Algebras with Parametrized Iterativity

Jiří Adámek [a] Stefan Milius [a,*] Jiří Velebil [b,1]

[a] *Institute of Theoretical Computer Science, Technical University, Braunschweig, Germany*

[b] *Faculty of Electrical Engineering, Czech Technical University, Prague, Czech Republic*

Abstract

Iterative algebras, as studied by E. Nelson and J. Tiuryn, are generalized to algebras whose iterativity is parametrized in the sense that only some variables can be used for iteration. For example, in case of one binary operation, the free iterative algebra is the algebra of all rational binary trees; if only the left-hand variable is allowed to be iterated, then the free iterative algebra is the algebra of all right-wellfounded rational binary trees. In order to express such parametrized iterativity, we work with parametrized endofunctors of Set, i.e., finitary endofunctors $H\colon \mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}$, and introduce the concept of iterativity for algebras for the endofunctor $X \mapsto H(X, X)$. We then describe free iterative H-algebras.

Key words: Parametrized endofunctor, algebra for an endofunctor, rational trees.

One for the money
Two for the show
Three to get ready
And four to go.

Nursery rhyme

* Corresponding author.
Email addresses: `adamek@iti.cs.tu-bs.de` (Jiří Adámek), `s.milius@tu-bs.de` (Stefan Milius), `velebil@math.feld.cvut.cz` (Jiří Velebil).

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1 Introduction

The concept of iterative theory introduced by Calvin Elgot [E] led Evelyn Nelson [N] and Jerzy Tiuryn [T] to a simpler notion: iterative algebra. This resulted in a fundamental simplification of the description of free iterative theories: these are theories of free iterative algebras. In our recent work [AMV_1], [AMV_2] we showed, using a coalgebraic approach, that iterative algebras can be naturally introduced over an arbitrary finitary endofunctor H of Set, and again, the theory (or monad) of free iterative H-algebras is a free iterative monad on H.

In the present paper we follow the footsteps of Tarmo Uustalu [U] and generalize iterative algebras to the case where iteration is performed in some variables only; the choice of these variables is a (freely chosen) parameter. We speak about algebras with parametrized iterativity. We present examples demonstrating the generality that parametrized iterativity provides, and prove that free algebras with parametrized iterativity exist and describe them as certain tree algebras. In subsequent work we intend to prove that free algebras with parametrized iterativity yield a finitary [2] parametrized monad in the sense of Tarmo Uustalu [U]. We use the word "base" instead of finitary parametrized monad because our main concepts are "base algebra" and "iterative base algebra", and they need a short adjective. A base on Set is a finitary functor from Set to $\mathsf{FM}(\mathsf{Set})$, the category of all finitary monads on Set. The base derived from free parametrized iterative algebras is characterized by a universal property generalizing Elgot's free iterative theories. Our present paper shows the motivation: we introduce parametrized signatures and bases, and explain why parametrized iterativity presents a valuable enrichment of the original concept of iterative algebras. The technical part concerning (free) bases on a locally finitely presentable category is postponed to a future paper, announced in [AMV_3].

Let us explain the idea of parametrized iterativity on the simple case of algebras with a single binary operation (denoted by $*$):

Case 1: Full iterativity. This is the concept of iterative algebra of Evelyn Nelson [N]: An algebra $(A, *)$ is iterative if and only if every system

$$\begin{aligned} x_1 &\approx t_1 \\ &\vdots \\ x_m &\approx t_m \end{aligned} \tag{1.1}$$

[2] "Finitary" means: preserving filtered colimits.

of finitely many equations in variables $X = \{x_1, \dots, x_m\}$ and with right-hand sides which are terms on $X + A$, none a single variable in X, has a unique solution. The free iterative algebra on a set Y is the algebra of all rational binary trees on Y. (*Rational* means that the tree has only finitely many subtrees, up to isomorphism. And *on* Y means that leaves are labelled in Y.)

Case 2: Restricted iterativity. Here we require that the variables are only allowed to occur on the left-hand position of $*$. Thus, an iterative algebra is one in which every system (1.1) with right-hand sides of the form $t = y_1 * (y_2 * \dots * (y_n * a) \dots)$ for $y_1, \dots, y_n \in X$ has a unique solution. The free iterative algebra on Y, for iterativity w.r.t these systems of equations, is the algebra of all right-wellfounded rational binary trees on Y; *right-wellfounded* are those trees which have the right-most path from every node finite. (Choosing the left-hand position of $*$ for iteration is, by symmetry, equivalent to choosing the right-hand one.)

Case 3: No iterativity. Here no variable is allowed to occur on right-hand sides of systems (1.1), i.e., we are left with the "trivial" systems in which all right-hand sides are terms on A. Every algebra is then iterative.

It turns out that these various forms of parametrized iterativity are captured by moving from finitary functors $H\colon \mathsf{Set} \longrightarrow \mathsf{Set}$, used for "classical" algebra, to *parametrized endofunctors*, i.e., finitary functors

$$H\colon \mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}.$$

Then the classical H-algebras (of an endofunctor H) are replaced by morphisms of the form

$$\alpha\colon H(A, A) \longrightarrow A.$$

In the case of one binary operation the "classical" polynomial endofunctor $H\colon \mathsf{Set} \longrightarrow \mathsf{Set}$, $HX = X \times X$, is now substituted by three parametrized endofunctors: $H(X, A) = X \times X$ for Case 1, $H(X, A) = X \times A$ for Case 2, and $H(X, A) = A \times A$ for Case 3. All these three parametrized functors yield of course the same algebras but not the same iterative algebras! Let us denote by

$$X \mathbin{\square} A \qquad \text{(read "}X\text{ box }A\text{")}$$

a free $H(X, -)$-algebra on A (for all pairs of sets X, A). More precisely, for every set X we denote by $X \mathbin{\square} -$ the free monad on the endofunctor $H(X, -)$ (which, as proved by Michael Barr [B], is just the monad of the free algebras for $H(X, -)$); it has a simple description, see 2.17 below). This yields a *base*, viz. a finitary functor from Set to $\mathsf{FM}(\mathsf{Set})$ given by $X \mapsto X \mathbin{\square} -$.

In the present paper we prove that free iterative algebras exist, and describe them for all parametrized polynomial functors. In a future paper, announced in [AMV$_3$], we prove that the base of free iterative algebras is "free" in a sense which generalizes free iterative theories of Elgot.

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2 Parametrized Endofunctors

Assumption 2.1. *Throughout this section H denotes a parametrized endofunctor on* Set*, i.e., a finitary functor*

$$H\colon \mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}$$

is given. (This is equivalent to being finitary in both variables.)

Example 2.2.

(i) The projections $H(X,A) = X$ and $H(X,A) = A$ are parametrized endofunctors.
(ii) A finite product of parametrized endofunctors is a parametrized endofunctor. Examples: $H(X,A) = X \times A$ or $H(X,A) = X \times X \times X$.
(iii) A coproduct of parametrized endofunctors is a parametrized endofunctor. Example:
$$H(X,A) = X \times A + X \times X \times X.$$

Notation 2.3. For a parametrized endofunctor H we denote by

$$\mathsf{Alg}\, H$$

the category of H*-algebras*, i.e., pairs consisting of an underlying set A and a morphism $\alpha\colon H(A,A) \longrightarrow A$. Morphisms of H-algebras, called *homomorphisms*, are morphisms f of Set such that the square

$$\begin{array}{ccc} H(A,A) & \xrightarrow{\alpha} & A \\ {\scriptstyle H(f,f)}\downarrow & & \downarrow{\scriptstyle f} \\ H(A',A') & \xrightarrow[\alpha']{} & A' \end{array} \tag{2.1}$$

commutes.

Remark 2.4. Thus, for the diagonal functor $\triangle\colon \mathsf{Set} \longrightarrow \mathsf{Set}\times\mathsf{Set}$, H-algebras are the "classical" algebras for the endofunctor $H{\cdot}\triangle$. For example, for both of the projection functors of 2.2 (i) we have the same category $\mathsf{Alg}\, H$, viz. the usual algebras on one unary operation.

