitary subfunctor $m : M' \hookrightarrow M$ such that $M = M' + \mathrm{Id}$ with injections $m$ and $\eta$, and a natural transformation $\mu' : M'M \longrightarrow M'$ such that the square below commutes:

$$\begin{array}{ccc} M'M & \xrightarrow{\ \mu'\ } & M' \\ {\scriptstyle mM}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ MM & \xrightarrow[\ \mu\ ]{} & M \end{array} \tag{2.2}$$

**Remark 2.4.**

1. Since $\mathcal{A}$ is extensive, the equation $M = M' + \mathrm{Id}$ determines $M'$ uniquely up to natural isomorphism.

2. Recall that coproduct injections in extensive categories are monic.

**Example 2.5.**

1. The monads $MX = X \times \mathbb{N}$ (of one unary operation), $MX = X^+$ (of semigroups), $MX =$ binary trees over $X$ (of one binary operation) are ideal.

2. For every finitary endofunctor $H$ the monad $\mathbb{M}$ of free $H$-algebras is ideal, see [3].

3. In contrast, the monads $MX = X^*$ (of monoids) and $MX = \mathcal{P}X$ (of complete join-semilattices) are not ideal.

**Definition 2.6.**

(1) By a *(finitary) equation morphism* we mean a morphism

$$e : X \longrightarrow M(X + A)\,,$$

where $X$ is a finitely presentable object (of "variables") and $A$ an object (of "parameters").

(2) We call $e$ *guarded* provided that it factorizes through the summand $M'(X + A) + A$ of $M(X + A) = M'(X + A) + X + A$:

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & M(X+A) \\ & \searrow^{e_0} & \uparrow {\scriptstyle [m_{X+A},\eta_{X+A}\cdot \mathsf{inr}]} \\ & & M'(X+A)+A \end{array} \tag{2.3}$$

**Remark 2.7.** Recall that if $\mathcal{A} = \mathsf{Set}$ then for every finitary monad $\mathbb{M}$ there exists an equational presentation such that $\mathbb{M}$ is the associated free-algebra monad. That is, for every set $Z$ we can consider $MZ$ as the set of all terms of the equational presentation with the free variables in $Z$.

(1) If we put $X = \{x_1, \ldots, x_n\}$ in Definition 2.6, then the equation morphism $e$ can be regarded as the system of recursive equations

$$\begin{aligned} x_1 &\approx t_1(x_1, \ldots, x_n, a_1, \ldots, a_k) \\ &\vdots \\ x_n &\approx t_n(x_1, \ldots, x_n, a_1, \ldots, a_k) \end{aligned}$$

whose right-hand sides $t_i = e(x_i)$ are $\mathbb{M}$-terms in the variables from $X$ and parameters $a_1, \ldots, a_k \in A$.

(2) The concept of a guarded equation morphism forbids equations such as $x_1 \approx x_1$. Evelyn Nelson [8] introduced iterative algebras for a signature as those algebras in which guarded systems of equations have unique solutions, see also the related concept by Jerzy Tiuryn [9]. We now formulate this concept categorically:

**Definition 2.8.** We say that the algebra $a : MA \longrightarrow A$ is *iterative* provided that every guarded equation morphism $e : X \longrightarrow M(X + A)$ has a unique solution, i. e. a unique morphism $e^\dagger : X \longrightarrow A$ for which the square below commutes:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle a} \\ M(X+A) & \xrightarrow[M[e^\dagger, A]]{} & MA \end{array} \tag{2.4}$$

**Remark 2.9.** The following was proved in [5]:

(1) Iterative algebras form a full subcategory of $\mathcal{A}^{\mathbb{M}}$. The reason why we consider the usual homomorphisms as the "right" morphisms of iterative algebras is that homomorphisms automatically preserve solutions.

(2) Every object $X$ generates a free iterative $\mathbb{M}$-algebra which we denote by $\widehat{M}X$ with the structure and the universal arrow denoted by

$$\rho_X : M\widehat{M}X \longrightarrow \widehat{M}X \qquad \text{and} \qquad \widehat{\eta}_X : X \longrightarrow \widehat{M}X$$

respectively. In other words, the forgetful functor of the category of iterative $\mathbb{M}$-algebras has a left adjoint $X \longmapsto (\widehat{M}X, \rho_X)$.

(3) We obtain a new monad $\widehat{\mathbb{M}} = (\widehat{M}, \widehat{\eta}, \widehat{\mu})$ and a monad morphism $\lambda : \mathbb{M} \longrightarrow \widehat{\mathbb{M}}$ with the components

$$\lambda_X \equiv MX \xrightarrow{M\widehat{\eta}_X} M\widehat{M}X \xrightarrow{\rho_X} \widehat{M}X \tag{2.5}$$

(4) We also proved that every ideal monad $\mathbb{M}$ has an iterative reflection—and in the present paper we prove that this is $\lambda : \mathbb{M} \longrightarrow \widehat{\mathbb{M}}$.

In [5] we worked with ideal (rather than guarded) equation morphisms. However, all the results remain valid under our present assumption. In particular, our proof of the existence of an iterative reflection (Theorem 2.13 and Remark 2.14 of [5]) uses split epimorphisms and does not use extensivity. The proof is based on the fact (proved in the Appendix) that iterative algebras are closed under limits—by inspecting the proof one sees immediately that this is true for iterativity based, as in Definition 2.8 above, on guarded equation morphisms.

**Examples 2.10.** Algebras with unary operations.

(1) The monad

$$MX = X \times \Sigma^*$$

of free unary $\Sigma$-algebras yields the monad

$$\widehat{M}X = X \times \Sigma^* + \Sigma^*(\Sigma^*)^\omega$$

obtained by adding to $MX$ all infinite words in $\Sigma^\omega$ that are eventually periodic (and the unary operations are given by concatenation), see [8].

(2) In contrast, let $K$ be a nontrivial commutative monoid and consider the monad

$$MX = X \times K$$

of free actions of $K$. Then $\widehat{M}$ is obtained by adding a single element which is a joint fixed point of the unary operations:

$$\widehat{M}X = X \times K + 1$$

Here is a proof, suggested to us by Bruno Courcelle. In the free iterative algebra $\widehat{M}X$ the equation $x \approx ax$, where $a \in K$, has a unique solution. That is, a unique $t_a \in \widehat{M}X$ with $t_a = at_a$. Since $K$ is commutative, $bt_a$ solves $x \approx ax$ for every $b \in K$:

$$bt_a = b(at_a) = (ab)t_a = a(bt_a).$$

However, $t_a$ was the unique solution, thus, we conclude

$$bt_a = t_a \qquad \text{for all } a, b \in K.$$

If $e$ denotes the neutral element of $K$, we thus see that $t_e$ is a common fixed point of all the unary operations. It is easy to verify that the subalgebra $MX \cup \{t_e\}$ of $\widehat{M}X$ is iterative, therefore, it is all of $\widehat{M}X$.

(3) In the case of just one unary operation

$$MX = X \times \mathbb{N}$$

we get from (1) also the addition of a unique fixed point:

$$\widehat{M}X = X \times \mathbb{N} + 1.$$

This remains true for all varieties. For example, consider one idempotent unary operation (where the free algebra on $X$ is $X + X$ with the operation $[\mathsf{inr}, \mathsf{inr}]$ on it), that is:

$$MX = X + X,$$

then

$$\widehat{M}X = X + X + 1.$$

**Example 2.11.** Algebras with binary operations.

(1) One binary operation without equation yields[2]

$$MX = \text{finite binary trees on } X.$$

This is an ideal monad since it is free on the endofunctor $HX = X \times X$. And from [3] we know that

$$\widehat{M}X = \text{rational binary trees on } X.$$

Recall that a tree is *rational* iff it has only finitely many subtrees up to isomorphism, see [10].

[2]Trees are supposed to be rooted, directed, ordered trees with a given labeling of the nodes throughout the paper. We consider trees always up to (label-preserving) isomorphism.

(2) Commutativity leads to a completely analogous example, except that whereas in (1) the trees were supposed to be ordered, here we (exceptionally) work with non-ordered trees. The monad $\mathbb{M}$ of one commutative binary operation, given by finite non-ordered binary trees, is ideal, since this also is a free monad on an endofunctor: consider $HX$ given by all unordered pairs in $X$. And $\widehat{M}$ is given by rational non-ordered binary trees, see [4].

(3) In contrast, associativity "trivializes" the passage from $\mathbb{M}$ to $\widehat{\mathbb{M}}$. Consider one binary associative operation, leading to the monad

$$MX = X^+$$

of finite nonemtpy words. This is an ideal monad with $M'X$ denoting the set of all words of length at least 2. The free iterative algebra is given by adding a single element $t$ that is absorbing, i. e., $w \cdot t = t = t \cdot w$ for all $w \in X^+$:

$$\widehat{M}X = X^+ + 1.$$

Here is the proof (already presented in [5]): The equation $x \approx xx$ has a unique solution in the iterative algebra $\widehat{M}X$, i. e., that algebra has a unique idempotent, $t$. To prove that for every $a \in \widehat{M}X$ we have $ta = t$, consider the equation $x \approx xa$. It has a unique solution, say, $\bar{a} = \bar{a}a$. However, $\bar{a}\bar{a}$ is also a solution:

$$(\bar{a}\bar{a})a = \bar{a}(\bar{a}a) = \bar{a}\bar{a},$$

consequently, $\bar{a} = \bar{a}\bar{a}$ is an idempotent. This proves $\bar{a} = t$, hence, $t = ta$. Analogously for $at$. It is easy to see that the subalgebra $X^+ + \{t\}$ of $\widehat{M}X$ is iterative, hence, this is all of $\widehat{M}X$.

(4) The monad

$$MX = X^*$$

of free monoids does not fit into our framework: it is not ideal. This is because the monad multiplication $\mu : (X^*)^* \longrightarrow X^*$ (concatenation of words) does not restrict to $M'(X^*) \longrightarrow M'X$ as required. Indeed, $M'X$ contains all words that are not of length 1, but $\mu$ maps $(x, \varepsilon) \in M'(X^*)$ to $x$ which is not in $M'X$ for $x \in X$. In [6] we claimed by mistake that $X^*$ is an ideal monad.

(5) Consider two commutative and associative operations $+$ and $\cdot$ satisfying the distributive law

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

The corresponding monad is

$$MX = \text{finite sums of monomials } x_1^{n_1}, x_1^{n_1}x_2^{n_2}, x_1^{n_1}x_2^{n_2}x_3^{n_3}, \ldots$$

where $x_i \in X$ and $n_i = 1, 2, 3, \ldots$.

Once again, the free iterative algebras are given by adding a single element absorbing w.r.t. both operations:

$$\widehat{M}X = MX + 1$$

To prove this, use the argument of (3) to prove that $+$ has an absorbing element $t$ and $\cdot$ has an absorbing element $s$. Then $s \cdot t = t$ because $t$ is unique solution of $x \approx x + x$ and $s \cdot t$ solves this equation:

$$s \cdot t = s \cdot (t + t) = (s \cdot t) + (s \cdot t).$$

And this proves $s = t$ because $s$ is the unique solution of $x \approx s \cdot x$. Consequently, $t$ is absorbing for both operations. Once again, $MX + \{t\}$ is an iterative subalgebra of $\widehat{M}X$, thus, it is all of $\widehat{M}X$.

**Example 2.12.** Here are some variations on the above examples we find worth mentioning.

(1) The monad of bags (= finite multisets) is not ideal because of the emtpy bag. But the monad

$$MX = \text{nonemtpy bags in } X$$

yields

$$\widehat{M}X = MX + 1.$$

This is completely analogous to Example 2.11(3), since $M$ is the monad of free commutative semigroups.

(2) The monad of all finite trees

$$MX = \text{finite trees on } X$$

yields

$$\widehat{M}X = \text{rational trees on } X$$

This is completely analogous to Example 2.11(1), since $M$ is the free monad on $HX = X^*$.

(3) Generalizing Example 2.11(2), consider an equational class of $\Sigma$-algebras in which every equation has the form $\sigma(x_1, \dots, x_n) = \tau(y_1, \dots, y_k)$ for $\sigma, \tau \in \Sigma$ and $x_i, y_i$ variables (not necessarily distinct). The free-algebra monad is obvious:

$$MX = \text{finite } \Sigma\text{-trees on } X \text{ modulo } \sim_X$$

where $t \sim_X t'$ means that we can obtain the tree $t'$ by finitely many applications of the given equations starting with $t$. In [4] it was proved that

$$\widehat{M}X = \text{rational } \Sigma\text{-trees on } X \text{ modulo } \approx_X$$

where $\approx_X$ allows finitely or infinitely many applications of the given equations.

**Definition 2.13.** [1] An ideal monad $\mathbb{M}$ is called *iterative* if every guarded equation morphism $e : X \longrightarrow M(X + A)$ has a unique *solution*, which means a morphism $e^\dagger : X \longrightarrow MA$ such that the square below commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & MA \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_A} \\ M(X+A) & \xrightarrow[M[e^\dagger, \eta_A]]{} & MMA \end{array} \tag{2.6}$$

**Definition 2.14.** Suppose we have two ideal monads $\mathbb{M} = (M, \eta, \mu, M', m, \mu')$ and $\overline{\mathbb{M}} = (\overline{M}, \overline{\eta}, \overline{\mu}, \overline{M}', \overline{m}, \overline{\mu}')$. By an *ideal monad morphism* we understand a monad morphism $h : (M, \eta, \mu) \longrightarrow (\overline{M}, \overline{\eta}, \overline{\mu})$ such that there exists a domain-codomain restriction $h' : M' \longrightarrow \overline{M}'$ of $h$ with $\overline{m} \cdot h' = h \cdot m$ (which is necessarily unique, recall that $\overline{m}$ is a monomorphism).

**Remark 2.15.** In the category of all finitary monads on $\mathcal{A}$ we now consider

(a) the non-full subcategory of all ideal monads and ideal monad morphisms, denoted by

$$\mathsf{FM}_{\text{id}}(\mathcal{A}),$$

(b) its full subcategory of all iterative monads, denoted by

$$\mathsf{IFM}(\mathcal{A}).$$

## 3. A Construction of Free Iterative Algebras

**Assumption 3.1.** Throughout the rest of the paper $\mathbb{M}$ denotes an ideal monad on a category $\mathcal{A}$ (satisfying the assumptions of 2.1).

**Remark 3.2.** In [3] we described for every endofunctor $H$ of a locally finitely presentable category, the free iterative $H$-algebra on an object $Y$ as a colimit

$$\widehat{M}Y = \operatorname{colim} \mathrm{Eq}_Y$$

of the diagram $\mathrm{Eq}_Y$ of all *flat equation morphisms*

$$e : X \longrightarrow HX + Y \qquad (X \text{ finitely presentable}).$$

The connecting morphisms of that diagram $\mathrm{Eq}_Y$ are simply the coalgebra homomorphisms for the endofunctor $H(-) + Y$. The fact that $\mathrm{Eq}_Y$ is a filtered diagram whose colimit is the free iterative $H$-algebra on $Y$ turned out to be the basic step for describing the rational monad of $H$. The proof of this fact was technically rather involved.

In the present section we prove an analogous result for algebras for an ideal monad $\mathbb{M}$: we form the diagram of all guarded equation morphisms

$$e : X \longrightarrow M(X + Y) \qquad (\text{X finitely presentable}).$$

In lieu of coalgebra homomorphisms for $M(- + Y)$ we need more general "solution homomorphisms" here. To make sure that $\mathrm{Eq}_Y$ is a filtered diagram we, unfortunately, need the restrictive side condition of the splitting of strong epimorphisms.

**Notation 3.3.** Given an equation morphism $e : X \longrightarrow M(X + A)$ every morphism $h : A \longrightarrow B$ yields a new equation morphism (by changing parameters)

$$h \bullet e \equiv X \xrightarrow{\;e\;} M(X + A) \xrightarrow{M(X+h)} M(X + B).$$

In particular, use the universal arrow

$$\widehat{\eta}_Y : Y \longrightarrow \widehat{M}Y$$

of Remark 2.9 to turn every "abstract" equation morphism $e : X \longrightarrow M(X+Y)$ into a "concrete" equation morphism

$$\widehat{\eta}_Y \bullet e : X \longrightarrow M(X + \widehat{M}Y)$$

in the free iterative algebra $\widehat{M}Y$. The latter has, whenever $e$ is guarded, a unique solution in $\widehat{M}Y$ which, by abuse of notation, we denote by $e^\dagger : X \longrightarrow \widehat{M}Y$. Thus, for every guarded equation morphism $e : X \longrightarrow M(X + Y)$ we define $e^\dagger$ by the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & \widehat{M}Y \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle \rho_Y} \\ M(X+Y) & \xrightarrow[M[e^\dagger,\widehat{\eta}_Y]]{} & M\widehat{M}Y \end{array} \tag{3.1}$$

**Definition 3.4.** Let $e : X \longrightarrow M(X+Y)$ and $f : Z \longrightarrow M(Z+Y)$ be guarded equation morphisms. By a *solution homomorphism* is meant a morphism $h : X \longrightarrow Z$ in $\mathcal{A}$ for which the triangle below commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad h \quad} & Z \\ & {\scriptstyle e^\dagger}\searrow \quad \swarrow{\scriptstyle f^\dagger} & \\ & \widehat{M}Y & \end{array}$$

**Notation 3.5.** For every object $Y$ we denote by

$$\mathsf{EQ}_Y$$

the category of all guarded equation morphisms in $Y$ and all solution homomorphisms.

We also denote by $\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathcal{A}$ the forgetful functor assigning to $e : X \longrightarrow M(X + Y)$ the object $X$.