Example 2.5. Recall that for a finitary signature Σ, i.e., a collection $\Sigma(n)$ of sets ($n \in \mathbb{N}$), the category of Σ-algebras can be expressed as the category of H_Σ-algebras for the *polynomial endofunctor* $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$, $H_\Sigma(X) = \coprod_{n\in\mathbb{N}} \Sigma(n) \times X^n$, where $\Sigma(n)$ is the set of all symbols of arity n. This generalizes to parametric endofunctors as follows:

Definition 2.6. By a **parametrized signature** is meant a finitary signature Σ together with a function it (**iterativity**) assigning to every operation symbol $\sigma \in \Sigma(n)$ a number $\mathsf{it}(\sigma) = 0, 1, 2, \dots, n$.

Notation 2.7. We denote by $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ the *parametrized polynomial functor*

$$H_\Sigma(X, A) = \coprod_{i,p\in\mathbb{N}} \Sigma(i,p) \times X^i \times A^p$$

where $\Sigma(i,p)$ is the set of all operation symbols of iterativity i and arity $n = i + p$.

Example 2.8. Binary Algebras.

One binary operation $*$ corresponds to three parametrized signatures: the iterativity of $*$ can be $2, 1$ or 0. The corresponding parametrized endofunctors are

$$H(X, A) = X \times X \qquad \big(\text{iterativity } 2\big)$$
$$H(X, A) = X \times A \qquad \big(\text{iterativity } 1\big)$$

and

$$H(X, A) = A \times A \qquad \big(\text{iterativity } 0\big).$$

All these functors yield the same category $\mathsf{Alg}\, H$.

Notation 2.9. For every parametrized endofunctor H the (finitary!) endofunctor $H(X, -) \colon \mathsf{Set} \longrightarrow \mathsf{Set}$ has free algebras, see [A]. That means that the forgetful functor of the category of algebras for $H(X, -)$ into Set has a left adjoint. We denote by

$$X \mathbin{\square} A$$

the free algebra for $H(X, -)$ on A.

Explicitly, given objects X and A, we have an object $X \mathbin{\square} A$ together with a morphism

$$f_A^X \colon H(X, X \mathbin{\square} A) \longrightarrow X \mathbin{\square} A$$

forming a free algebra for $H(X,-)$ w.r.t. a universal arrow

$$u_A^X \colon A \longrightarrow X \,\square\, A.$$

The universal property states that for every algebra $\beta\colon H(X,B) \longrightarrow B$ and every morphism $h\colon A \longrightarrow B$ there exists a unique extension to a homomorphism; more precisely, there exists a unique morphism $h'\colon X \,\square\, A \longrightarrow B$ such that the diagram below commutes:

$$\begin{array}{ccccc} H(X, X \,\square\, A) & \xrightarrow{f_A^X} & X \,\square\, A & \xleftarrow{u_A^X} & A \\ {\scriptstyle H(X,h')}\downarrow & & \downarrow{\scriptstyle h'} & \swarrow{\scriptstyle h} & \\ H(X,B) & \xrightarrow[\beta]{} & B & & \end{array} \tag{2.2}$$

Remark 2.10. We obtain a new parametrized endofunctor $\square$ defined on objects by $X \,\square\, A$ above. For morphisms

$$h : X \longrightarrow Y \quad \text{and} \quad p : A \longrightarrow B$$

the definition of

$$h \,\square\, p : X \,\square\, A \longrightarrow Y \,\square\, B$$

is very "natural": $Y \,\square\, B$ can be considered as an algebra for $H(X,-)$ via

$$H(X, Y \,\square\, B) \xrightarrow{H(h, Y \square B)} H(Y, Y \,\square\, B) \xrightarrow{f_B^Y} Y \,\square\, B$$

and then $h \,\square\, p$ is the unique homomorphism w.r.t. $H(X,-)$ extending $u_X^Y \cdot p : A \longrightarrow Y \,\square\, B$:

$$\begin{array}{ccccccc} H(X, X \,\square\, A) & & \xrightarrow{\qquad f_A^X \qquad} & & X \,\square\, A & \xleftarrow{u_A^X} & A \\ {\scriptstyle H(X,h\square p)}\downarrow & & & & \downarrow{\scriptstyle h\square p} & & \downarrow{\scriptstyle p} \\ H(X, Y \,\square\, B) & \xrightarrow[H(h,Y\square B)]{} & H(Y, Y \,\square\, B) & \xrightarrow[f_B^Y]{} & Y \,\square\, B & \xleftarrow[u_B^Y]{} & B \end{array}$$

The fact that $\square$ is indeed a well-defined functor is easy to derive from the universal property of free algebras. In particular, given morphisms $k : Y \longrightarrow Z$ and $q : B :\longrightarrow C$ we have the equation

$$(k \cdot h) \,\square\, (q \cdot p) = (k \,\square\, q) \cdot (h \,\square\, p) : X \,\square\, A \longrightarrow Z \,\square\, C.$$

Example 2.11. Let Σ be a parametrized signature. By fixing X in the parametrized polynomial endofunctor $H_\Sigma(X,A) = \coprod_{i,p} \Sigma(i,p) \times X^i \times A^p$ we obtain a non-parametrized polynomial endofunctor

$$A \mapsto \coprod_{i,p} \Sigma(i,p) \times X^i \times A^p$$

The (non-parametrized) signature to which this functor corresponds is called a *derived signature* of Σ w.r.t. the set X, and it is denoted by $\Sigma(X)$. In other words, we have

$$H_{\Sigma(X)}A = H_\Sigma(X, A) \quad \text{for all sets } A$$

and analogously

$$H_{\Sigma(X)}f = H_\Sigma(\mathrm{id}_X, f) \quad \text{for all morphisms } f.$$

Since $H_{\Sigma(X)}A = \coprod_{p\in\mathbb{N}}\Big(\coprod_{i\in\mathbb{N}} \Sigma(i,p) \times X^i\Big) \times A^p$, the p-ary symbols of the derived signature $\Sigma(X)$ are simply elements of $\coprod_{i\in\mathbb{N}} \Sigma(i,p) \times X^i$. This means that

> a p-ary operation symbol of $\Sigma(X)$ is an (i,p)-ary symbol of Σ together with an i-tuple in X.

A "classical" free algebra for $\Sigma(X)$ on a set A is denoted by

$$X \,\square_\Sigma\, A$$

Let us illustrate this on the simple case of one binary operation:

Example 2.12. Binary Algebras (continued).

(i) In case of iterativity 2 where

$$H(X, A) = X \times X$$

the functor $H(X, -)$ is constant with value $X \times X$. Thus $\Sigma(X)$ has nullary operations indexed by $X \times X$. We thus get

$$X \,\square\, A = X \times X + A$$

Here

$$f_A^X \equiv X \times X \xrightarrow{\mathrm{inl}} X \times X + A$$

and

$$u_A^X \equiv A \xrightarrow{\mathrm{inr}} X \times X + A.$$

(ii) In case of iterativity 1 where

$$H(X, A) = X \times A$$

the functor $H(X, -)$ is the polynomial functor of the (non-parametrized) signature of unary operations indexed by X. The free algebra is

$$X \,\square\, A = X^* \times A$$

where X^* is the set of all finite lists on X (with the concatenation map $c\colon X \times X^* \longrightarrow X^*$ and the empty list $\eta : 1 \longrightarrow X$). Here

$$f_A^X \equiv X \times X^* \times A \xrightarrow{c \times A} X^* \times A$$

and

$$u_A^X \equiv A = 1 \times A \xrightarrow{\eta \times A} X^* \times A$$

(iii) In case of iterativity 0 where

$$H(X, A) = A \times A$$

the functor $H(X, -)$ corresponds to one (non-parametrized) binary operation. Thus

$$X \,\square\, A = \text{free binary algebra on } A$$

that can be described as the algebra of all finite binary trees with leaves labelled in A. Here

$$f_A^X : (X \,\square\, A) \times (X \,\square\, A) \longrightarrow X \,\square\, A$$

is the usual tree tupling, and

$$u_A^X : A \longrightarrow X \,\square\, A$$

is the map of single-node trees.

Remark 2.13.

(i) The monad of free K-algebras is clearly finitary for every finitary (non-parametrized) endofunctor $K\colon \mathsf{Set} \longrightarrow \mathsf{Set}$. As proved by Michael Barr [B], this monad is a free monad on K. In other words, the forgetful functor from $\mathsf{FM}(\mathsf{Set})$, the category of all finitary monads on Set, to the category of all finitary endofunctors of Set, also has a left adjoint. It assigns to a finitary endofunctor K the monad induced by free K-algebras.