**Example 3.6.** Whenever $h : X \longrightarrow Z$ is a coalgebra homomorphism, i. e., whenever the square

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & M(X+Y) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle M(h+Y)} \\ Z & \xrightarrow[f]{} & M(Z+Y) \end{array}$$

commutes, then $h$ is a solution homomorphism. Indeed, $f^\dagger \cdot h = e^\dagger$ follows from the uniqueness of solutions since $f^\dagger \cdot h$ solves $e$. To see this consider the diagram below:

$$\begin{array}{ccccc} & & \overset{f^\dagger \cdot h}{\frown\!\!\longrightarrow} & & \\ X & \xrightarrow{\quad h \quad} & Z & \xrightarrow{\quad f^\dagger \quad} & \widehat{M}Y \\ {\scriptstyle e}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle \rho_Y} \\ M(X+Y) & \xrightarrow[M(h+Y)]{} & M(Z+Y) & \xrightarrow[M[f^\dagger,\widehat{\eta}_Y]]{} & M\widehat{M}Y \\ & & \underset{M[f^\dagger \cdot h,\widehat{\eta}_Y]}{\smile\!\!\longrightarrow} & & \end{array}$$

The right-hand square commutes by (3.1), the left-hand one by assumption, and the upper and lower parts obviously do. So the outside of the diagram commutes, showing that $f^\dagger \cdot h$ is a solution of $e$ as desired.

**Remark 3.7.** In the coalgebraic construction of the free iterative monad on an endofunctor $H$ in [3] we used the category $\mathsf{EQ}_Y$ of all flat equation morphisms. This category is trivially filtered because it is closed under finite colimits in the category of all coalgebras, and so the corresponding forgetful functor $\mathrm{Eq}_Y$ is a filtered diagram whose colimit yields the object assignment of the desired free iterative monad.

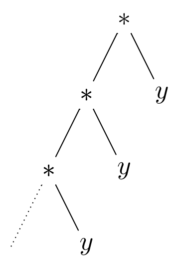
Our present setting is more subtle: here we cannot work with coalgebra homomorphisms (for $M(-+Y)$) because they are insufficient to relate all equations with the same solution in the corresponding diagram. To see this we consider the example of a signature with one binary operation $*$. The associated free monad on that signature is the finite binary tree monad $\mathbb{M}$. Now let, just in this example, $\mathsf{EQ}'_Y$ denote the category of guarded equations and coalgebra homomorphisms. Consider the two recursive equations (trees are written as terms here)

$$x \approx x * y \qquad \text{and} \qquad x \approx (x * y) * y$$

which give rise to two different equation morphisms

$$e, f : \{\, x \,\} \longrightarrow M(\{\, x \,\} + \{\, y \,\}).$$

These two equations specify the same rational binary tree:



However, the above two equations will lead to two distinct elements in the colimit of the diagram given by the forgetful functor on $\mathsf{EQ}'_Y$—this is due to the fact that any morphism in $\mathsf{EQ}'_Y$ preserves the height of the binary trees on the right-hand side of recursive equations.

**Lemma 3.8.** *The category* $\mathsf{EQ}_Y$ *is filtered.*

*Proof.* Firstly, given two objects $(X, e)$ and $(Z, f)$ of $\mathsf{EQ}_Y$ form their coproduct in $\mathsf{Coalg}\, M(- + Y)$ to obtain a cospan

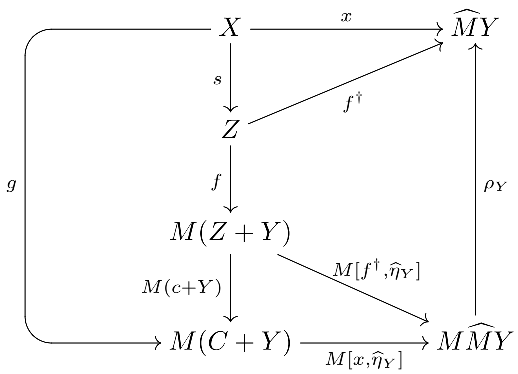
$$(X, e) \xrightarrow{\mathsf{inl}} (X + Z, g) \xleftarrow{\mathsf{inr}} (Z, f)$$

of coalgebra homomorphisms whence morphisms of $\mathsf{EQ}_Y$. Secondly, suppose we have a parallel pair $h, k : (X, e) \longrightarrow (Z, f)$ of solution homomorphisms. Take their coequalizer $c : Z \longrightarrow C$ in $\mathcal{A}$. Since every regular epi is strong we can, by Assumption 2.1(2), choose some splitting $s : C \longrightarrow Z$, $c \cdot s = \mathrm{id}$. Since the equations $f^\dagger \cdot h = e^\dagger = f^\dagger \cdot k$ hold, there exists a unique morpism $x : C \longrightarrow \widehat{M}Y$

with $x \cdot c = f^\dagger$ and thus $x = f^\dagger \cdot s$. The object $C$ is finitely presentable since $X$ and $Z$ are, and we obtain the equation morphism

$$g \equiv C \xrightarrow{\ s\ } Z \xrightarrow{\ f\ } M(Z+Y) \xrightarrow{M(c+Y)} M(C+Y)\,.$$

It is guarded since $f$ is. Finally, from the commutativity of the following diagram we conclude that $g^\dagger = x$:




Thus, $c : (Z, f) \longrightarrow (C, g)$ is a solution homomorphism with $c \cdot h = c \cdot k$. $\square$

**Theorem 3.9.** *The free iterative algebra* $\widehat{M}Y$ *is a filtered colimit of the diagram* $\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathcal{A}$ *of all guarded equation morphisms in* $Y$*:*

$$\widehat{M}Y = \operatorname{colim} \mathrm{Eq}_Y\,.$$

**Remark 3.10.** The proof will be performed by a series of auxiliary results. For these results we denote by $\widetilde{M}Y$ a colimit of the filtered diagram $\mathrm{Eq}_Y$ in $\mathcal{A}$ with colimit morphisms

$$e^\sharp : X \longrightarrow \widetilde{M}Y$$

for all $e : X \longrightarrow M(X+Y)$ in $\mathsf{EQ}_Y$. We will prove that

(1) there is a morphism $\widetilde{\rho} : M\widetilde{M}Y \longrightarrow \widetilde{M}Y$ turning $\widetilde{M}Y$ into a free iterative $\mathbb{M}$-algebra on $Y$, shortly, $\widetilde{M}Y = \widehat{M}Y$, and

(2) the cocone $e^\sharp : X \longrightarrow \widetilde{M}Y$ is formed by the solution morphisms $e^\dagger : X \longrightarrow \widehat{M}Y$ (cf. (3.1)).

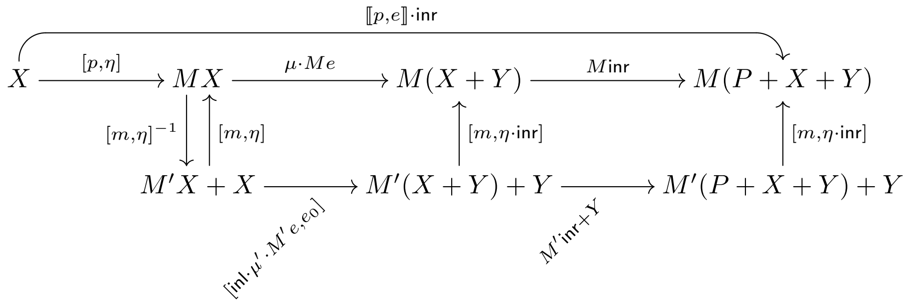
**Notation 3.11.** Let $e : X \longrightarrow M(X+Y)$ be a guarded equation morphism and let $p : P \longrightarrow MX$ be a morphism with $P$ finitely presentable. We denote by $[\![p, e]\!]$ the following guarded equation morphism

$$\begin{aligned}[\![p,e]\!] = (P + X \xrightarrow{[p,\eta]} MX &\xrightarrow{Me} MM(X+Y) \\ &\xrightarrow{\mu_{X+Y}} M(X+Y) \xrightarrow{M\mathsf{inr}} M(P+X+Y)).\end{aligned} \tag{3.2}$$

In particular, we have

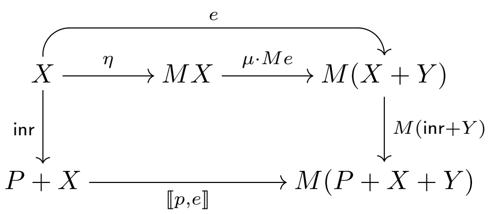
$$[\![p, e]\!] \cdot \mathsf{inr} = M\mathsf{inr} \cdot e. \tag{3.3}$$

**Remark 3.12.** It is not difficult to see that $[\![p, e]\!]$ is indeed guarded. Since $e$ is guarded we have by assumption $e_0 : X \longrightarrow M'(X + Y) + Y$ such that $e = [m_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \cdot e_0$. Now we establish the commutativity of the diagram below (notice that from now on we shall frequently drop the subscripts of natural transformations in diagrams):




Indeed, the right-hand square commutes by naturality of $m$ and $\eta$. For the left-hand square consider the components of $M'X + X$ separately. The right-hand component commutes due to $e = [m_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \cdot e_0$. For the left-hand one use the naturality of $m$ and Diagram (2.2).

**Remark 3.13.** We see that $\mathsf{inr} : X \longrightarrow P + X$ is a coalgebra homomorphism:



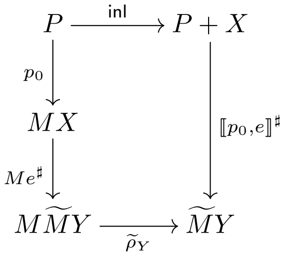

(3.4)

Then $\mathsf{inr}$ is, by Example 3.6, a solution homomorphism; thus, the equation

$$e^\dagger = [\![p, e]\!]^\dagger \cdot \mathsf{inr} \tag{3.5}$$

holds. Consequently we have $e^\sharp = [\![p, e]\!]^\sharp \cdot \mathsf{inr}$.

**Proposition 3.14.** *$\widetilde{M}Y$ is an algebra for the monad $\mathbb{M}$ whose algebra structure $\widetilde{\rho}_Y : M\widetilde{M}Y \longrightarrow \widetilde{M}Y$ is the unique morphism such that the square*



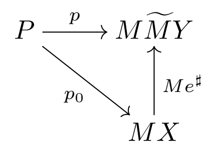

(3.6)

*commutes for all $e : X \longrightarrow M(X + Y)$ in $\mathsf{EQ}_Y$ and all $p_0$ in $\mathcal{A}_{\mathrm{fp}}/MX$.*

*Proof.* (1) Definition of $\widetilde{\rho}_Y$. Recall that since $\mathcal{A}$ is locally finitely presentable, the object $M\widetilde{M}Y$ is a colimit of the canonical diagram

$$\mathcal{A}_{\mathrm{fp}}/M\widetilde{M}Y \longrightarrow \mathcal{A}$$

of all morphisms $p : P \longrightarrow M\widetilde{M}Y$ with $P$ finitely presentable. Thus, we can define a morphism $\widetilde{\rho}_Y : M\widetilde{M}Y \longrightarrow \widetilde{M}Y$ by specifying its composites $\widetilde{\rho}_Y \cdot p$ with every $p \in \mathcal{A}_{\mathrm{fp}}/M\widetilde{M}Y$. Since $M$ is finitary and $P$ is finitely presentable, for every $p$ there exists $e : X \longrightarrow M(X+Y)$ in $\mathsf{EQ}_Y$ and a factorization

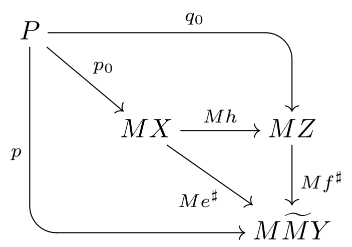



(3.7)

(1a) Let us first prove that in (3.6) the composite

$$P \xrightarrow{\mathsf{inl}} P + X \xrightarrow{[\![p_0, e]\!]^\sharp} \widetilde{M}Y \qquad (3.8)$$

is independent of the choice of factorization (3.7). Indeed, suppose we chose another factorization $q_0 : P \longrightarrow MZ$ for some $f : Z \longrightarrow M(Z+Y)$ in $\mathsf{EQ}_Y$, i. e., suppose $p = Mf^\sharp \cdot q_0$. Then we prove $[\![p_0, e]\!]^\sharp = [\![q_0, f]\!]^\sharp$. Since the diagram $M\mathrm{Eq}_Y$ is filtered, we can assume without loss of generality that a solution homomorphism $h$ exists from $e$ to $f$ in $\mathsf{EQ}_Y$ such that the diagram

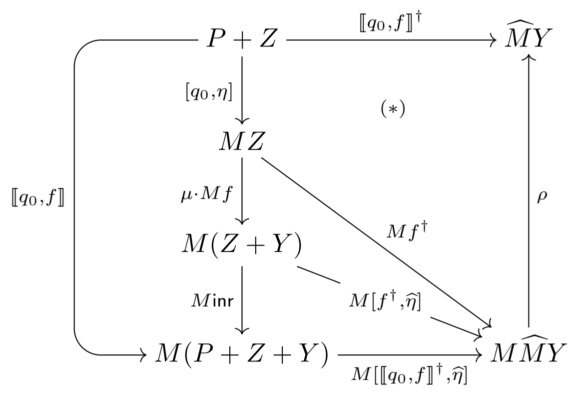



(3.9)

commutes. We are going to show that $P + h : P + X \longrightarrow P + Z$ is a solution homomorphism from $[\![p_0, e]\!]$ to $[\![q_0, f]\!]$, i. e., we prove

$$[\![p_0, e]\!]^\dagger = [\![q_0, f]\!]^\dagger \cdot (P + h). \qquad (3.10)$$

To this end we first consider the diagram




(3.11)

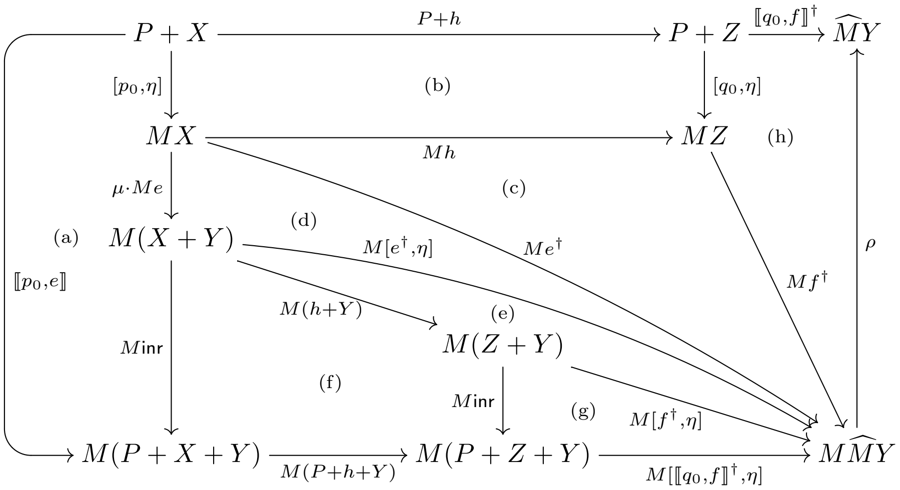
The outside commutes (see (3.1)), and the left-hand part does by (3.2). The lowest triangle commutes by (3.5). We do not claim that the middle triangle commutes. However, it does when extended by $\rho_Y$. Indeed, notice that all morphisms in the extended triangle are algebra homomorphisms. Since $MZ$ is the free $\mathbb{M}$-algebra on $Z$, it suffices to check that the two morphisms in question agree when precomposed with $\eta_Z : Z \longrightarrow MZ$, that is we verify:

$$\rho_Y \cdot Mf^\dagger \cdot \eta_Z = \rho_Y \cdot M[f^\dagger, \eta_Y] \cdot \mu_{Z+Y} \cdot Mf \cdot \eta_Z.$$

Indeed, the left-hand side is $f^\dagger$, and the right-hand side is $\rho_Y \cdot M[f^\dagger, \eta_Y] \cdot f$ (due to the naturality of $\eta$ and the unit law $\rho_Y \cdot \eta_{\widehat{M}Y} = \mathrm{id}$ and one unit law for the monad $\mathbb{M}$). So we obtain an equation which says that $f^\dagger$ solves $f$, which holds. From all this we conclude that the upper part labelled $(*)$ commutes, too. We thus proved

$$[\![q_0, f]\!]^\dagger = \rho_Y \cdot Mf^\dagger \cdot [q_0, \eta_Z] : P + Z \longrightarrow \widehat{M}Y\,. \tag{3.12}$$

We are prepared to prove Equation (3.10) by showing that its right-hand side is a solution of $[\![p_0, e]\!]$. That is, we now prove that the outside of the diagram below commutes:

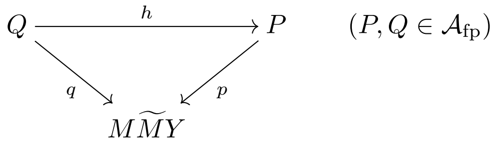



Indeed, part (a) is the definition of $[\![p_0, e]\!]$, (b) commutes due to Diagram (3.9), (c) and (e) commute since $h$ is a morphism of $\mathsf{EQ}_Y$, for (d) use the extension by $\rho_Y$ precisely as in the argument concerning Diagram (3.11) above. The commutativity of (f) is obvious, for (g) use Equation (3.5). Finally, part (h) is Equation (3.12).

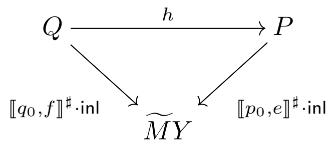
(1b) The squares (3.6) define a unique morphism $\widetilde{\rho}_Y : M\widetilde{M}Y \longrightarrow \widetilde{M}Y$. Indeed, $M\widetilde{M}Y$ is a canonical colimit of the diagram of all

$$p : P \longrightarrow M\widetilde{M}Y, \qquad P \text{ finitely presentable},$$

and, by (1a), for each $p$ the morphism $[\![p_0, e]\!]^\sharp \cdot \mathsf{inl} : P \longrightarrow \widetilde{M}Y$ is independent of the choice of factorization (3.7). So all we have to verify is that these morphisms form a cocone of the canonical diagram of $M\widetilde{M}Y$. In other words, we need to prove that given a morphism $h$ of that canonical diagram:

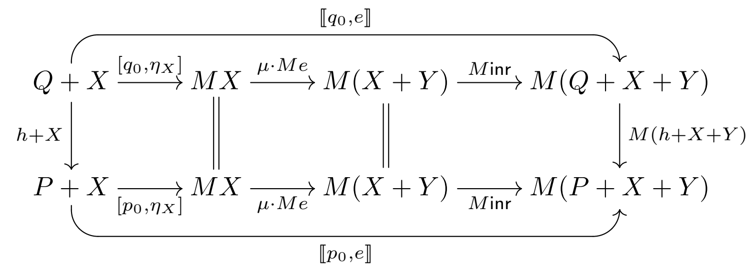



then the triangle




(3.13)

also commutes; here $p = Me^\sharp \cdot p_0$ and $q = Mf^\sharp \cdot q_0$ are arbitrary factorizations as in Diagram (3.7). But by (1a) we can assume that $f = e$ and $q_0 = p_0 \cdot h$. Then we observe that $h + X : Q + X \longrightarrow P + X$ is a coalgebra homomorphism from $[\![q_0, e]\!]$ to $[\![p_0, e]\!]$:

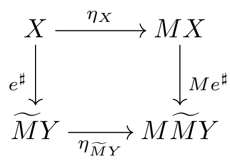



Thus $h + X$ is a solution homomorphism, see Example 3.6. This establishes (3.13):

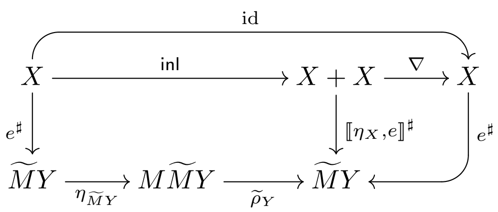
$$[\![p_0, e]\!]^\sharp \cdot \mathsf{inl} \cdot h = [\![p_0, e]\!]^\sharp \cdot (h + X) \cdot \mathsf{inl} = [\![q_0, e]\!]^\sharp \cdot \mathsf{inl} = [\![q_0, f]\!]^\sharp \cdot \mathsf{inl}.$$

(2) $(\widetilde{M}Y, \widetilde{\rho}_Y)$ is an algebra for the monad $\mathbb{M}$. We now verify the two laws of an Eilenberg-Moore algebra:

(2a) $\widetilde{\rho}_Y$ satisfies the unit law $\widetilde{\rho}_Y \cdot \eta_{\widetilde{M}Y} = \mathrm{id}$. For every $e$ of $\mathsf{EQ}_Y$ we have by naturality of $\eta$ a commutative square

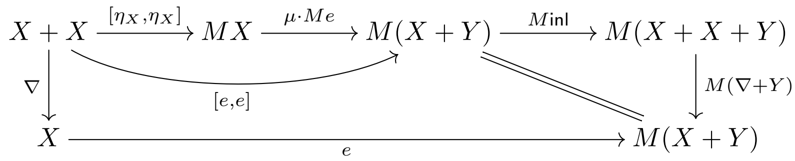



which means that for $p = \eta_{\widetilde{M}Y} \cdot e^\sharp$ we can choose in (3.7) the factorization $p_0 = \eta_X$. We will now prove that the diagram below commutes:




(3.14)

Indeed, the left-hand part commutes by definition of $\widetilde{\rho}_Y$. The right-hand triangle commutes since the codiagonal $\nabla$ is a coalgebra homomorphism (thus, a solution homomorphism):



Since the colimit injections $e^\sharp$ form a jointly epimorphic family, we obtain from the commutativity of the outside of Diagram (3.14) the desired result.

(2b) $\widetilde{\rho}_Y$ satisfies the associativity law

$$\begin{array}{ccc} MM\widetilde{M}Y & \xrightarrow{\mu_{\widetilde{M}Y}} & M\widetilde{M}Y \\ {\scriptstyle M\widetilde{\rho}_Y}\downarrow & & \downarrow{\scriptstyle \widetilde{\rho}_Y} \\ M\widetilde{M}Y & \xrightarrow[\widetilde{\rho}_Y]{} & \widetilde{M}Y \end{array} \tag{3.15}$$

To prove this, use the fact that $MM\widetilde{M}Y$ is a colimit of the canonical diagram $\mathcal{A}_{\mathrm{fp}}/MM\widetilde{M}Y \longrightarrow \mathcal{A}$. Thus, it suffices to show that Diagram (3.15) commutes when precomposed by any morphism $p : P \longrightarrow MM\widetilde{M}Y$, where $P$ is finitely presentable:

$$\widetilde{\rho}_Y \cdot \mu_{\widetilde{M}Y} \cdot p = \widetilde{\rho}_Y \cdot M\widetilde{\rho}_Y \cdot p\,. \tag{3.16}$$

Firstly, observe that $\widetilde{M}Y = \operatorname{colim} \mathrm{Eq}_Y$ implies $MM\widetilde{M}Y = \operatorname{colim} MM\mathrm{Eq}_Y$ because $\mathrm{Eq}_Y$ is a filtered diagram, and $MM$ is a finitary functor. Thus, we have a factorization

$$\begin{array}{ccc} P & \xrightarrow{p} & MM\widetilde{M}Y \\ & {\scriptstyle p_0}\searrow & \uparrow{\scriptstyle MMe^\sharp} \\ & & MMX \end{array} \tag{3.17}$$

for some $e : X \longrightarrow M(X+Y)$ in $\mathsf{EQ}_Y$. Analogously, since $MX$ is a filtered colimit of $\mathcal{A}_{\mathrm{fp}}/MX$ and $M$ is finitary, we have a factorization

$$\begin{array}{ccc} P & \xrightarrow{p_0} & MMX \\ & {\scriptstyle p_1}\searrow & \uparrow{\scriptstyle Mq_0} \\ & & MQ \end{array} \tag{3.18}$$

for some $q_0 : Q \longrightarrow MX$ with $Q$ finitely presentable.

Using Notation 3.11, we form the following three equation morphisms (where $\mathsf{inl} : Q \longrightarrow Q + X$ is the coproduct injection):

$$\begin{array}{rcl} q & = & [\![q_0, e]\!] : Q + X \longrightarrow M(Q+X+Y), \\ \widehat{q} & = & [\![\mu_X \cdot p_0, e]\!] : P + X \longrightarrow M(P+X+Y), \text{ and} \\ \overline{q} & = & [\![M\mathsf{inl} \cdot p_1, q]\!] : P + Q + X \longrightarrow M(P+Q+X+Y). \end{array}$$

The left-hand side of Equation (3.16) is the passage from $P$ to $\widetilde{M}Y$ in the diagram

$$\begin{array}{ccccc}
P & & \xrightarrow{\quad \mathsf{inl} \quad} & & P+X \\
\downarrow{\scriptstyle p_0} & & & & \\
MMX & \xrightarrow{\mu_X} & MX & & \downarrow{\scriptstyle \widehat{q}^\sharp} \\
\downarrow{\scriptstyle MMe^\sharp} & & \downarrow{\scriptstyle Me^\sharp} & & \\
MM\widetilde{M}Y & \xrightarrow[\mu_{\widetilde{M}Y}]{} & M\widetilde{M}Y & \xrightarrow[\widetilde{\rho}_Y]{} & \widetilde{M}Y
\end{array}
\qquad (p : P \to MM\widetilde{M}Y) \tag{3.19}$$

This diagram commutes: the leftmost part is Diagram (3.17), the lower square commutes by naturality of $\mu$, and the right-hand part does by the definition of $\widetilde{\rho}_Y$ (see (3.6)). Observe further that (3.6) for $e = q$ and $p_0 = M\mathsf{inl} \cdot p_1$ yields, by definition of $\overline{q}$:

$$\overline{q}^\sharp \cdot \mathsf{inl} = \widetilde{\rho}_Y \cdot Mq^\sharp \cdot (M\mathsf{inl} \cdot p_1). \tag{3.20}$$

The right-hand side of Equation (3.16) is the corresponding passage in the diagram

$$\begin{array}{ccccc}
P & & \xrightarrow{\quad \mathsf{inl} \quad} & & P+Q+X \\
\downarrow{\scriptstyle p_1} & & & & \\
MQ & \xrightarrow{M\mathsf{inl}} & M(Q+X) & & \\
\downarrow{\scriptstyle Mq_0} & & & & \downarrow{\scriptstyle \overline{q}^\sharp} \\
MMX & & \downarrow{\scriptstyle Mq^\sharp} & & \\
\downarrow{\scriptstyle MMe^\sharp} & & & & \\
MM\widetilde{M}Y & \xrightarrow[M\widetilde{\rho}_Y]{} & M\widetilde{M}Y & \xrightarrow[\widetilde{\rho}_Y]{} & \widetilde{M}Y
\end{array}
\qquad (p : P \to MM\widetilde{M}Y) \tag{3.21}$$

This diagram also commutes: For the leftmost part paste together Diagrams (3.17) and (3.18), and the right-hand part is (3.20). For the middle square remove $M$ and consider Diagram (3.6) for $q = [\![q_0, e]\!]$.

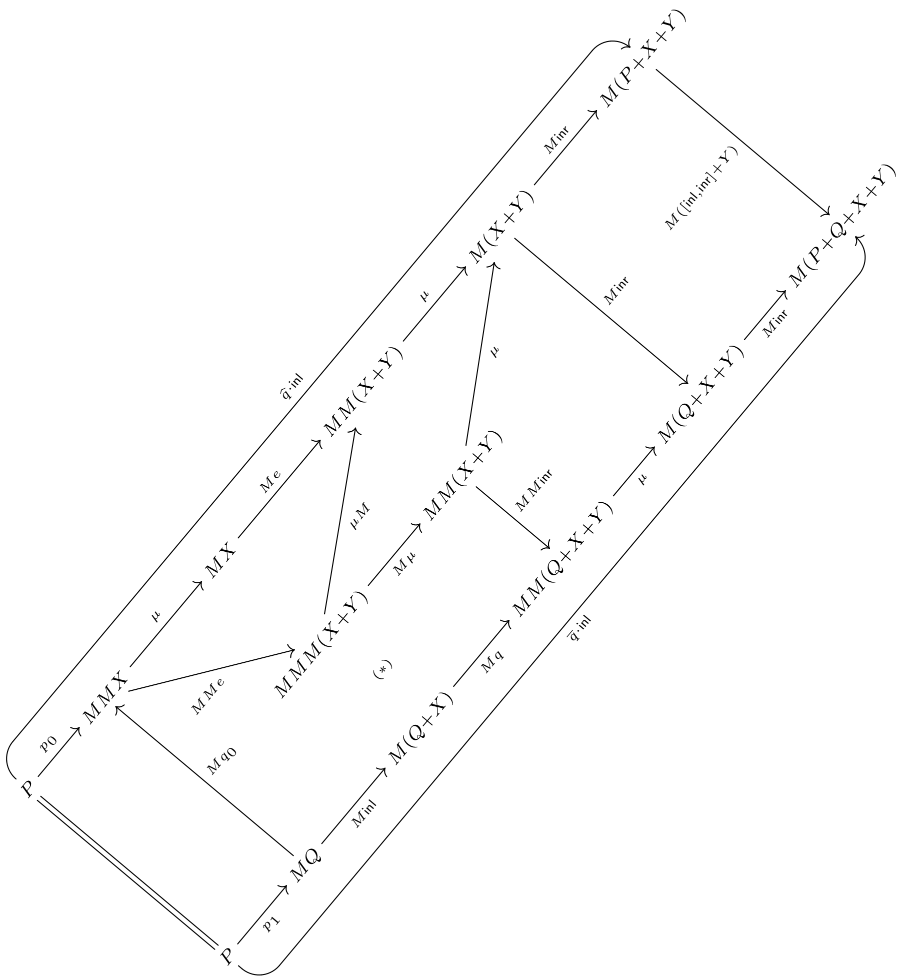
To complete the proof of Equation (3.16) it is thus sufficient to compare (3.19) and (3.21) and verify that the equation

$$\overline{q}^\sharp \cdot \mathsf{inl} = \widehat{q}^\sharp \cdot \mathsf{inl} \tag{3.22}$$

holds. We do this by showing that $[\mathsf{inl}, \mathsf{inr}] : P+X \longrightarrow P+Q+X$ is a coalgebra homomorphism, whence a solution homomorphism from $\widehat{q}$ to $\overline{q}$. That is, we have to verify that the square

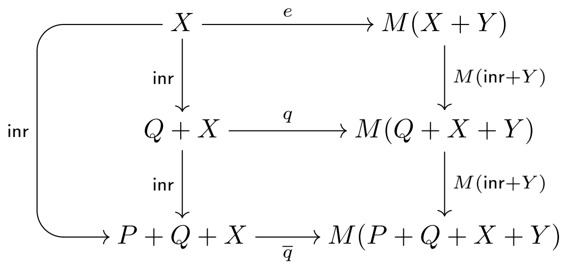
$$\begin{array}{ccc}
P+X & \xrightarrow{\widehat{q}} & M(P+X+Y) \\
\downarrow{\scriptstyle [\mathsf{inl},\mathsf{inr}]} & & \downarrow{\scriptstyle M([\mathsf{inl},\mathsf{inr}]+Y)} \\
P+Q+X & \xrightarrow[\overline{q}]{} & M(P+Q+X+Y)
\end{array} \tag{3.23}$$

commutes, and we do this for the coproduct components separately. The left-hand components commute due to the following commutative diagram:




The upper and lower parts commute due to the definitions of $\widehat{q}$ and $\overline{q}$, respectively. The leftmost square is Diagram (3.18). For the part labelled with $(*)$ remove $M$ and observe that the resulting diagram commutes due to $q = [\![q_0, e]\!]$, see Diagram (3.2). All other parts commute due to naturality and the monad laws for $\mathbb{M}$.

For the right-hand component of Diagram (3.23), observe that $\mathsf{inr} : Q + X \longrightarrow P+Q+X$ is a coalgebra homomorphism from $q$ to $\overline{q}$ (cf. Diagram (3.4)). Similarly, $\mathsf{inr} : X \longrightarrow Q+X$ is a coalgebra homomorphism from $e$ to $q$. Compose the two resulting squares to obtain

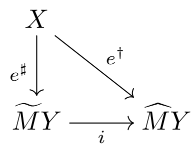



and note that, due to (3.3), the upper and right-hand edges compose to the upper and right-hand edges of (3.23) precomposed with $\mathsf{inr}$. So this proves that (3.23) commutes from which (3.22) immediately follows. This completes the proof of Equation (3.16), and so $(\widetilde{M}Y, \widetilde{\rho}_Y)$ is an algebra for $\mathbb{M}$. $\square$

**Notation 3.15.** The morphisms $e^\dagger : X \longrightarrow \widehat{M}Y$ form a cocone of the diagram $\mathrm{Eq}_Y$, thus, there exists a unique morphism

$$i : \widetilde{M}Y \longrightarrow \widehat{M}Y$$

for which the triangles




$(e \in \mathsf{EQ}_Y)$ (3.24)

commute. We are going to prove that $i$ is an isomorphism.

**Notation 3.16.** Let $Y$ be a finitely presentable object. For the "trivial" guarded equation morphism

$$Y \xrightarrow{\mathsf{inr}} Y + Y \xrightarrow{\eta} M(Y+Y) \tag{3.25}$$

we denote by

$$\widetilde{\eta}_Y = (\eta_{Y+Y} \cdot \mathsf{inr})^\sharp : Y \longrightarrow \widetilde{M}Y \tag{3.26}$$

the corresponding colimit injection.

Recall from Remark 2.9 that $\rho_Y$ and $\widehat{\eta}_Y$ are the structure and universal morphism, respectively, of the free iterative $\mathbb{M}$-algebra $\widehat{M}Y$.

**Lemma 3.17.** *For every finitely presentable object $Y$, the morphism*

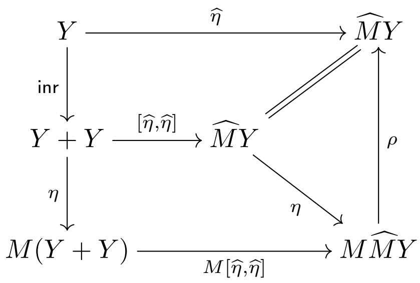
$$i : (\widetilde{M}Y, \widetilde{\rho}_Y) \longrightarrow (\widehat{M}Y, \rho_Y)$$

*is a homomorphism of $\mathbb{M}$-algebras with $\widehat{\eta}_Y = i \cdot \widetilde{\eta}_Y$.*

*Proof.* Consider the guarded equation morphism $\eta_{Y+Y} \cdot \mathsf{inr}$ from (3.25). We are now going to prove that

$$\widehat{\eta}_Y = (\eta_{Y+Y} \cdot \mathsf{inr})^\dagger : Y \longrightarrow \widehat{M}Y\,. \tag{3.27}$$

Indeed, the following diagram commutes:




Thus, we obtain the equation

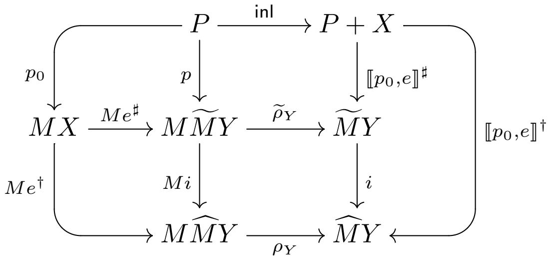
$$i \cdot \widetilde{\eta}_Y = i \cdot (\eta_{Y+Y} \cdot \mathsf{inr})^\sharp = (\eta_{Y+Y} \cdot \mathsf{inr})^\dagger = \widehat{\eta}_Y \tag{3.28}$$

by the definition of $\widetilde{\eta}_Y$ and $i$, see Notations 3.15 and 3.16.