(ii) Applied to $H(X, -)$ the above tells us that the notation $X \,\square\, A$ is well-defined, and for fixed X we get a monad $X \,\square\, -$ in $\mathsf{FM}(\mathsf{Set})$. Moreover, by varying X we obtain a finitary functor from Set to $\mathsf{FM}(\mathsf{Set})$, given on objects X by $X \,\square\, -$. In fact, the definition of this functor on morphisms $p\colon X \longrightarrow X'$ uses the universal property of $X \,\square\, -$ being the free monad on $H(X, -)$. Therefore, the natural transformation

$$H(p, -)\colon H(X, -) \longrightarrow H(X', -)$$

yields a unique monad morphism to be denoted by

$$p \,\square\, -\colon (X \,\square\, -) \longrightarrow (X' \,\square\, -).$$

This is a special case of the following general

Definition 2.14. By a **base** on Set is understood a finitary functor from Set to $\mathsf{FM}(\mathsf{Set})$.

Example 2.15. Every parametrized endofunctor H defines a base $\Box_H$ (or $\Box$, if there is no danger of confusion) as above: $X \Box_H A$ is the free $H(X,-)$-algebra on A. We call this base the *free base* on H. For example, algebras on one binary operation yield the following bases on Set:

$$\begin{array}{ll} X \Box A = X \times X + A & (\text{for } H(X,A) = X \times X), \\ X \Box A = X^* \times A & (\text{for } H(X,A) = X \times A), \\ X \Box A = \text{free binary algebra on } A & (\text{for } H(X,A) = A \times A). \end{array}$$

See Example 2.12.

Example 2.16. Every finitary monad S on Set defines a *trivial base*

$$X \Box A = SA.$$

More precisely: the curried version $\mathsf{Set} \longrightarrow \mathsf{FM}(\mathsf{Set})$ is the constant functor with value S. We have seen the special case where S is the free-binary-algebra monad in 2.15. In general, a trivial base is not free on any parametrized endofunctor.

Remark 2.17. Recall from [A] that given a finitary (non-parametrized) endofunctor K, then the free K-algebra on an object A is a colimit of the following ω-chain, in which $\mathrm{inr}\colon A \longrightarrow KA + A$ denotes the right-hand coproduct injection

$$A \xrightarrow{\mathrm{inr}} KA + A \xrightarrow{K\,\mathrm{inr} + A} K(KA + A) + A \xrightarrow{K(K\,\mathrm{inr} + A) + A} \cdots$$

Consequently, the free base $X \Box A$ can be described as a colimit of the ω-chain

$$A \xrightarrow{\mathrm{inr}} H(X,A) + A \xrightarrow{H(X,\mathrm{inr}) + A} H\big(X, H(X,A) + A\big) + A \xrightarrow{H(X,H(X,\mathrm{inr})+A)+A} \cdots$$

More precisely:

Proposition 2.18. *We can describe $X \Box_H A$ as a colimit of the ω-chain W defined on objects by*

$$W_0 = A \quad \textit{and} \quad W_{n+1} = H(X, W_n) + A.$$

If $c_n : W_n \longrightarrow X \Box_H A$ denotes the colimit cocone then $u_A^X = c_0 : A \longrightarrow X \Box_H A$ and the algebraic structure $f_A^X\colon H(X, X \Box A) \longrightarrow X \Box A$ is given by the *commutativity of the following squares*

$$\begin{array}{ccc} H(X, W_n) & \xrightarrow{\ \mathrm{inl}\ } & W_{n+1} = H(X, W_n) + A \\ {\scriptstyle H(X, c_n)}\downarrow & & \downarrow{\scriptstyle c_{n+1}} \\ H(X, X \,\square\, A) & \xrightarrow[\ f_A^X\ ]{} & X \,\square\, A \end{array}$$

for all $n \in \mathbb{N}$.

Proof. In fact, since $H(X, -)$ is finitary, we know that

$$H(X, X \,\square\, A) = \operatorname*{colim}_{n \in \mathbb{N}} H(X, W_n)$$

for the unique ω-chain with objects defined as above and with connecting morphisms

$$\begin{aligned} w_0 &= \mathrm{inr} \colon A \longrightarrow H(X, A) + A \\ w_{n+1} &= H(X, w_n) + A \colon H(X, W_n) + A \longrightarrow H(X, W_{n+1}) + A. \end{aligned}$$

Since $c_{n+1} \cdot \mathrm{inl}$, $n \in \mathbb{N}$, form a cocone, the above squares thus determine f_A^X uniquely. ☐

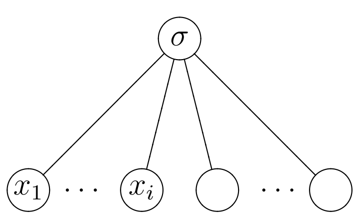
Example 2.19. In particular, for the initial object 0 we get

$$X \,\square\, 0$$

as a colimit of the following chain, where ! denotes the unique morphism

$$0 \xrightarrow{\ !\ } H(X, 0) \xrightarrow{H(X,!)} H\big(X, H(X, 0)\big) \xrightarrow{H(X, H(X,!))} \cdots$$

Example 2.20. Free base $\square_\Sigma$ on the polynomial endofunctor $H_\Sigma(X, A)$, see Example 2.11. The derived signature has operation symbols indexed by elements of $\Sigma(i, p) \times X^i$. We can depict them in the following form



for $\sigma \in \Sigma(i, p)$ and $(x_1, \ldots, x_i) \in X^i$ (2.3)

Consequently, the free algebra

$$X \,\square_\Sigma\, A$$

of the derived signature on A is the algebra of all finite trees labelled in $\Sigma + X + A$ as follows:

(a) every node with $n > 0$ successors is labelled by an operation symbol in $\Sigma(i,p)$ for some i,p with $i+p=n$, so that the first i children are leaves labelled in X

and

(b) every leaf is either labelled in X according to (a), or it is labelled in $\Sigma(0,0)+A$.

We call such trees *parametrized Σ-trees on the pair* (X,A).

Example 2.21. One ternary operation. Besides the (two) cases completely analogous to Example 2.12, namely:

(i) Iterativity 3 where

$$H(X,A) = X \times X \times X \qquad \text{with } X \,\Box\, A = X \times X \times X + A,$$

and

(ii) Iterativity 0 where

$$H(X,A) = A \times A \times A \qquad \text{with } X \,\Box\, A = \text{free ternary algebra on } A,$$

we have two new nice free bases:

(iii) $H(X,A) = X \times X \times A$, i.e., iterativity 2. Here $H(X,-)$ corresponds to a unary signature with operations indexed by $X \times X$, thus

$$X \,\Box\, A = (X \times X)^* \times A.$$

(iv) $H(X,A) = X \times A \times A$, i.e., iterativity 1. Here $H(X,-)$ corresponds to a binary signature with operations indexed by X, thus,

$$X \,\Box\, A = \text{all finite binary trees with inner nodes labelled in } X \text{ and leaves labelled in } A.$$

Example 2.22. Commutativity in a ternary operation. Denote by Q the endofunctor assigning to every set X the set QX of all unordered pairs in X (a quotient functor of $X \mapsto X \times X$). Then the parametrized endofunctor

$$H(X,A) = QX \times A$$

has as H-algebras ternary algebras with the first two variables commutative. More precisely, $\mathsf{Alg}\, H$ is given by a ternary operation σ and the commutative law

$$\sigma(x,y,z) = \sigma(y,x,z).$$

The corresponding iterative algebras allow iterativity in the first two (commutative) variables. More precisely, since $H(X, -)$ is the polynomial endofunctor of unary operations indexed by QX, we see that

$$X \,\square\, A = (QX)^* \times A$$

(a quotient functor of $(X \times X)^* \times A$ of Example 2.21 (iii)).

3 Iterative Algebras

We introduce, for an arbitrary parametrized endofunctor H, the concept of iterativity for H-algebras, which is very simple, and corresponds in case of non-parametrized polynomial functors $H = H_\Sigma$ precisely to the concept of parametrized iterativity in the Introduction. We demonstrate first the idea by considering one binary operation as in Example 2.12.

Example 3.1.