To verify that $i$ is a homomorphism we use the fact that $M\widetilde{M}Y$ is a colimit of the canonical diagram $\mathcal{A}_{\mathrm{fp}}/M\widetilde{M}Y$. Thus in order to verify

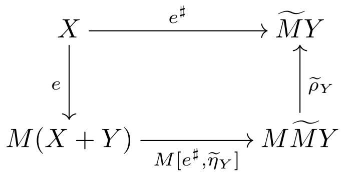
$$i \cdot \widetilde{\rho}_Y = \rho_Y \cdot Mi : M\widetilde{M}Y \longrightarrow \widehat{M}Y$$

we only need to prove that the equation holds when precomposed with a morphism $p : P \longrightarrow M\widetilde{M}Y$ with $P$ finitely presentable. Consider for an arbitrary factorization (3.7) the diagram




The upper left-hand part commutes (see Diagram (3.7)). The upper middle square commutes due to the definition of $\widetilde{\rho}_Y$ (see Diagram (3.6)), and the right-hand part and the lower left-hand one commute due to (3.24). Now the outside of the diagram commutes, too; indeed, repeat the argument for part $(*)$ of Diagram (3.11) with $f = e$ and $q_0 = p_0$. Thus, we conclude that the desired middle lower square commutes. □

**Lemma 3.18.** *Let $Y$ be a finitely presentable object and let $e : X \longrightarrow M(X+Y)$ be a guarded equation morphism. Then the square*




(3.29)

*commutes. Moreover, $e^\sharp$ is the unique morphism from $X$ to $\widetilde{M}Y$ with that property.*

*Proof.* (1) Form the coproduct of the equation morphisms $e$ and $\eta_{Y+Y} \cdot \mathsf{inr}$ (cf. (3.25)) in $\mathsf{EQ}_Y$. We first show that this coproduct is the equation morphism

$$f = (\,X + Y \xrightarrow{[e,\eta\cdot\mathsf{inr}]} M(X+Y) \xrightarrow{M[\mathsf{inl},\mathsf{inr}]} M(X+Y+Y)\,).$$

Clearly, we have $f \cdot \mathsf{inl} = M(\mathsf{inl} + Y) \cdot e$ and, for the right-hand component use naturality of $\eta$ to obtain the commutative diagram below:

$$\begin{array}{ccccc}
Y & \xrightarrow{\mathsf{inr}} & Y+Y & \xrightarrow{\eta} & M(Y+Y) \\
{\scriptstyle \mathsf{inr}}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} & & \\
X+Y & \xrightarrow{[\mathsf{inl},\mathsf{inr}]} & X+Y+Y & & \downarrow{\scriptstyle M(\mathsf{inr}+Y)} \\
{\scriptstyle \eta}\downarrow & & & \searrow{\scriptstyle \eta} & \\
M(X+Y) & & \xrightarrow[M[\mathsf{inl},\mathsf{inr}]]{} & & M(X+Y+Y)
\end{array}$$

Now let us prove that the triangle

$$\begin{array}{ccc}
X & \xrightarrow{\mathsf{inl}} & X+X+Y \\
& {\scriptstyle e^\sharp}\searrow \quad \swarrow{\scriptstyle [\![e,f]\!]^\sharp} & \\
& \widehat{M}Y &
\end{array} \tag{3.30}$$

commutes. This follows from the equality $e^\dagger = [\![e,f]\!]^\dagger \cdot \mathsf{inl}$, which we prove by verifying that

$$[e^\dagger, e^\dagger, \widehat{\eta}_Y] = [\![e,f]\!]^\dagger : X + X + Y \longrightarrow \widehat{M}Y\,. \tag{3.31}$$

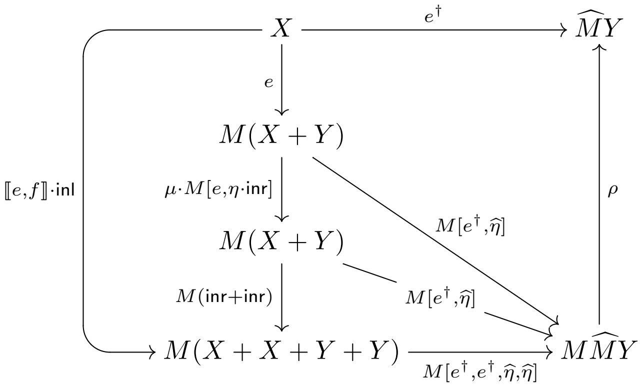
By Equation (3.5), we have $f^\dagger = [\![e,f]\!]^\dagger \cdot \mathsf{inr} : X + Y \longrightarrow \widehat{M}Y$. Moreover, we know that $f^\dagger = [e^\dagger, \widehat{\eta}_Y] : X + Y \longrightarrow \widehat{M}Y$ since $\mathsf{inl}$ and $\mathsf{inr}$ are solution homomorphisms from $e$ and $\eta_{Y+Y} \cdot \mathsf{inr}$ to their coproduct $f$, respectively (see Equation (3.27)). Consequently,

$$[e^\dagger, \widehat{\eta}_Y] = f^\dagger = [\![e,f]\!]^\dagger \cdot \mathsf{inr}\,. \tag{3.32}$$

To establish (3.31) we will prove that $[e^\dagger, e^\dagger, \widehat{\eta}_Y]$ is a solution of the guarded equation morphism $[\![e,f]\!]$, i.e. we will verify that the square below commutes:

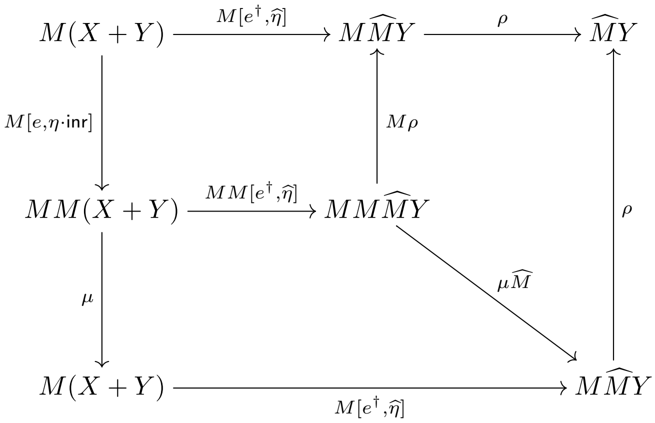
$$\begin{array}{ccc}
X+X+Y & \xrightarrow{[e^\dagger,e^\dagger,\widehat{\eta}]} & \widehat{M}Y \\
{\scriptstyle [\![e,f]\!]}\downarrow & & \uparrow{\scriptstyle \rho} \\
M(X+X+Y+Y) & \xrightarrow[M[e^\dagger,e^\dagger,\widehat{\eta},\widehat{\eta}]]{} & M\widehat{M}Y
\end{array}$$

For the right-hand component $X+Y$ this follows from (3.32) and the fact that $\mathsf{inr} : X + Y \longrightarrow X + X + Y$ is a coalgebra homomorphism from $f$ to $[\![e,f]\!]$ (cf. Diagram (3.4)). For the left-hand component $X$ we consider the diagram



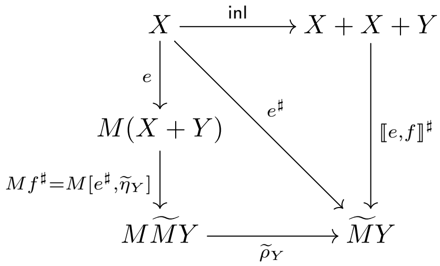

(3.33)

The left-hand part commutes by definition of $f$ and Diagram (3.2), using the naturality of $\mu$. All other parts clearly commute, except, perhaps, the middle triangle. We do not claim that the middle triangle commutes, but it does when extended by $\rho_Y$. Indeed, this follows from the commutative diagram




For the lower part use naturality of $\mu$, the right-hand part is associativity of the algebra $(\widehat{M}Y, \rho_Y)$, and for the upper part remove $M$ and consider the two coproduct components separately: the left-hand one expresses that $e^\dagger$ is a solution of $e$, and for the right-hand one use the naturality of $\eta$ and the unit law for $\rho_Y$. This proves that the outside of Diagram (3.33) commutes, and we obtain (3.30) as desired.

(2) The square (3.29) commutes. Indeed, the outside of the diagram

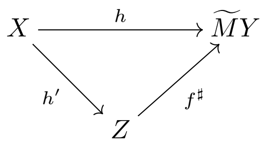


commutes by the definition of $\widetilde{\rho}_Y$ (cf. Diagram 3.6), the upper right-hand triangle commutes by (3.30), and from $f^\dagger = [e^\dagger, \widehat{\eta}_Y]$ in (3.32) we get

$$f^\sharp = [e^\sharp, \widetilde{\eta}_Y]$$

using (3.27), (3.26) and the fact that $\mathsf{inl}$ and $\mathsf{inr}$ are solution homomorphisms from $e$ and $\eta_{Y+Y} \cdot \mathsf{inr}$ to their coproduct $f$, respectively. Thus, the lower left-hand triangle commutes and yields (3.29).

(3) Unicity of $e^\sharp$: Suppose that $h : X \longrightarrow \widetilde{M}Y$ is another morphism with $h = \widetilde{\rho}_Y \cdot M[h, \widetilde{\eta}_Y] \cdot e$. Since $X$ is finitely presentable we can choose an $f : Z \longrightarrow M(Z + Y)$ in $\mathsf{EQ}_Y$ and a morphism $h' : X \longrightarrow Z$ such that the triangle

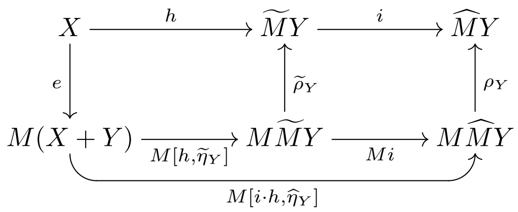



commutes. We show that $h'$ is a solution homomorphism, i. e., that the equation

$$e^\dagger = f^\dagger \cdot h' \tag{3.34}$$

holds. It then follows that $e^\sharp = f^\sharp \cdot h' = h$ proving the desired uniqueness.

In order to establish (3.34) observe the commutativity of the diagram



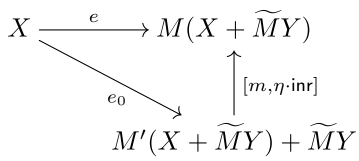

Indeed, the lower and right-hand parts follow from Lemma 3.17, and the left-hand part commutes by assumption. This implies the commutativity of the outside of the diagram, which shows that $i \cdot h$ is the unique solution of $e$. Thus, we get

$$e^\dagger = i \cdot h = i \cdot f^\sharp \cdot h' = f^\dagger \cdot h' \,,$$

where the last equation follows from Diagram (3.24). □

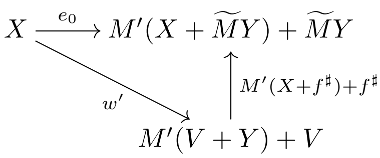
**Proposition 3.19.** *For every finitely presentable object $Y$ the algebra $(\widetilde{M}Y, \widetilde{\rho}_Y)$ is iterative.*

*Proof.* Suppose we are given a guarded equation morphism



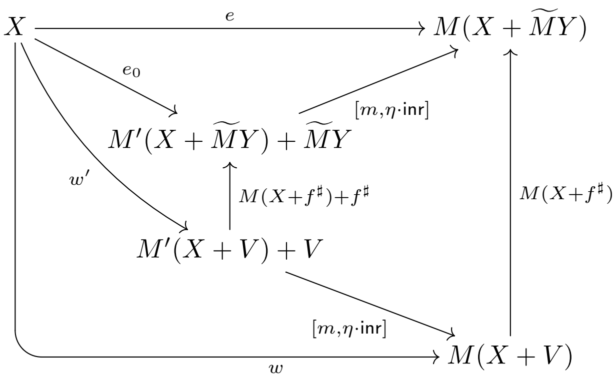

Since $M'$ is finitary, the object $M'(X+\widetilde{M}Y)+\widetilde{M}Y$ is a colimit of $M'(X+\mathrm{Eq}_Y)+\mathrm{Eq}_Y$. Thus, as $X$ is finitely presentable, we can choose some equation morphism

$f : V \longrightarrow M(V+Y)$ in $\mathsf{EQ}_Y$ and a factorization $w' : X \longrightarrow M'(V+Y)+V$ such that the triangle



(3.35)

commutes. For $w = [m_{X+V}, \eta_{X+V}] \cdot w'$ we get a commutative diagram



(3.36)

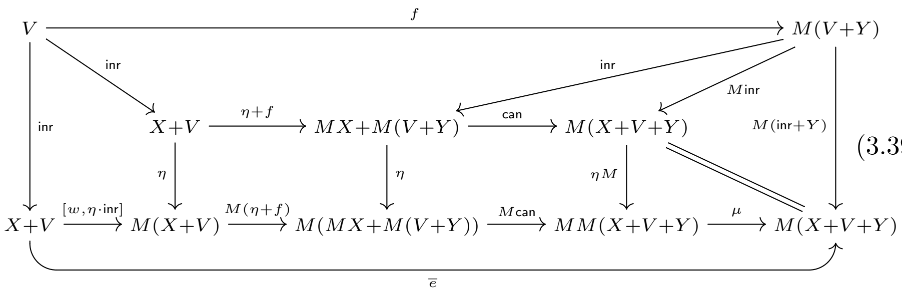
Define the equation morphism

$$\begin{aligned} \overline{e} = (X+V \xrightarrow{[w,\eta\cdot\mathsf{inr}]} & M(X+V) \\ & \downarrow M(\eta_X + f) \\ & M(MX + M(V+Y)) \\ & \downarrow M\mathsf{can} \\ & MM(X+V+Y) \xrightarrow{\mu} M(X+V+Y)), \end{aligned} \tag{3.37}$$

where $\mathsf{can} : MX + M(V+Y) \longrightarrow M(X+V+Y)$ is the canonical morphism $[M\mathsf{inr}, M\mathsf{inl}]$. To see that $\overline{e}$ is guarded use Diagram (3.35), the naturality of $m$ and $\eta$ and that $f$ is guarded. We will show that the unique solution of $e$ is the morphism

$$\overline{e}^\dagger = (X \xrightarrow{\mathsf{inl}} X+V \xrightarrow{\overline{e}^\sharp} \widetilde{M}Y). \tag{3.38}$$

(1) $\overline{e}^\dagger$ is a solution. Before proving this, observe that $\mathsf{inr} : V \longrightarrow X+V$ is a coalgebra homomorphism from $f$ to $\overline{e}$:

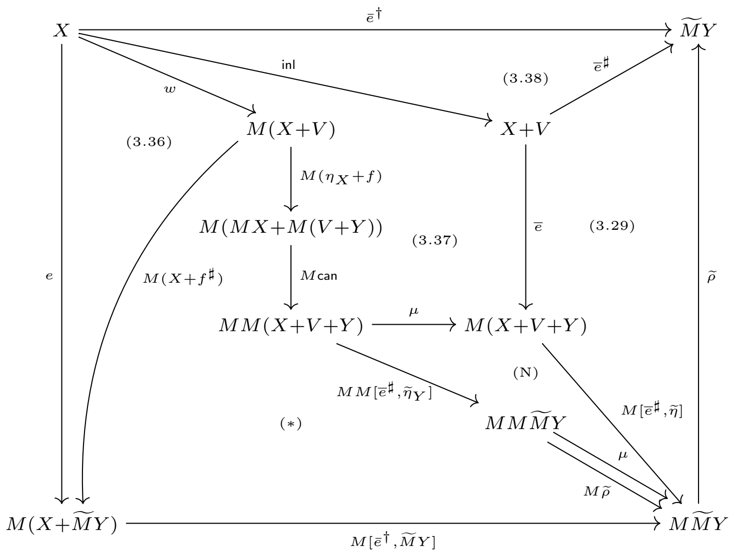


(3.39)

Thus, inr is a solution homomorphism, see Example 3.6, and we obtain

$$f^\sharp = \overline{e}^\sharp \cdot \mathsf{inr}\,. \tag{3.40}$$

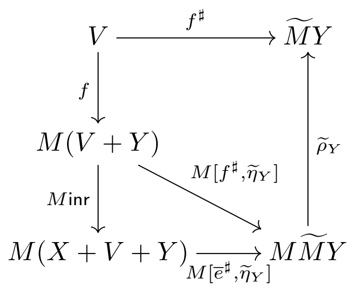
It is our task to prove that the outside of the following diagram




(3.41)

commutes. The part labelled by (N) commutes by naturality of $\mu$. Notice that the two parallel morphisms on the lower right-hand corner are merged by $\widetilde{\rho}_Y$, see (3.15). Except for $(*)$ all other parts commute as indicated, so let us prove that the part $(*)$ commutes. To this end remove $M$ and consider the coproduct components separately.

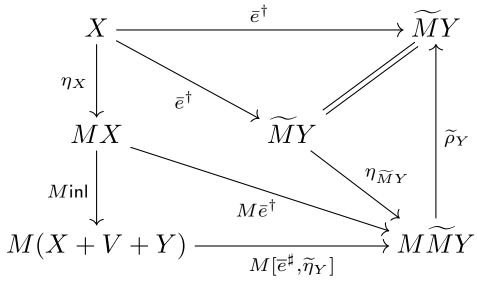
For the right-hand component we get the diagram




(3.42)

Its upper part commutes by Lemma 3.18, and its lower triangle does by (3.40).

For the left-hand component of part $(*)$ consider the commutative diagram




(3.43)

For its lower part recall (3.38), and the other inner parts are obvious.

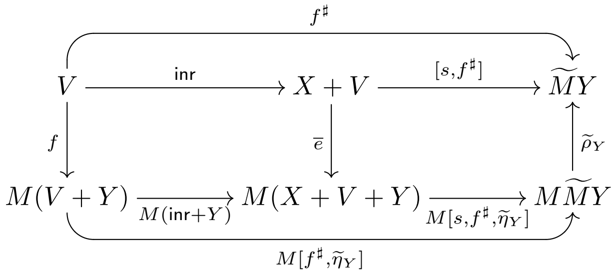
(2) Uniqueness of solutions. Given any solution $s : X \longrightarrow \widetilde{M}Y$ of $e$ we prove that the square

$$\begin{array}{ccc} X+V & \xrightarrow{[s,f^\sharp]} & \widetilde{M}Y \\ {\scriptstyle \overline{e}}\downarrow & & \uparrow{\scriptstyle \widetilde{\rho}_Y} \\ M(X+V+Y) & \xrightarrow[M[s,f^\sharp,\widetilde{\eta}_Y]]{} & M\widetilde{M}Y \end{array} \tag{3.44}$$

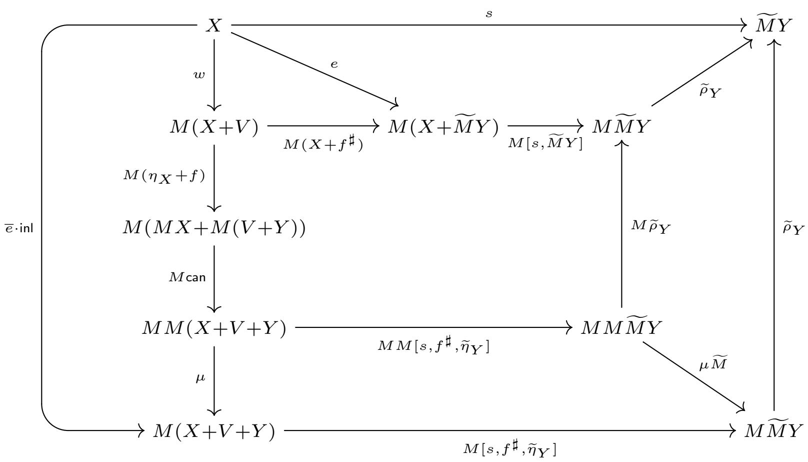
commutes. By Lemma 3.18, it follows that $\overline{e}^\sharp = [s, f^\sharp]$, hence

$$\overline{e}^\dagger = \overline{e}^\sharp \cdot \mathsf{inl} = [s, f^\sharp] \cdot \mathsf{inl} = s$$

as desired. Consider the coproduct components of Diagram (3.44) separately. For the right-hand component see Diagrams (3.29) and (3.39):



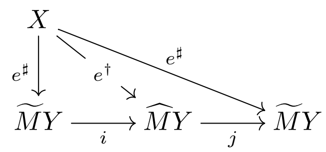

For the left-hand component we obtain the diagram



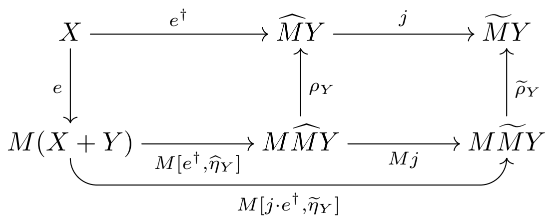

The upper left-hand triangle commutes due to Diagram (3.36), and the right-hand part does since $\widetilde{\rho}_Y$ is an $\mathbb{M}$-algebra structure. The lowest part commutes by the naturality of $\mu$, and for the left-hand one see (3.37). For the middle square remove $M$ and consider the components separately. The right-hand component commutes due to Diagram (3.42) with $[s, f^\sharp]$ in lieu of $\overline{e}^\sharp$, and the left-hand component yields $s$ on both paths similarly as in Diagram (3.43). $\square$

*Proof of Theorem 3.9.* It is sufficient to prove that morphism $i : \widetilde{M}Y \longrightarrow \widehat{M}Y$ of Notation 3.15 is an isomorphism.

(a) Let $Y$ be finitely presentable. Since $\widetilde{M}Y$ is an iterative algebra by Propostion 3.19, there exists an algebra homomorphism $j : \widehat{M}Y \longrightarrow \widetilde{M}Y$ such that $\widetilde{\eta}_Y = j \cdot \widehat{\eta}_Y$. From the freeness of $\widehat{M}Y$ and Lemma 3.17 it follows immediately that $i \cdot j = \mathrm{id}$. To see that $j \cdot i = \mathrm{id}$ consider the diagram



whose left-hand triangle is Diagram (3.24). Thus, it suffices to prove that the right-hand triangle commutes. To this end use the following commutative diagram



Indeed, the left-hand square commutes since $e^\dagger$ is a solution of $e$, and the right-hand one does since $j$ is an algebra homomorphism. By Lemma 3.18 we get $j \cdot e^\dagger = e^\sharp$.

(b) For arbitrary objects $Y$ we extend the result by using filtered colimits. For that we first observe that the functor $\widehat{M}$ is finitary because it is the composite of the free-iterative-algebra functor (a left adjoint) and the forgetful functor of the category of iterative algebras; the latter is finitary by Theorem 2.13 in [5]. Express $Y = \operatorname*{colim}_{t \in T} Y_t$ as a filtered colimit of finitely presentable objects. It is easy to see that $\operatorname{colim} \mathrm{Eq}_Y$ is a filtered colimit of $\operatorname*{colim}_{t\in T} \mathrm{Eq}_{Y_t}$. Thus,

$$
\begin{aligned}
\widehat{M}Y &\simeq \operatorname*{colim}_{t\in T} \widehat{M}Y_t && \text{(since } \widehat{M} \text{ is finitary)}\\
&\simeq \operatorname*{colim}_{t\in T} \operatorname{colim} \mathrm{Eq}_{Y_t} && \text{(by part (a))}\\
&\simeq \operatorname{colim} \operatorname*{colim}_{t\in T} \mathrm{Eq}_{Y_t} && \text{(colimits commute with colimits)}\\
&\simeq \operatorname{colim} \mathrm{Eq}_Y\,.
\end{aligned}
$$

□

## 4. Rational Equation Morphisms

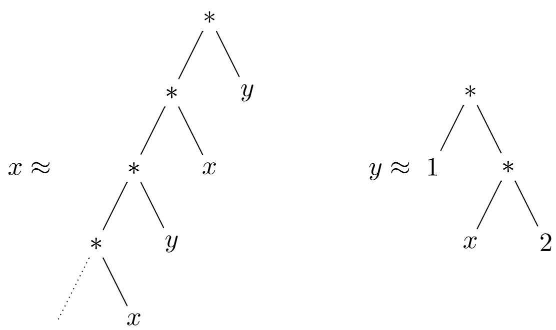
In this section we prove that iterative algebras have a stronger property of solving equations than stated in their definition. As an example consider the monad $\mathbb{M}$ of finite binary trees, for which an algebra is a set $A$ with a binary operation. The algebra $A$ is iterative iff every guarded system of equations

$$x_i \approx t_i(x_1, \ldots, x_n, a_1, \ldots, a_k) \qquad i = 1, \ldots, n,$$

where each $t_i$ is a finite binary tree on $\{\, x_i \mid i = 1, \dots n \,\} + \{\, a_j \mid j = 1, \dots, k \,\}$ has a unique solution. However, in lieu of finite trees we can as well take rational infinite trees on the right-hand sides. That is, in lieu of equation morphisms of the form $e : X \longrightarrow M(X+A)$ we are allowed to consider all $e : X \longrightarrow \widehat{M}(X+A)$, where $\widehat{\mathbb{M}}$ is the monad of free iterative $\mathbb{M}$-algebras (as constructed in Section 3). We generalize this in the following way:

**Definition 4.1.** By a *rational equation morphism* is meant a morphism $X \longrightarrow \widehat{M}(X + A)$ with $X$ finitely presentable.

**Example 4.2.** Consider the monad $\mathbb{M}$ of finite binary trees. As mentioned in Example 2.11, here $\widehat{\mathbb{M}}$ is the monad of rational binary trees. Consider the rational equation morphism $e$ with variables $X = \{x, y\}$ and parameters $A = \mathbb{N}$ given as follows:




We will use this as a running example in this section.

The concept of a solution in an iterative $\mathbb{M}$-algebra is based on the following

**Notation 4.3.** For an iterative $\mathbb{M}$-algebra $(A, a)$ we denote by $\widehat{a} : \widehat{M}A \longrightarrow A$ the unique homomorphism extending the identity:

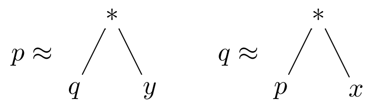
$$\begin{array}{ccccc} M\widehat{M}A & \xrightarrow{\rho_A} & \widehat{M}A & \xleftarrow{\widehat{\eta}_A} & A \\ {\scriptstyle M\widehat{a}}\downarrow & & \downarrow{\scriptstyle \widehat{a}} & \swarrow\!\!= & \\ MA & \xrightarrow[a]{} & A & & \end{array} \tag{4.1}$$

Using the universal property of free iterative algebras it is easy to prove that $(A, \widehat{a})$ is an algebra for the monad $\widehat{\mathbb{M}}$.

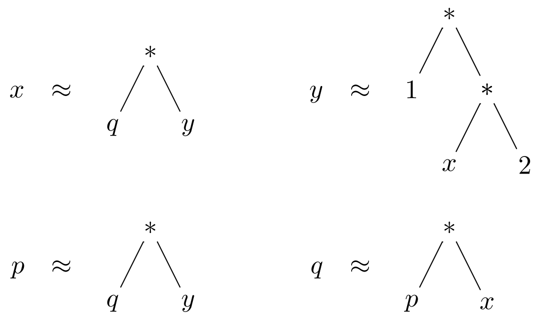
**Definition 4.4.** By a *solution* of a rational equation morphism $e : X \longrightarrow \widehat{M}(X + A)$ in an iterative $\mathbb{M}$-algebra $(A, a)$ is meant a morphism $e^\ddagger$ such that the square below commutes:

$$\begin{array}{ccc} X & \xrightarrow{e^\ddagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widehat{a}} \\ \widehat{M}(X + A) & \xrightarrow[\widehat{M}[e^\ddagger, A]]{} & \widehat{M}A \end{array}$$

**Example 4.5.** The rational equation morphism $e$ of Example 4.2 has a unique solution. This can be seen directly by considering the individual levels of the trees $e^\ddagger(x)$ and $e^\ddagger(y)$. Or indirectly as follows: the right-hand side of $x$ is a rational tree that we can obtain by solving the finitary flat equations



Thus, instead of the given rational system $e$ we can work with the finitary system



This is the main idea of the proof of Theorem 4.13.

**Remark 4.6.** In order to state the theorem about unique solutions of rational equation morphisms $e$, we need to introduce the concept of $e$ being guarded. This would be easy if we knew that the monad $\widehat{\mathbb{M}}$ is ideal. Although this is actually true, and we prove this below (see Theorem 5.9), we are in no position for proving this now. In lieu of the desired equality $\widehat{M} = \widehat{M}' + \mathrm{Id}$, we will now simply introduce a (seemingly arbitrary) subfunctor

$$\widehat{m} : \widehat{M}' \longrightarrow \widehat{M}$$

of $\widehat{M}$ and relate our concept of guarded equation morphism to $\widehat{M}'$—for distinction from the "real thing" we call this notion "weakly guarded" equation morphism. At the end of our paper we will indeed verify $\widehat{M} = \widehat{M}' + \mathrm{Id}$ and thus "guarded" is the same concept as "weakly guarded".

**Notation 4.7.**

(1) We denote by $\rho : M\widehat{M} \longrightarrow \widehat{M}$ the natural transformation whose components are the algebra maps $\rho_Y : M\widehat{M}Y \longrightarrow \widehat{M}Y$ of the free iterative $\mathbb{M}$-algebras $\widehat{M}Y$, see Remark 2.9.

(2) Recall from Remark 2.9 that the monad $\widehat{\mathbb{M}}$ of free iterative $\mathbb{M}$-algebras has the unit $\widehat{\eta}$ given by universal morphisms $\widehat{\eta}_Y : Y \longrightarrow \widehat{M}Y$. The multiplication $\widehat{\mu}$ given by extending $\mathrm{id}_{\widehat{M}Y}$ to the unique $\mathbb{M}$-homomorphism $\widehat{\rho_Y}$ (cf. Notation 4.3):

$$\begin{array}{ccccc} M\widehat{M}\widehat{M}Y & \xrightarrow{\rho_{\widehat{M}Y}} & \widehat{M}\widehat{M}Y & \xleftarrow{\widehat{\eta}_{\widehat{M}Y}} & \widehat{M}Y \\ {\scriptstyle M\widehat{\mu}_Y}\downarrow & & \downarrow{\scriptstyle \widehat{\mu}_Y} & \swarrow\!\!= & \\ M\widehat{M}Y & \xrightarrow[\rho_Y]{} & \widehat{M}Y & & \end{array} \tag{4.2}$$

**Remark 4.8.** Recall from [11] that in every locally finitely presentable category every morphism can be factorized as a strong epimorphism followed by a monomorphism.

**Definition 4.9.** We define the subfunctor $\widehat{M}'$ of $\widehat{M}$ to be the image of the natural transformation $\rho \cdot m\widehat{M} : M'\widehat{M} \longrightarrow \widehat{M}$. More precisely, for every object $Y$ we have a strong epimorphism $\gamma_Y$ and a monomorphism $\widehat{m}_Y$ such that the diagram

$$\begin{array}{ccccc} M'\widehat{M}Y & \xrightarrow{m_{\widehat{M}Y}} & M\widehat{M}Y & \xrightarrow{\rho_Y} & \widehat{M}Y \\ & {\scriptstyle\gamma_Y}\searrow & & \nearrow{\scriptstyle\widehat{m}_Y} & \\ & & \widehat{M}'Y & & \end{array} \tag{4.3}$$

commutes. Obviously, $\gamma : M'\widehat{M} \longrightarrow \widehat{M}'$ and $\widehat{m} : \widehat{M}' \longrightarrow \widehat{M}$ are natural transformations with $\widehat{m} \cdot \gamma = \rho \cdot m\widehat{M}$.

**Definition 4.10.** A rational equation morphism $e : X \longrightarrow \widehat{M}(X + A)$ is called *weakly guarded* if it factorizes through $[\widehat{m}_{X+A}, \widehat{\eta}_{X+A} \cdot \mathsf{inr}] : \widehat{M}'(X + A) + A \longrightarrow \widehat{M}(X + A)$ as shown below:

$$\begin{array}{ccc} X & \xrightarrow{e} & \widehat{M}(X+A) \\ & {\scriptstyle e'}\searrow & \uparrow{\scriptstyle [\widehat{m},\widehat{\eta}\cdot\mathsf{inr}]} \\ & & \widehat{M}'(X+A)+A \end{array} \tag{4.4}$$

**Remark 4.11.** Recall from Notation 3.3 our convention that for every morphism $e : X \longrightarrow M(X + Y)$ in $\mathsf{EQ}_Y$ we denote by $e^\dagger : X \longrightarrow \widehat{M}Y$ the unique solution in the free iterative $\mathbb{M}$-algebra $(\widehat{M}Y, \rho_Y)$. Recall also the notation $h \bullet e$.

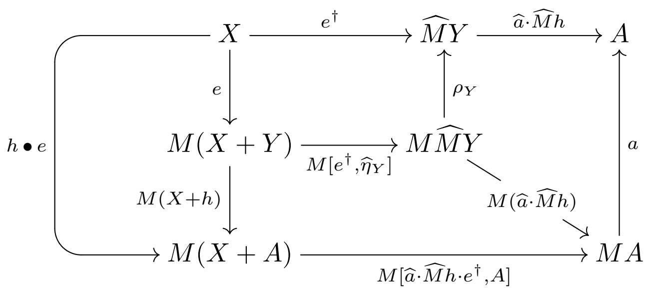
**Lemma 4.12.** *Let $(A, a)$ be an iterative $\mathbb{M}$-algebra, and let $h : Y \longrightarrow A$ be a morphism. Then for every guarded equation morphism $e : X \longrightarrow M(X + Y)$ of $\mathsf{EQ}_Y$ the triangle below commutes:*

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & \widehat{M}Y \\ & {\scriptstyle (h\bullet e)^\dagger}\searrow & \downarrow{\scriptstyle \widehat{a}\cdot\widehat{M}h} \\ & & A \end{array}$$

*Proof.* Observe first that $\widehat{a} \cdot \widehat{M}h$ is the unique homomorphism with

$$\widehat{a} \cdot \widehat{M}h \cdot \widehat{\eta}_Y = h\,. \tag{4.5}$$

This implies that $\widehat{a} \cdot \widehat{M}h \cdot e^\dagger$ is a solution of $h \bullet e$ since the following diagram commutes:




Indeed, the upper left-hand square expresses that $e^\dagger$ solves $e$, the right-hand part commutes since $\widehat{a} \cdot \widehat{M}h$ is a homomorphism, and the lower part is due to Equation (4.5). The desired result now follows from the uniqueness of solutions. □

**Theorem 4.13.** *Let $(A, a)$ be an iterative $\mathbb{M}$-algebra. Every weakly guarded rational equation morphism $e : X \longrightarrow \widehat{M}(X + A)$ has a unique solution.*

*Proof.* Suppose we are given a weakly guarded rational equation morphism $e$ as in (4.4). Since $\gamma_{X+A} : M'\widehat{M}(X + A) \longrightarrow \widehat{M}'(X + A)$ is a strong epimorphism, we have, by Assumption 2.1, a morphism $s : \widehat{M}'(X+A) \longrightarrow M'\widehat{M}(X+A)$ with

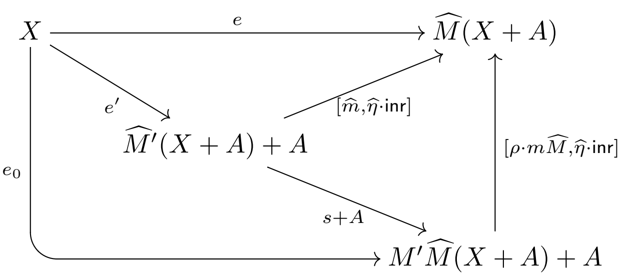
$$\gamma_{X+A} \cdot s = \mathrm{id}. \tag{4.6}$$

We define

$$e_0 = (X \xrightarrow{e'} \widehat{M}'(X + A) + A \xrightarrow{s+A} M'\widehat{M}(X + A) + A). \tag{4.7}$$

For example, consider the equation morphism $e$ of Example 4.2. Since $\gamma_{X+\mathbb{N}}$ is the obvious concatenation of trees, we can choose $s$ to be the inclusion map: every nontrivial rational tree $t$ on $X+\mathbb{N}$ is also a nontrivial finite tree on $\widehat{M}(X+\mathbb{N})$ by considering the two maximum subtrees $t_1$ and $t_2$ of $t$ (the concatenation of which is $t$). Thus, in this example, $e_0$ takes the same values as $e$, but now considered as elements of $M'\widehat{M}(X + \mathbb{N})$.