(i) Full iterativity: $H(X, A) = X \times X + A$. Here iterative algebras are precisely algebras $\alpha : A \times A \longrightarrow A$ of Case 1 in the introduction. As already observed by Evelyn Nelson [N], in lieu of solving general systems (1.1) it is sufficient to solve the *flat* ones defined as those whose right-hand sides are

$$t_i = y * z \text{ for } y, z \in X \quad \text{or} \quad t_i = a \text{ for } a \in A.$$

In fact, every system (1.1) has an obvious "flattening" with the same solution. For example, the system

$$\begin{aligned} x_1 &\approx x_2 * a \\ x_2 &\approx x_1 * b \end{aligned}$$

can be flattened by using new variables y_1, y_2:

$$\begin{aligned} x_1 &\approx x_2 * y_1 \\ y_1 &\approx a \\ x_2 &\approx x_1 * y_2 \\ y_2 &\approx b \end{aligned}$$

Observe that a flat equation morphism can be presented as a morphism

$$e : X \longrightarrow X \times X + A$$

where $X = \{x_1, \dots, x_n\}$ are the left-hand variables, and the right-hand sides $e(x_i) = t_i$ lie either in $X \times X$ or in A. A *solution* of e can be presented as a morphism $e^\dagger : X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha,\mathrm{id}]} \\ X \times X + A & \xrightarrow[e^\dagger \times e^\dagger + \mathrm{id}]{} & A \times A + A \end{array}$$

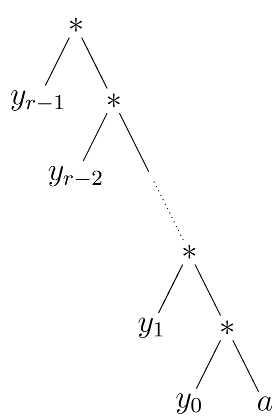
commutes.

(ii) Restricted iterativity 1: $H(X, A) = X \times A$. This means that in expressions like $y * z$ just the left-hand variable y is used for iteration. We could also choose just z, of course. (Due to symmetry, we consider the first case only. Also later, for operations of larger arities: given the parameter i, we always assume that the *first* i variables from the left are those used for iteration. But any other choice of the i variables among the n possible ones would also work, of course.) What does this choice of the left-hand-iteration-only mean in terms of the equations we are solving? Consider the system (1.1) in the Introduction: the right-hand sides t_j are finite binary trees with leaves labelled in $X + A$—and we now additionally request that

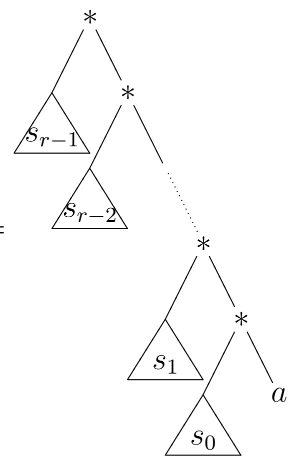
every leaf labelled in X is the left-hand child of its parent. (3.1)

Thus, iterativity with parameter $i = 1$ means that every system (1.1) of recursive equations whose right-hand sides satisfy (3.1) has a unique solution. Algebras with this property are called *parametrized iterative.*

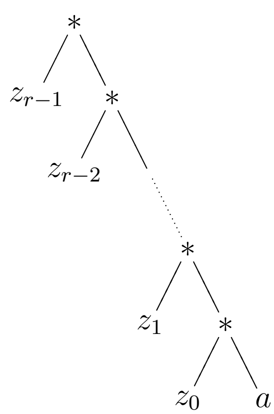
The flat variation here has the following form: the right-hand sides are trees

$t_j =$  (3.2)

for $y_0, \ldots, y_{r-1} \in X$ and $a \in A$. In fact, whenever a binary algebra has unique solutions of all systems (1.1) with right-hand sides of the form (3.2), then it is iterative in the present sense. The flattening of a right-hand side t_j can be performed recursively (by adding new variables) as follows: it is clear that (3.1) implies that t_j has the form

$t_j =$  (3.3)

for a in A and trees $s_0, \ldots, s_{r-1}$ satisfying (3.1). Introduce new variables $z_0, \ldots, z_{r-1}$, replace t_j on the right-hand side of equations by



add the equations

$$z_j \approx s_j \quad j = 0, \ldots, r-1$$

and continue with flattening of these latter equations. Consequently, an algebra is parametrized iterative if and only if every system (1.1) of recursive equations with right-hand sides of the form (3.2) has a unique solution. This is strictly weaker than full iterativity, e.g., the empty algebra is iterative in the parametrized sense (but not iterative in the full sense where $x \approx x * x$ must have a unique solution). The free iterative algebra on a set Y of generators is the algebra of all binary rational trees that are *right-wellfounded*, i.e., the right-most path from every node is finite, see 4.6.

Observe that the above trees (3.2) are elements of

$$X^* \times A$$

so that the equation systems (1.1) we consider here can be represented by morphisms of the form

$$e : X \longrightarrow X^* \times A, \quad X \text{ finite.}$$

A *solution* of e is represented by a morphism

$$e^\dagger : X \longrightarrow A$$

such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widehat{\alpha}} \\ X^* \times A & \xrightarrow[(e^\dagger)^* \times A]{} & A^* \times A \end{array} \tag{3.4}$$

commutes, where $\widehat{\alpha}$ extends the binary operation $*$ of A:

$$\widehat{\alpha}(y_{r-1}y_{r-2}\cdots y_0, a) = y_{r-1} * (y_{r-2} * (\cdots * (y_0 * a))\dots).$$

(iii) Iterativity 0. If in $y * z$ neither y nor z can be used for iteration, then the right-hand sides of (1.1) are terms from $F(A)$, a free binary algebra on A. Then every algebra A is trivially iterative: every equation morphism $e : X \longrightarrow F(A)$ has the unique solution $e^\dagger : X \longrightarrow A$ composed of e and the canonical homomorphism from $F(A)$ to A.

Remark 3.2. In all three cases above the form of an equation morphism was the same: $e : X \longrightarrow X \,\square\, A$, where X is a finite set. To be able to express the concept of a solution $e^\dagger : X \longrightarrow A$ we need the following

Notation 3.3. For every H-algebra

$$\alpha \colon H(A, A) \longrightarrow A$$

denote by

$$\widehat{\alpha} \colon A \,\square\, A \longrightarrow A$$

the unique homomorphism of $H(A, -)$-algebras extending id_A. More precisely, $\widehat{\alpha}$ is the unique morphism such that the diagram

$$\begin{array}{ccccc} H(A, A \,\square\, A) & \xrightarrow{f_A^A} & A \,\square\, A & \xleftarrow{u_A^A} & A \\ {\scriptstyle H(A,\widehat{\alpha})}\downarrow & & \downarrow{\scriptstyle \widehat{\alpha}} & \swarrow\!\!= & \\ H(A, A) & \xrightarrow[\alpha]{} & A & & \end{array} \tag{3.5}$$

commutes. Observe that α determines $\widehat{\alpha}$ uniquely, and also conversely:

$$\alpha \equiv H(A, A) \xrightarrow{H(A, u_A^A)} H(A, A \,\square\, A) \xrightarrow{f_A^A} A \,\square\, A \xrightarrow{\widehat{\alpha}} A. \tag{3.6}$$

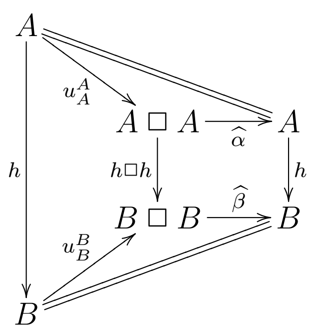
Lemma 3.4. *Given algebras* (A,α) *and* (B,β)*, a morphism* $h\colon A \longrightarrow B$ *is a homomorphism if and only if the square below commutes:*

$$\begin{array}{ccc} A \,\square\, A & \xrightarrow{\widehat{\alpha}} & A \\ {\scriptstyle h\square h}\downarrow & & \downarrow{\scriptstyle h} \\ B \,\square\, B & \xrightarrow[\widehat{\beta}]{} & B \end{array} \tag{3.7}$$

Proof. If h is a homomorphism, see (2.1), then it is a homomorphism of $H(A,-)$-algebras, where the algebra structure of B is

$$H(A,B) \xrightarrow{H(h,B)} H(B,B) \xrightarrow{\beta} B\,.$$

Then both $h{\cdot}\widehat{\alpha}$ and $\widehat{\beta}{\cdot}(h \,\square\, h)$ are easily seen to be homomorphisms w.r.t. $H(A,-)$. Therefore the above square commutes because $A \,\square\, A$ is a free algebra for $H(A,-)$ and the diagram



commutes.