In general, for $e_0$ we obtain the commutative diagram




(4.8)

The upper triangle commutes by (4.4), the left-hand part does by (4.7), and the right-hand part does due to Equations (4.3) and (4.6):

$$(\rho \cdot m\widehat{M})_{X+A} \cdot s = (\widehat{m} \cdot \gamma)_{X+A} \cdot s = \widehat{m}_{X+A}\,.$$

Now apply Theorem 3.9 and use the fact that $M'$ is finitary to see that $M'\widehat{M}(X+A)+A = \mathrm{colim}(M'\mathrm{Eq}_{X+A}+A)$. Thus, by the finite presentability of $X$, there exists an object $g : W \longrightarrow M(W+X+A)$ in $\mathsf{EQ}_{X+A}$ and a factorization $w'$ of $e_0$ through the colimit injection $M'g^\sharp + A$:

$$\begin{array}{ccc} X & \xrightarrow{e_0} & M'\widehat{M}(X+A)+A \\ & \searrow^{w'} & \uparrow{\scriptstyle M'g^\sharp+A} \\ & & M'W+A \end{array}$$

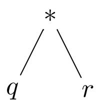
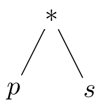
We define the morphism $w$ by

$$w = (\,X \xrightarrow{w'} M'W+A \xrightarrow{m_W+A} MW+A). \tag{4.9}$$

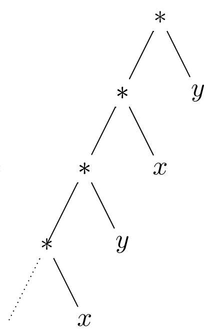
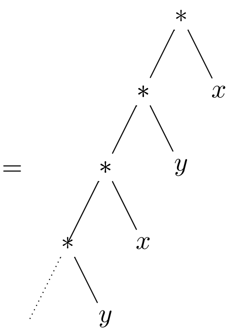
In the case of the equation morphism $e$ of Example 4.5 we form $g$ by using, besides the variables $p$ and $q$ there, the variables $r$ and $s$ representing $x$ and $y$ and the variables $u$, $v$ representing the parameters 1 and 2 of $e$:

$$W = \{p, q, r, s, u, v\}$$

where

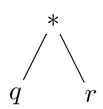
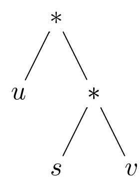
$g(p) =$  , $g(q) =$  , $\begin{array}{rclrcl} g(r) & = & y, & g(s) & = & x, \\ g(u) & = & 1, & g(v) & = & 2. \end{array}$

Then

$g^\sharp(p) =$  , $g^\sharp(q) =$  ,

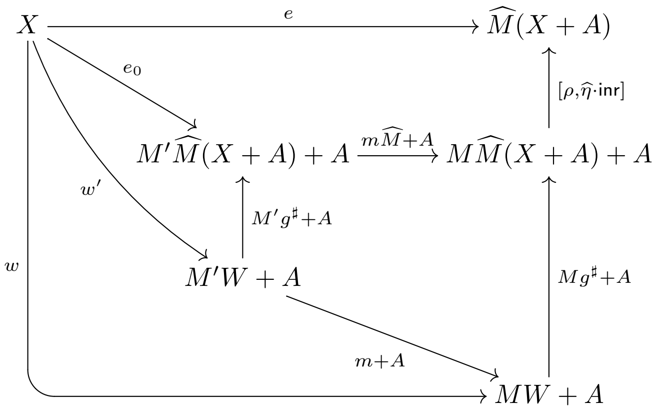
$$g^\sharp(r) = y, \qquad g^\sharp(s) = x, \qquad g^\sharp(u) = 1, \qquad g^\sharp(v) = 2.$$

Consequently, we can choose $w'$ by

$w'(x) =$  , $\quad w'(y) =$  .

And $w$ will have the same values.

Use the naturality of $m : M' \longrightarrow M$ and Diagram (4.8) to obtain the commutative diagram

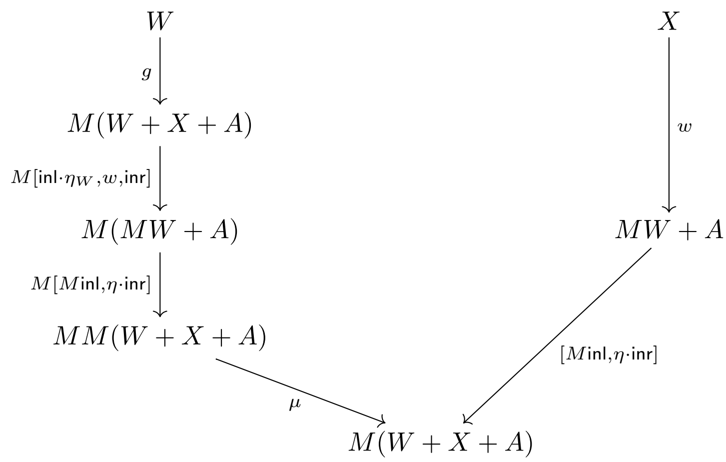



(4.10)

Next we define an equation morphism

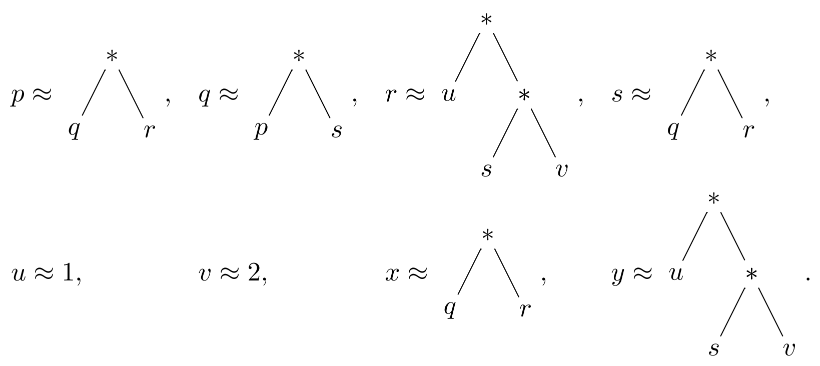
$$\langle e \rangle : W + X \longrightarrow M(W + X + A)$$

in $\mathsf{EQ}_A$ by its coproduct components

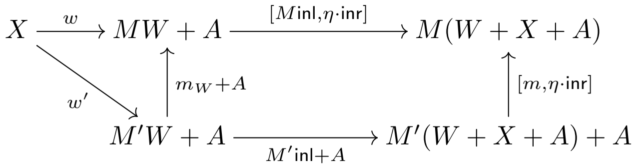



(4.11)

In our running example of $e$ the equation morphism $\langle e \rangle$ is given by $W + X = \{p, q, r, s, u, v, x, y\}$ and



We check that $\langle e \rangle$ is a guarded equation morphism. This can be done for the components separately. For the right-hand component of $\langle e \rangle$ consider the diagram




Its left-hand triangle is (4.9), and the right-hand coproduct component of the right-hand part commutes trivially, while its left-hand coproduct component commutes by the naturality of $m$.

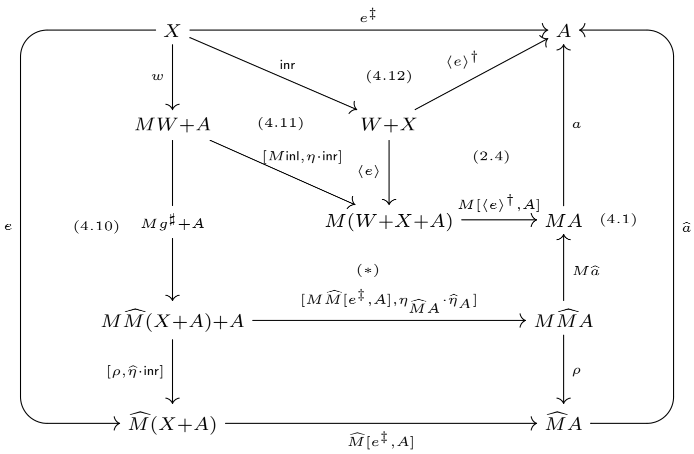
Checking the guardedness of $\langle e \rangle$ for the left-hand component results in a rather big diagram whose commutativity is nevertheless easy to establish using guardedness of $g$, naturality of $m$ and $\eta$ as well as the laws of the given ideal monad $\mathbb{M}$. We leave the verification to the reader.

Since $\langle e \rangle$ is guarded, there exists a unique solution $\langle e \rangle^\dagger : W + X \longrightarrow A$, and we define

$$e^\ddagger = (\, X \xrightarrow{\mathsf{inr}} W + X \xrightarrow{\langle e \rangle^\dagger} A \,). \tag{4.12}$$

We will prove that this provides the desired unique solution of $e$.

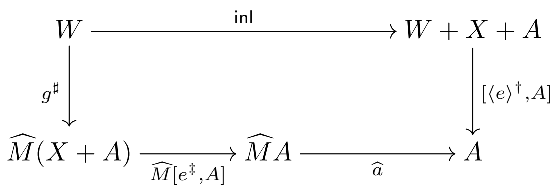
(1) $e^\ddagger$ is a solution of $e$. To see this consider the diagram




(4.13)

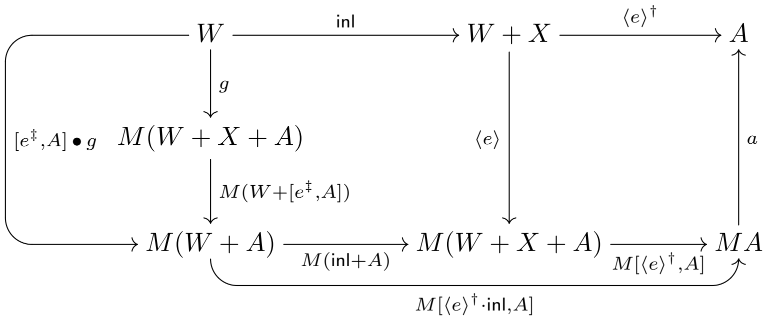
The lowest square commutes by the unit law $\rho_A \cdot \eta_{\widehat{M}A} = \mathrm{id}$ and the naturality of $\rho$ and $\widehat{\eta}$. All the other inner parts except $(*)$ commute as indicated. We will now prove that $(*)$ commutes. We consider the two components of $MW + A$ separately.

(1a) The left-hand component of $(*)$ of Diagram (4.13). Remove $M$ to obtain the square



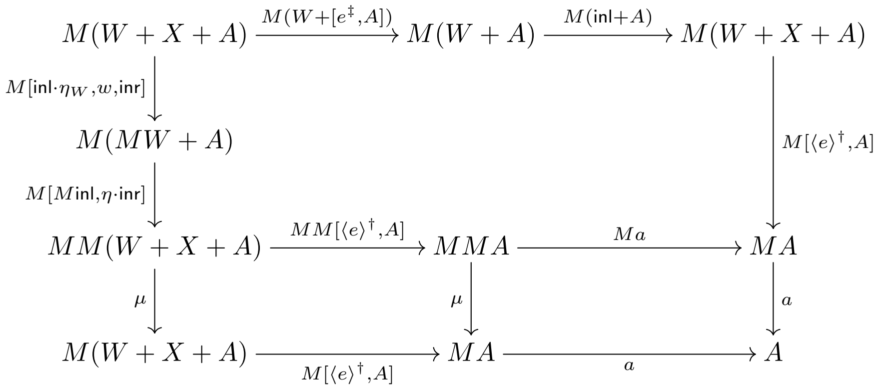

(4.14)

Notice that, by Remark 3.10(2), $g^\sharp : W \longrightarrow \widehat{M}(X+A)$ is the solution of $g$ in the free iterative algebra. Thus, by Lemma 4.12 applied to $g$ and $h = [e^\ddagger, A] : X + A \longrightarrow A$ we know that $\widehat{a} \cdot \widehat{M}[e^\ddagger, A] \cdot g^\sharp$ is the unique solution of the equation morphism $[e^\ddagger, A] \bullet g$. Hence, to prove that (4.14) commutes, we establish that also $\langle e \rangle^\dagger \cdot \mathsf{inl}$ is a solution of $[e^\ddagger, A] \bullet g$. That is, we have to show that the diagram

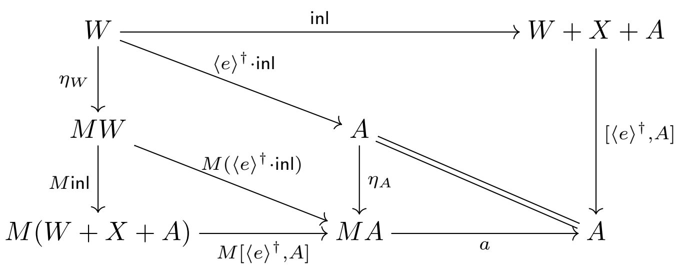


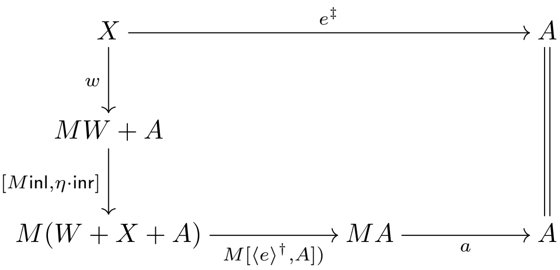

(4.15)

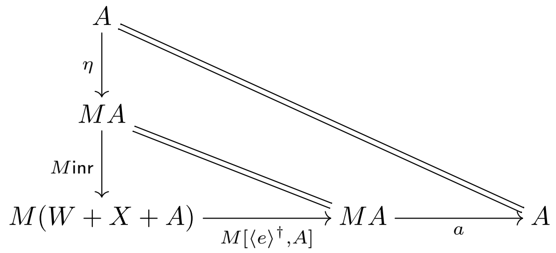
commutes. Indeed, the right-hand square expresses that $\langle e \rangle^\dagger$ is a solution of $\langle e \rangle$, and the lower and left most parts are obvious. We do not claim that the middle part commutes. However, it does when extended by $a \cdot M[\langle e \rangle^\dagger, A]$. To see this recall the left-hand component of $\langle e \rangle$ of Diagram (4.11), and observe that the two morphisms we have to prove equal both start with $g$. We shall now prove that they are equal even if we remove $g$, more precisely, we prove (4.15) by verifying that the following diagram commutes:




Indeed, the lower left-hand square commutes due to the naturality of $\mu$, and the lower right-hand one is the associativity of the algebra $(A, a)$. For the upper square remove $M$ and consider the three coproduct components separately. We obtain, from left to right, the following three diagrams:

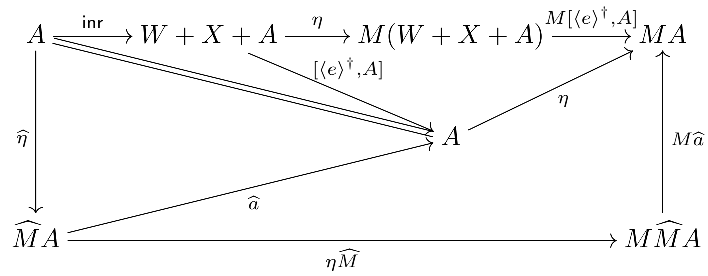





The first and the third diagrams clearly commute; use naturality of $\eta$ and the unit law of the algebra $(A, a)$. The second diagram is the same as the upper part of Diagram (4.13). This proves (4.15) and thus establishes (4.14).

(1b) The right-hand component of $(*)$ of Diagram (4.13). We just observe the commutativity of the diagram




(4.16)

using the naturality of $\eta$ and the right-hand triangle of Diagram (4.1). This completes the proof of Diagram (4.13).

(2) Uniqueness of solutions. Given a solution $s : X \longrightarrow A$ of the equation morphism $e$, we will show that for the morphism

$$u = (W \xrightarrow{g^{\sharp}} \widehat{M}(X + A) \xrightarrow{\widehat{M}[s,A]} \widehat{M}A \xrightarrow{\widehat{a}} A) \tag{4.17}$$

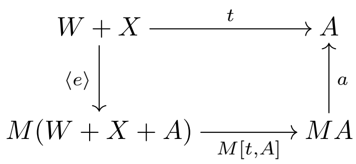
we obtain a solution

$$t = (W + X \xrightarrow{[u,s]} A) \tag{4.18}$$

of $\langle e \rangle$ in the iterative algebra $(A, a)$. Thus, we get from Equation (4.12) that

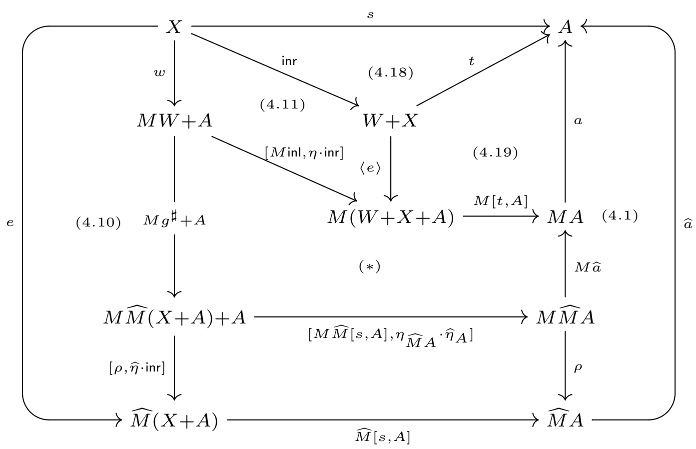
$$e^{\ddagger} = \langle e \rangle^{\dagger} \cdot \mathsf{inr} = t \cdot \mathsf{inr} = s\,.$$

It is our task to prove that the square



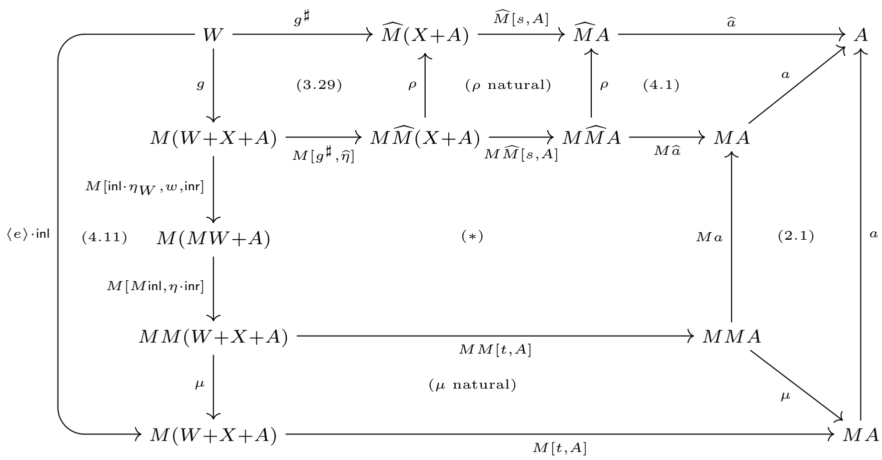

(4.19)

commutes; it suffices to consider the coproduct components separately.
(2a) The right-hand component of Diagram (4.19). Here we consider the diagram

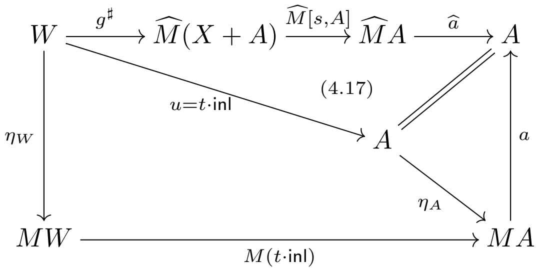


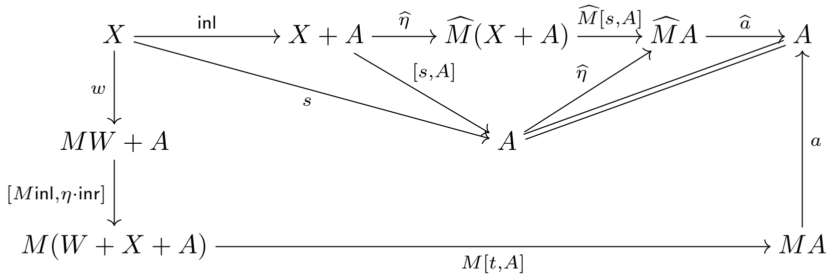

(4.20)

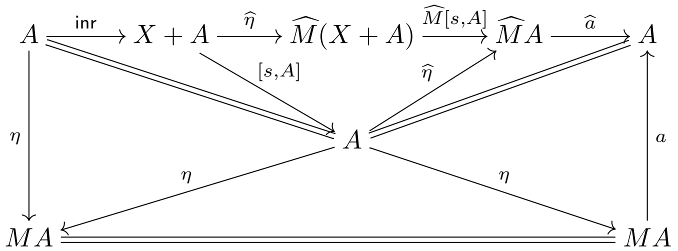
The lowest square is the same as in Diagram (4.13). All the other inner parts except two, the desired one (4.19) and $(*)$, commute as indicated. For part $(*)$ we consider the coproduct components separately. The left-hand component yields (since $t\cdot\mathsf{inl} = u$), after $M$ is removed, the morphism $u = \widehat{a}\cdot\widehat{M}[s, A]\cdot g^\sharp$ (see (4.17)) on both paths. The right-hand component commutes: indeed, consider the analogue of Diagram (4.16) with $\langle e\rangle$ replaced by $t$. Now the outside of Diagram (4.20) commutes since $s$ is a solution of $e$. Thus, it follows that the right-hand component of the remaining inner part (4.19) commutes, too.
(2b) The left-hand component of (4.19). We have to establish that the following diagram commutes:



All inner parts except $(*)$ commute as indicated. For that part remove $M$ and consider the three coproduct components separately. We get from left to right the following three diagrams:




The first and the third diagrams clearly commute; use the definitions (4.17) and (4.18) of $u$ and $t$, the naturality of $\eta$ and $\widehat{\eta}$, the unit law of the algebra $(A, a)$, and Diagram (4.1). Now consider the second diagram. The three triangles also clearly commute (use naturality of $\widehat{\eta}$ and (4.1)). For the lower part see Diagram (4.20).

We have established that $t$ is a solution of $\langle e\rangle$, thus $\langle e\rangle^\dagger = t$, which completes the proof. $\square$

## 5. The Iterative Reflection

We are ready to prove that for every ideal monad $\mathbb{M}$ the monad $\widehat{\mathbb{M}}$ of free iterative algebras (see Remark 2.9) is the free iterative reflection. More detailed:

(1) $\widehat{M} = \widehat{M}' + \mathrm{Id}$ with coproduct injections $\widehat{m}$ (Remark 4.6) and $\widehat{\eta}$,

(2) the multiplication $\widehat{\mu}$ has a restriction $\widehat{\mu}' : \widehat{M}'\widehat{M} \longrightarrow \widehat{M}'$,

(3) every guarded equation morphism $e : X \longrightarrow \widehat{M}(X + A)$ has a unique solution,

(4) the natural transformation

$$\kappa = (\, M \xrightarrow{M\widehat{\eta}} M\widehat{M} \xrightarrow{\ \rho\ } \widehat{M}\,) \tag{5.1}$$

is an ideal monad morphism, and

(5) $\kappa$ has the universal property that for every ideal monad morphism from $\mathbb{M}$ to an iterative monad there exists a unique extension along $\kappa$ to an ideal monad morphism.

We have to leave (1) to the end and prove the other properties first. We will use the same terminology as in Section 4: in (3) we speak about weakly guarded equation morphisms meaning those with a factorization as in (4.4). In (4) and (5) we use the following notion of weakly ideal monads.

**Definition 5.1.**

(1) A *weakly ideal monad* consists of a finitary monad $\mathbb{M} = (M, \eta, \mu)$, a finitary subfunctor $m : M' \hookrightarrow M$, and a natural transformation $\mu'$ such that the square below commutes:

$$\begin{array}{ccc} M'M & \xrightarrow{\ \mu'\ } & M' \\ {\scriptstyle mM}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ MM & \xrightarrow[\ \mu\ ]{} & M \end{array} \tag{5.2}$$

(2) Suppose we have two weakly ideal monads $\mathbb{M} = (M, \eta, \mu, M', m, \mu')$ and $\overline{\mathbb{M}} = (\overline{M}, \overline{\eta}, \overline{\mu}, \overline{M}', \overline{m}, \overline{\mu}')$. By a *weakly ideal monad morphism* we understand a monad morphism $h : (M, \eta, \mu) \longrightarrow (\overline{M}, \overline{\eta}, \overline{\mu})$ such that there exists a domain-codomain restriction $h' : M' \longrightarrow \overline{M}'$ of $h$ with $\overline{m} \cdot h' = h \cdot m$.

(3) A weakly ideal monad is called *weakly iterative* if every weakly guarded equation morphism has a unique solution.

**Remark 5.2.** Every ideal monad (see Definition 2.3) is, of course, weakly ideal. But the converse does not hold; for example, every monad $(S, \eta^S, \mu^S)$ is weakly ideal with $S' = S$, $\mathrm{id} : S' \longrightarrow S$ and $\mu' = \mu^S$.

**Lemma 5.3.** *The monad* $\widehat{\mathbb{M}}$ *of free iterative algebras for* $\mathbb{M}$ *is weakly ideal w.r.t.* $\widehat{m}$ *of Remark 4.6.*

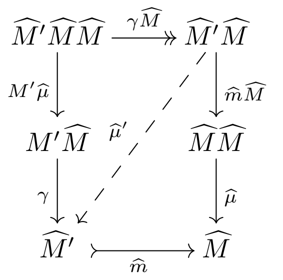
*Proof.* We only need to supply the restriction $\widehat{\mu}' : \widehat{M}'\widehat{M} \longrightarrow \widehat{M}'$ of the monad multiplication $\widehat{\mu} : \widehat{M}\widehat{M} \longrightarrow \widehat{M}$. Then $\widehat{\mathbb{M}} = (\widehat{M}, \widehat{\eta}, \widehat{\mu}, \widehat{M}', \widehat{m}, \widehat{\mu}')$ is a weakly ideal monad.

Observe first that the diagram

$$\begin{array}{ccccc} & & \xrightarrow{\ \widehat{m}\widehat{M}\cdot\gamma\widehat{M}\ } & & \\ M'\widehat{M}\widehat{M} & \xrightarrow{\ m\widehat{M}\widehat{M}\ } & M\widehat{M}\widehat{M} & \xrightarrow{\ \rho\widehat{M}\ } & \widehat{M}\widehat{M} \\ {\scriptstyle M'\widehat{\mu}}\big\downarrow & & {\scriptstyle M\widehat{\mu}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\mu}} \\ M'\widehat{M} & \xrightarrow[\ m\widehat{M}\ ]{} & M\widehat{M} & \xrightarrow[\ \rho\ ]{} & \widehat{M} \\ & & \xrightarrow[\ \widehat{m}\cdot\gamma\ ]{} & & \end{array}$$

commutes. Indeed, the left-hand square commutes by naturality of $m$, the rest follows from Diagrams (4.2) and (4.3).

Thus, by diagnonal fill-in there exists a unique natural transformation $\widehat{\mu}' : \widehat{M}' \longrightarrow \widehat{M}$ such that the diagram

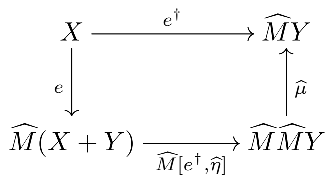



(5.3)

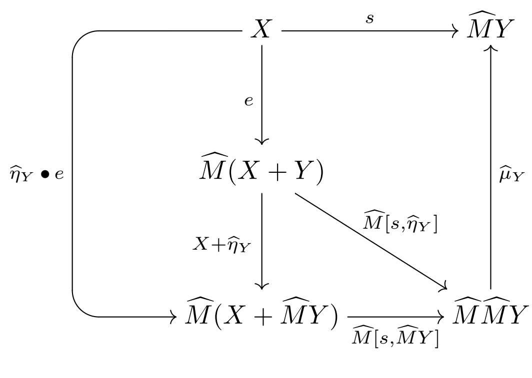
commutes. The lower triangle shows that $\mu'$ is the required restriction of $\mu$ (cf. (5.2)). □

**Lemma 5.4.** *The monad* $\widehat{\mathbb{M}}$ *of the free iterative algebras for* $\mathbb{M}$ *is weakly iterative.*

*Proof.* We will show that every weakly guarded equation morphism $e : X \longrightarrow \widehat{M}(X+Y)$ has a unique solution, i. e., there exists a unique morphism $e^\dagger : X \longrightarrow \widehat{M}Y$ such that the following square commutes:




Indeed, apply Theorem 4.13 to $A = \widehat{M}Y$ and the equation morphism $\widehat{\eta}_Y \bullet e$ (see Notation 3.3). To see the result observe that solutions of $e$ are in a 1–1–correspondence to solutions of $\widehat{\eta}_Y \bullet e$:




Indeed, notice that $\widehat{\mu}_Y = \widehat{\rho_Y}$ (see Notation 4.3). Thus, a solution $s$ of $\widehat{\eta}_Y \bullet e$ makes the outside this diagram commutative. Equivalently, since the lower triangle trivially commutes, the upper part commutes, which is to say that $s$ is a solution of $e$. □

**Remark 5.5.** Notice that the equation

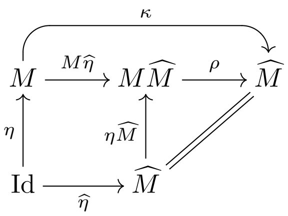
$$\rho = \widehat{\mu} \cdot \kappa\widehat{M} : M\widehat{M} \longrightarrow \widehat{M} \tag{5.4}$$

holds. Indeed, we have

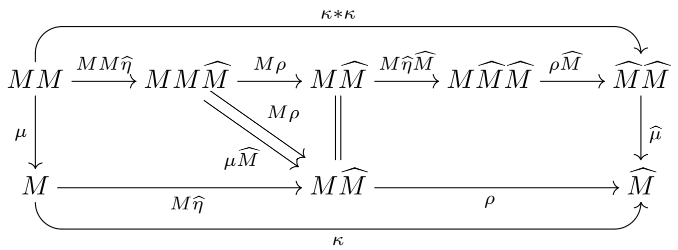
$$\begin{aligned} \widehat{\mu} \cdot \kappa\widehat{M} &= \widehat{\mu} \cdot \rho\widehat{M} \cdot M\widehat{\eta}\widehat{M} && \text{(definition of } \kappa) \\ &= \rho \cdot M\widehat{\mu} \cdot M\widehat{\eta}\widehat{M} && \text{(definition of } \widehat{\mu}) \\ &= \rho \cdot M(\widehat{\mu} \cdot \widehat{\eta}\widehat{M}) && \\ &= \rho && \text{(unit law of the monad } \widehat{M}) \end{aligned}$$

**Lemma 5.6.** *The natural transformation $\kappa : M \longrightarrow \widehat{M}$ is a weakly ideal monad morphism.*

*Proof.* (1) The preservation of the unit follows from the unit law of the algebras $(\widehat{M}Y, \rho_Y)$, for every object $Y$, and from the naturality of $\eta : \mathrm{Id} \longrightarrow M$:




(2) For the preservation of multiplication consider the diagram

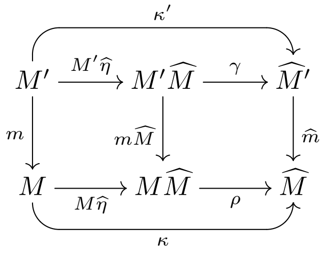



Its right-hand square commutes due to Equations (5.1) and (5.4), and its left-hand part by the naturality of $\mu$. The two parallel morphisms $M\rho$ and $\mu\widehat{M}$ are merged by $\rho : M\widehat{M} \longrightarrow \widehat{M}$ since $\rho$ is componentwise an algebra structure. Since the remaining upper and lower parts obviously commute, so does the outside of the diagram.

(3) To see that $\kappa$ has the restriction as required consider the natural transformation

$$\kappa' = (M' \xrightarrow{M'\widehat{\eta}} M'\widehat{M} \xrightarrow{\gamma} \widehat{M}').$$

Thus, the diagram




(5.5)

commutes: its left-hand square does by the naturality of $m$, and its right-hand one by Diagram (4.3). $\square$

**Lemma 5.7.** *Let $\mathbb{S} = (S, \eta^S, \mu^S, S', s, (\mu^S)')$ be an iterative monad. For every weakly ideal monad morphism $\lambda : \mathbb{M} \longrightarrow \mathbb{S}$ there exists a unique weakly ideal monad morphism $\overline{\lambda} : \widehat{\mathbb{M}} \longrightarrow \mathbb{S}$ with $\lambda = \overline{\lambda} \cdot \kappa$.*

**Remark 5.8.** Notice that $\mathbb{S}$ is assumed to be an ideal monad, thus $[s, \eta^S] : S' + \mathrm{Id} \longrightarrow S$ is an isomorphism. We will use this fact in the proof below.

*Proof of Lemma 5.7.* (1) For every object $Y$, we prove that $SY$ is an iterative $\mathbb{M}$-algebra. Indeed, since $\lambda : M \longrightarrow S$ is a monad morphism we obtain an $\mathbb{M}$-algebra

$$MSY \xrightarrow{\lambda_{SY}} SSY \xrightarrow{\mu^S} SY .$$

It is our task to show that those algebras are iterative. Given a weakly guarded equation morphism

$$\begin{array}{ccc} X & \xrightarrow{e} & M(X+SY) \\ & {\searrow}^{e_0} & \uparrow{\scriptstyle [m,\eta\cdot\mathsf{inr}]} \\ & & M'(X+SY)+SY \end{array} \tag{5.6}$$

we can form an equation morphism $\overline{e}$ with respect to the iterative monad $\mathbb{S}$ as follows:

$$\begin{array}{rl} \overline{e} = (X \xrightarrow{e} & M(X+SY) \\ & \downarrow{\scriptstyle \lambda*(\eta^S_X + SY)} \\ & S(SX+SY) \\ & \downarrow{\scriptstyle S\mathsf{can}} \\ & SS(X+Y) \xrightarrow{\mu^S} S(X+Y)\,). \end{array} \tag{5.7}$$

To see that $\overline{e}$ is guarded consider the commutative diagram in Figure 1. Indeed, its upper left-hand triangle commutes due to Diagram (5.6). The two lower squares, the right most part and the lower rigth-hand triangle all commute obviously. The upper left-hand square commutes since $\lambda$ is a weakly ideal monad morphism and by the naturality of $\eta^S$. The upper middle square commutes by the naturality of $s$. To see that the remaining part $(*)$ commutes we consider the components separately: the left-hand one commutes since $\mathbb{S}$ is an ideal monad, see the upper part of Diagram (2.2), the middle and right-hand components by the monad law $\mu^S \cdot \eta^S S = \mathrm{id}$ and the naturality of $s$.

There is a 1–1–correspondence between solutions of $e$ in the algebra $SY$ and solutions of $\overline{e}$ with respect to $\mathbb{S}$. To see this consider the diagram in Figure 2. All its inner parts except part $(*)$ are easily seen to commute. (For part $(+)$ remove $S$ and consider the coproduct components separately: the left-hand one commutes by the naturality of $\eta^S$ and the right-hand one is obvious.)

Now $e^\dagger$ is a solution of $e$ if and only if part $(*)$ commutes. Equivalently, the outside of the diagram commutes, which is the case if and only if $e^\dagger$ is a solution of $\overline{e}$ with respect to the monad $\mathbb{S}$.