If square (3.7) commutes, compose it with the above morphism

$$f_A^A{\cdot}H(A,u_A^A)\colon H(A,A) \longrightarrow A \,\square\, A$$

from (3.6) to conclude that h is a homomorphism. □

Definition 3.5.

(i) By a (**finitary**) **equation morphism** is meant a morphism of the form $e\colon X \longrightarrow X \,\square\, A$ where X is a finite set.

(ii) An H-algebra A is said to be **iterative** if for every equation morphism $e\colon X \longrightarrow X \,\square\, A$ there exists a unique **solution**, i.e., a unique morphism $e^\dagger\colon X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widehat{\alpha}} \\ X \,\square\, A & \xrightarrow[e^\dagger \square A]{} & A \,\square\, A \end{array} \tag{3.8}$$

commutes.

Example 3.6. One binary operation.

(i) For the case $H(X, A) = X \times X + A$, an equation morphism $e\colon X \longrightarrow X \times X + A$ is precisely a flat equation morphism as in Case 1 of the Introduction. Thus, an algebra is iterative if and only if it is iterative in the sense of Evelyn Nelson.

(ii) For the case $H(X, A) = X \times A$, an equation morphism

$$e\colon X \longrightarrow X^* \times A$$

has the right-hand sides of the form

$$(\varepsilon, a), \quad (x, a), \quad (xy, a), \quad (xyz, a), \quad \dots$$

Iterative algebras are precisely the binary algebras with restricted iterativity, see Example 3.1(ii).

Example 3.7. Non-parametrized endofunctors. For every finitary endofunctor $H\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ we obtain a parametrized endofunctor (by considering the full iterativity) with

$$H(X, A) = HX.$$

The corresponding base is, obviously,

$$X \square A = HX + A,$$

the free algebra on A of the constant endofunctor $H(X, -) = HX$. Observe that given an algebra $\alpha\colon HA \longrightarrow A$ then $\widehat{\alpha} = [\alpha, A] = HA + A \longrightarrow A$.

An H-algebra $\alpha\colon HA \longrightarrow A$ is iterative in the above sense if and only if for every finitary equation morphism $e\colon X \longrightarrow HX + A$ there exists a unique morphism $e^\dagger$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ HX + A & \xrightarrow[\;He^\dagger + A\;]{} & HA + A \end{array}$$

commutes. This is precisely the concept of "fully" iterative algebras studied in [AMV$_2$].

Proposition 3.8. *A limit or a filtered colimit of iterative H-algebras in* $\mathsf{Alg}\,H$ *is always iterative.*

The proof is left to the reader because it is a straightforward analogy of the proof of Proposition 2.20 in [AMV$_2$]. For example, a product of iterative algebras $\alpha_i : H(A_i, A_i) \longrightarrow A_i$ is iterative because given an equation morphism $e : X \longrightarrow X \,\Box\, (\prod_{i\in I} A_i)$ then the solutions $e_i^\dagger$ of the equation morphisms $(\mathrm{id}_X \,\Box\, \pi_i)\cdot e : X \longrightarrow X \,\Box\, A_i$ yield the unique solution $e^\dagger = \langle e_i^\dagger \rangle_{i\in I} : X \longrightarrow \prod_{i\in I} A_i$.

Notation 3.9. We denote for every parametrized endofunctor H by

$$\mathsf{Alg}_{it}\, H$$

the category of all iterative H-algebras, a full subcategory of $\mathsf{Alg}\, H$.

Remark 3.10. The above notation indicates that we consider "ordinary" homomorphisms as the right choice of morphisms between iterative algebras. The reason is that homomorphisms are precisely the morphisms that preserve solutions in the following "natural" sense.

Let $e\colon X \longrightarrow X \,\Box\, A$ be an equation morphism in A. Every morphism $h\colon A \longrightarrow B$ in Set defines an equation morphism

$$h \bullet e \equiv X \xrightarrow{\;e\;} X \,\Box\, A \xrightarrow{\;X \Box h\;} X \,\Box\, B$$

in B. If A and B are iterative H-algebras, we say that h *preserves solutions* provided that for every finitary equation morphism $e\colon X \longrightarrow X \,\Box\, A$, the solution of $h \bullet e$ in B is obtained from that of e in A by the following commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{(h\bullet e)^\dagger} & B \\ & {\scriptstyle e^\dagger}\searrow \quad \nearrow{\scriptstyle h} & \\ & A & \end{array} \tag{3.9}$$

Lemma 3.11. *Let (A, α) and (B, β) be iterative H-algebras. Then a morphism $h\colon A \longrightarrow B$ in Set is a homomorphism if and only if it preserves solutions.*

Proof. (1) Sufficiency. If h preserves solutions, we are to verify that the square (3.7) commutes. Express $A \,\Box\, A$ as a canonical colimit of the filtered diagram of all arrows $p\colon P \longrightarrow A \,\Box\, A$, P finitely presentable. It suffices to show that the equation

$$h\cdot\widehat{\alpha}\cdot p = \widehat{\beta}\cdot(h \,\Box\, h)\cdot p$$

holds for all $p\colon P \longrightarrow A \,\Box\, A$. Now the functor $Z \mapsto Z \,\Box\, Z$ is finitary, therefore $A \,\Box\, A$ is a filtered colimit of the diagram of all $q \,\Box\, q\colon X \,\Box\, X \longrightarrow A \,\Box\, A$ with X finitely presentable. Since P is finitely presentable, there exists a factorization

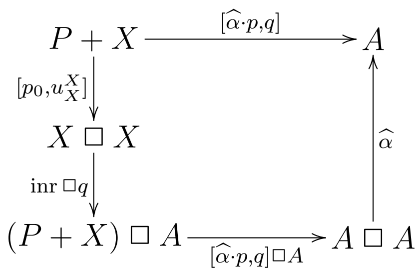
$$\begin{array}{ccc} P & \xrightarrow{\;p\;} & A \,\Box\, A \\ & {\scriptstyle p_0}\searrow & \uparrow{\scriptstyle q \Box q} \\ & & X \,\Box\, X \end{array} \tag{3.10}$$

where $q \colon X \longrightarrow A$ is a morphism with X finitely presentable.

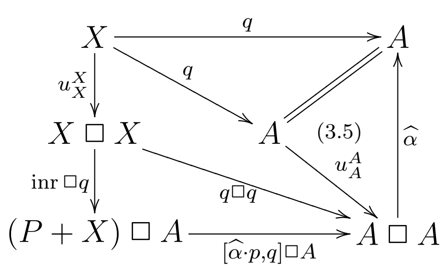
Define an equation morphism

$$e \equiv P + X \xrightarrow{[p_0, u_X^X]} X \,\square\, X \xrightarrow{\mathrm{inr}\,\square q} (P+X) \,\square\, A.$$

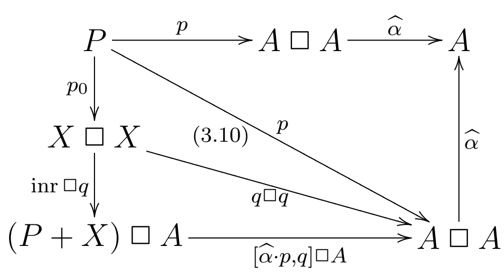
We will show that $e^\dagger = [\widehat{\alpha}{\cdot}p, q]$ holds. To this end it suffices to prove that the square



commutes. In fact, its right-hand component (with domain X) commutes:



and so does the left-hand one (with domain P):



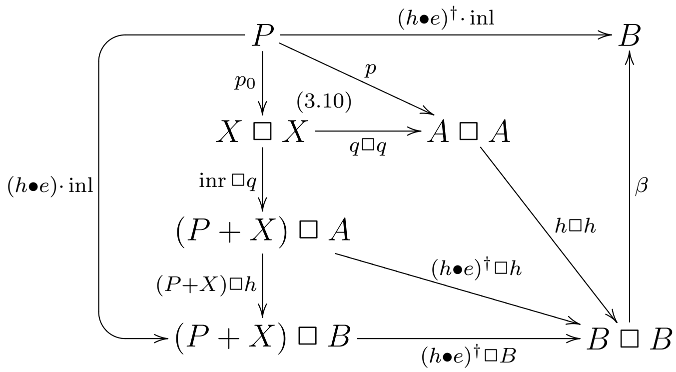
Since h is assumed to preserve solutions we have

$$(h \bullet e)^\dagger = h{\cdot}e^\dagger = h{\cdot}[\widehat{\alpha}{\cdot}p, q] = [h{\cdot}\widehat{\alpha}{\cdot}p, h{\cdot}q].$$

Next we prove the equation

$$(h \bullet e)^\dagger = \Big[\beta{\cdot}(h \,\square\, h){\cdot}p, h{\cdot}q\Big] \colon P + X \longrightarrow B. \tag{3.11}$$

The verification of the right-hand component, with domain X, is analogous to the above verification of the right-hand-component case. For the left-hand component consider the diagram

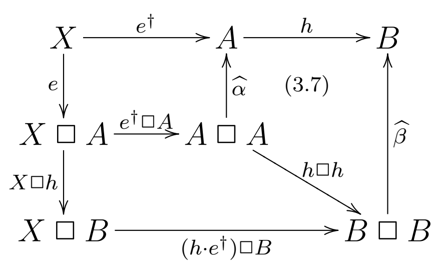


Since its outer shape commutes and all inner parts except for the right-hand one commute, so does the desired right-hand part. We have established the desired equality $(h{\cdot}\widehat{\alpha}){\cdot}p = (\beta{\cdot}(h \mathbin{\square} h)){\cdot}p$ for every $p\colon P \longrightarrow A \mathbin{\square} A$.

(2) Necessity: given a homomorphism

$$h\colon (A, \alpha) \longrightarrow (B, \beta),$$

then the diagram



obviously commutes, which shows that $h{\cdot}e^{\dagger}$ is the solution of $h \bullet e$. $\square$

Lemma 3.12. *$\mathsf{Alg}\, H$ is a locally finitely presentable category and its full subcategory $\mathsf{Alg}_{it}\, H$ of iterative H-algebras is reflective.*

Proof. The category $\mathsf{Alg}\, H$ is locally finitely presentable due to Remark 2.4 and Corollary 2.75 of [AR]. Since the full subcategory $\mathsf{Alg}_{it}\, H$ is closed under limits and filtered colimits by Proposition 3.8, it is a reflective subcategory, see the Reflection Theorem 2.48 in [AR]. $\square$

Corollary 3.13. *Every object of Set generates a free iterative H-algebra. That means that the forgetful functor $\mathsf{Alg}_{it}\, H \longrightarrow \mathsf{Set}$ is a right adjoint.*

In fact, since $\mathsf{Alg}\, H \longrightarrow \mathsf{Set}$ is a right adjoint (see Remark 2.13), this follows from Lemma 3.12.

Notation 3.14. Given a parametrized endofunctor H we denote by

$$RZ$$

a free iterative H-algebra on an object Z with the universal arrow

$$\eta_Z \colon Z \longrightarrow RZ.$$

Example 3.15. Binary algebras in Set.

(i) For $H(X, A) = X \times X$, i.e., full iterativity, a free iterative algebra RA was described by Susanna Ginali [G] as the algebra of all rational binary trees on A, see Introduction, Case 1.

(ii) For $H(X, A) = X \times A$ we will show below that a free iterative algebra can be described as the algebra of all *right-wellfounded* rational binary trees on A, see Introduction, Case 2. This is similar to Example 3.14 of [U].

(iii) For $H(X, A) = A \times A$ all algebras are iterative. Thus, RA is the free algebra (of all finite binary trees) on A.

Example 3.16. Ternary algebras in Set.

(i) For $H(X, A) = X \times X \times X$, see Example 2.21 (i), we see that RA is the algebra of all rational ternary trees on A.

(ii) For $H(X, A) = X \times X \times A$, see Example 2.21 (iii), we obtain, analogously to 3.15 (ii), the algebra RA of all right-wellfounded ternary rational trees.

(iii) For $H(X, A) = QX \times A$, see Example 2.22, the free iterative algebra is obtained from the algebra of all rational right-wellfounded trees by dropping the linear ordering on the first and second child of every node.

4 Free Iterative Algebras

We are going to describe free iterative Σ-algebras in Set for all parametrized signatures Σ. Recall the "classical" case of full iterativity: given a non-parametrized signature Σ, by a Σ*-tree on* Z (where Z is a set of generators) is meant a tree with leaves labelled in $Z + \Sigma(0)$ and inner nodes with n children labelled in $\Sigma(n)$. The tree is called *rational* if it has, up to isomorphism, only finitely many subtrees. The algebra $\overline{R}_\Sigma Z$ of all rational Σ-trees on Z is a free iterative algebra on Z, see [N]. The algebraic structure of $\overline{R}_\Sigma Z$ is given by tree tupling, and the universal arrow $Z \longrightarrow \overline{R}_\Sigma Z$ by forming single-node trees labelled in Z.

Let Σ be a parametrized signature. By a *rational* Σ*-tree* on Z we mean the above concept (obtained by ignoring the iterativities). Given a tree t, we can take an arbitrary path in t, and consider it "non-iterative" provided that

$$\text{every node } d \text{ on the path which is not a leaf has the property that if } d \text{ is labelled in } \Sigma(i,p), \text{ then the next node on the path is one of the } p \text{ right-hand children of } d. \tag{4.1}$$

If all "non-iterative" paths are finite, we call the tree Σ-wellfounded (analogously to the concept of a wellfounded tree which is a tree with all paths finite):

Definition 4.1. Let Σ be a parametrized signature. A Σ-tree on a set Z is called **Σ-wellfounded** provided that no infinite path takes at every node "the turn to the right", or, equivalently, every path satisfying (4.1) is finite. We denote by

$$R_\Sigma Z$$

the set of all rational Σ-wellfounded trees on Z.

Example 4.2. For parametrized signatures with full iterativity every Σ-tree is, of course, Σ-wellfounded. Thus $R_\Sigma Z$ is the algebra of all rational Σ-trees on Z. For the signature Σ with one binary operation of iterativity 1 the concept of right-wellfounded of Example 3.15 is precisely that of Σ-wellfounded. And in case of iterativity 0,

$$\Sigma\text{-wellfounded} = \text{finite}.$$

In fact, by König's Lemma, every finitely branching tree with no infinite path is finite. Thus, here $R_\Sigma Z$ is the "usual" free algebra on Z.

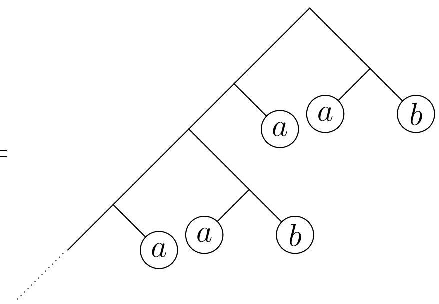
Construction 4.3. Let t be a Σ-wellfounded, rational Σ-tree on Z containing at least one symbol from Σ. Denote by X_t the (finite) set of all subtrees of t. Denote further by $\widehat{t}$ the tree obtained from t by substituting, for every inner node d labelled in $\Sigma(i,p)$, the first i children of d by the names (in X_t) of the corresponding subtrees of d. Since t is Σ-wellfounded, $\widehat{t}$ has no infinite path. Therefore, $\widehat{t}$ is a finite tree, by König's Lemma again. It is obvious from this construction that

$$\widehat{t} \in X_t \,\Box_\Sigma\, Z.$$

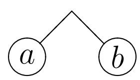
We thus obtain a finitary equation morphism

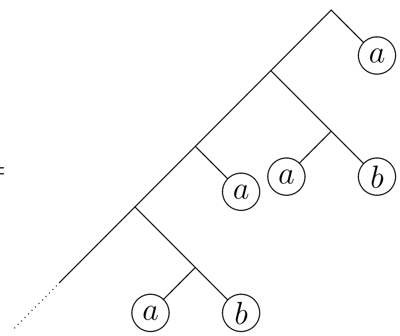
$$e_t \colon X_t \longrightarrow X_t \,\Box_\Sigma\, Z, \qquad e_t(s) = \widehat{s} \text{ for all } s \in X_t.$$

Let us illustrate this on the case of one binary operation of iterativity 1, i.e., $H(X,A) = X \times A$: every left-hand child is substituted by a variable. For example the tree

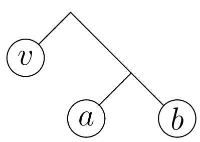
$t =$ 

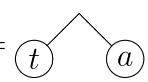
on $Z = \{a, b\}$ has, besides t, a, b, only two subtrees:

$u =$ 

$v =$ 

Thus, $X_t = \{t, u, v, a, b\}$, and we have

$\widehat{t} =$ 

$\widehat{u} = u$ $\widehat{v} =$ 

$\widehat{a} = a$ $\widehat{b} = b.$

Notation 4.4. The free Σ-algebra on Y is denoted by

$$\bar{\eta}_Y : Y \longrightarrow \overline{F}_\Sigma Y.$$

Thus, $\overline{F}_\Sigma Y$ the algebra of all finite Σ-trees on Y, and $\bar{\eta}_Y$ takes $y \in Y$ to the single-node tree labelled by y. Given a Σ-algebra A and a function $f : Y \longrightarrow A$ we denote by

$$f^\sharp : \overline{F}_\Sigma Y \longrightarrow A$$

the corresponding homomorphism with $f^\sharp \cdot \bar{\eta}_Y = f$.