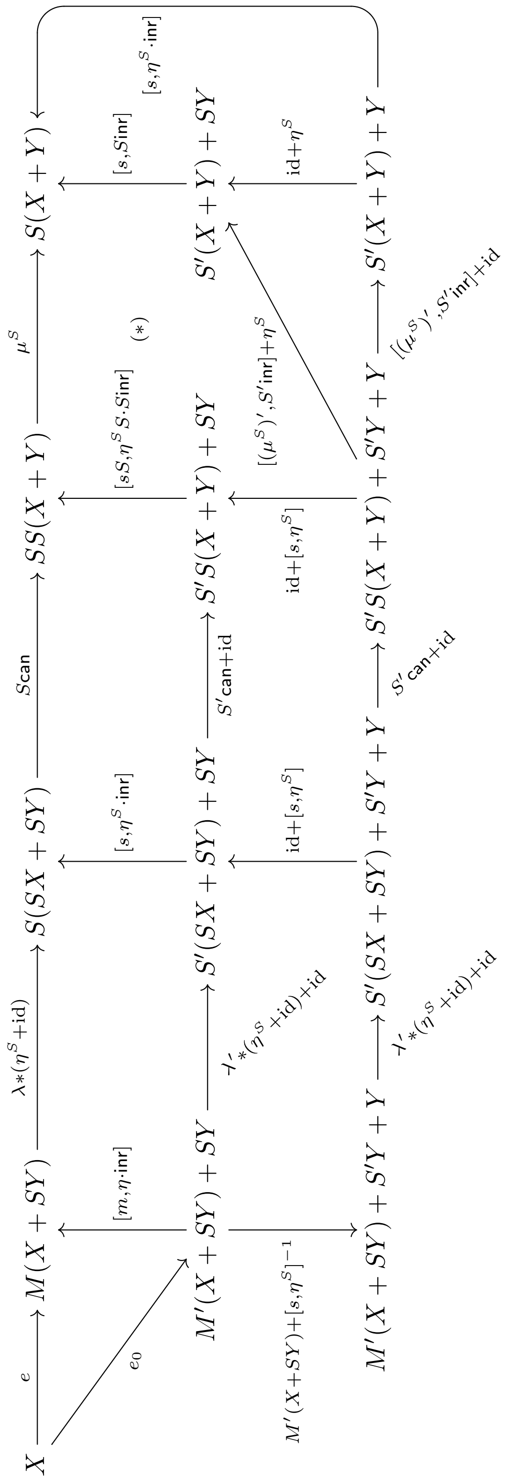



Figure 1: Proving that $\overline{e}$ is guarded

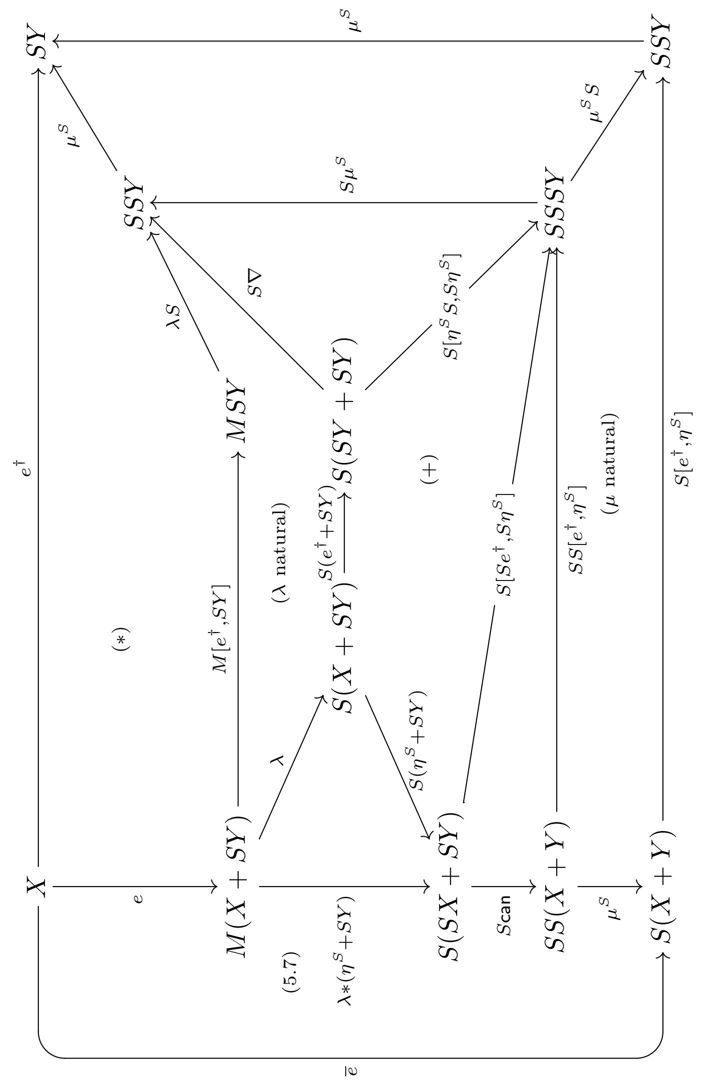



Figure 2: 1–1–correspondence between solutions of $e$ and $\overline{e}$

(2) The freeness of $\widehat{M}Y$ as an iterative algebra implies, by (1), the existence of a unique morphism $\overline{\lambda}_Y : \widehat{M}Y \longrightarrow SY$ such that the diagram

$$
\begin{array}{ccccc}
M\widehat{M}Y & \xrightarrow{\quad\rho_Y\quad} & \widehat{M}Y & \xleftarrow{\widehat{\eta}_Y} & Y \\
{\scriptstyle M\overline{\lambda}_Y}\downarrow & & {\scriptstyle \overline{\lambda}_Y}\downarrow & \swarrow{\scriptstyle \eta^S_Y} & \\
MSY \xrightarrow[\lambda_{SY}]{} SSY & \xrightarrow[\mu^S_Y]{} & SY & &
\end{array}
\tag{5.8}
$$

commutes. We first observe that $\overline{\lambda}$ is a natural transformation. Given a morphism $h : Y \longrightarrow Z$, then $Sh$ is a homomorphism of $\mathbb{M}$-algebras from $SY$ to $SZ$:

$$
\begin{array}{ccccc}
MSY & \xrightarrow{\lambda_{SY}} & SSY & \xrightarrow{\mu^S_Y} & SY \\
{\scriptstyle MSh}\downarrow & & {\scriptstyle SSh}\downarrow & & \downarrow{\scriptstyle Sh} \\
MSZ & \xrightarrow[\lambda_{SZ}]{} & SSZ & \xrightarrow[\mu^S_Z]{} & SZ
\end{array}
\tag{5.9}
$$

Indeed, the left-hand square commutes due to the naturality of $\lambda$, and the right-hand one does due to the naturality of $\mu^S$. Thus, we have two parallel algebra homomorphisms

$$Sh \cdot \overline{\lambda}_Y, \overline{\lambda}_Z \cdot \widehat{M}h : \widehat{M}Y \longrightarrow SZ,$$

which agree when precomposed with $\widehat{\eta}_Y$:

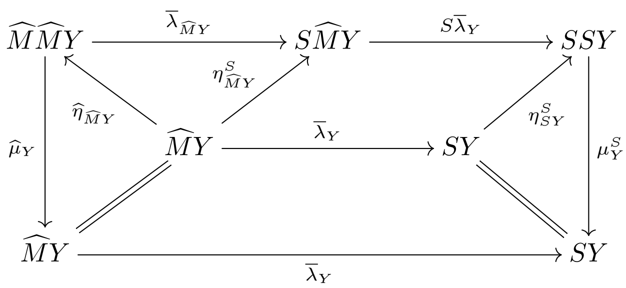
$$
\begin{array}{ccccc}
\widehat{M}Y & & \xrightarrow{\quad\overline{\lambda}_Y\quad} & & SY \\
 & \nwarrow{\scriptstyle \widehat{\eta}_Y} & & {\scriptstyle \eta^S_Y}\nearrow & \\
{\scriptstyle \widehat{M}h}\downarrow & & Y & & \downarrow{\scriptstyle Sh} \\
 & & \downarrow{\scriptstyle h} & & \\
 & & Z & & \\
 & \swarrow{\scriptstyle \widehat{\eta}_Z} & & {\scriptstyle \eta^S_Z}\searrow & \\
\widehat{M}Z & & \xrightarrow[\quad\overline{\lambda}_Z\quad]{} & & SZ
\end{array}
$$

Since $SZ$ is an iterative algebra by (1) above it follows from the universal property of $\widehat{M}Y$ that the outside of the square above commutes, proving the naturality of $\overline{\lambda}$.

(3) Let us prove now that $\overline{\lambda}$ is a monad morphism. The unit law $\overline{\lambda} \cdot \widehat{\eta} = \eta^S$ holds by definition (see Diagram (5.8)). Thus, it remains to show that $\overline{\lambda}$ preserves the monad multiplication, i. e., we prove that the diagram

$$
\begin{array}{ccccc}
\widehat{M}\widehat{M}Y & \xrightarrow{\overline{\lambda}_{\widehat{M}Y}} & S\widehat{M}Y & \xrightarrow{S\overline{\lambda}_Y} & SSY \\
{\scriptstyle \widehat{\mu}_Y}\downarrow & & & & \downarrow{\scriptstyle \mu^S_Y} \\
\widehat{M}Y & & \xrightarrow[\quad\overline{\lambda}_Y\quad]{} & & SY
\end{array}
\tag{5.10}
$$

commutes. Indeed, use (5.9) with $h = \overline{\lambda}_Y$ to see that $S\overline{\lambda}_Y$ is an algebra homomorphism. By the universal property of $\widehat{\eta}_{\widehat{M}Y} : \widehat{M}Y \longrightarrow \widehat{M}\widehat{M}Y$ is suffices to show that Diagram (5.10) commutes when precomposed with it:

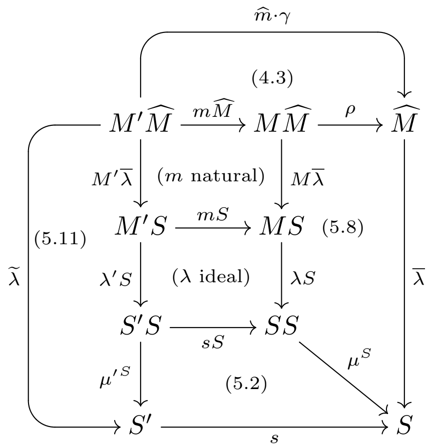



To see that all the inner parts of this diagram commutes use (5.8) the unit laws of the monads $\widehat{M}$ and $S$, the preservation of units by $\overline{\lambda}$, and the naturality of $\eta^S$.

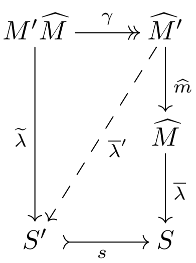
(4) Next we show that $\overline{\lambda}$ is a weakly ideal monad morphism. Indeed, define first

$$\widetilde{\lambda} = (\, M'\widehat{M} \xrightarrow{\lambda' * \overline{\lambda}} S'S \xrightarrow{\mu'^S} S' \,) \tag{5.11}$$

and consider the commutative diagram



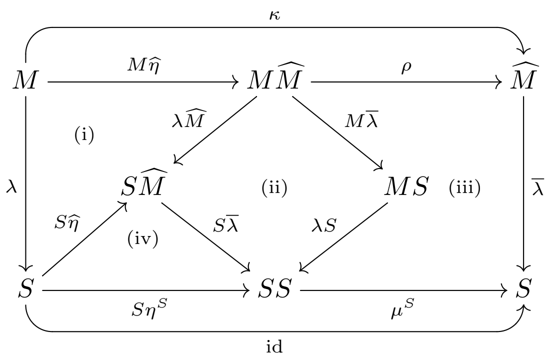

Since $s : S' \hookrightarrow S$ is a monomorphism, we obtain by diagonalization a unique natural transformation $\overline{\lambda}' : \widehat{M}' \longrightarrow S'$ such that the diagram




(5.12)

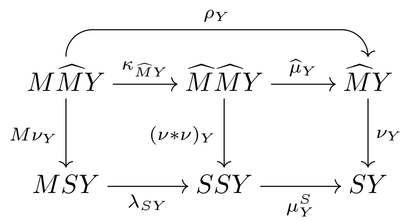
commutes. Its right-hand triangle shows that $\overline{\lambda}$ restricts to $\overline{\lambda}'$ establishing that $\overline{\lambda}$ is a weakly ideal.

(5) The equation $\overline{\lambda} \cdot \kappa = \lambda$ follows from the commutativity of the diagram



The uppermost part is the definition of $\kappa$ (see (5.1)), parts (i) and (ii) commute by the naturality of $\lambda$, part (iii) commutes by the definition of $\overline{\lambda}$ (see Diagram (5.8)), and for part (iv) remove $S$ and use (5.8). Finally, use the unit law of the monad $S$ for the lowest part.

(6) Uniqueness of $\overline{\lambda}$. Suppose that $\nu : \widehat{M} \longrightarrow S$ is a weakly ideal monad morphism with $\nu \cdot \kappa = \lambda$. Now in order to prove $\nu = \overline{\lambda}$ we show that for every object $Y$ the morphism $\nu_Y : \widehat{M}Y \longrightarrow SY$ is a homomorphism of $\mathbb{M}$-algebras such that $\nu_Y \cdot \widehat{\eta}_Y = \eta^S_Y$. Indeed, the last equation holds since the monad morphism $\nu$ preserves units, and $\nu_Y$ is an algebra homomomorphism since the diagram



commutes; for the uppermost part recall Equation (5.4), for the right-hand square use that $\nu$ preserves multiplication of monads, and for the left-hand one use naturality and the equation $\nu \cdot \kappa = \lambda$. $\square$

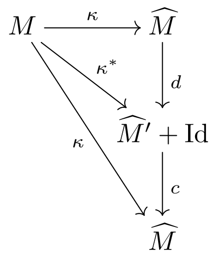
**Theorem 5.9.** *The iterative reflection of an ideal monad is the monad* $\widehat{\mathbb{M}}$ *of free iterative* $\mathbb{M}$*-algebras.*

*Proof.* In view of the preceeding results this amounts to proving that $\widehat{\mathbb{M}}$ is ideal, that is, $\widehat{M} = \widehat{M'} + \mathrm{Id}$ with injections $\widehat{m}$ and $\widehat{\eta}$.

It is known that every weakly ideal monad $\mathbb{S}$ has an ideal coreflection $c : \mathbb{S}^* \longrightarrow \mathbb{S}$ (see [12], Proposition 5.11). Moreover, whenever $\mathbb{S}$ is weakly iterative, then $\mathbb{S}^*$ is iterative; the proof of this fact is completely analogous to Lemma 5.13 in [12]. More detailed: let $\mathbb{S}$ be weakly ideal with the corresponding subfunctor $s : S' \hookrightarrow S$. Then for the functor $S^* = S' + \mathrm{Id}$ there is a structure of a monad $\mathbb{S}^*$ with unit $\mathsf{inr} : \mathrm{Id} \longrightarrow S' + \mathrm{Id}$ and multiplication $\mu^* : S^*S^* \longrightarrow S^*$ such that the morphism $c = [s, \eta] : S' + \mathrm{Id} \longrightarrow S$ is a weakly ideal monad morphism from $\mathbb{S}^*$ to $\mathbb{S}$. Moreover, every weakly ideal monad morphism from an ideal monad into $\mathbb{S}$ uniquely factorizes through $c$. We now apply this to $\mathbb{S} = \widehat{\mathbb{M}}$: we obtain an iterative monad $\widehat{\mathbb{M}}^* = (\widehat{M'} + \mathrm{Id}, \mathsf{inr}, \widehat{\mu}^*)$ and a weakly ideal monad morphism

$c = [\widehat{m}, \widehat{\eta}] : \widehat{\mathbb{M}}^* \longrightarrow \widehat{\mathbb{M}}$. We prove that $c$ is an isomorphism—this implies the desired statement $\widehat{M} = \widehat{M'} + \mathrm{Id}$.

Since $\mathbb{M}$ is an ideal monad, the weakly ideal monad morphism $\kappa : \mathbb{M} \longrightarrow \widehat{\mathbb{M}}$ factorizes as $\kappa = c \cdot \kappa^*$ for a weakly ideal monad morphism $\kappa^* : \mathbb{M} \longrightarrow \widehat{\mathbb{M}}^*$ using the universal property of the coreflection $\widehat{M}^*$. By the universal property of Lemma 5.7 we obtain a weakly ideal monad morphism $d : \widehat{\mathbb{M}} \longrightarrow \widehat{\mathbb{M}}^*$ such that the diagram




commutes. We immediately conclude that $c \cdot d = \mathrm{id}$. Now, $d \cdot c$ is an ideal monad endomorphism on the ideal coreflection $\widehat{M'} + \mathrm{Id}$ of $\widehat{M}$. Thus, the equality $c \cdot d \cdot c = c$ proves that $d \cdot c = \mathrm{id}$. □

**Corollary 5.10.** *The full embedding of the category* $\mathsf{IFM}(\mathcal{A})$ *of iterative monads to the category* $\mathsf{FM}_{\mathrm{id}}(\mathcal{A})$ *of ideal monads forms an adjoint situation*

$$\mathsf{IFM}(\mathcal{A}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{FM}_{\mathrm{id}}(\mathcal{A}) .$$

## 6. Conclusions

The purpose of the present paper was the step from establishing that every ideal monad $\mathbb{M}$ has an iterative reflection to a description of this reflection. We have achieved this goal for set-like base categories. For Set it has been already proved by Evelyn Nelson [8] that every set $X$ generates a free iterative algebra $\widehat{M}X$ for the monad $\mathbb{M}$. (Nelson used the language of universal algebra.) Consequently, we obtain a monad $\widehat{\mathbb{M}}$ of free iterative algebras for $\mathbb{M}$. Unfortunately, it does not seem obvious that the monad $\widehat{\mathbb{M}}$ is iterative. We presented a proof that this is the case, and moreover, $\widehat{\mathbb{M}}$ is the iterative reflection of $\mathbb{M}$. We thus derive a number of examples of iterative monads:

(1) For the finite non-empty list monad $MX = X^+$ we obtain the iterative reflection $\widehat{M}X = X^+ \cup \{t\}$ where $t$ is an absorbing element.

(2) Analogously, for the finite bag monad $\mathbb{M}$ we have $\widehat{M}X = MX + \{t\}$ where $t$ is an absorbing element.

(3) For the finite tree monad $\mathbb{M}$, the reflection is the monad $\widehat{\mathbb{M}}$ of rational trees.

(4) An analogous example works for non-ordered finite trees: here $\widehat{\mathbb{M}}$ is the monad of rational unordered trees. This follows from results in [4].

(5) The iterative reflection of the unary algebra monad $MX = X \times \Sigma^*$ is the monad $\widehat{M}X = X \times \Sigma^* + \Sigma^*(\Sigma^*)^\omega$.

The existence of iterative reflections for all ideal monads was established in [5] for a rather wide range of base categories: all locally finitely presentable categories in which every object is a coproduct of connected objects. We do not know whether in this generality it is true that the iterative reflection of every ideal monad is the monad of its free iterative Eilenberg-Moore algebras.

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