Observation 4.5. Let t be a Σ-wellfounded rational tree on Z containing at least one symbol from Σ. The formation $\widehat{s}$ for subtrees $s \in X_t$, see Construction 4.3, defines an equation morphism

$$e_t : X_t \longrightarrow X_t \,\square_\Sigma\, Z.$$

Since $X_t \,\square_\Sigma\, Z$ is contained in the free algebra $\overline{F}_\Sigma(X_t + Z)$, we can compose e_t with the inclusion map and obtain a function

$$\bar{e}_t : X_t \longrightarrow \overline{F}_\Sigma(X_t + Z).$$

Observe that the image of $\bar{e}_t$ is disjoint from the inclusion $X_t \hookrightarrow \overline{F}_\Sigma(X_t + Z)$ of single variables from X_t to the free algebra, thus, $\bar{e}_t$ is guarded in the sense of [AMV$_2$]. The algebra $\overline{R}_\Sigma Z$ of all rational Σ-trees on Z is iterative, thus, by [AMV$_2$] there exists a unique solution $\bar{e}_t^\dagger \colon X_t \longrightarrow \overline{R}_\Sigma Z$. In other words, a unique morphism such that the triangle

$$\begin{array}{ccc} X_t & \xrightarrow{\ \bar{e}_t^\dagger\ } & \overline{R}_\Sigma Z \\ {\scriptstyle \bar{e}_t}\downarrow & \nearrow{\scriptstyle [\bar{e}_t^\dagger, \bar{\eta}_Z]^\sharp} & \\ \overline{F}_\Sigma(X_t + Z) & & \end{array}$$

commutes. In our case, the solution is simply the inclusion map:

$$\bar{e}_t^\dagger(s) = s \qquad \text{for all subtrees } s \text{ of } t.$$

To prove this, observe that for the inclusion the above triangle commutes (since $[\bar{e}^\dagger, \bar{\eta}_Z]$ is also an inclusion, thus, $[\bar{e}^\dagger, \bar{\eta}_Z]^\sharp$ just substitutes variables of X_t with the Σ-trees they represent).

Proposition 4.6. *Let Σ be a parametrized signature. For every set Z the algebra $R_\Sigma Z$ of all rational Σ-wellfounded trees (w.r.t. tree tupling) is a free iterative Σ-algebra on Z.*

Proof. (i) It is clear that a tree obtained by tupling of rational, Σ-wellfounded trees is itself rational and Σ-wellfounded.

(ii) $R_\Sigma Z$ is an iterative algebra. In fact, for every finitary equation morphism

$$e \colon X \longrightarrow X \,\Box_\Sigma\, (R_\Sigma Z)$$

we have a unique solution $e^\dagger$ in the algebra $\overline{R}_\Sigma Z$ of rational Σ-trees on Z. It remains to check that each tree $e^\dagger(x)$ is Σ-wellfounded. Recall the definition of $X \Box_\Sigma (R_\Sigma Z)$ from Example 2.17: given a tree $t = e(x)$ in $X \Box_\Sigma (R_\Sigma Z)$, then the leaves labelled by elements of X do not lie on any path with property (4.1) from the root of t. Thus, such paths of the trees $t = e(x)$ and $t' = e^\dagger(x)$ (obtained from t by recursively substituting any leaf labelled in X by the corresponding solution) are the same. Since t is obviously Σ-wellfounded so is t'.

(iii) $R_\Sigma Z$ is a free iterative Σ-algebra w.r.t.

$$\eta_Z \colon Z \longrightarrow R_\Sigma Z$$

assigning to every $z \in Z$ the single-node tree labelled by z. In fact, let A be an iterative algebra and $f \colon Z \longrightarrow A$ a function. We define a function $h\colon R_\Sigma Z \longrightarrow A$ as follows: for every tree t in $R_\Sigma Z$ containing at least one symbol from Σ form the equation morphism

$$f \bullet e_t \colon X_t \longrightarrow X_t \,\square\, A$$

(see Remark 3.10 and Observation 4.5) and put

$$h(t) = \big(f \bullet e_t\big)^{\dagger}(t),$$

and if t is a single node tree labelled by $z \in Z$ put

$$h(t) = f(z).$$

It is not difficult to verify that h is a homomorphism extending f. The uniqueness follows from Remark 3.10: if h is a homomorphism extending t, then it preserves the solution of $\eta_Z \bullet e_t \colon X_t \longrightarrow X_t \,\square\, R_\Sigma Z$ which, by an argument analogous to Observation 4.5, assigns t to t, thus, $h(t) = (f \bullet e_t)^{\dagger}(t)$. $\square$

5 Flattening of Equations

In the nonparametric world, where an endofunctor H is given, equation morphisms can be flattened, see, e.g., Example 3.1(i). In general, we can either consider the free-H-algebra monad $\overline{F}$, which is just the free monad on H (as proved by Michael Barr [B]) and for which we have a natural isomorphism $\overline{F} \cong H\overline{F} + \mathrm{Id}$. Then the general equation morphisms are the morphisms

$$e \colon X \longrightarrow \overline{F}(X + A)$$

where $\overline{F}Z$ denotes the free H-algebra on Z. Or we just consider *flat* equation morphisms

$$e \colon X \longrightarrow HX + A.$$

In the first case e is called *guarded* if it factors through the coproduct injection

$$H\overline{F}(X + A) + A \hookrightarrow \overline{F}(X + A) = H\overline{F}(X + A) + (X + A).$$

We proved in [AMV_2] that an algebra is iterative (i.e., every flat equation morphism has a unique solution) if and only if every guarded equation morphism has a unique solution. Do we have an analogous result in the parametrized situation? The answer, as we demonstrate here on the simplest case of $H(X, A) = X \times A$, is negative. Although the formulation of the "flat iterativity" is straightforward, it is *not* true that it implies iterativity, in general. For this reason we do not use the "flat iterativity" beyond the present section.

We first make an observation about iterative algebras (A, α) of an arbitrary parametrized endofunctor H: they are "flat iterative", i.e., every *finitary flat* *equation morphism*

$$e\colon X \longrightarrow H(X,A)+A$$

has a unique *solution*, i.e., a unique morphism $e^\dagger\colon X \longrightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\Big\downarrow & & \Big\uparrow{\scriptstyle [\alpha,A]} \\ H(X,A)+A & \xrightarrow[H(e^\dagger,A)+A]{} & H(A,A)+A \end{array}$$

commutes. In fact, denote

$$\kappa_A^X \equiv H(X,A) \xrightarrow{H(X,u_A^X)} H(X, X \,\square\, A) \xrightarrow{f_A^X} X \,\square\, A$$

(see Notation 2.9) and form the corresponding finitary equation morphism

$$\bar{e} \equiv X \xrightarrow{\ e\ } H(X,A)+A \xrightarrow{[\kappa_A^X, u_A^X]} X \,\square\, A.$$

We prove that a morphism is a solution of $\bar{e}$ if and only if it is a solution of e—then $e^\dagger$ is unique. We first observe that the diagram

$$\begin{array}{ccccc} H(A,A) & & \xrightarrow{\quad \kappa_A^A \quad} & & A \,\square\, A \\ \Big\| & {\scriptstyle H(A,u_A^A)}\searrow & & \nearrow{\scriptstyle f_A^A} & \Big\downarrow{\scriptstyle \widehat{\alpha}} \\ \Big\| & & H(A, A \,\square\, A) & & \Big\downarrow \\ \Big\| & \swarrow{\scriptstyle H(A,\widehat{\alpha})} & & & \Big\downarrow \\ H(A,A) & & \xrightarrow[\quad \alpha \quad]{} & & A \end{array}$$

commutes because $\widehat{\alpha}$ is a homomorphism of $H(A,-)$-algebras extending the identity. Consequently, $\alpha = \widehat{\alpha}\cdot\kappa_A^A$ which implies

$$[\alpha, A] = \widehat{\alpha}\cdot[\kappa_A^A, u_A^A].$$

Let $\bar{e}^\dagger : X \longrightarrow A$ be the unique solution of $\bar{e}$, then the following diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\qquad \bar{e}^\dagger \qquad} & & A \\ {\scriptstyle e}\Big\downarrow & & & \nearrow{\scriptstyle [\alpha,A]} & \Big\uparrow \\ H(X,A)+A & \xrightarrow{H(\bar{e}^\dagger,A)+A} & H(A,A)+A & & \Big\uparrow{\scriptstyle \widehat{\alpha}} \\ {\scriptstyle [\kappa_A^X,u_A^X]}\Big\downarrow & & & \searrow{\scriptstyle [\kappa_A^A,u_A^A]} & \Big\uparrow \\ X \,\square\, A & & \xrightarrow[\qquad \bar{e}^\dagger \square A \qquad]{} & & A \,\square\, A \end{array}$$

commutes, which proves that $\bar{e}^\dagger$ is a solution of e. Conversely, if in the above diagram the upper part commutes, i.e., if $\bar{e}^\dagger$ is a solution of e, then the whole diagram commutes, thus, $\bar{e}^\dagger$ is also a solution of e.

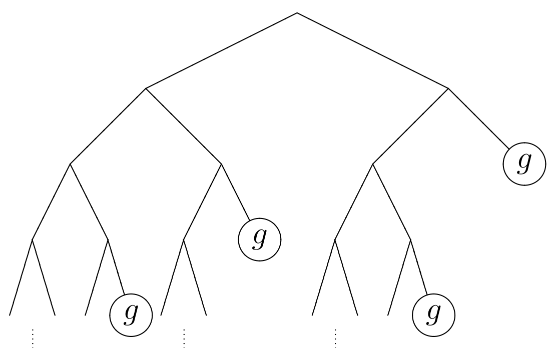
Example 5.1. We give an example of a "flat-iterative" algebra for $H(X, A) = X \times A$ that is not iterative. Recall that a free iterative algebra $R1$ on one generator g is the algebra of all binary, rational, right-wellfounded trees. The following subalgebra

$$B \subseteq R1$$

is not iterative although it is flat-iterative: B consists of all trees t in $R1$ for which the number of right-hand edges in all paths is bounded. More precisely: there is a natural number $\ell(t)$ such that every path of t has at most $\ell(t)$ right-hand edges. It is obvious that B is a subalgebra containing all finite trees, and not containing all trees in $R1$. An example of a tree in $R1 \setminus B$ is the unique solution of

$$x \approx (xx, g),$$

see Example 2.12(ii). This is the following tree t:



To see that t does not lie in B observe the right-most node n labelled by g in the picture above. This node is reached by a path with two right-hand edges. Notice that the subtree of t rooted at the left-hand sibling of node n is isomorphic to the tree t. Thus, the corresponding path with two right-hand edges in the subtree yields a path with three right-hand edges in t. This path in the subtree yields a path with four right-hand edges in t, etc. Hence, there can be no bound $\ell(t)$.

Suppose that B is an iterative algebra. Then, since $g \in B$, we have the above equation morphism

$$e : \{x\} \longrightarrow \{x\} \,\square\, B, \quad x \mapsto (xx, g)$$

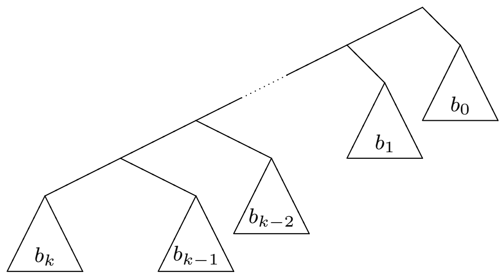
Its solution will be, by Lemma 3.11, the same tree, t, as the corresponding solution in $R1$. This contradicts $t \notin B$.

However, B is flat-iterative. In fact, let $e\colon X \longrightarrow X \times B + B$ be a finitary flat equation morphism. Since $B \subseteq R1$ we have a unique solution $e^\dagger\colon X \longrightarrow R1$, and it is sufficient to prove that each of the trees $e^\dagger(x)$ for $x \in X$ lies in B. We consider two types of variables $x \in X$:

(1) Suppose that x "reaches" B, i.e., there exist $x = x_0, x_1, \ldots, x_k$ in X and b_0, b_1,..., b_k in B such that

$$e(x_i) = (x_{i+1}, b_i) \qquad \text{for } i < k \qquad \text{and} \qquad e^\dagger(x_k) = b_k.$$

The solution $t = e^\dagger(x)$ is then the following tree

$t =$ 

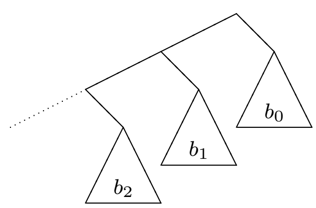
which lies in B: we can put

$$\ell(t) = \max\Big\{\ell(b_0) + 1, \ell(b_1) + 1, \ldots, \ell(b_{k-1}) + 1, \ell(b_k)\Big\}.$$

(2) Suppose that x does not reach B, i.e., there exists an infinite sequence of variables $x = x_0, x_1, x_2, \ldots$ in X and trees $b_0, b_1, b_2, \ldots$ in B with

$$e(x_i) = (x_{i+1}, b_i) \qquad i = 0, 1, 2, \ldots$$

The solution is the following tree $t = e^\dagger(x)$:



Since X is a finite set, there exist $i < j$ with $x_i = x_j$, and then the trees $b_0, b_1, \ldots, b_{j-1}$ are all the subtrees one finds in t when moving along the left-most path. Consequently, we can put

$$\ell(t) = \max\Big\{\ell(b_0) + 1, \ldots, \ell(b_{j-1}) + 1\Big\}.$$

Remark 5.2. The above example explains that the concept of "flat iterativity" is not useful in the parametrized world. It can also be argued that this is simply a wrong approach: in case we view $X \,\Box\, A$ as the branching structure of the trees under investigation, then there is no urge to flatten an equation morphism of the form $e : X \longrightarrow X \,\Box\, A$. This is particularly obvious if a non-parametrized endofunctor H should be viewed as the parametrized one via $(X, A) \mapsto HX$ in which case $X \,\Box\, A = HX + A$.

6 Summary and Future Research

In the present paper algebras for parametrized endofunctors $H\colon \mathsf{Set}\times\mathsf{Set} \longrightarrow \mathsf{Set}$ are studied having the property that certain recursive finitary equations have unique solutions. In order to formalize the concept of these equations and their solutions, a base, i.e., a parametrized endofunctor $\Box\colon \mathsf{Set}\times\mathsf{Set} \longrightarrow \mathsf{Set}$ that is an uncurried version of a finitary functor from Set to $\mathsf{FM}(\mathsf{Set})$, is introduced. We proved that every object generates a free algebra with the above parametrized iterativity, and we described the corresponding monad R on Set, called the rational monad of H.

In future research, announced in [AMV$_3$], we will study algebras for an arbitrary base $\Box$ (not necessarily associated with a parametrized endofunctor) on an arbitrary locally finitely presentable category (see [AR]) and we will describe the rational monad R of that base. Then we will prove, analogously to the non-parametrized case [AMV$_2$], that every iterative algebra has unique solutions of much more general recursive equations: the guarded rational equations, where the right-hand sides are taken from the rational monad. Also the main result of [AMV$_2$] that the rational monad of an endofunctor H is the free iterative theory on H in the sense of Calvin Elgot [E], will be generalized to bases. But this needs introducing the concept of a module and of an iterative module for a general base.

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